Exploring Advanced Euclidean Geometry with *Geometer's Sketchpad*

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Contents

Preface

This book can be used either as a computer laboratory manual to supplement a course in the foundations of geometry or as a stand-alone introduction to advanced topics in Euclidean geometry. The geometric content is substantially the same as that of the the first half of the classic text *Geometry Revisited* by Coxeter and Greitzer [3]; the organization and method of study, however, are quite different. This book utilizes dynamic geometry software, specifically *Geometer's Sketchpad*™, to explore the statements and proofs of many of the most interesting theorems in advanced Euclidean geometry. The text consists almost entirely of exercises that guide students as they discover the mathematics and then come to understand it for themselves.

GEOMETRIC CONTENT

The geometry studied in this book is Euclidean geometry. Euclidean geometry is named for Euclid of Alexandria, who lived from approximately 325 BC until about 265 BC. The ancient Greeks developed geometry to a remarkably advanced level and Euclid did his work during the later stages of that development. He wrote a series of books, called the *Elements*, that organize and summarize the geometry of ancient Greece. Euclid's *Elements* became by far the best known geometry text in history and Euclid's name is universally associated with geometry as a result.

Roughly speaking, *elementary* Euclidean geometry is the geometry that is contained in Euclid's writings. Most readers will already be familiar with a good bit of elementary Euclidean geometry since all of high school geometry falls into that category. *Advanced* Euclidean geometry is the geometry that was discovered later—it is geometry that was done after Euclid's death but is still built on Euclid's work. It is to be distinguished from *non-Euclidean geometry*, which is geometry based on axioms that are different from those used by Euclid. Throughout the centuries since Euclid lived, people have continued to develop Euclidean geometry and have discovered large numbers of interesting relationships. Those discoveries constitute advanced Euclidean geometry and are the subject matter of this course.

Many of the results of advanced Euclidean geometry are quite surprising. Most people who study them for the first time find the theorems to be amazing, almost miraculous, and value them for their aesthetic appeal as much as for their utility. It is hoped that users of this book will come to appreciate the elegance and beauty of Euclidean geometry and better understand why the subject has captivated the interest of so many people over the past two thousand years.

The book ends with a study of the Poincaré disk model for hyperbolic geometry. Since this model is built within Euclidean geometry, it is an appropriate topic for study in a course on Euclidean geometry. Euclidean constructions, mostly utilizing inversions in circles, are used to illustrate many of the standard results of hyperbolic geometry.

COMPUTER SOFTWARE

This book is not designed to be read passively. Instead the user of this book should be actively involved in the exploration and discovery process. The main technical tool used to facilitate that active involvement is the software package Geometer's Sketchpad (GSP). This computer program enables users to explore the theorems of advanced Euclidean geometry, to discover many of the results for themselves, and to see the remarkable relationships with their own eyes.

The book consists mostly of exercises, tied together by short explanations. The user of the book should work through the exercises while reading the book. That way he or she will be guided through the discovery process. Any exercise that is marked with a star (*) is meant to be worked at the computer using GSP, while the remaining exercises should be worked using pencil and paper. No prior knowledge of GSP is assumed; complete instructions on how to use GSP are included in Chapters 1 and 3.

One of the best features of GSP is how easy it is to use. Even a beginner can quickly produce intricate diagrams that illustrate complicated geometric relationships. Users soon learn to make useful tools that automate parts of the constructions. In order to ensure that every user of this book has the opportunity to experience that first hand, the reader is expected to produce essentially all the diagrams and illustrations. For that reason the number of figures in the text is kept to a minimum and no disk full of professionally-produced GSP documents is supplied with the book.

GSP is the most popular and most widely used of the dynamic software packages for geometry, but it is not the only one available. Such programs as Cabri Geometry, Cinderella, and Geometry Expressions can also be used to accompany this text. The instructions that are included in Chapters 1 and 3 are specific to GSP, but the rest of the book can be studied using any one of the other programs mentioned above.

PROOF

One of the major accomplishments of the ancient Greeks was to introduce logic and rigor into geometry. They *proved* their theorems from first principles and thus their results are more certain and lasting than are mere observations from empirical data. This logical, deductive aspect of geometry is epitomized in Euclid's *Elements* and proof continues to be one of the hallmarks of geometry to this day.

Until recently, all those who worked on advanced Euclidean geometry followed in Euclid's footsteps and did geometry by proving theorems, using only pencil and paper. Now that computer programs such as GSP are available to be used as tools in the study of geometry, we must reexamine the place of proof in geometry. There are those who expect the use of dynamic software to replace the deductive approach to geometry, but there is no reason the two approaches cannot enhance each other. It is hoped that this book will demonstrate that proof and computer exploration can coexist quite comfortably in geometry and that each can support the other.

The exercises in this book will guide the student to use GSP to explore and discover the statements of the theorems and then will go on to use GSP to better understand the proofs of the theorems as well. At the end of this process of discovery the student should be able to write a proof of the result that has been discovered. In this way students will come to understand the material in a depth that would not be possible if just computer exploration or just pencil and paper proof were used and should come to appreciate the fact that proof is an integral part of exploration, discovery, and understanding in mathematics.

Not only is proof an important part of the process by which we come to discover and understand geometric results, but the proofs also have a subtle beauty of their own. The author hopes that the experience of writing the proofs for themselves will help students to appreciate this aesthetic aspect of the subject as well.

In this text the word verify will be used to describe the kind of confirmation that is possible with GSP. Thus to *verify* that the angle sum of a triangle is 180° will mean to use GSP to construct a triangle, measure the three angles, calculate the sum of those measures, and then to observe that the sum is always equal to 180° regardless of how the shape of the triangle is changed. On the other hand, to *prove* that the angle sum is 180° will mean to supply a written argument based on the axioms and previously proved theorems of Euclidean geometry.

TWO WAYS TO USE THIS BOOK

This book can be used as a manual for a computer laboratory that supplements a course in the foundations of geometry. The notation and terminology used here are consistent with **THE FOUNDATIONS OF GEOMETRY** [16], but this manual is designed to be used along side any textbook on axiomatic geometry. The review chapter that is included at the beginning of the book establishes all the necessary terminology and notation.

