ALFINIO FLORES AND TROY P. REGIS

How Many Times Does a Radius Square Fit into the Circle?

THIS ARTICLE PRESENTS ALTERNATIVE AND INTERESTING approaches to exploring π. The first part of the article describes an empirical method that allows students to estimate the area of a circle with surprising accuracy. This method can be used as an extension of the approach of using a grid to approximate the area of a circle. The use of a grid is conceptually illuminating; it emphasizes the idea that the area is measured in square units and that areas of shapes that do not have straight sides can nevertheless be measured with square units. An example of such an approach is described in "Covering a Circle" (Lappan et al. 1998); however, when students actually count squares, the values they find may not be exact. For example, in a sixth-grade classroom, when students counted the squares in a grid containing a circle with

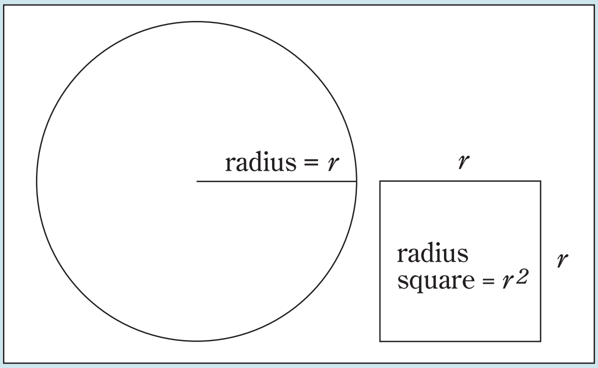

Fig. 1 The circle and the radius square

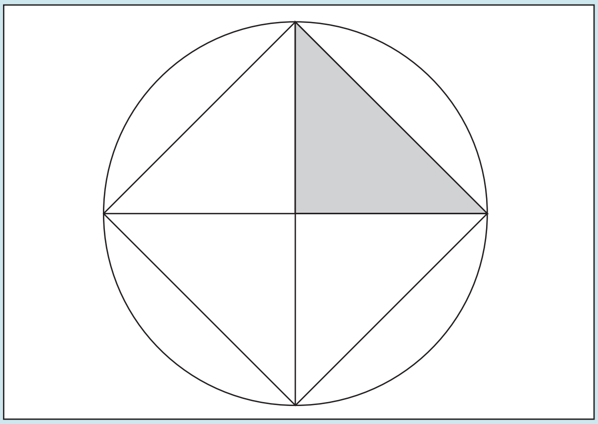

Fig. 2 The shaded triangle is one-half of the radius square.

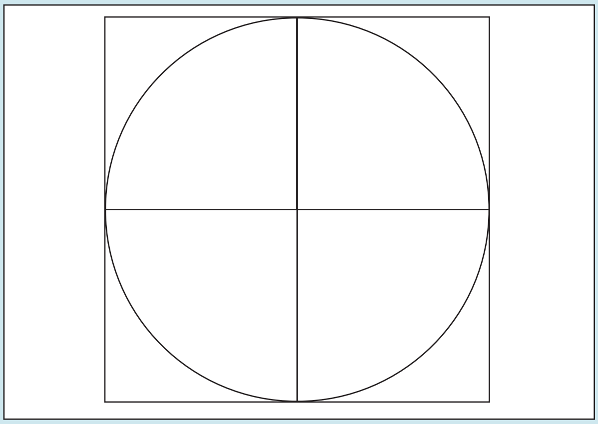

Fig. 3 The area of the circle is less than the area of four radius squares.

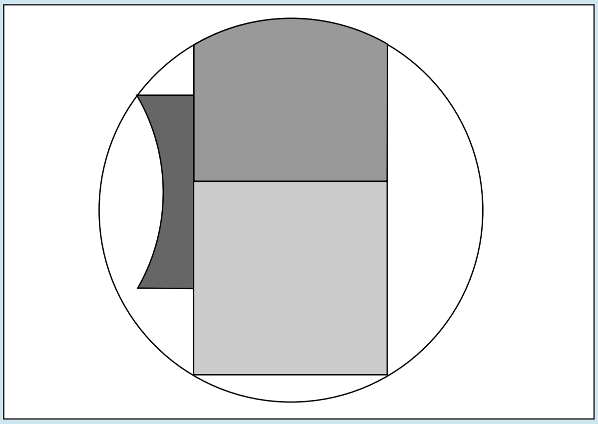

Fig. 4 Students fit two radius squares into the circle before using the third one.

radius 7, their answers ranged from 142 to 178 square units for the approximate area of the circle. Dividing such numbers by the square of the radius, that is, 49, we obtain values ranging from 2.9 to 3.6. The area approximation was not exact enough to determine the value of π with two significant digits. In the first activity we describe, the goal is to help students understand that the area of the circle is about 3.1 times the area of the radius square (that is, a square with a side length that is equal to the radius).

