Teaching Geometry With Problems: Negotiating Instructional Situations and Mathematical Tasks

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Two questions are asked that concern the work of teaching high school geometry with problems and engaging students in building a reasoned conjecture: What kinds of negotiation are needed in order to engage students in such activity? How do those negotiations impact the mathematical activity in which students participate? A teacher’s work is analyzed in two classes with an area problem designed to bring about and prove a conjecture about the relationship between the medians and area of a triangle. The article stresses that to understand the conditions of possibility to teach geometry with problems, questions of epistemological and instructional nature need to be asked—not only whether and how certain ideas can be conceived by students as they work on a problem but also whether and how the kind of activity that will allow such conception can be summoned by customary ways of transacting work for knowledge.

Key words: Classroom interaction; Geometry; High school, 9–12; Instructional intervention; Planning, decision-making; Proof in geometry; Reasoning; Teaching practice

To teach mathematics, a teacher must enable and put to work a relationship between students and mathematical ideas; in so doing, problems and questions can be useful instruments to this end. Indeed, the notion that students’ inquiry into problems can motivate the development of mathematical knowledge has long been an element of the conventional wisdom of mathematics teaching. The notion of “teaching with problems” has helped envision contexts for authentic mathematical activity and spurred inquiry on the work of the teacher (Lampert, 1990, 2001). Indeed, teaching mathematics with problems is complex but not just because it requires a teacher to have knowledge and skills instrumental to envision and foster students’ opportunities to learn. It is complex because collective work on those problems occurs in a context (the mathematics classroom) that imposes often unspoken

The collection of the data reported in this article was supported by a seed grant from the Office of the Vice-President for Research at the University of Michigan. The analysis and writing were partially supported by the National Science Foundation (NSF) CAREER Grant REC 0133619. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation. The author acknowledges valuable comments by Magdalene Lampert, Vilma Mesa, Talli Nachlieli, and anonymous reviewers.
but no less real conditions and constraints on the mathematical activity that teachers may enable.

I examine one aspect of the work of teaching mathematics with problems in American high school geometry classrooms: the use of a problem to engage students in building a reasoned conjecture. By “reasoned conjecture” I mean a statement whose plausibility hinges on rational argument and is not contingent on perception or experience (see Reid, 2002). The work of building reasoned conjectures is arguably central to one of the primary purposes of high school geometry: to afford students a chance to transition to a theoretical way of knowing mathematics. In this class, if anywhere, American high school students have the chance to experience the power of finding out what could or should be true by exercising mathematical reasoning. Furthermore, problems that involve students in finding the conceptual grounds on which something could be true are good vehicles to grasp the meaning of abstract mathematical concepts (and thus differentiate them from concrete percepts; see Tall, Thomas, Davis, Gray, and Simpson, 2000).

To make a problem do that job, a teacher must establish conditions for students’ intellectual engagement as well as conditions under which the teacher and the students can value that engagement. To establish those working conditions, the teacher has to engage in negotiations not just of the mathematical meanings involved but also of the instructional context that frames the possibility to make those meanings. I illustrate this claim through the analysis of a case in which a teacher uses students’ work in a problem about area of triangles to bring to the surface a connection between medians and area and to introduce some of the basic postulates of area. The case shows the work of a teacher with a problem originally designed to involve students in building a reasoned conjecture. I propose a way to think about the demands of teaching with this problem and how we can explain the transformations observed.

THEORETICAL FRAMEWORK

There is a growing interest in examining the work of the mathematics teacher in classroom instruction. Because of increased demands on the mathematical quality of the opportunities to learn that teachers create in their classrooms (e.g., NCTM, 2000), research is needed to understand the conditions in which teachers work and how those conditions impact the mathematical work that teachers can sustain. To understand the actions of teachers that shape the mathematical ideas at play in classroom interaction, I consider a mathematics classroom as both an institutional context for the procurement of cultural finalities regarding the communication of disciplinary knowledge and a microsocial context for joint mathematical activity where meanings emerge and are negotiated. The school institution operates on the assumption that the mathematics class is the place where its subjects go to transact (or to trade on) mathematical knowledge. As a space where knowledge is traded, the classroom is a symbolic economy (Bourdieu, 1998). However, the classroom is different from other economies in that the “currency” that students use to lay claim
to what they acquire is academic work—mathematical work, weaved into action in context, with others, and over time. The work of the teacher includes creating conditions for students to do mathematical work while bearing witness of the trade value of that work vis-à-vis the knowledge at stake.

A central element of this theoretical perspective is the postulate that the relationship between teacher and students with regard to the study of mathematics is regulated by a contract that exists between them—the didactical contract (Brousseau, 1997; Herbst, 2002a, 2003). The assumption that a didactical contract exists, establishing global responsibilities among teacher, student, and subject of study, helps one understand why teacher and students act as if they have permission but are also under the obligation to work together, dividing labor and trading on the subject matter. This postulate invites inquiry into the negotiation of the terms of this contract between participants.

Task and Situation

To study the teacher’s actions managing students’ engagement in problems, I introduce a conceptual distinction between problem, task, and situation (a distinction that builds on work by Brousseau, 1997; Doyle, 1988; Goffman, 1964/1997; and Lampert, 1990). For Brousseau (1997), a mathematical problem is a question whose answer hinges on bringing to bear a mathematical theory within which a concept, formula, or method involved in answering the question is warranted. A problem thus constitutes a representation or an embodiment of a piece of knowledge, in that it points to some of the meaning of that piece of knowledge. Brousseau suggested that this notion of problem still leaves the cognizing agent (i.e., the student and the intellectual context of his or her actions dealing with the problem) out of the picture. To understand how students might have the chance to learn from working on problems requires coming to grips not just with how problems embody knowledge but also with how students’ knowing of that knowledge may spring from the interaction between a cognizing agent and the problem. Building on Doyle’s (1988) work on tasks (see also Herbst, 2003; Stein, Grover, & Henningsen, 1996), I use task to refer to the specific units of meaning (i.e., the actions and interactions with the symbolic environment) that constitute the intellectual context in which individuals think about the mathematical ideas at stake in a problem. A task is the universe of possible operations that an individual might or might not take while working on a problem, toward a certain product, with certain resources, the feedback that the problem can provide on those operations, and the operations adapted to the feedback that may ensue (Doyle, 1988).1 Students’ engagement with a particular problem might precipitate the task of building a reasoned conjecture. But choosing a problem that can potentially engage students in building a reasoned

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1 Certain tasks where the problem provides feedback that leads to the adaptation of the subject’s knowings are called “adidactical situations” (Brousseau, 1997), but not all tasks are of this kind. The word task is thus a more general way of referring to an enacted problem as an embodiment of mathematics and can help describe units of mathematical work in intact as well as experimental settings.
conjecture is no guarantee that such a task will be enacted: Intellectual demands in
tasks like these have a tendency to decline (Stein et al., 1996). Therefore, it is impor-
tant to understand what is at play in a teacher’s capacity to sustain the intellectual
demands of the task of building a reasoned conjecture.

To understand how problems are and can be used in classroom teaching requires
more than foreseeing and observing the tasks in the context of which children interact
with those problems. If problems are to be used to fulfill the didactical contract, we
need theoretical tools to understand how the enactment of problems by students may
serve an instructional purpose and be identified as such in the interactions between
students and their teacher. Students’ work has to be accountable: There must be a
way for participants to establish how the doing of it helps fulfill the didactical
contract (i.e., its trade value). The labels (Mason, 1999) used by teachers and
students to describe what they do (e.g., “discovery lesson,” “giving notes”) point
to how students’ work is traded for claims on the fulfillment of the contract. Conceivably the same problem might prompt different tasks depending on how
classroom work on the problem is framed.

In his critique of how researchers had mishandled the relationships between talk
and social structure, Erving Goffman (1964/1997) spoke of how they had “neglected
[the] situation” (p. 229). For Goffman, situations are “environments of mutual moni-
toring possibilities” entailing “constraint and organization” and “structuring of
conduct,” where two or more people “jointly ratify one another as co-sustainers of
a single, albeit moving, focus of visual and cognitive attention” (p. 231). The situ-
ation construct helped Goffman make the point that talk is neither just an emergent
of the encounter of individuals in the moment nor just an application of macrosoc-
ial variables such as class or gender. For the researcher on mathematics teaching
in classrooms, Goffman’s point suggests an analogy between talk and mathe-
matical work and points to the need to understand mathematical work in classrooms
as framed by situations that structure what participants implicitly expect from each
other when they find themselves in those situations. I use instructional situation to
refer to any one of the customary ways in which classroom actions are framed into
units of work so as to be traded in for (or accounted to) claims over the knowledge
at stake (and, reciprocally, any one of the customary ways in which the teaching
or learning of objects of knowledge is deployed as classroom work).