A class that meets for one computer lab session per week should be able to lightly cover the entire text in one semester. When the book is used as a lab manual, Chapter 0 is not covered separately, but serves as a reference for notation, terminology, and statements of theorems from elementary Euclidean geometry. Most of the other chapters can be covered in one laboratory session. The exceptions are Chapters 6 and 10, which are quite short and could be combined, and Chapter 11, which will require two or three sessions to cover completely.

A course that emphasizes Euclidean geometry exclusively will omit Chapter 13 and probably Chapter 12 as well, since the main purpose of Chapter 12 is to develop the tools that are needed for Chapter 13. On the other hand, most instructors who are teaching a course that covers non-Euclidean geometry will want to cover the last chapter; in order to do so it will probably be necessary to omit many of the applications of the Theorem of Menelaus. A thorough coverage of Chapter 13 will require more than one session.

At each lab session the instructor should assign an appropriate number of GSP exercises, where the appropriate number is determined by the background of the students and the length of the laboratory session. It should be possible for students to read the short explanations and work through the exercises on their own. A limited number of the written proofs can be assigned as homework following the lab session.

A second way in which to use the book is to use it as a text for a short inquiry-based course in advanced Euclidean geometry. Such a course would be taught in a modified Moore style in which the instructor does almost no lecturing, but students work out the proofs for themselves.[1] A course based on these notes would differ from other Moore-style courses in the use of computer software to facilitate the discovery and proof phases of the process. Another difference between this course and the traditional Moore-style course is that students should be encouraged to discuss the results of their GSP explorations with each other. Class time is used for student computer exploration and student presentations of solutions to exercises. The notes break down the proofs into steps of manageable size and offer the students numerous hints. It is the author's experience that the hints and suggestions offered are sufficient to allow the students to construct their own proofs of the theorems. The GSP explorations form an integral part of the process of discovering the proof as well as the statement of the theorem itself. This second type of course would cover the entire book, including Chapter 0 and all the exercises in all chapters.

THE PREPARATION OF TEACHERS

The basic recommendation in *The Mathematical Education of Teachers* [2] is that future teachers take courses that develop a deep understanding of the mathematics they will teach. There are many ways in which to achieve depth in geometry. One way, for example, is to understand what lies beneath high school geometry. This is accomplished by studying the foundations of geometry, by examining the assumptions that lie at the heart of the subject, and by understanding how the results of the subject are built on those assumptions. Another way in which to achieve depth is to investigate what is built on top of the geometry that is included in the high school curriculum. That is what this course is designed to do.

One direct benefit of this course to future high school mathematics teachers is that those who take the course will develop facility in the use of Geometer's Sketchpad. Dynamic geometry software such as GSP will undoubtedly become much more common in the high school classroom in the future, so future teachers need to know how to use it and what it can do. In addition, software such as GSP will likely lead to a revival of interest in advanced Euclidean geometry. When students learn to use GSP they will have the capability to investigate geometric relationships that are much more intricate than those studied in the traditional high school geometry course. A teacher who knows some advanced Euclidean geometry will have a store of interesting geometric results that can be used to motivate and excite students.

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[1]For more information on R. L. Moore and the Moore method, see the Legacy of R. L. Moore Project at `http://www.discovery.utexas.edu/rlm/`.

DO IT YOURSELF!

As mentioned above, the philosophy of these notes is that students can and should work out the geometry for themselves. But you will soon discover that many of the GSP tools you are asked to make in the exercises can be found on the World Wide Web. It is my opinion that students should be encouraged to make use of the mathematical resources that are available on the web, but that they also benefit from the experience of making the tools for themselves. Just downloading a tool that someone else has made, and using it, is too passive an activity. By working through the constructions for yourself and seeing how the intricate constructions of advanced Euclidean geometry are based on the simple constructions from high school geometry, you will achieve a much deeper understanding than you would if you simply used ready-made tools.

I believe it is especially important that future high school mathematics teachers have the experience of doing the constructions for themselves. Only in this way do they come to know that they can truly understand mathematics for themselves and that they do not have to rely on others to work it out for them.

The issue of whether or not to rely on tools made by others comes up most especially in the last chapter. There are numerous tools available on the web that can be used to make constructions in the Poincaré disk. But I still think students should work through the constructions for themselves so that they clearly understand how the hyperbolic constructions are built on Euclidean ones.[2] After they have built rudimentary tools of their own, students might want to find more polished tools on the web and add those to their toolboxes.

PRELIMINARY EDITION

This is a preliminary edition of the manuscript. The final edition may contain more diagrams, although I intend to hold to the philosophy that students should construct the diagrams for themselves. This preliminary version of the manuscript is being made freely available on the World Wide Web. Copies may be made for personal use. Instructors who intend to reproduce all or part of the manuscript for classroom use are requested to inform the author.

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[2] Many of the constructions in Chapter 14 were inspired by those in the beautiful paper [9] by Chaim Goodman-Strauss.

CHAPTER 0

A quick review of elementary Euclidean geometry

This preliminary chapter lays out the basic results from elementary Euclidean geometry that will be assumed in the course. Those readers who are using this book as a supplement to a course in the foundations of geometry should omit the chapter and simply refer to it as needed for a summary of the notation and terminology that will be used in the remainder of the book. Readers who are using this book as a stand-alone text in Euclidean geometry should study the chapter carefully because the material in this chapter will be used in later chapters.

The theorems stated in this chapter are to be assumed without proof; they may be viewed as an extended set of axioms for the subject of advanced Euclidean geometry. The results in the exercises in the chapter should be proved using the theorems stated in the chapter. All the exercises in the chapter are results that will be needed later in the course.

We will usually refer directly to Euclid's *Elements* when we need a result from elementary Euclidean geometry. Several current editions of the *Elements* are listed in the bibliography (see [4], [5], or [10]). The *Elements* are in the public domain and are freely available on the world wide web. Euclid's propositions are referenced by book number followed by the proposition number within that book. Thus, for example, Proposition III.36 refers to the 36th proposition in Book III of the *Elements*.

0.1 MEASUREMENT AND CONGRUENCE

For each pair of points A and B in the plane there is a nonnegative number AB, called the *distance* from A to B. The *segment* from A to B, denoted $\overline{AB}$, consists of A and B together with all the points between A and B. The *length* of $\overline{AB}$ is the distance from A to B. Two segments $\overline{AB}$ and $\overline{CD}$ are *congruent*, written $\overline{AB} \cong \overline{CD}$, if they have the same length. There is also a *ray* $\overrightarrow{AB}$ and a *line* $\overleftrightarrow{AB}$.