Frequently, even after students conduct an empirical exploration of the area of the circle and its circumference, they are not helped to understand why the same number, π, appears in the formulas for both the area and the circumference of a circle. The second activity in this article is geared to help students see the connection. This second activity may be more appropriate for revisiting the topic in seventh or eighth grade. We assume that students have done an activity for finding the ratio of the circumference to the diameter, for example, by measuring around circular objects, measuring along the corresponding diameters, and computing the ratio. An example of such an activity is "Surrounding a Circle" (Lappan et al. 1998). Also, we assume that students know and understand the formulas for the areas of a triangle and a parallelogram.

Activity 1: The Ratio of the Area of the Circle to the Radius Square

STUDENTS ARE GIVEN A CIRCLE AND A SQUARE with a side length that is equal to the radius—the radius square (see **fig. 1**). They are asked to guess how many times the radius square fits into the circle. Students in sixth grade figured out fairly quickly that the answer had to be more than two but less than four. **Figure 2** shows a way to help students visualize that the radius square fits more than two times in the circle. The area of the square inscribed in the circle is equal to two radius squares because the area of the shaded part is one-half of the radius square, and four such halves fit in the circle. **Figure 3** shows that the radius square fits fewer than four times because four radius squares cover and extend beyond the circle.

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Cutting and pasting radius squares

Copy the four radius squares and the circle from the materials section in **Appendix 1** on page 368 and distribute the copies to students. Note that **figure 4** shows one possible response; the pieces of the squares fit in other ways. Then, give students the following instructions:

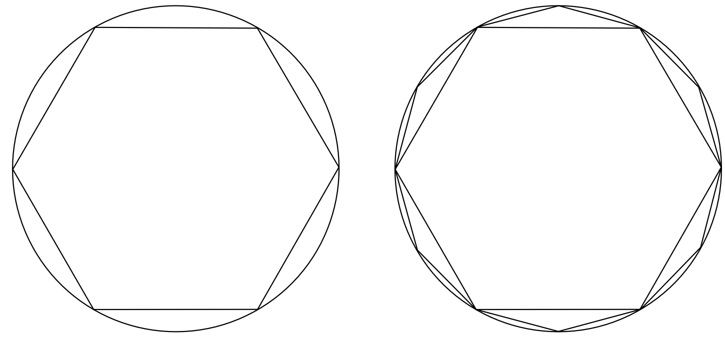
> Color the four radius squares with different colors. Cut out the circle. Cut out the first square and fit it entirely inside the circle. Cut out the second square and fit it inside without overlapping the previous color. You will have to cut parts of the second square so that they fit inside the circle without extending beyond it. Use all of the second square before using the third square. Continue with the third and fourth squares in the same manner. Save the remainder of the fourth square and use the grid to estimate how much of the fourth square you were able to fit.

Results of the activity

Students see that they can fit three squares completely and a little of the fourth one. That is, the radius square fits "three and a little more" times into the circle. Using a radius of ten units and the corresponding ten-by-ten grid to draw the fourth radius square is convenient for this activity. By counting the number of small unit squares of the fourth radius square that were actually used, students can see that the ratio of the area of the circle to the area of the radius square is about 3.1 to 1, or about 3.1.

This activity required about two 45-minute sessions to complete in a sixth-grade class. The teacher may want to have wax paper, patty paper, or other translucent paper available to allow students to trace areas that are not yet covered on the circle and use the tracing to cut the corresponding parts on the colored radius squares. Students have different strategies to fit parts of the fourth square, some of which lead to better estimates. However, in the sixth-grade class mentioned, students' estimates of the ratio of the area of the circle to the area of the radius square were all between 3.10 and 3.20.

By using circles of other sizes, students can also be convinced that the ratio between the area of the circle and the area of the radius square is the same for circles of any size. At this point, some students will notice that this value of 3.1 is the same as the ratio of the circumference to the diameter and may guess that this fact is not a coincidence. The next part of the activity helps students see why the same constant, π, appears in the formulas for both the circumference and the area of the circle.