Situation (short for instructional situation) identifies each of the frames that partic-i-
pants use to know who has to do what and when, so that whatever they do can be
used to claim the fulfillment of their contractual obligations. Participation in
instructional situations often involves the enactment of implicit roles, norms, and
scripts that shape the mathematical task in which students are involved (Stigler &
Hiebert, 1999; Voigt, 1985). Therein lies a key to identifying the difficulties that

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2 This use of situation is more general than the way Brousseau (1997) talks about didactical situa-
tions because I aim it as a term that can describe knowledge transactions in a wide range of mathematics
classrooms (including intact classrooms). Yet this use of situation can be shown to include Brousseau’s
examples of didactical situations of devolution and institutionalization as particular cases.
a teacher might have in using problems to engage students in building a reasoned conjecture: Situations are like marketplaces in the classroom’s symbolic economy (Bourdieu, 1998)—they allow some intellectual work to exist and enable it to trade as mathematics in exchange for a claim on the contract; in doing so, situations may transform that work, even denaturalize it.

The distinction between task and situation helps construct the object of study of this article, which is the work of teaching using a problem, in the following way: The teachers’ actions that make students responsible for a problem can be considered in terms of the management of a task. Those actions enable students to come in contact with specific mathematical ideas. The same actions can be considered in terms of the management of the situation that can make room for the assignment and solution of a problem. Those actions build on customary patterns of interaction over time and accountability about content in order to warrant a place for their existence within classroom life. In particular, to make students responsible for a problem while expecting that they will use it as context to build a reasoned conjecture, a teacher would need to craft a situation that makes it possible and legitimate for students to take the problem on, engage in intellectual work, and be accountable for its proceeds. How would a teacher do that? And how might it impact the mathematical activity thus enabled? The answers to those questions seem essential to understand the viability of teaching geometry with problems.

**Conjecturing and Proving in Geometry**

The process of making students responsible for a problem is a delicate one. The question is not only whether students can enact a task where they take on a problem and come in contact with a new idea but also whether it is possible for a teacher to frame a situation in which students can be held responsible to engage in that task and at the same time for teachers to recognize that engagement as a claim on some of the stakes in the didactical contract. These ideas are particularly helpful in studying the use of problems to teach high school geometry. As a subject of study, geometry is structured not as a sequence of problems and their solutions but as a system of abstract concepts and general properties (Moise, 1975). How might engagement in problems be used to bring about conjectures about and proofs of those properties?

Conjecturing and proving are concurrent in the actual work of mathematicians. Lakatos (1976) demonstrated that the work of justifying a naïve conjecture involves the mathematician in a process of simultaneous formulation and calibration of the proposition that can be asserted, as well as in uncovering and delimiting the meaning of the ideas at play. But in classrooms, conjecturing and proving have often been used as labels for different kinds of mathematical work. Conjecturing and proving have found their way into geometry instruction but often in ways that are more attentive to their essential differences in logical form than to their functional kinship in creating mathematical substance. Students often work inductively to produce conjectures, then later work deductively to produce proofs, and often the
means at their disposal for these activities are different (e.g., Eggleton, 2001; Lapp, 1999). Chazan (1995) has critiqued this traditional separation of situations for conjecturing from situations for proving. He argued that to expect conjectures to emerge inductively from students’ experiences without allowing students access to the reasons that make a conjecture true leaves eventual decisions on the correctness or incorrectness of the conjecture to the authority of the teacher. Students may thus be caught in a bind between reasoning inductively from experience and second-guessing the intentions of the teacher (Christiansen, 1997).

One would think that a way to empower students to conjecture a geometric property is to engage them in problems that summon the ideas that would allow them to formulate and conceive the sources for the reasonableness of the conjecture—that is, the arguments that make it true—not just to perceive that conjecture as a fact (see van Dormolen, 1977). Similarly, Boero and colleagues (Boero, Garuti, Lemut, & Mariotti, 1996; Boero, Garuti, & Mariotti, 1996; Mariotti, Bartolini-Bussi, Boero, Ferri, & Garuti, 1997) have argued for teaching geometry in ways that preserve the cognitive unity of theorems (conjecturing, proving, and theorizing) and have actually shown how teenagers can engage in such mathematical work. Hence, my expression “building a reasoned conjecture” refers to the process of reducing a problem to a system of deductively articulated, more elementary problems whose solutions could suggest not only a conjecture that solves that problem but also why that conjecture is reasonable (see Ball & Bass, 2003; Simon, 1996). This activity would thus produce not only a geometric proposition but also the substance of the proof. But, whereas the foregoing argument may make a case on epistemological grounds for the value of using problems to teach, questions on the instructional viability of such work remain. What management demands impinge upon the teacher’s work promoting students’ building of a reasoned conjecture?

The aforementioned question makes sense within a perspective that recognizes teaching not just as the application of an individual’s will, knowledge, and beliefs but as an activity that belongs to a practice, adapted to how this practice has historically operated, by different actors who nevertheless share a practical rationality (Bourdieu, 1998; Herbst & Chazan, 2003). The task of engaging students in building reasoned conjectures could unfold against the background of two customary situations: “conjecturing” and “doing proofs.” Students occasionally engage in empirical activities that lead them to make conjectures. They also, at times, engage in deductive reasoning. These situations are framed differently.

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3 This argument is not meant to ignore the cognitive distance between proving as producing a rational argument and proving as providing a formal derivation, which has been documented in the work of Duval (1991, 1992) as well as in work related to the van Hiele levels (Fuys, Geddes & Tischler, 1988; Senk, 1989; see also Balacheff, 1987). The notion that proving (as a rational argument) can be a tool for students to know with (in particular, to conjecture) might indeed constitute an epistemological obstacle for students to understand the notion of proof as formal derivation. The argument presented, however, suggests that for a teacher to make possible students’ understanding of abstract concepts and propositions, he or she might need to foster customs of proving as rational argument.
From various studies of “doing proofs” in classrooms (Herbst, 2002b; Herbst & Brach, 2006; Lampert, 1993; Schoenfeld, 1987, 1988; Sekiguchi, 1991) the following thumbnail sketch emerges of what “doing proofs” is like in the prototypical geometry classroom: Students produce proofs that consist of matched statements and reasons written neatly into two parallel columns, in response to prompts from the textbook or the teacher that provide a specified “given” and a truth about a particular configuration; a truth that the student is explicitly required to prove but often not expected to remember or use later. The textbook or the instructor typically provide diagrams that suggest the content of the proof and the student is left with the task of figuring out the correct order of steps, steps that are to be justified by reasons drawn from the collection of theorems and definitions already learned in class. Because of its particular division of labor (who must do what) and organization of interactive turn-taking (when things are done), I have identified “doing proofs” as an example of an instructional situation (see Herbst & Brach, 2006):

“Doing proofs” frames student and teacher exchanges in such a way that some kinds of mathematical work (notably, some enacted tasks) can take place and be given a value in relation to the contractual obligation of teaching students how to do proofs.

**Negotiating the Didactical Contract**

The didactical contract constitutes a classroom as a space for trade of work for knowledge. *Situation* and *task* are two constructs that point to things that matter in how this trade over knowledge takes place. What thinking a problem can elicit (what kind of task) and which frame actors can use to interpret and value such work (what kind of situation) are objects of negotiation: Participants interact as if they were constructing agreed-upon responses to those questions, responses that would let them preserve and fulfill the contract. Negotiation of the contract is thus negotiation over some of the many rules and objects that exist under the aegis of the contract, particularly negotiation of task and situation. This article shows how the teacher’s need to negotiate an instructional situation can explain how the task of building a reasoned conjecture changed.

**METHOD OF INQUIRY**

In this article I make an argument that helps explain epistemological phenomena observed in classrooms from a perspective grounded in the work of teaching. To ground a theoretical argument as well as to argue for its plausibility, I make use of empirical data. The data consist of records of classroom practice (Lampert & Ball, 1998) as well as conversations with practitioners about those records for purposes of triangulation. The argument thus engages the data as experiential grounds to explore an intellectual problem but intends not as much to provide a comprehensive explanation of the data as to create a coherent, general argument that the data help illustrate and probe.
An Instructional Experiment on Area of Plane Figures

I created a context that shares some of the logistical characteristics of “collaborative teaching experiments” (Cobb, 2000). Insofar as the purpose of the work was to study how the need to make instruction possible conditions and constrains the mathematical work that teachers and students do together, I call the brand of experiments I do instructional experiments.

I collaborated with an experienced high school geometry teacher and her student teacher in the design of a replacement unit on area of triangles, inspired by a few principles supported by current reform ideas in mathematics education. I brought to the design team a perspective inspired by the notion that problems should be used as contexts for learning and that conjecturing and proving should be a recurrent feature of students’ mathematical activity (NCTM, 2000). As I observed the teacher working to sustain a viable, working relationship with her students in that learning environment I strived to understand the demands and constraints of using a problem to engage students in the building of a reasoned conjecture. The intervention was thus an instrument in the inquiry on what piecemeal changes to the didactical contract might be needed to make possible the targeted mathematical work: The intervention helped the inquiry by inducing a global perturbation on the customary ways of going about the business of teaching and learning geometry. By looking at how the instructional system would maintain (or achieve) stability, I would have the opportunity to see which components of the contract are called to play in using problems to teach.