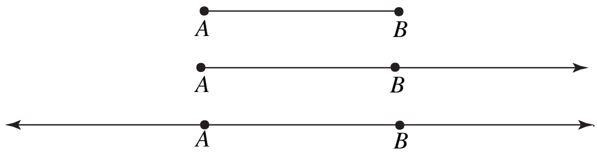

FIGURE 0.1: A segment, a ray, and a line

For each triple of points A, B, and C with $A \neq B$ and $A \neq C$ there is an *angle*, denoted $\angle BAC$, that is defined by $\angle BAC = \overrightarrow{AB} \cup \overrightarrow{AC}$. The *measure* of the angle is a number $\mu(\angle BAC)$. We will always measure angles in degrees and assume that $0 \leq \mu(\angle BAC) \leq 180^\circ$. The measure is 0° if the two rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are equal; the measure is 180° if the rays are opposite; otherwise it is between 0° and 180°. An angle is *acute* if its measure is less than 90°, it is *right* if its measure equals 90°, and it is *obtuse* if its measure is greater than 90°. Two angles are *congruent* if they have the same measure.

The *triangle* with *vertices* A, B, and C consists of the points on the three segments determined by the three vertices; i.e.,

$$\triangle ABC = \overline{AB} \cup \overline{BC} \cup \overline{AC}.$$

The segments $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ are called the *sides* of the triangle $\triangle ABC$. Two triangles are *congruent* if there is a correspondence between the vertices of the first triangle and the vertices of the second triangle such that corresponding angles are congruent and corresponding sides are congruent.

Notation. It is understood that the notation $\triangle ABC \cong \triangle DEF$ means that the two triangles are congruent under the correspondence $A \leftrightarrow D$, $B \leftrightarrow E$, and $C \leftrightarrow F$. The assertion that two triangles are congruent is really the assertion that there are six congruences, three angle congruences and three segment congruences. Specifically, $\triangle ABC \cong \triangle DEF$ means $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $\overline{AC} \cong \overline{DF}$, $\angle ABC \cong \angle DEF$, $\angle BCA \cong \angle EFD$, and $\angle CAB \cong \angle FDE$. In high school this is often abbreviated CPCTC (corresponding parts of congruent triangles are congruent).

0.2 PASCH'S AXIOM AND THE CROSSBAR THEOREM

The two results stated in this section specify how one-dimensional lines separate the two-dimensional plane. Neither of these results is stated explicitly in Euclid's

Elements. They are the kind of foundational results that Euclid took for granted. The first statement is named for Moritz Pasch (1843–1930).

Pasch's Axiom. *Let $\triangle ABC$ be a triangle and let ℓ be a line such that none of the vertices A, B, and C lie on ℓ. If ℓ intersects $\overline{AB}$, then ℓ also intersects either $\overline{BC}$ or $\overline{AC}$ (but not both).*

Let A, B, and C be three noncollinear points. A point P is in the *interior* of $\angle BAC$ if P is on the same side of $\overleftrightarrow{AB}$ as C and on the same side of $\overleftrightarrow{AC}$ as B.

Note that the interior of $\angle BAC$ is defined provided $0^\circ < \mu(\angle BAC) < 180^\circ$. It would be reasonable to define the interior of $\angle BAC$ to be the empty set in case $\mu(\angle BAC) = 0^\circ$, but there is no interior for an angle of measure 180°. The segment $\overline{BC}$ is called a *crossbar* for $\angle BAC$.

Crossbar Theorem. *If D is in the interior of $\angle BAC$, then there is a point G such that G lies on both $\overrightarrow{AD}$ and $\overline{BC}$.*

0.3 LINEAR PAIRS AND VERTICAL PAIRS

Angles $\angle BAD$ and $\angle DAC$ form a *linear pair* if A, B, and C are collinear and A is between B and C.

Linear Pair Theorem. *If angles $\angle BAC$ and $\angle CAD$ form a linear pair, then $\mu(\angle BAC) + \mu(\angle CAD) = 180^\circ$.*

Two angles whose measures add to 180° are called *supplementary angles* or *supplements*. The Linear Pari Theorem asserts that if two angles form a linear pair, then they are supplements.

Angles $\angle BAC$ and $\angle DAE$ form a *vertical pair* (or are *vertical angles*) if rays $\overrightarrow{AB}$ and $\overrightarrow{AE}$ are opposite and rays $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are opposite or if rays $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are opposite and rays $\overrightarrow{AC}$ and $\overrightarrow{AE}$ are opposite.

Vertical Angles Theorem. *Vertical angles are congruent.*

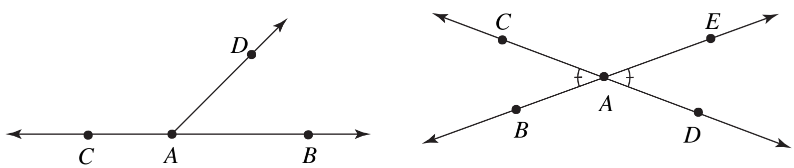

FIGURE 0.2: A linear pair and a vertical pair

The linear pair theorem is not found in the *Elements* because Euclid did not use angle measure; instead he simply called two angles "equal" if, in our terminology, they have the same measure. The vertical angles theorem is Euclid's Proposition I.15.

0.4 TRIANGLE CONGRUENCE CONDITIONS

If you have two triangles and you know that three of the parts of one are congruent to the corresponding parts of the other, then you can usually conclude that the other three parts are congruent as well. That is the content of the triangle congruence conditions.

Side-Angle-Side Theorem (SAS). *If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\overline{AB} \cong \overline{DE}$, $\angle ABC \cong \angle DEF$, and $\overline{BC} \cong \overline{EF}$, then $\triangle ABC \cong \triangle DEF$.*

Euclid used his "method of superposition" to prove SAS (Proposition I.4), but it is usually taken to be a postulate in modern treatments of geometry. The next two results (ASA and AAS) are both contained in Euclid's Proposition I.26 and the third (SSS) is Euclid's Proposition I.8.