(a)
The initial hexagon

(b)
New vertexes at the midpoints of the arcs

Fig. 5 Regular polygons that approximate a circle

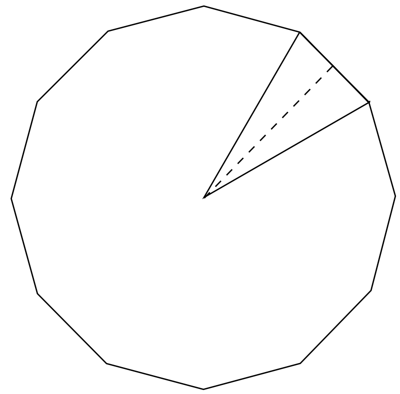


Fig. 6 A triangle with a base that is one side of the polygon

Activity 2: The Relation to π in the Formula of the Circumference

STUDENTS NEED TO RECALL THAT THE RATIO OF circumference to diameter is approximately 3.1. The exact value of the ratio is called π, and its first digits are 3.14, which for most practical purposes, gives enough accuracy. Many calculators have a key for π in the event that more digits are needed.

Approximating the circle by regular polygons (a thought experiment)

Imagine that you have a family of regular polygons inscribed in the same circle constructed in the following way: Starting with a regular hexagon (see **fig. 5a**), the next polygon will have twelve sides. Six of the vertexes will be common with the hexagon; the additional vertexes will be the midpoints of the arcs (see **fig. 5b**). In the same way, each successive term of the family of polygons has twice as many sides. The more sides contained in the regular polygon, the closer the perimeter of the polygon is to the circumference of the circle. Furthermore, by using a polygon with a large

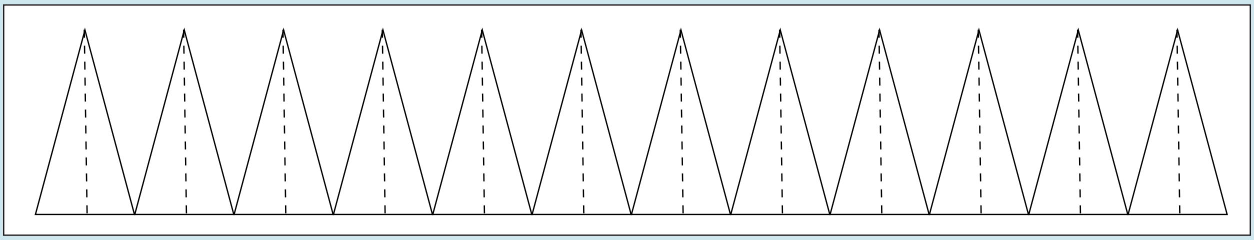


Fig. 7 The polygon is broken into triangles.