The site selection and the design in which I engaged were instrumental to address the research questions. I worked in “traditional” high school geometry classes that followed a standard, commercial textbook, and I engaged in the design of a replacement unit (rather than a yearlong or 1-day intervention). Those circumstances were expected to enhance the chances for the unit to act as a perturbation on the system of customary practices of geometry instruction. The unit was thus not meant to change the clauses of the didactical contract as much as create a context for observing what those clauses were that mattered in engaging students in building a reasoned conjecture.

Setting

I worked collaboratively with Megan, an experienced teacher, and Norah, her student teacher. Megan was teaching three regular geometry classes of mostly 10th graders in a very large and diverse comprehensive high school in a midsized city in the American Midwest. The regular geometry class was one among three versions of the geometry course being offered, reportedly not as fast paced or covering as many difficult problems as in the “accelerated” course with which it shared the same book (Boyd, Burrill, Cummins, Kanold, & Malloy, 1998). A third version of the geometry class was described to me as “geometry without proofs” and used the textbook by Hoffer et al. (1998).

4 All names are pseudonyms.
Megan was particularly fond of teaching geometry and had an extensive background in mathematics and education, powered by an interest in connecting students’ mathematical experiences to other aspects of life (such as everyday arguments or work ethic). Her students had been exposed to working in groups, although many of her lessons, especially when she would introduce new material, would be carried out in a whole-class format. She was a strong presence in the room and led a rich and logical development of the subject, constantly seeking student input in the form of focused questions, oriented toward making connections and anticipating problems. The class climate was friendly, but Megan and her students worked very hard and it appeared that she was trusted as an intellectual and moral leader. As for Norah, she was an inquisitive and reflective mathematics major enrolled in an undergraduate teacher education program and had already been placed with Megan for a semester when we started collaborating.

I had approached Megan with the idea of doing a study on the role that proof could play in the study of area. I shared with Megan that my interest was to understand what is involved in the work of a teacher engaging students in conjecturing and proving. Her students had had frequent experiences doing “two-column” proofs. They also had experienced making conjectures about figures. I shared with her my interest in inspecting conjecturing and proving in the domain of area because I suspected that, unlike in other geometric domains, visual inspection of diagrams would not be helpful for students to find out what is true and that such limitation might create a context for students to answer questions using proof. Megan was not used to having students make conjectures or proofs in the study of area but was willing to try new ideas and interested in having her student teacher and students have a unique experience.

*The Instructional Unit*

We designed a 12-day replacement unit structured as a series of problems that would give students the opportunity to develop key ideas about area of plane figures. We expected that the process of solving the problems would help create opportunities for students to mathematize, conjecture, and prove. We also intended to use the activity of proving a conjecture to develop students’ awareness of the basic properties of the area function (see Moise, 1963). Our discussions over the design of the unit started from the assumption that most students already knew that the formula for the area of a triangle is half the base times the height, which we took as a given on which to build new knowledge. The problems we chose would have students use that knowledge in making connections between areas of elementary figures. In addition, many of the problems were built on the expectation that students would work in groups. We planned together how to use those problems to bring to the fore the various targeted ideas, and the problems themselves were adapted as we revised future lessons day after day on the basis of what had been done. The problems were also discussed and then reworded. We developed scaf-
folds together; identified moments when students might need minilectures; and agreed to switch the order of, or discard, some problems.

Sources of Data

This article focuses on the Triangle Problem, which was used at the very beginning of the unit in the foreground to emphasize the importance of the area-addition postulate as well as to apply connections between medians and area:

Say you have a triangle $ABC$, how can you find a point $O$ inside $ABC$ so that the three new triangles $AOB$, $BOC$, and $AOC$ have the same area?

I examine Megan’s work in the two class periods in which she was leading instruction (2nd and 3rd period); both teachers were present and participated in these classes each day (see also Herbst, 2005).

Megan’s work on the Triangle Problem constitutes appropriate grounds for the argument made in the article for two reasons. First, the Triangle Problem had been chosen because it could help engage students in building a reasoned conjecture—that the point sought is the intersection of medians. In the contrast between what was expected of these lessons and what actually happened, one gets an important context to think about the work of building reasoned conjectures. Second, because the problem was used at the beginning of the unit, it constituted a perturbation to the didactical contract in the sense noted previously and provided a view on a teacher’s accommodation to such a perturbation. Because these lessons differed across the two classes, especially in how the teacher managed making and proving a conjecture, and because these lessons had taken place at the beginning of the unit, the differences were grounds to explore the teacher’s search for ways to handle the novel kind of activities planned in the unit. Records used for this study include notes from design and planning meetings, videotaped lessons for the Triangle Problem, and individual and group interviews with Megan and Norah throughout the following summer.

Data Analysis

The data were analyzed in three stages. The first consisted of an analysis a priori (Comiti, Grenier, & Margolinas, 1995; Herbst, 2002a, 2003) of the task that could develop around the problem chosen. This stage of analysis was aimed at anticipating moments that might require negotiation of the task; these moments were later used to observe actual instruction. The second stage consisted of using those moments in the devolution of the problem to inspect the recorded interactions between teacher and students. Action was divided into episodes that corresponded to moments in the devolution of the problem identified in the analysis a priori. Then each of those episodes in each class was parsed into smaller segments identifying self-contained interactions between the teacher and (all or some of) her class. Changes in conversational partners when moving from group to group or in the nature of the task when addressing the whole class were used to mark endpoints
for segments. Each of those segments, lasting anywhere from 30 seconds to 6 minutes, was described and compared with what had been expected in the a priori analysis.

Within each of those segments, actions and discourse were searched for traces of the negotiation of the didactical contract. One of the ways in which negotiation was identified was by looking at participants’ talk in reference to the normality or abnormality of what they were doing: if participants talked in ways that pointed to the oddness of the task or the situation or if they engaged in repairing the task or the situation to make them more similar to what was customary for them (Mehan & Wood, 1975). That inspection permitted fleshing out the anticipations produced by the first analysis by describing how each of the segments contributed to the actual work in each class.

The third stage of analysis consisted of comparing the work across classes. The analysis proceeded by contrasting how the teacher’s actions changed across classes for given episodes. Purposeful changes in how Megan structured students’ work were taken as warrant for the existence of tensions she had to cope with in managing the work. In spelling out those tensions and showing how they operated, the purpose was to propose a plausible rationale for why it would have been sensible for a teacher to act in the ways observed.

**ANALYSIS: BUILDING A REASONED CONJECTURE AND THE TRIANGLE PROBLEM**

The discussion of this case is articulated into two parts. First, I discuss the planned lessons. I specify ideas of area that were expected to surface and identify moments when I anticipated specific moments when the didactical contract would need negotiation. Later, I use those moments to describe the work done in Megan’s classes around the triangle problem.

*The Triangle Problem as it Had Been Conceived and Planned*

Building on what students knew about areas of triangles (the dimensions, base and height, and the formula), the first lesson of the unit challenged students to anticipate relationships between areas based on the geometric features of figures. One problem asked them to draw a line through the vertex of a triangle to divide the triangle into two triangles whose areas were in a given part-to-part ratio. For the 1:1 ratio, students had recognized that the line was what they knew as a *median* of a triangle.⁵ For the following 2 days, students worked on the Triangle Problem. The problem was an opportunity for students to use knowledge of a construction to make and justify area claims. It was expected that as they worked on solving this problem and justifying its solution, it would become meaningful for students that the properties of equality of a quantity apply to area (if one adds or subtracts equal areas,

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⁵ All these special segments in a triangle had been studied in the 1st semester. Whereas points of intersection had been constructed and named, none of the intersections had been proved unique.
or multiplies equal areas by a scalar, one gets equal areas). Figure 1 shows an argument for one way the point sought could be found.6

Various tasks could be enacted as students worked on the Triangle Problem. Students would have to interpret the problem and anticipate what a solution might be like, then they would actually have to produce a solution to the problem. The problem could be interpreted as one of (1) finding a solution for a particular

[Diagram: Draw median \( CM \). By definition of a median, \( MB \) and \( MA \) are congruent. Since \( AMC \) and \( BMC \) also share the same altitude with respect to vertex \( C \), triangles \( AMC \) and \( BMC \) have equal areas. For any point \( O \) on \( CM \), and for the same reason as above, \( AMO \) and \( BMO \) would also have the same area. Thus subtracting \( AMO \) from \( AMC \) and \( BMO \) from \( BMC \) gives that triangles \( AOC \) and \( BOC \) have equal areas. Draw now median \( BP \) and restrict \( O \) to be the point of intersection of \( BP \) and \( CM \); for that \( O \) it still holds that \( AOC \) and \( BOC \) have equal areas. And also the same argument can be made for the median \( BP \) and thus obtain that \( BOC \) and \( BOA \) have equal areas as well. Hence, if point \( O \) is defined as the intersection of two medians \( CM \) and \( BP \), all three triangles \( BOC \), \( BOA \), and \( AOC \) have equal areas. Further, the argument is independent of which two medians are chosen to define \( O \). Suppose therefore that two pairs of medians (e.g., \( CM \) and \( BP \), \( CM \) and \( AN \)) determined two different intersection points, \( O \) and \( O' \). In that case \( O \) and \( O' \) would split the triangle into three equal areas and one would have that all six triangles \( AOC \), \( BOC \), \( AOB \), \( AO'C \), \( BO'C \), and \( AO'B \) would be of equal area. As \( O' \) is interior to \( ABC \) it could either coincide with \( O \) or not. If not, it would belong to the interior or a side of at least one of the triangles determined by \( O \) (\( AOC \), \( BOC \), \( AOB \)). Assume \( O' \) belonged to \( AOB \) but was not coincident with \( O \), then triangle \( AO'B \) would be strictly included in triangle \( AOB \), which contradicts the assertion their areas are equal. Thus \( O' \) and \( O \) must coincide and the point of intersection of the medians is unique. A similar argument would show that any other point \( O' \) that splits the triangle into three equal areas would have to coincide with point \( O \) determined as the intersection of two medians.