Angle-Side-Angle Theorem (ASA). *If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\angle CAB \cong \angle FDE$, $\overline{AB} \cong \overline{DE}$, and $\angle ABC \cong \angle DEF$, then $\triangle ABC \cong \triangle DEF$.*

Angle-Angle-Side Theorem (AAS). *If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\angle ABC \cong \angle DEF$, $\angle BCA \cong \angle EFD$, and $\overline{AC} \cong \overline{DF}$, then $\triangle ABC \cong \triangle DEF$.*

Side-Side-Side Theorem (SSS). *If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\overline{CA} \cong \overline{FD}$, then $\triangle ABC \cong \triangle DEF$.*

There is no Angle-Side-Side condition, except in the special case in which the angle is a right angle.

Hypotenuse-Leg Theorem (HL). *If $\triangle ABC$ and $\triangle DEF$ are two right triangles with right angles at the vertices C and F, respectively, $\overline{AB} \cong \overline{DE}$, and $\overline{BC} \cong \overline{EF}$, then $\triangle ABC \cong \triangle DEF$.*

EXERCISES

0.4.1. Use SAS to prove the following theorem (Euclid's Proposition I.5).
Isosceles Triangle Theorem. *If $\triangle ABC$ is a triangle and $\overline{AB} \cong \overline{AC}$, then $\angle ABC \cong \angle ACB$.*

0.4.2. Draw an example of two triangles that satisfy the ASS condition but are not congruent.

0.4.3. The *perpendicular bisector* of a segment $\overline{AB}$ is a line ℓ such that ℓ intersects $\overline{AB}$ at its midpoint and $\ell \perp \overline{AB}$. Prove the following theorem.
Pointwise Characterization of Perpendicular Bisector. *A point P lies on the perpendicular bisector of $\overline{AB}$ if and only if $PA = PB$.*

0.4.4. The *angle bisector* of $\angle BAC$ is a ray $\overrightarrow{AD}$ such that $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and $\mu(\angle BAD) = \mu(\angle DAC)$. The distance from a point to a line is measured along a perpendicular. Prove the following theorem.
Pointwise Characterization of Angle Bisector. *A point P lies on the bisector of $\angle BAC$ if and only if P is in the interior of $\angle BAC$ and the distance from P to $\overleftrightarrow{AB}$ equals the distance from P to $\overleftrightarrow{AC}$.*

0.5 THE EXTERIOR ANGLE THEOREM

There is an inequality regarding the angles in a triangle that is of fundamental importance in many of the proofs of elementary geometry. The theorem is known as the Exterior Angle Theorem and it is Euclid's Proposition I.16.

Let $\triangle ABC$ be a triangle. At each vertex of the triangle there is an *interior angle* and two *exterior angles*. The interior angle at A is the angle $\angle BAC$. The two angles $\angle CAD$ and $\angle BAE$ shown in Figure 0.3 are the exterior angles at A. Note that the two exterior angles at a vertex form a vertical pair and are therefore congruent.

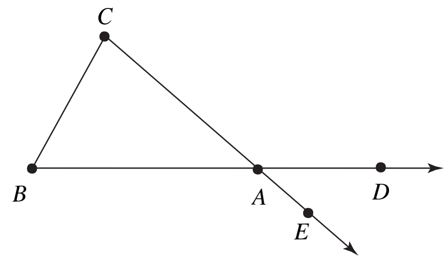

FIGURE 0.3: At each vertex there is one interior angle and there are two exterior angles

Exterior Angle Theorem. *The measure of an exterior angle for a triangle is strictly greater than the measure of either remote interior angle.*

0.6 PERPENDICULAR LINES AND PARALLEL LINES

Two lines ℓ and m are *perpendicular*, written $\ell \perp m$, if they intersect at right angles. If ℓ is a line and P is any point, then there is exactly one line m such that P lies on m and $m \perp \ell$. The point at which m intersects ℓ is called the *foot* of the perpendicular from P to ℓ. In case P lies on ℓ, P itself is the foot of the perpendicular. The process of constructing the perpendicular m is called *dropping a perpendicular*—see Figure 0.4. Euclid proved that it is possible to construct the unique perpendicular with compass and straightedge (Proposition I.12).

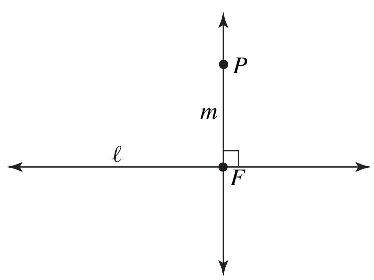

FIGURE 0.4: F is the foot of the perpendicular from P to ℓ

Two lines ℓ and m in the plane are *parallel*, written $\ell \parallel m$, if they do not intersect. It is the existence and uniqueness of parallels that distinguishes Euclidean geometry from non-Euclidean geometries. The Euclidean parallel property is stated most succinctly in the following postulate.

Playfair's Postulate. *For every line ℓ and for every point P that does not lie on ℓ there exists exactly one line m such that P lies on m and $m \parallel \ell$.*

In the presence of the other axioms of geometry, Playfair's Postulate is equivalent to Euclid's Fifth Postulate. The next two theorems relate parallelism to angle congruence. The two theorems are a standard part of high school geometry and are also Propositions I.27, and I.29 in Euclid. It is in the proof of Proposition I.29 that Euclid first uses his fifth postulate.

Let ℓ and ℓ' denote two lines in the plane. A *transveral* for the two lines is a line t such that t intersects ℓ and ℓ' in distinct points. The transversal makes a total of eight angles with the two lines—see Figure 0.5. The two pairs $\{\angle ABB', \angle BB'C'\}$ and $\{\angle A'B'B, \angle B'BC\}$ are called *alternate interior angles*. The angles $\{\angle ABB', \angle A'B'B''\}$ are *corresponding angles*. There are three other pairs of corresponding angles defined in the obvious way.

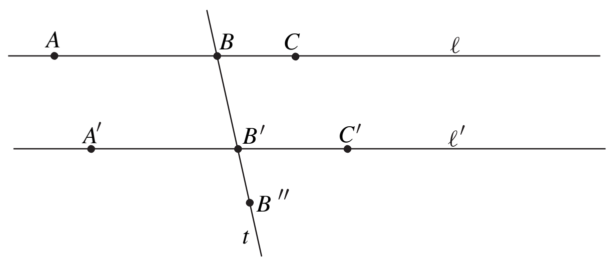

FIGURE 0.5: Angles formed by two lines and a transversal

Alternate Interior Angles Theorem. *If ℓ and ℓ' are two lines cut by a transversal t in such a way that a pair of alternate interior angles is congruent, then ℓ is parallel to ℓ'.*

Converse to the Alternate Interior Angles Theorem. *If two parallel lines are cut by a transversal, then both pairs of alternate interior angles are congruent.*

EXERCISES

0.6.1. Prove the following theorem (Euclid's Proposition I.28).
Corresponding Angles Theorem. *If ℓ and ℓ' are lines cut by a transversal t in such a way that two corresponding angles are congruent, then ℓ is parallel to ℓ'.*

0.6.2. Prove the following theorem (Euclid's Proposition I.32).
Angle Sum Theorem. *For every triangle, the sum of the measures of the interior angles of the triangle is* $180°$.
[Hint: Let $\triangle ABC$ be a triangle. Draw a line through C that is parallel to the line through A and B. Then apply the Converse to Alternate Interior Angles.]