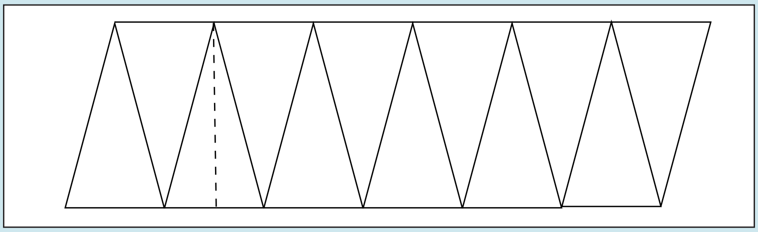


Fig. 8 The polygon rearranged into a parallelogram

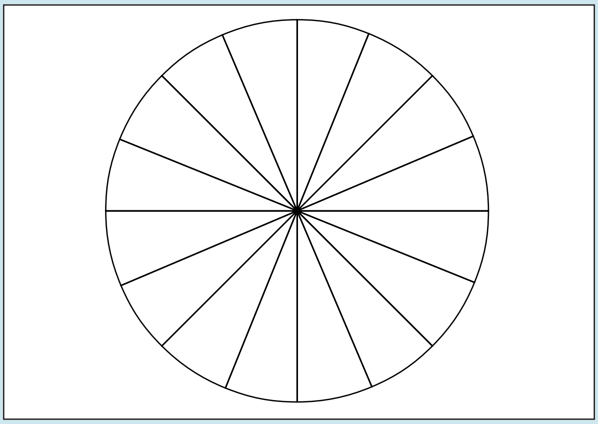


Fig. 9 A circle cut into sixteen slices

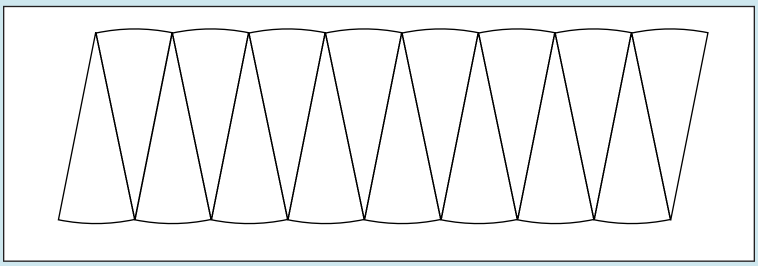


Fig. 10 The slices of the circle rearranged

enough number of sides, we can make the difference between the perimeter of the polygon and the circumference as small as we want. The areas of the regular polygons, then, offer an increasingly close approximation to the area of the circle. The difference between the area of the circle and the area of one of the polygons can also be made as small as we want by choosing a polygon with a sufficiently large number of sides.

The area of the regular polygon can be computed by multiplying the perimeter by the height of one of the triangles forming the regular polygon (see **fig. 6**) and dividing by 2. This formula can be proved in several ways. One is to imagine all the triangles laid out side by side (see **fig. 7**). The total area of the polygon is the sum of the areas of the triangles. One method to obtain the total area is to compute the area of each triangle by multiplying the base times the height, then dividing by 2, and adding the areas. Alternatively, we can add all the bases first, which gives us the perimeter; then multiply by the height; and divide by 2. As the number of sides on the polygon increases, the sum of the bases will be very close to the circumference of the circle ($2\pi r$), and the height of the triangle will be very close to the radius (r). Therefore, the area of the polygon will be close to

$$\frac{\text{circumference} \times \text{radius}}{2}.$$

Because the area of the circle and that of the polygon can be made as close to each other as we want, we can find the area of the circle by this calculation:

$$\frac{\text{circumference} \times \text{radius}}{2} = \frac{\text{diameter} \times \pi \times \text{radius}}{2}$$

$$= \frac{2 \times \text{radius} \times \pi \times \text{radius}}{2} = \pi \times \text{radius}^2$$

We can also arrange the triangles that form the regular polygon into a parallelogram (see **fig. 8**). Its base will be close to half of the circumference, or $1/2 \times d \times \pi = 1/2 \times 2 \times r \times \pi$, that is, $r \times \pi$, and its height will be close to the radius of the circle. The area of the parallelogram will, therefore, be close to πr^2. As the number of sides of the regular polygon increases, the height of the corresponding parallelogram gets closer to the radius of the circle, and its base gets closer to πr. The area of the circle is given by $\pi r \times r = \pi r^2$.

Activity 3: Approximating the Area of the Circle Using Parallelograms

FIGURE 9 SHOWS A CIRCLE OF RADIUS 5 CENTIMEters. Its circumference can be computed using the formula $2 \times r \times \pi$; in this example, the circumference of the circle is $2 \times 5 \times 3.14 = 31.4$ cm. Imagine that you cut out the circle into sixteen sections and rearranged them to form a shape that resembles a parallelogram, as shown in **figure 10**. The area of the circle is the same as the area of this shape. To compute the area of this shape, we will use the fact that it resembles a parallelogram