Figure 1. A complete mathematical solution for the Triangle Problem.

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6 Other solutions are possible that do not draw on the idea of splitting a triangle by a given ratio. The one proposed here was one that we believed students could build by themselves given what they knew at the time.
triangle, (2) showing how a solution would be found for any triangle, or (3) treating the universe of all triangles as cases, and solving the problem for a representative of each case. The way the solution would be produced would depend on how the problem was interpreted. In the first case, students could decide where to plot the point based on perception or else plot an actual point $O$, measure bases and heights of three triangles, compare their actual areas, and choose another point if they were not equal. In the second case, students could select from known constructions of points inside a triangle one that appeared to work for one or more seemingly generic triangles, possibly verifying the procedure using measurement or perception. They could propose a known construction and prove that it worked. They could also select and adapt known constructions to subdivide a triangle in regions whose area relationships can be tracked and compared according to the construction performed. Finally, if the problem had been interpreted as one of finding solutions specific to triangle types (equilateral, acute, etc.), one could envision students prescribing various constructions (say, the orthocenter for equilateral triangles). They could arrive at or justify some of these deductively but resort to other solution methods for other kinds of triangles. These basic categories are useful when describing the tasks actually enacted.

In discussing the Triangle Problem in our planning meetings, Megan seemed to be aware that ideally we would want to get students to adapt known constructions to subdivide a triangle in regions whose area relationships could be tracked and compared according to the construction performed, possibly using the triangle area formula in an algebraic way. And to do so she had to be very careful when assigning the problem. In our discussions, we had devised scaffolds that Megan could use to move students along the work on the problem. Each of four scaffolds consisted of a more restricted version of the problem. The first two scaffolds were aimed at steering students to interpret the problem as one of specifying how to find a point in any triangle and away from one of finding an actual point for a particular triangle. In the first, Megan would ask students to sketch a triangle and what the solution would look like for that triangle. To move away from the particulars of one triangle, Megan would ask students to give instructions for somebody to plot the point sought inside a triangle that one could not see. The other two scaffolds were resources that Megan could use to help students connect known constructions with area claims. The third would ask students to examine whether the midpoint of a line joining two side midpoints could be the point sought, which might lead students to think about what a solution could look like. The fourth scaffold would ask students to find a point that, when united to the vertices, splits a triangle into three triangles, two of which are of equal area—a simpler version of the problem that students could then modify to solve the more general triangle problem. We expected that these scaffolds might help Megan use the problem to have students build a reasoned conjecture connecting area to medians.
How the Triangle Problem Perturbed What Was Customary

The expectation that the Triangle Problem be used to engage students in making a reasoned conjecture constituted a perturbation to the (unspoken) didactical contract between Megan and her students. Whereas they had previously worked on problems, the Triangle Problem was a foreseeable perturbation on the situations of students solving problems: It was unusual for Megan to assign problems whose solution method students had to figure out. Megan and Norah noticed that the Triangle Problem did not just aim at engaging students in a novel task but that students might be miscued by it, thinking the task to be “familiar” (Doyle, 1988): Anywhere one picked a point, one could construct triangles and find their areas. Thus, the problem could be seen as an invitation to get busy and practice the area formula. The real task, hidden under the given problem, was not to find but to problematize the solution of that familiar task, realizing that pursuing a completely different, general solution might be interesting and fruitful. Yet, the very elements of knowledge that students would need to invest in order to appreciate a general solution (i.e., a theoretical notion of area-equivalence) did not pre-exist the problem. They were to be developed as the problem was solved.

The Triangle Problem was also a foreseeable perturbation of the didactical contract insofar as doing it might be a way of developing new knowledge (i.e., the postulates of area). In the past, students had been introduced to postulates and basic theorems first and then asked to solve problems or prove explicitly stated claims where they only had to use postulates and theorems they already knew. Furthermore, the introduction of new material had usually involved the teacher in a much more active delivery role. The expectation that Megan use the problem to bring about new knowledge hardly seemed to fit with other instances in which she had introduced new knowledge.

Finally, as observed earlier, the Triangle Problem lesson was a foreseeable perturbation on the didactical contract in regard to conjecturing and proving. Megan’s students had previously been involved in “doing proofs” in “two columns.” In these proving situations, they had always been told what they needed to prove and had been given diagrams with the auxiliary lines needed to build the argument. Students had had occasional lessons where they were prompted to inspect figures and make conjectures, and some of those conjectures had been identified as theorems to be proved. But it had not been customary for a proving situation to unfold seamlessly from students’ development of a conjecture, nor had it been customary for students to build reasoned conjectures as opposed to making guesses based on measurement and perception.

Thus, it was predictable that the devolution of the Triangle Problem would require negotiation of the didactical contract. The theoretical framework proposed herein suggests that to make room for the task of students’ building of a reasoned conjecture, Megan would have to make that task part of a situation. Students would engage in that task if doing that was part of the work they could exchange for a claim laid on objects of the contract; they would do that by playing their role and expecting
the teacher to play her role in some customary pattern of interaction. Because the problem was a perturbation to each of several situations that might accommodate that exchange, I anticipated that implicit normative elements of each of those situations would become explicit as participants negotiated and managed their interaction around this problem.

**Working on the Triangle Problem**

Some of the norms of the existing didactical contract that might be perturbed by the Triangle Problem were addressed directly by Megan on the 1st day of work on the unit. She indicated to the 2nd period class that their work was going to be structured differently than was usual:

Pretty much all we’re going to do is a lot of group-work. We are not going to use the book a whole lot, so you don’t even need to bring it. . . . You still need to bring your notebook, 'cause I’m still going to give you some notes, just not as much as normal. The idea is you are going to come up with some stuff within your group, on your own, and then you’ll write it down on your own instead of me putting it on the board all the time.7

The 2nd and 3rd days of work on the area unit were dedicated to the Triangle Problem. Each of the four following subsections shows how Megan did her share in negotiating with students in each class a viable situation for them to work on the problem.

**Assigning the Triangle Problem**

On Day 2, Megan introduced the work for the day by saying to her 2nd hour students that they would be spending “pretty much the whole hour” on one problem. She gave several indications to both classes that the task ahead was not trivial. Yet when working with her 3rd period class, Megan kept a share of the responsibility, telling them that she would stop their work part way through to talk with people so that she could provide direction and some hints. To introduce the triangle problem to the 2nd period class, Megan said,

If I have a triangle [Megan walks up to the board to draw, but suddenly stops, you know, I am not gonna draw it. . . . [Megan faces class again.] If I have a triangle and I wanna divide it. . . . Yesterday we divided the triangle up into two triangles that had equal areas, but what if I wanted to divide the triangle up into three—three triangles that all had equal area? [Megan moves her fingers as if she was drawing in the air.] Where do you think that point would have to be? That’s basically the problem today.8

With both classes Megan left it for students to draw the triangle. But there were differences in how she described the cognitive operations involved in the task. To

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7 Megan introduced the unit similarly in 3rd period, adding that students’ work would include making conjectures about figures, as they had done earlier in the year.

8 Text within brackets includes descriptions of actions that contextualize speech or interpolations of speech. Ellipses without brackets indicate short pauses whereas long pauses are indicated by the number of seconds within parentheses. Excisions from the actual speech are indicated by ellipsis within brackets.
the 2nd period class she explained that the instruction to “propose how to find such point $O$,” meant that they were to say “where would that point be, how would you get it, if you’ve got to tell somebody.” In contrast, the 3rd period students were asked to “guess where [they thought the point] would be, draw it, . . . then share your guessings [sic].” Thus, in introducing the problem, Megan conveyed that the students had to understand a problem about a geometric figure and draw a diagram of that figure. The product of the task made the task sound “familiar,” but the operations they were expected to carry out made the intended task “novel” (Doyle, 1988). Megan tried different ways of relaying that to students.

The plan had stipulated that students would work for a few minutes on the problem, drawing their own diagram and using it to grasp what the requirement of having three triangles of equal area might mean. After a few minutes, the teacher was to bring the class back together and push students to turn that naïve problem into a general one by asking them to “imagine that [their] friend has a triangle that [they] can’t see and [they] have to give them instructions to find the point [sought].”