0.7 THE PYTHAGOREAN THEOREM

The Pythagorean theorem is probably the most famous theorem in all of geometry; it is the one theorem that every high school student remembers. For Euclid it was the culmination of Book I of the *Elements*. The theorem is named for Pythagoras of Samos who lived from about 569 to 475 BC. Few details about the life of Pythagoras are known, so it is difficult to determine whether Pythagoras really did prove the theorem that bears his name or what kind of proof he might have used.

Notation. Let $\triangle ABC$ be a triangle. It is standard to use lower case letters to denote the lengths of the sides of the triangle: $a = BC$, $b = AC$, and $c = AB$.

Pythagorean Theorem. *If $\triangle ABC$ is a right triangle with right angle at vertex C, then $a^2 + b^2 = c^2$.*

Euclid gave two kinds of proofs of the Pythagorean theorem; the first one based on area and then later another based on similar triangles.

0.8 SIMILAR TRIANGLES

The similar triangles theorem is one of the most useful in elementary Euclidean geometry. Euclid did not prove it, however, until Book VI of the *Elements*. (The similar triangles theorem is Euclid's Proposition VI.4.) The reason he waited so long is that the ancient Greeks had trouble dealing with the irrational ratios that can arise when similar triangles are compared. It is believed that Eudoxus of Cnidus (408–305 BC) was the first to give a complete proof of the theorem.

Triangles $\triangle ABC$ and $\triangle DEF$ are *similar* if $\angle ABC \cong \angle DEF$, $\angle BCA \cong EFD$, and $\angle CAB \cong \angle FDE$. Write $\triangle ABC \sim \triangle DEF$ if $\triangle ABC$ is similar to $\triangle DEF$. As with congruence of triangles, the order in which the vertices are listed is significant.

Similar Triangles Theorem. *If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\triangle ABC \sim \triangle DEF$, then*

$$\frac{AB}{AC} = \frac{DE}{DF}.$$

EXERCISES

0.8.1. Prove the following theorem. It is a special case of the Parallel Projection Theorem [16, Theorem 7.3.1] and will prove to be very useful later.

Euclid's Proposition VI.2. *Let $\triangle ABC$ be a triangle, and let D and E be points on the sides $\overline{AB}$ and $\overline{AC}$, respectively. Then $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$ if and only if $AD/AB = AE/AC$.*
[Hint for converse: Assume $AD/AB = AE/AC$. Let ℓ be the line through D such that $\ell \parallel \overleftrightarrow{BC}$. Use Pasch's Axiom to prove that there is a point E' where ℓ intersects $\overline{AC}$. Prove that $E' = E$.]

0.8.2. Prove the following theorem (Euclid's Proposition VI.6).

SAS Similarity Criterion. *If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\angle CAB \cong \angle FDE$ and $AB/AC = DE/DF$, then $\triangle ABC \sim \triangle DEF$.*

[Hint: If $AB = DE$, the proof is easy. Otherwise it may be assumed that $AB > DE$ (explain). Choose a point B' between A and B such that $AB' = DE$ and let m be the line through B' that is parallel to $\overleftrightarrow{BC}$. Prove that m intersects $\overline{AC}$ in a point C' such that $\triangle AB'C' \cong \triangle DEF$.]

0.9 QUADRILATERALS

Four points A, B, C, and D such that no three of the points are collinear determine a *quadrilateral*, which we will denote by $\square ABCD$. Specifically,

$$\square ABCD = \overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}.$$

It is usually assumed that the sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ intersect only at their endpoints, but we will relax that requirement later in the course.

The four segments are called the *sides* of the quadrilateral and the points A, B, C, and D are called the *vertices* of the quadrilateral. The sides $\overline{AB}$ and $\overline{CD}$ are called *opposite sides* of the quadrilateral as are the sides $\overline{BC}$ and $\overline{AD}$. Two quadrilaterals are *congruent* if there is a correspondence between their vertices so that all four corresponding sides are congruent and all four corresponding angles are congruent.

There are several special kinds of quadrilaterals that have names. A *trapezoid* is a quadrilateral in which at least one pair of opposite sides is parallel. A *parallelogram* is a quadrilateral in which both pairs of opposite sides are parallel. It is obvious that every parallelogram is a trapezoid, but not vice versa. A *rhombus* is a quadrilateral in which all four sides are congruent. A *rectangle* is a quadrilateral in which all four angles are right angles. A *square* is a quadrilateral that is both a rhombus and a rectangle.

EXERCISES

0.9.1. Prove the following theorem.
Euclid's Proposition I.34. *The opposite sides of a parallelogram are congruent.*
[Hint: Draw a diagonal and use ASA.]

0.10 CIRCLES AND INSCRIBED ANGLES

Let r be a positive number and let O be a point. The *circle* with *center* O and *radius* r is defined by $\mathcal{C}(O, r) = \{P \mid OP = r\}$. The *diameter* of the circle is $d = 2r$. While the radius of a circle is usually thought of as a number, it is often convenient to refer to one of the segments $\overline{OP}$, $P \in \mathcal{C}(O, r)$, as a radius of the circle $\mathcal{C}(O, r)$. In the same way, a segment $\overline{PQ}$ such that P and Q lie on the circle and $O \in \overline{PQ}$ is called a diameter of $\mathcal{C}(O, r)$.

Let γ be a circle and let P be a point on γ. A line t is *tangent to γ at P* if $t \cap \gamma = \{P\}$.