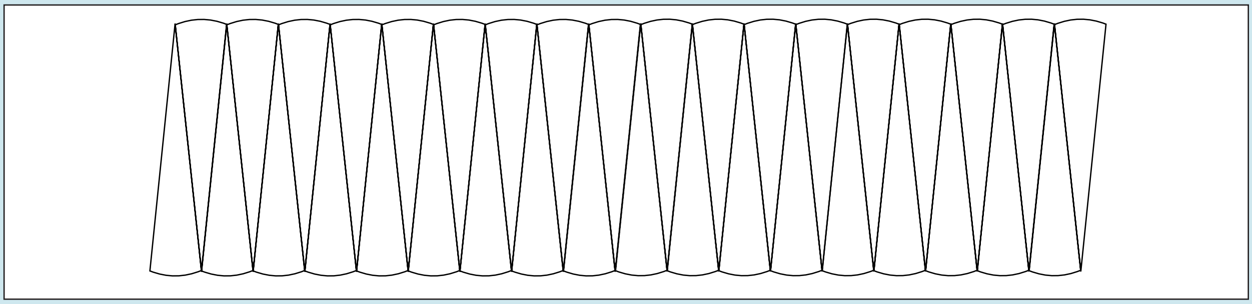


Fig. 11 A better approximation to a parallelogram of height *r*

and refer to it as a parallelogram for our purposes, although its "base" is not quite a straight line. We will use the formula for the area of a parallelogram, area = base $\times$ height. The length of the base of this parallelogram is half the circumference of the original circle because half of the sections point down and the other half point up. Therefore, the length of the base is close to 5π. If you measure along the base on a straight line, you will find that the length is about 15.7 centimeters. The "height" of the parallelogram is close to the length of the radius of the original circle, that is, 5. The area of this parallelogram will be close to the length of the base times the height, that is, $5 \times 5\pi = 25\pi$. Notice that 25 is the value of the radius squared. Because the area of the parallelogram is equal to the area of the circle, we can compute the area of the circle by squaring the radius and multiplying by π. The ratio of the area of the circle to the area of the radius squared, then, is precisely π. The fact that in our first activity, we obtained 3.1 for the ratio of the area of the circle to the area of the radius square, the same value as the ratio of the circumference to the diameter, was not just a coincidence.

Instead of cutting the circle into sixteen parts, suppose that we cut it into more sections, say, thirty-six, and rearranged the sections as before; the new shape will resemble a parallelogram even more (see **fig. 11**). The length of the base of this new parallelogram is half the circumference, that is, $5 \times \pi$, and it is even closer to being a straight line, and the height of the parallelogram is even closer to the radius, that is, 5. We can conclude that the area of the circle is given by 25π, that is, $5^2\pi$.

We can imagine the same process with circles of different radii. If the radius of the circle is 4, the circumference would be $2 \times 4 \times \pi$; half the circumference would be $4 \times \pi$, which would be approximately the base of the parallelogram. The height of the parallelogram would be approximately 4; therefore, its area would be $4 \times 4 \times \pi$, or $4^2\pi$. In general, if the circle has a radius of length r, its circumference will be $2r\pi$. Half the circumference will be $r\pi$. If we cut the circle into thin slices and rearrange them to form a parallelogram, the length of its base will be $r\pi$ and its height will be close to r. Its area, therefore, will be close to $r^2\pi$. The ratio of the area of the circle to the radius squared, $\pi r^2/r^2$, is precisely π.

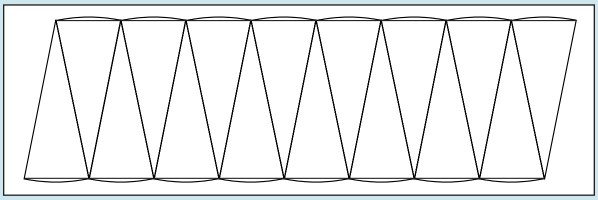


Fig. 12 A real parallelogram and the circle rearranged

The two methods discussed in the second part of this article are closely related (see **fig. 12**). Some students may prefer working with real parallelograms that offer closer approximations to the area of the circle. Others may prefer working with families of shapes, all of which have the same area as the circle and more closely resemble a real parallelogram.

Concluding Remarks

ACCORDING TO VAN HIELE (1999), STUDENTS progress through levels of geometric thinking. Learning opportunities and guidance from the teacher will help them make the transition from one level to the next. In the middle grades, a particularly important transition is from the level of development, in which students rely heavily on empirical verification, to a level at which deductive and more abstract thinking plays an increasingly important role. Our approach to mathematics, although still based on concrete and visual representations, should, therefore, gradually rely less on empirical measurement and more on thought experiments and convincing arguments. By allowing students to explore the topic of the area of the circle, first through an empirical approach and, later, through a more deductive one, we hope to help students in the middle grades make that important transition.

References

Lappan, Glenda, James T. Fey, William M. Fitzgerald, Susan N. Friel, and Elizabeth D. Phillips. *Covering and Surrounding.* Connected Mathematics—Geometry. Menlo Park, Calif.: Dale Seymour Publications, 1998.

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Appendix 1

Activity 1: The Ratio of the Area of a Circle to the Radius Square

Materials

The squares on this page are radius squares of the given circle. Color each square with a different color. Cut out the squares and the circle. Follow the instructions for Activity 1.