In 2nd period, Megan and Norah remained mostly silent after introducing the problem, letting students work on their own for almost 9 minutes before starting to visit the groups and looking at what students were doing. As Megan walked around, she saw that many students had answered that $O$ should be placed “in the middle.” Megan noticed that students’ answers “in the middle” seemed unambiguous to most of them because most of them had chosen to draw equilateral triangles. Thus, it was intuitively obvious to students not only that these triangles had “a middle” but also that the three triangles thus determined could be proved congruent, hence equal in area. The matter-of-fact, even ironic, way in which some students answered that the point was “in the middle [. . .], obviously” suggests that students would not consider the problem as giving an opportunity to develop an elaborated, novel idea, in spite of Megan’s indications of the magnitude of the task.

A plausible explanation for this mismatch of expectations regarding students’ work appeals to what students would customarily do in a situation where they would be accountable for solving a problem. What they were held responsible for doing, what choices they were allowed to make, and how much time they were given to work were three important variables that had been included by the teacher to promote a certain kind of mathematical thinking by students. But these expectations arose in a situation where, customarily, such kind of thinking would not take place. Such a circumstance could be a source of stress for the teacher because it would, in fact, suggest to him or her that the chances students would indeed use the problem as a context to build a reasoned conjecture would be slim unless she gave more clear indications of what was at stake. Thus, one can provide a plausible rationale for the changes observed in the 3rd period class. The encouragement that students “guess” where the point should be could be seen as a reduction of the responsibility to know what the point would be and a way to encourage students to draw any triangle. Similarly, the promise that Megan would help them in their work could reinforce the indications that the task was not familiar. Finally, Megan’s shorter wait time (only 2 minutes after stating the problem) to start visiting students’
groups put her in a better position to negotiate what kind of solution students should be looking for.

Moving to Solve a General Problem: Producing a Conjecture

We had planned to ask students to develop a set of instructions for somebody who had an arbitrary triangle to find the desired “middle.” However, Megan’s moves to push for a general approach to the problem departed dramatically from what was planned. Megan challenged students’ naïve response that the point was “in the exact middle” by asking questions such as, “How do you get the “exact middle?” Is it from the angle bisectors?” After visiting several groups, Megan addressed the whole class, taking issue with the fact that most students had drawn an equilateral triangle:

Let’s look at one that’s equilateral. [Draws a seemingly equilateral triangle freehand.] OK, like Amber said, she just looked at it for where the center of the triangle looked like it was [shakes head] . . . OK . . . [(3 secs)]. How do I find the center . . . you know . . . [to Amber] you just looked? Or did you do something?

One student meekly ventured “centroid,” and Megan then engaged the class in a review of the various segments inside a triangle and the names of the points at which they intersect (“Who remembers what the orthocenter was?” “What was the circumcenter?”). She continued thus:

OK, my point is here, we had a lot of centers, when we studied those things [. . .]. What’s the problem with drawing one of these [points at the equilateral triangle at the board]? Where do the angle bisectors [meet]? [. . .] OK the problem is . . . on a equilateral triangle [sic] . . . the angle bisectors are also the what? {Mandy: altitudes} Altitudes. {Mandy: Why is that a problem?} Because I don’t know which one, which one I am getting here, which center I am actually using. If I draw a triangle like this [draws an obtuse scalene triangle] then the angle bisectors will still be concurrent, they’ll still meet at a point inside the triangle but it won’t be the same point where all the other stuff meets, like the altitudes of this one are gonna meet outside the triangle.

After showing how orthocenter, incenter, and centroid would differ in the new triangle, Megan engaged students in a new task. The planned problem was meant to have students think about a construction common to all triangles by constraining the control they had on the diagram, but Megan moved to bringing about a general conjecture in a different way. She led students to think that the solution was one of the centers of a triangle they had studied before and that they could figure out what this general solution was by visual inspection of triangle-types:

I want you to spend some more time with your groups. Listen to me: Draw different triangles. If you drew some that were close to being equilateral, you can see there are some problems with that. So, maybe, try with your group to draw some of these skewed type triangles and see if you have a different idea from that—or if, maybe, your old idea still works. Does it look like all those triangles you get have the same area? Then maybe you need to start working on why would that be true.

Megan’s intervention, engaging in a review of the various centers of a triangle and in a critique of the choice to draw an equilateral triangle, was a bid to replace students’ misperception of the difficulty of the task with a clearer indication that
students should have thought of the problem differently. Her repair of the task also included a repair of the situation—students had to produce a general conjecture; in exchange she took responsibility for spelling out the ideas they had to apply, thus turning the task into a more customary problem-solving situation.

Megan intervened earlier in 3rd period, questioning groups’ initial approaches to the problem and steering them away from naïve responses. This showed her efforts to preempt a mismatch of expectations and to preserve students’ opportunities to understand the requirements for the problem through making constructions inside their own triangles. Both classes were similarly treated, however, in regard to what general problem students were eventually asked to solve. The new task involved them in solving a general problem by making a conjecture grounded on perception—to inspect which of four known centers (centroid, circumcenter, incenter, orthocenter) appeared to split a nonspecial triangle into three triangles of equal area. This was useful to enable students to make a general, true conjecture, but it sanctioned sources for that conjecture unrelated to the reasons that would make such a conjecture true—namely, the presumptions that one construction would work for all triangles, that it had to be one of four known centers, and that perception would help identify it.

The task thus changed. At least in part, this change can be attributed to the need for Megan to find a situation within which it would be possible to hold students accountable to make a conjecture. The task of developing a reasoned argument could fit, in Megan’s customary practices, in a situation of “doing proofs” but a characteristic of those situations was that an admittedly true statement would be given to students. Yet, if students were expected to conjecture this true statement, that activity would more likely fit in a situation of “solving problems,” more open to the use of a variety of resources and focused on the production of a best answer. In enabling students to make a perceptually grounded conjecture, Megan allocated to herself a role where she could sanction the conjecture as true and could then make students responsible to prove it. Near the end of the hour, many groups in each of the two classes had decided that the centroid would split a triangle into three equal areas, and Megan had certified to those groups that they were correct.

On Which Grounds Is the Conjecture Reasonable: Asking to Consider Somebody Else’s Solution

The third scaffold, known as Last Year’s Conjecture, consisted of asking students to decide whether a solution allegedly proposed by students of another class had merit. This conjecture suggested that the solution to the problem might be the midpoint of the segment connecting the midpoints of two sides. The conjecture is false for any triangle, but the task of inspecting this conjecture could arguably help students think about how to create equal areas (see Figure 2).

None of Megan’s classes used Last Year’s Conjecture to engage in the task envisioned. Megan assigned it, but she did so in ways that made it apparent that it was problematic for her to hold students accountable for considering the conjecture as
a possible solution. Many students in each class indicated that they did not think it would work and argued so on the basis of perceptual differences of size in a diagram. Rather than using the conjecture to encourage students to think about what areas were equal in the proposed construction and what was still missing, Megan asked students to think whether the conjecture was true for a special case. Some students (Saul in 2nd period and Michael in 3rd period) proposed it would be true in an equilateral triangle. Although Saul realized immediately that “it still wouldn’t work,” Megan asked, “What do you know is the same in those triangles?” Saul made the argument that two triangles $[AOC$ and $BOC$ in Figure 3] would be congruent by “side-side-side.”

Last year’s conjecture did not play a role in finding the grounds on which a solution to the triangle problem would be reasonable. Students constructed point $O$ with great care and, by visual inspection, decided that it was not the point sought.
Whereas Megan attempted to challenge the perceptual grounds for that decision, she eventually compromised and steered their work to an ad hoc exploration of the equilateral triangle case. In this case, some pieces were demonstrated equal but on the grounds of triangle congruence (which was old knowledge to them) rather than by arguing area equivalence. Thus, although the conjecture was rejected, this rejection could not serve to build a different, improved conjecture or to move to the foreground the properties of area equivalence. It was as if the qualities of Last Year’s Conjecture as a proposition about areas were difficult to see on account of a more pressing need to decide what it meant for the class to be working on it, given that the work could not be defended as solving the Triangle Problem. In later conversations about these episodes with Megan and Norah, both commented that in spite of the mathematical value they saw in it, for them to ask students to consider it as a possible solution was an unfair use of the trust that students were depositing on them. The change of the task into showing that in an equilateral triangle two of the three triangles are congruent was a way of creating conditions to attribute some trade value to the otherwise useless work done in that segment of time. The “conjecture” did not help students to solve the problem, but at least it provided an opportunity to practice doing triangle congruence proofs.

On Which Grounds Is the Conjecture Reasonable: Asking Students to Solve a Simpler Problem

The last scaffold (the Two-Equal-Area Split Problem), designed to help students make a reasoned conjectured, asked students to solve a simpler version of the Triangle Problem:

Find $O$ inside $ABC$ such that two triangles have equal area. (You should suggest where to locate point $O$ for any triangle $ABC$ and explain why your suggestion works.)

The problem was included in their homework on Day 2 and discussed as a whole class on Day 3. Megan spent substantial amounts of time on this problem on Day 3. Emerging from the simultaneous negotiation of task and situation was how the work on the simpler problem could help build the argument that would eventually prove the centroid conjecture. Two main elements of that argument were produced: that it was relevant to think about medians to divide a triangle into two triangles of equal area and that if a third triangle had to be carved inside the original, the addition property of equality$^9$ for areas could help maintain the equality of two triangles ($AO′B$ and $BO′C$ are equal if $O′$ is on median $BO$; Figure 4). These developments were different in the two classes.