Tangent Line Theorem. *Let $\gamma = \mathcal{C}(O, r)$ be a circle and let ℓ be a line that intersects γ at P. Then ℓ is tangent to γ at P if and only if $\ell \perp \overleftrightarrow{OP}$.*

Let $\gamma = \mathcal{C}(O, r)$ be a circle. An *inscribed angle* for γ is an angle of the form $\angle PQR$, where P, Q, and R all lie on γ. The *arc intercepted* by the inscribed angle $\angle PQR$ is the set of points on γ that lie in the interior of $\angle PQR$.

Inscribed Angle Theorem. *If two inscribed angles intercept the same arc, then the angles are congruent.*

The inscribed angle theorem is Euclid's Proposition III.21.

EXERCISES

0.10.1. Prove the following theorem. It is Euclid's Proposition III.3.
Secant Line Theorem. *If $\gamma = \mathcal{C}(O, r)$ is a circle and ℓ is a line that intersects γ at distinct points P and Q, then O lies on the perpendicular bisector of the chord $\overline{PQ}$.*

0.10.2. Prove the following theorem.
External Tangents Theorem. *If $\gamma = \mathcal{C}(O, r)$ is a circle and ℓ and m are two nonparallel lines that are tangent to γ at the points P and Q, and A is the point of intersection of ℓ and m, then $PA = QA$.*

0.10.3. The following theorem can be viewed as a special case of the Inscribed Angle Theorem. Give a proof that does not use the Inscribed Angle Theorem. The theorem is named for Thales of Miletus (624–547 BC); it is Euclid's Proposition III.31.
Thales' Theorem. *If the vertices of $\triangle ABC$ lie on a circle and $\overline{AB}$ is a diameter of that circle, then $\angle ACB$ is a right angle.*
[Hint: Let O be the midpoint of $\overline{AB}$. Observe that $AO = BO = CO$ and apply the Isosceles Triangle Theorem along with the Angle Sum Theorem.]

0.10.4. Prove the following theorem.
Converse to Thales' Theorem. *If $\angle ACB$ is a right angle, then the vertices of $\triangle ABC$ lie on a circle and $\overline{AB}$ is a diameter of that circle.*
[Hint: Again let O be the midpoint of $\overline{AB}$. There is a point C' such that C' lies on $\overrightarrow{OC}$ and $OC' = OA$. Show that the assumption $C \neq C'$ leads to a contradiction.]

0.10.5. Use the angle sum theorem and the linear pair theorem to prove the following result: *If $\triangle ABC$ is a right triangle with right angle at C and O is the midpoint of $\overline{AB}$, then $\mu(\angle BOC) = 2\mu(\angle BAC)$.* This theorem is a special case of the Central Angle Theorem [16, Theorem 10.4.9]. It will be used repeatedly in later chapters.

0.11 AREA

A *polygon* is a generalization of triangle and quadrilateral. A polygon P has a finite set of *vertices* $A_1, A_2, \ldots, A_n$. The polygon is defined by

$$P = \overline{A_1A_2} \cup \overline{A_2A_3} \cup \cdots \cup \overline{A_{n-1}A_n} \cup \overline{A_nA_1}.$$

The segments $\overline{A_1A_2}$, etc., are called the *sides* of the polygon. The sides of a polygon are one-dimensional and have no area. Corresponding to each polygon in the plane there is a *region*, which consists of the points of the polygon itself together with the points inside the polygon. It is the region that is two-dimensional and has area. The distinction between the polygon and the corresponding polygonal region will be important in this course because Geometer's Sketchpad treats the two as different objects that must be constructed separately.

For each polygonal region in the plane there is a nonnegative number called the *area* of the region. The area of a region R is denoted by $\alpha(R)$. The area of a triangular region is given by the familiar formula

$$\text{area} = (1/2)\ \text{base} \times \text{height}.$$

The other important property of area is that it is *additive*, which means that the area of a region that is the union of two nonoverlapping subregions is the sum of the areas if the subregions.

EXERCISES

0.11.1. Prove that the area of a triangle $\triangle ABC$ is given by the formula

$$\alpha(\triangle ABC) = \frac{1}{2} AB \cdot AC \cdot \sin(\angle BAC).$$

C H A P T E R 1

The elements of GSP

In this chapter we take a quick tour of *Geometer's Sketchpad* (GSP). The objective is to become acquainted with the basic capabilities of the software. GSP is relatively easy use to use and experienced computer users will pick it up quickly—you should simply rely on your intuition as you try out functions and observe what happens. You will continue to learn more about various features as you use them throughout in the course.

The basic tools and commands that are explained in this chapter are enough for you to begin creating sketches of your own. Other GSP techniques will be introduced in later chapters. In particular, the use of custom tools and action buttons will be discussed in Chapter 3. In case you find you want more specific information on these or other features, you should consult the GSP Help menu or the documentation that came with the software.

Throughout this chapter and the remainder of the book, exercises that are to be worked using GSP are marked with *.

1.1 THE TOOLBOX

Start the GSP program. Click anywhere to make the GSP logo go away. A blank document should be open; if not, choose New Sketch under the File menu. Note that a GSP document is called a *sketch*.

On the upper left you should see a tool bar containing six icons. (If the tool bar is not visible, choose Show Toolbox under the Display menu.) Three of these tools allow you to make the geometric constructions that Euclid made with his compass and straightedge; the others allow you to select items in your document, to add text to your document, and to create your own custom tools. Here is a quick description of each, starting at the top of the tool bar. You should try out each of the tools as you read the description of how it works.

- Selection Tool. Click on the arrow icon to activate the tool. The selection tool is used to select or deselect items in a sketch. Items that have been selected

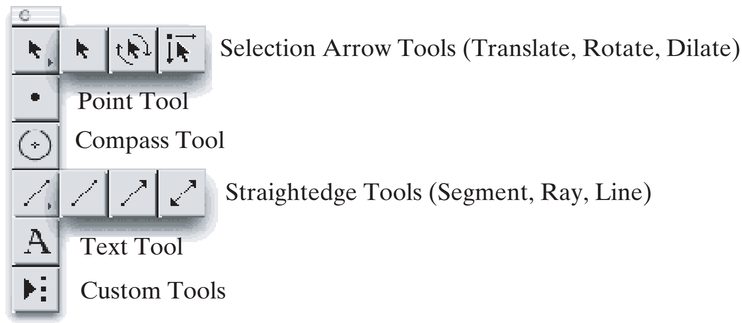

FIGURE 1.1: The toolbox

remain selected until they are deselected or the select tool is clicked on a blank portion of the sketch. The selection tool also allows you to move objects. To do so you simply select the objects and then drag them in the sketch window. There are three variations of this tool that allow three different kinds of motion: drag-translate (the plain arrow), drag-rotate or turn, and drag-dilate or shrink. To access the other variants, hold down the mouse button on the tool icon and slide over to the one you want. For now it is enough to use just the plain arrow tool.