From homework task to scaffold for a problem. In both classes, there were students who had done the homework problem by placing $O$ on the midpoint of a

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$^9$ As stated by Megan to her 3rd period students: “We never had any theorem or postulate that says that [the properties of equality] will carry over to area, but we’re gonna accept that now. If we have equal areas, then we can add equal areas, get equal areas, we can subtract equal areas off, still have equal areas.”
side and drawing a median to produce two equal-area triangles, thus overlooking
that $O$ had to be inside the triangle. Amber contributed that solution in 2nd period
when Megan started reviewing the homework (Figure 4). Against protests by those
who realized that Amber’s solution had overlooked the problem and yet had not
themselves found a solution to the problem, Megan used Amber’s contribution to
engage students in building a rationale for the conjecture that the centroid would
solve the triangle problem. In 3rd period, Megan stressed that the homework
problem was to give them an idea for how to approach the Triangle Problem,
marking it as legitimate mathematical work (“Oftentimes in math, it’s easier to look
at a simpler problem and then see, OK, can I use any of that idea to carry over to
this other problem?”). Megan contributed the idea that a median would divide a
triangle into two equal areas and then connected that to her reading of the home-
work problem: “What this homework problem was saying though is, where else
could $O$ be? Does it have to be here on the bottom [side]?”.

Getting the work started. In regard to how the two classes worked on the correct
version of the simpler problem, the 3rd period class went sooner and smoother
toward placing $O$ on a median and into looking for pairs of equal areas. In 2nd period,
Megan had posed a rather open question (“Who had an idea what they could do to
[a triangle with one median drawn]?”) that was met with a solitary and ambiguous
response from Damon who suggested “moving” $O$ up the median (see $O'$ in Figure
4). Megan tried to take charge of justifying why a point “anywhere on a median”
would create equal area triangles, asking students’ input in making that argument.
She stopped herself when she realized that she was doing most of the talking. She
distributed the day’s assignment and only much later got back to working with the
class in developing the solution for the simpler problem.

In 3rd period, however, after stressing that the median of a triangle splits the
triangle into two equal areas and labeling that “the median rule,” Megan acted on
John’s suggestion to pick point $O$ on a median (see Figure 5). She then said, “Say
I picked $O$ here,” connected it to the vertices, and asked, “who sees a pair of trian-
gles that if I use the median rule, they’ll be equal?” Her question elicited responses pointing to the two small triangles at the bottom \([AOM \text{ and } COM]\) in Figure 5, so that after only 4 minutes of work on this problem, she had obtained from Karl the answer she had not been able to obtain from Damon after almost 10 minutes of work in 2nd period. This was understandable, since the questions posed to the classes were different. That change of questions reflects Megan’s struggle to find the means to make the Two-Equal-Area Split Problem work both as a problem to solve and as a vehicle to develop an argument. Megan’s management of the situation in 2nd period seemed oriented by the expectation that students discover a justifiable solution to a simpler, interesting problem. Megan’s management of the situation in 3rd period seemed oriented by the expectation that students apply known ideas to solve a given problem.

Getting the intermediate problem solved. Megan handed the 2nd period students the day’s worksheet with the assignment to work on the triangle problem again and went back to Amber’s idea and Damon’s diagram (Figure 4), insisting, “There is more I can get from this picture!” Once again, she reviewed why a median would split a triangle in equal areas and asked, “Which other triangles in that picture can I say are equal because of that?” and, eventually, “The median divides the triangle into equal areas. . . . But what we are saying now is that there are some little triangles that are like that too.” Saul then identified the bottom triangles as equal in area, and Megan sanctioned this observation while covering the piece of \(ABC\) over \(O'\) with one hand:

These two here on the bottom—. . . . I’ve got a little triangle here [outlines \(AO'C\), Figure 4] with a median [outlines \(O'O\)] drawn in. . . . So these two [outlines \(AO'O\) and \(CO'O\)] are equal [in area]. (3 seconds) OK, listen to me. If I know the big ones are equal [in area] ’cause Amber told us that . . . , and I know these little ones are equal [points to \(AO'O\) and \(CO'O\), Figure 4] . . . , then how can I get to that these [points to \(AO'B\) and \(CO'B\)] are?
Megan welcomed Connor’s response that “they could subtract equal areas” and sent the class back to work in their groups to “think about that [and] write that down.” The same conclusion was arrived at differently in the 3rd period class. Differences had to do with what Megan said and when she said it and her management of the direction of the discussion. There were also important differences in the sense of accomplishment or purpose that transpired from the work.

The 3rd period class’s arrival at the addition property of equality for areas was a more prominent moment than in 2nd period. After revoicing Karl’s observation that triangles $AOM$ and $COM$ (Figure 5) were equal, Megan asked, “What does that do for you?” and refrained from giving away the answer for almost 2 minutes while Mark, Karl, Kyle, and Veronica tried different ideas toward a conclusion. Mark eventually claimed that triangle $AOB$ was equal to triangle $BOC$ (Figure 5) and justified it on the grounds that he was subtracting equal things. Megan thus took time to revisit what Mark had done and to officially introduce the addition property of equality for areas.

It was difficult for Megan to enable students to reason about equal areas in this new way. Students were expected to use mathematical reasoning as an alternative source of credibility to perception. They were also expected to build part of the tool kit that would enable that reasoning. Each of Megan’s classes experienced a mix of open and targeted questions, which suggests that figuring what-to-ask-when was one of the key decisions Megan had to make as she managed students’ work. Should she expect students to easily subtract equal areas to justify a given claim (as in 2nd period), pretending they already knew the addition postulate? Or should she give students more room to reason on their own and then revisit the reasoning to underline that they had introduced a new tool (as in 3rd period)? It was as if what was customary to do in situations of solving problems and in situations of developing new knowledge was providing different cues for Megan’s actions as she tried to negotiate with students a share of accountability for them.

Organizing the Proving of the Conjecture

At the beginning of Day 3, Megan commented to each class that though many had made “the right” conjecture on the previous day, “they had no idea how to prove it.” The production of a conjecture had been unrelated to the ideas at stake in the lesson. In both of her classes, Megan gave reassurance that “[the day before] more than half of this class knew which segments they should be looking at to see where this point $O$ is” in exchange for her expectation that they would work on the same problem again, to “come up with why [their conjecture] works.” In both classes, Megan primed the production of an argument by collecting students’ memories on how they had solved the problem.

Connor’s argument—doing a proof for a sanctioned conjecture. The 2nd period class worked for about 24 minutes on collecting their thoughts on the Triangle Problem and then coming up with a proof for the conjecture that if $O$ was the centroid of a triangle, the three triangles it would form with the vertices would all have equal
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areas. Megan asked students to look at triangles with all their medians drawn in, focusing on one median at a time and looking for triangles that, based on the Two-Equal-Area Split Problem, would be equal in area, then suggested that students “go to the next median.” In her interactions with groups, she pointed out that the task for them was to think about “how will you write that down” to prove why all three triangles would be equal in area.

Near the end of this episode, Megan called on Connor to put his argument up. Although Connor only spoke his argument, he prepared for and demonstrated his work in a manner that strongly resembled how students in Megan’s classes ordinarily presented proofs at the board. He started by silently drawing a triangle, marking midpoints on its sides, and noting that they were midpoints by using an equal number of hash marks on each side. He drew the three medians and labeled all points in order (vertices, midpoints, centroid). Then he dashed the segments from the centroid to midpoints (see Figure 6) Connor then explained his proof:

Connor: This triangle here $ABF$ is congruent to triangle $F \ldots$ or the areas are congruent, $ABF$ to $FBC$ because of the median rule . . . and then you have this $AOF$ and the $FOC$, which are also, the areas are the same because of the median rule and then what you do is you just say that these two here [points to $AOB$ and $BOC$] are congruent because you take these [points to $AOF$ and $FOC$] from the whole. So these two [points to $AOB$ and $BOC$] are congruent. And then what you do is you just [moves wrist indicating turning the triangle] turn it sideways and you get that $ABE$ has the same area as $AEC$.

Megan: OK, which median are you using now?

Connor: This one [traces $AE$] right here.

Megan: OK, so now he is using the slanty one.

Connor: And then you get that these two big triangles [points to $ABE$ and $ACE$] are . . . have the same area and then you do $BOE$ and $EOC$ have the same area because of the median rule again and then you have these two [points to $AOB$ and $AOC$] big triangles equal each other. And then as these two [$AOB$ and $COB$] equal each other and these two [$AOB$ and $AOC$] equal each other then all three areas are equal.

![Figure 6. Connor’s diagram.](image-url)
Megan: By what—in your proof what would it say? If those—if the first two triangles . . .
Connor: Transitive.
Megan: Transitive or substitution, that’s right. You still can use that stuff.

Connor’s performance witnesses some of the norms associated with the “doing proofs” situation that was customary in Megan’s class. His performance shows the tendency to use a diagram with all the features needed for the argument already drawn before starting a proof, the tendency to produce a proof consisting of statements that describe a figure, the tendency to justify every statement made about the figure, and the tendency to refer to objects in the figure by labels (neither conceptually nor by indexical or gestural references; see Herbst, 2004; Herbst & Brach, 2006). Connor was the first person to use “the median rule” in public as a label for the notion that a median divides a triangle into two equal areas; this label helped him provide a concise reason for his statement. Likewise, the creation of key ideas for the argument (e.g., why it was convenient to look for another median in order to make two other equal area triangles) was not reported in the argument. Moreover, Megan repaired Connor’s invitation to “turn [the triangle] sideways,” asking him instead to say “which median he was using [then],” which downplays the possibility to interact generatively with the figure and requests instead a more descriptive interaction (Herbst, 2004).

To make students responsible for proving the conjecture in 2nd period, Megan did the customary job of a teacher in situations of “doing proofs.” She sanctioned the conclusion to be proved before the proving activity started, made available the ideas to be used in the argument but held students accountable for organizing those in writing, and supported discursively the students’ work of articulating in sequence statements justified by reasons. Megan’s actions can be understood by making the hypothesis that she was seeing that the work could count as “doing [a] proof.”

John’s explanation—rediscovering the conjecture through making the argument. Students in 3rd period spent 33 minutes on this part of the work. In this class, a public argument was also made that the intersection of medians splits the triangle into three equal areas, although this argument contrasted with that produced in 2nd period. To begin, memories of having had the correct conjecture the day before were not as strong. It was not apparent that they had understood how the addition postulate of equality had helped Mark and Karl solve the Two-Equal-Area Split Problem. Those circumstances had an effect on Megan’s interactions with the groups as she tried to help them work through the Triangle Problem: She relied less on what students had thought or knew from before and more on what they could do with specific hints she would give them. This meant that Megan played a crucial role designing the solution of the problem and that the kind of collective discourse in which she involved students was different than in 2nd period. Rather than articulating the production of the argument as a sequence of descriptive statements about the diagram backed up by reasons,

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10 Later that day, Megan herself introduced the label “median rule” for the 3rd period class.
Megan helped produce an argument where reasons were strategically implemented to *generate* statements about the diagram. Thus, instead of making students responsible for producing a proof of the kind they had been used to, Megan appeared to assign them a peripheral role in the more proactive work of collectively building an argument.

The argument made on the board for why the intersection of medians solved the problem also featured generative statements about the diagram. Following on Alyssa’s diagram (see Figure 7a) showing how a point on one median could be used to solve the simpler problem, John proposed to draw another median and just “flip it.” He drew a triangle that seemed like Alyssa’s, next to hers, and shaded a piece of it different from the shaded region in Alyssa’s (Figure 7b). Megan continued,

*Megan:* Stop for a minute.

*John:* It’s the same exact thing, I mean—

*Megan:* OK, you are saying now draw in a different median.

*John:* Yeah and there’s—

*Megan:* OK, here, I’m gonna erase them. [Megan erases the shading on John’s triangle, $COA$; Figure 7b\(^{11}\)]

![Alyssa’s diagram.](image-a)

![John’s diagram.](image-b)

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\(^{11}\) John’s own diagram (the “flipped” one) had no labels. These labels (in square brackets) have been added to ease the reader’s understanding of what he says.
John: Yeah, so you draw a different median and then you say—
Megan: Here, let’s draw in this one now [redraws medians, Figure 7b]. . . . Okay, now you prove, now you’ve got—
John: Now this one and this one [points to BCO and ACO; see Figure 7b] are equal . . . because you have the median.
Megan: [Seemingly confused] These two are equal [points to ABO and CBO].
John: No, these two are equal [points to BCO and ACO]. I don’t—
Megan: OK, you are doing the same median [Alyssa] did if you do the down one.
John: No, I flipped it over.
Norah: [From the back] He’s drawn it already flipped. [Norah comes to the board and shades again the triangle that John had shaded originally.] He drew the same picture he had over here [Alyssa’s] and the reason he had this [ACO] colored in is because he’s shown you the turned triangle.
Megan: OK, OK, I didn’t see that this was what you were doing John.
Norah: And these are the two that he was now saying are the same.
John: And like I’m saying, this is a new median [points to CD] and these are equal [points to BCO and ACO]. Now this one [ACO] is equal to that one [BCO] and these [BCO and ABO] are equal, then they are all equal.
Megan: Then they are all equal to each other. What’s that called? We use that in geometry—Several Students: Substitution.
Megan: Substitution, that’s right. You’re just substituting . . . If they are both equal to that shaded one then they are all equal to each other.

As the class jointly produced this argument, their actions deviated from what is customary when “doing proofs.” Megan tried to use reasoning to help students build the centroid conjecture from the Two-Equal-Area Split Problem. In doing that, the building of the conjecture included making arguments for equal areas as well as drawing new features or looking differently at diagrams. Alyssa was thus able to start a relevant argument with a figure that was not final. And when John continued the argument, he evoked just two medians, bringing the second median into Alyssa’s diagram to produce an equality of areas. That, along with the use of shadings in two diagrams to refer to the original figure and the “flipped” figure, seems to deviate from the customary way in which diagrams are used in proving. John interacted with the diagram in such a way that the features and interpretation of the diagram became richer as the argument was developed—a pattern of interaction with the diagram that may be called generative (Herbst, 2004). John’s argument also deviated from the customary pattern of statements and reasons for a proof. His argument relied on a few statements, full of indexical references, and, other than his use of the informal notion of “flipping,” gives no explicit justifications for statements. Megan’s difficulties understanding what John was saying can conceivably be attributed to how much John’s use of diagrams deviated from what was customary in presenting a proof. Whereas her tolerance of the informality of his argument may have been related to John’s circumstances,12 Megan’s insistence

12 I learned after this episode that John was receiving special education services and that this had been the first time he had ever volunteered to come to the board in a mathematics class.
that he justify the last step can be seen as trying to make John’s argument stand next to Connor’s proof.

_Coda._ According to custom in Megan’s classes, Connor’s proof was evidently of higher status than John’s argument since its statements were justified. As a stand-alone product it might also be the one a mathematically educated observer would prefer. In contrast, John’s proof relied on intuitions whose translation into statements might require more work; it looked much more like an explanation of how he had solved the problem. Yet in terms of the connection between conjecturing and proving, John’s argument maintained closer, although still implicit, connections.

Each of the two classes experienced the public production of different kinds of arguments under different conditions. Connor’s performance fit more squarely with the customary practice of proving in Megan’s class—proving a conjecture the class had already made and the teacher had sanctioned. John’s performance built the making of a conjecture inside the explanation of his work solving the Triangle Problem. Megan’s allowance of both kinds of performance as she devolved responsibility for the argument attests to her need to manage a tension that concerned the relationships between conjectures, their arguments, and the situations within which each of those could be viable. This tension confronted custom in Megan’s class with the plan for the lessons, a confrontation that only came to the fore as the lessons were enacted.

In the 2nd period class, because a well-formulated conjecture was recalled, Megan could continue to build on the separation between making and proving a conjecture—a separation that she had enabled in Day 2. In making students responsible for producing a proof, she could rely upon the customary norms of “doing proofs” because the conjecture had already been formulated. If she had put students in a situation of having to reason through the development of a conjecture, such actions might have exposed a breach in the didactical contract (e.g., pointing at the previous day’s work as irrelevant). Thus, in the 2nd period class, maintaining the separation of conjecture and proof at the expense of the plan of developing the ties between conjecture and argument was instrumental in fulfilling the didactical contract. And it allowed Megan to enforce students’ accountability for proving. But, inasmuch as a conjecture still had to be developed, as was the case in the 3rd period class, Megan had the chance to come back to the original plan and use the simpler problem to push for the development of a reasoned conjecture. In this case, making students responsible for the production of an argument had very little in common with the situations of proving that were customary for Megan’s students.

The expectation that the Triangle Problem would be a context for both problem solving and proving had induced perturbations on the norms of what teacher and students did in various situations in Megan’s class. She attempted to deal with that perturbation on the first day as she split conjecture from proof (when she turned the problem of constructing $O$ into one of picking a center). The 3rd period students’ forgetfulness acted as a new, natural perturbation on the same system of practices. The low accountability for proving that Megan enforced in 3rd period confirmed
again how difficult it was for Megan to manage a situation where students were accountable for conjecturing and proving while solving a problem.

DISCUSSION

Earlier I posed this question: What kinds of negotiation of the didactical contract are needed in order to engage students in problems that aim at the building of a reasoned conjecture? The didactical contract makes teacher and students responsible for a trade over knowledge. In U.S. high school geometry classes, this trade involves a system of concepts and properties whose connections are substantiated by mathematical reasoning, and that creates a context for students to acquire this reasoning. Students and teacher flesh out that trade over knowledge by engaging in mathematical work and using that work to lay claim on the knowledge at stake.