- Point Tool. The point tool is used to create points. Click on the point icon to activate the tool. Then click anywhere in the sketch to create a point. The points you create should be labeled with upper case letters. If they are not, go to Preferences under either the Sketchpad or Edit menu. Click on the Text tab and then click on "Show labels automatically for all new points."
- Compass Tool. The compass tool is used to draw circles. Click on the circle icon to activate it. To draw a circle in your sketch, simply click in two different locations. The first click determines the center of the circle and the second click determines a point on the circle. Instead of clicking twice, you can also press the mouse button at the center, hold down the button, and drag—the point at which you first press the button will be the center and the point at which you release the mouse button will be on the circle.
- Straightedge Tools. These tools draw straight objects. There are three different kinds of straight objects (segments, rays, and lines) and there is one straightedge tool for each. To move from one of these straightedge tools to another, hold down the mouse button on the tool icon and slide over to the variant you want. To use the tool, simply click on the icon to activate it and then click on two points in the sketch. Again you have the option to press and drag rather than clicking twice.
- Text Tool. This tool allows you to add text to a sketch. Click on the text icon to activate the tool, then drag in the sketch window to create a text box in which you can type text. This tool can also be used to add (or remove) a label.

To use it this way, activate the tool and then click on the object you want to label. If you double click on an object, a box will come up and you can type in whatever label you want to use.

- Custom Tools. This tool allows you to create your own custom tools. The ability to create custom tools will ultimately prove to be one of the most useful and powerful features of GSP, but creating custom tools will not be discussed until after you have mastered the GSP basics.

1.2 PARENTS AND CHILDREN

There are three kinds of points in GSP: independent points, movable points, and dependent points. An *independent point* is one that is not constrained in any way. It is constructed without reference to any other previously defined object. A *movable point* is one that is constructed on a path object, such as a line or a circle. The point is "movable" in the sense that it is free to move along the path object, but it is constrained in the sense that it cannot move off the path object. If you activate the point tool and hold it over a line or segment, the line or segment will be highlighted. Clicking the tool while the path object is highlighted will construct a movable point. A *constructed point* is one that is completely determined by other objects; examples include the point of intersection of two lines and the midpoint of a segment. For example, if two intersecting lines are both highlighted when you click the point tool, the resulting point will be the intersection of the two lines. It is completely constrained in the sense that it cannot move unless the lines also move. The process of constructing such a point is called *marking* the point of intersection.

The relationship between a GSP object and other objects constructed from it is described using the parent-child metaphor. Thus a movable point on a circle is a *child* of the circle and the circle is a *parent* of the point. This parent-child relationship will become more important as we construct more complicated diagrams in which the relationships extend across many generations. One thing to note is that when an object is deleted from a sketch, all objects that are children of that object are also deleted (as are children of children, etc.). You can find a list of all an object's parents and children by selecting the object and then going to the Properties panel under the Edit menu.

EXERCISES

***1.2.1.** Construct a segment. Use the selection tool to choose one of the endpoints of the segment and then animate the point using the Animate Points command under the Display menu. While the point is moving, choose Trace Point under the Display menu and observe what happens.

***1.2.2.** Construct a movable point on a segment. Animate the point as in the previous exercise. (Be sure that only the movable point is selected; deselect other objects, if necessary.) Note what happens when this point is animated and contrast that with what happened in the previous exercise. Use the Properties panel to check the parent-child relationships.

***1.2.3.** Construct two intersecting lines. Mark the point of intersection. Make a prediction about what will happen when you animate the point of intersection. Then select the point of intersection (and nothing else) and use the Animate Intersection command to animate it. Note what happens. Is this what you predicted? Use the Trace Point command to trace the point while it is still animated. What happens to the trace of the point when the two lines pass through a configuration in which they are parallel?

1.3 CONSTRUCTIONS

The Construct menu allows you to construct various objects such as points, segments, and circles. When you use one of the commands under the Construct menu, you first select the objects that are to be the parents of the object you are constructing and then you choose the appropriate command under the Construct menu to cause the children to be born. This more or less reverses the order that was used when objects were constructed using the toolbox.

For example, you can construct a segment by first choosing the segment tool in the tool box and then clicking on two points. Alternatively, you could select two points that have already been constructed and then choose Segment under the Construct menu to accomplish the same thing. As another example, you can mark a point of intersection in two different ways. Assume two lines have already been constructed. If you activate the point tool, move the cursor to the point of intersection, and wait for both lines to be highlighted, then clicking will mark the point of intersection. The alternative way to accomplish this is to first select both lines and then choose Intersection under the Construct menu.

EXERCISES

***1.3.1.** Perform the following constructions in order to familiarize yourself with the workings of the Construct menu.

- **(a)** Construct segments in the two different ways described in the last paragraph.
- **(b)** Construct the midpoint of a segment.
- **(c)** Construct the perpendicular bisector of a segment. Check that the perpendicular bisector continues to be the perpendicular bisector when the segment is moved.
- **(d)** Construct a movable point on a segment, first using the point tool and then using the Point On Object command.
- **(e)** Construct two intersecting lines and mark the point of intersection in the two different ways described in the last paragraph of the section.
- **(f)** Construct a line ℓ and a point P that does not lie on ℓ. Use the Parallel Line command to construct the line through P that is parallel to ℓ. Use the Perpendicular Line command to construct the line through P that is perpendicular to ℓ.
- **(g)** Construct an angle $\angle BAC$ and use the Angle Bisector command to construct the bisector of $\angle BAC$. [Hint: An angle is the union of two rays with the same endpoint. To construct the bisector of $\angle BAC$, first select the points B, A, and C (in that order) and then use the Angle Bisector command.]

***1.3.2.** Construct a circle and a tangent line to the circle. Your construction should be robust enough so that the line remains tangent to the circle even when the circle and the point of tangency are moved. [Hint: Use the tangent line theorem.]

***1.3.3.** An *equilateral* triangle is a triangle in which all three sides have equal lengths. Euclid's very first proposition asserts that for every segment $\overline{AB}$, there is a point C such that $\triangle ABC$ is equilateral. Find Euclid's proof on the world wide web (or on page 6 of [16]) and reproduce it in GSP.