A variety of situations (or ways of organizing publicly the trade of work by knowledge) were customary in Megan’s classes. Megan introduced new concepts, postulates, and theorems with some input from the students; the students were responsible for solving problems involving ideas introduced before. Situations of high accountability for students included those of students doing proofs. Sometimes, Megan also engaged students in making conjectures. Those situations usually differed at least in regard to how students interacted with diagrams. (In situations of doing proofs, the diagram would be given by the teacher and described by the student, but in situations of making conjectures, the student could be expected to measure it and alter it.) Whenever students were to produce a proof, this expectation was communicated explicitly, and an endorsed statement to be proved was provided in advance.

The centroid theorem, which connects medians with areas, fits very well in that contract for geometry instruction. It was plausible to see it in a unit on area, thus giving students a chance to think about how the geometric properties of a figure can warrant inferences about area, before any measurement or calculation is needed. The Triangle Problem might have been a valuable instrument for a teacher to engage students in work that brought up that theorem. It might have involved students in acquiring the target knowledge actively, using the mathematical way of reasoning they had allegedly developed to conjecture that proposition as a solution to the problem as well as develop with it key properties of area. But, as shown, it was not easy for Megan to make students’ work on the problem have such trade value.

Students did work on the problem; they did have the chance to make a conjecture, to participate in the development of new knowledge about area, and to produce an argument for the conjecture. But, although students were engaged in each of those tasks at different moments, they were never engaged in the task of building a reasoned conjecture. What can be said about engaging students in building a reasoned conjecture in view of a case where such a task did not happen? The cognitive demands of mathematical tasks change from the time when they are conceived through the time they are launched and to the time when they are implemented.
Although important work has been done to understand the factors that help sustain those demands (Henningsen & Stein, 1997), it is equally important to understand where those pushes for change in the demands of a task come from. The “instructional situation” construct can help explain this phenomenon.

The comparison of how Megan managed the work in the two classes shows that it was not easy to use the Triangle Problem as a context for building a reasoned conjecture. It was a perturbation on customary ways of working in Megan’s classes. To understand why some tasks are more viable than others, it helps to look at how the class as a system handled that perturbation. The analysis shows that work on the Triangle Problem in Megan’s classes was made possible by drawing on customary ways of organizing trade between work and knowledge. At least four different instructional situations complemented one another in carving a place for the Triangle Problem: solving problems, making empirically based conjectures, developing new knowledge, and doing proofs. But the mathematical work demanded by the Triangle Problem, the task of building a reasoned conjecture, could hardly be contained in any one of those four customary situations. Megan’s changes from class to class through the four episodes in which students’ work on the problem unfolded attest to the claim that the Triangle Problem induced tensions in Megan’s work. These tensions were perceived in general by Megan at the moment, and commented on later in the year when we watched the videotapes together.

Changes in the ways of working on the Triangle Problem can be described in terms of two kinds of negotiation of the didactical contract. On the one hand, Megan and her students had to negotiate the situation in which they were interacting—or what they would say they were doing as they worked on the Triangle Problem—so as to find out how they were supposed to interact and what they were responsible for doing. Each possible way of organizing the situation, however, imposed conditions and constraints on the specific mathematical task that students could engage in (see Figure 8). To steer the work toward the ends planned, Megan also had to negotiate the task that students engaged in. The problem and each of the scaffolds used to steer the work toward the task of building a reasoned conjecture drew on one or more customary situations in order to be viable, but in turn the rules of each of those situations shaped the actual task that students engaged in. In turn, the mathematical entailments of the Triangle Problem imposed conditions on the task that perturbed the stability of each of those situations. Figure 8 shows the interrelated negotiations of task and of situation (building on Doyle’s work to describe the various tasks at play; see Doyle, 1988).

The “centroid theorem” was accommodated as a “solution to a problem” in a situation of “solving a (routine) problem” involving recalling and using various centers of the triangle and was accommodated as an “endorsed conclusion” in a situation close to what might be recognized as “doing a proof.” It is thus plausible that several instructional situations may be able to accommodate tasks that involve students in thinking about a given set of ideas, but each of those situations will shape differently what students can think about those ideas.
<table>
<thead>
<tr>
<th>Problem →</th>
<th>Triangle problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Possibly fitting in customary situation →</td>
<td>Solve an application problem.</td>
</tr>
<tr>
<td>Make a conjecture.</td>
<td>Do a proof.</td>
</tr>
<tr>
<td>Develop a new idea.</td>
<td></td>
</tr>
<tr>
<td>Taking the form of the task →</td>
<td>Find the point that splits a particular triangle into equal areas.</td>
</tr>
<tr>
<td>Make a perceptually grounded general conjecture.</td>
<td>Prove that the centroid of a triangle is such that the three triangles it makes with the vertices are of equal area.</td>
</tr>
<tr>
<td>Discover the addition principle of equality for areas.</td>
<td></td>
</tr>
<tr>
<td>With characteristics ↓</td>
<td>Split a triangle into a number of parts, some of which are equal in area.</td>
</tr>
<tr>
<td>Decide which center-construction produces the desired splitting point for all triangles.</td>
<td>Provide statements and reasons that show that three triangles are equal in area.</td>
</tr>
<tr>
<td>State the grounds on which two triangles have equal area.</td>
<td></td>
</tr>
<tr>
<td>Resources</td>
<td>Given that a median splits a triangle into two equal areas.</td>
</tr>
<tr>
<td>Given one or more actual triangles that represent a generic triangle (acute, scalene).</td>
<td>Given an actual triangle that represents a generic triangle, with its medians drawn in and all points labeled (vertices ABC, midpoints MNP, median intersection O). Known that a median splits a triangle in two equal areas. Known that triangles formed by drawing medians can be represented as combinations of triangles split by medians.</td>
</tr>
<tr>
<td>Operations</td>
<td>Draw a point on a median to ensure that it is “in the middle.”</td>
</tr>
<tr>
<td>Implement each known or proposed center-construction, produce each point, compare them perceptually as to which one seems to be in the middle of the triangle.</td>
<td>Sort statements of area equality between pieces of the given figure, using the principles of equality and the definition of median.</td>
</tr>
<tr>
<td>Perturbing the situation →</td>
<td>Solution too simple or too vague for the high student accountability assumed.</td>
</tr>
<tr>
<td>Student accountable for work unclearly related to introducing the ideas of area.</td>
<td>Postulates of area-equality not introduced before the proof perturb customary practice of finding reasons for statements.</td>
</tr>
<tr>
<td>Difficult to carve a share of labor for students.</td>
<td></td>
</tr>
</tbody>
</table>

*Figure 8. Negotiation of task and situation for the Triangle Problem*
CONCLUSION

The didactical contract, the set of implicit responsibilities that bind teacher and students in regard to the subject of studies, is a resource for teaching. It entitles the teacher to organize and manage students’ intellectual work with content—the tasks. The didactical contract is also a constraint for teaching. It imposes the need for a teacher to ensure that doing the work entitles the class to lay a claim on mathematical knowledge. Given a problem, a teacher must contend with the question of whether that problem might elicit interesting mathematical thinking. But they also must contend with the question of how such thinking about the problem serves the more general purposes and expectations set forth by the didactical contract. Not all problems are created equal: Whereas some may be deemed useful without question and others rejected without trial, a sizeable number may be like the Triangle Problem in Megan’s classes. These problems may be arguably valuable, but as their viability is negotiated, the proceeds of that negotiation may raise new questions as to their value. Epistemological and instructional questions need to be asked about whether and how certain ideas can be conceived by students as they work on a problem, and also, whether and how such work can unfold around the problem in the class where the problem is posed.

In making the argument for how those epistemological and instructional questions are related, I have argued for the notion that particular problems are accommodated into the stream of classroom action within customary, recurrent ways of doing academic work, which I called instructional situations. A task may become viable at the expense of being shaped by those situations. A task may also trigger adaptations in those situations, at the expense of isolating the task itself, perhaps compromising the extent to which the ideas at play receive the desired attention.

As for the specific work of using a problem as a context for building a reasoned conjecture, the previous analysis adds to the claim that customary, sometimes ritualistic, practices of proving in high school geometry classrooms stand in the way of engaging students in more authentic mathematical activity. If the only kind of situation where students are accountable to reason deductively is that of highly ritualized situations of “doing proofs,” the investment of this reasoning in conjecturing and problem solving will be incidental at best. Inversely, if in order to afford students an opportunity to grasp the meaning of mathematical ideas we recognize the need to afford access to how those ideas connect to and warrant each other, students should be exposed to tasks that require them to reason deductively to find things out. It seems important thus to develop customs of doing mathematical work that make students accountable for tasks that require reasoning deductively even if the product is not yet fully written as a mathematical proof. Tasks that demand building a conjecture about geometric objects in contexts where handling those objects empirically is inconvenient or leads to error may be viable examples of such mathematical work. Problems might be useful contexts for teaching students to be a different kind of student (Lampert, 2001), ones who will later invest reasoning and proving in finding solutions to problems. The teacher is a key agent in the
activity of teaching and learning mathematics in classrooms, but the activity of teaching is framed by mathematical tasks and instructional situations that constrain as well as enable the teacher’s agency. Understanding these structures is a key in developing a more empathetic discourse about what teachers can do to improve instruction.

REFERENCES


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