***1.3.4.** Construct a segment $\overline{AB}$. Construct a square that has $\overline{AB}$ as a side. [Hint: Construct a perpendicular line at A. Then construct a circle with center A and radius AB. Mark one of the points at which the perpendicular line intersects the circle. This is a third vertex of the square.]

1.4 MEASUREMENT AND CALCULATION

The commands in the Measure menu can be used to measure lengths, angles, and other quantities. The items to be measured must be selected first and then the measurement can be made. When an item has been measured, a text box containing the result appears in the sketch. The box can be moved to any convenient location in the sketch.

Once quantities have been measured, the Calculate command can be used to perform calculations on them. Choose Calculate under the Measure menu and a calculator will appear on the screen. You can enter quantities by typing them in or by clicking on previously measured quantities. One very nice and useful feature of the calculator is the fact that you can leave the measured and calculated quantities displayed while you do further constructions. As you move points around and make other changes to the sketch, the measurements and calculations are automatically updated and you can easily observe how they change.

EXERCISES

***1.4.1.** Go back to the equilateral triangle you constructed in Exercises 1.3.3. Measure the lengths of the three sides of the triangle and observe that they are equal. Measure the three angles in the triangle and observe that the angles too have equal measure.

***1.4.2.** Construct a pair of perpendicular lines. (Don't just construct lines that look approximately perpendicular, but make lines that are exactly perpendicular using the Perpendicular Line command.) Label the intersection point C and construct a point A on one line and B on the other. Now construct the right triangle $\triangle ABC$. Measure the lengths a, b, and c. Calculate $a^2 + b^2$ and c^2. Observe that $a^2 + b^2 = c^2$ regardless of how the points A and B are moved about on the two lines.

***1.4.3.** Construct a pair of parallel lines. Construct a point A on one line and construct points B and C on the other line. Construct the triangle $\triangle ABC$ and measure the area of $\triangle ABC$. [Hint: It is not the triangle itself that has area, but the triangle interior. To construct the triangle interior, select the three vertices and then use the Triangle Interior command under the Construct menu. Once the triangle interior has been constructed, it can be measured by choosing Area under the Measure menu.] Now animate the point A and observe that the area remains constant. [Make certain that the point moves but that the line does not.] Can you explain this in terms of the usual formula for the area of a triangle?

***1.4.4.** Construct a triangle, measure each of the interior angles, and calculate the sum of the measures of the three angles. Observe that the sum remains constant even when the vertices of the triangle are moved about in the plane.

***1.4.5.** Construct a triangle and an exterior angle for the triangle. Measure the exterior angle and compare with the measures of the two remote interior angles. What is the exact relationship between the measure of the exterior angle and the measures of the two remote interior angles? Verify your answer by calculating the sum of the measures of the two remote interior angles. Move the vertices of the triangle around and make sure that the relationship you discovered is correct for triangles of all shapes.

1.4.6. State and prove the theorem you discovered in the preceding exercise. It should improve on the Exterior Angle Theorem.

***1.4.7.** Construct a circle γ and three points A, B, and C on γ. Measure $\angle BAC$. Then animate A and observe what happens to $\mu(\angle BAC)$. How many different values are assumed? How are they related? Can you explain this relationship?

***1.4.8.** Construct a quadrilateral $\square ABCD$ and then construct the midpoints of the four sides. Label the midpoints E, F, G, H and construct the quadrilateral $\square EFGH$. This new quadrilateral is called the *midpoint quadrilateral* for $\square ABCD$. Regardless of the shape of the original quarilateral, the midpoint quadrilateral always has a special property. What special kind of quadrilateral is $\square EFGH$? Move the points A, B, C, and D around in the plane to change the shape of $\square ABCD$ in order to test your conjecture. Modify your conjecture as necessary.

***1.4.9.** Construct a quadrilateral, the two diagonals of the quadrilateral, and the midpoints of the diagonals of the quadrilateral. Experiment with the quadrilateral to determine answers to the following questions. (It might be helpful to review the names of the various kinds of quadrilaterals defined on page 8 before you begin.)

(a) For which quadrilaterals do the diagonals bisect each other? (Two segments *bisect* each other if they intersect at their midpoints.)

(b) For which quadrilaterals do the diagonals bisect each other and have the same length?

(c) For which quadrilaterals do the diagonals bisect each other and intersect at right angles?

The theorems you discovered in the last two exercises will be proved in Chapter 6.

1.5 TRANSFORMATIONS

Sketchpad can perform all the standard transformations of Euclidean geometry: translations, rotations, dilations, and reflections. Performing a transformation in GSP is a two-step process. First you must specify the lines and points that determine the transformation, and then you must apply the transformation to selected objects. The first step is called *marking the parameters* of the of the transformation.

For example, to reflect across a line ℓ, use the arrow tool to select a line and then choose Mark Mirror under the Transform menu. Next select an object or objects in your sketch. Finally, choose Reflect under the Transform menu. When you do that, the object you selected should be reflected across the line you marked.

To specify a translation, select two points A and B and then choose Mark Vector. The resulting translation will translate the selected objects a distance AB in the direction of the vector from A to B. To apply the translation, select the objects you want to translate and then choose Translate from the Transform menu. A dialog box will pop up with Marked chosen in the top row. Simply click Translate to apply the translation. You also have the option of clicking a different radio button in the top row of the dialog box; in that case you will be asked to specify the translation vector in terms of either rectangular or polar coordinates.

A rotation is determined by a point (the center of rotation) and an angle. You can either mark an angle in your sketch and use that as the angle of rotation, or you can enter an angle measure into the dialog box that appears when you choose Rotate from the Transform menu.

A dilation is determined by a point and a ratio (or scale factor). The point is marked in the usual way, but there are three different ways to mark the ratio. One option is to select two segments in your sketch and then choose Mark Segment Ratio from the Transform menu. The scale factor is then the ratio of the two segment lengths. A second method is to choose three collinear points A, B, and C (in order) and then choose Mark Ratio from the Transform menu. The dilation will then have a scale factor of AC/AB. The final option is to enter the ratio numerically after Dilate has been chosen.

EXERCISES

***1.5.1.** Construct two perpendicular lines ℓ and m. Construct a stick figure. Reflect the figure across ℓ and then reflect across m. Verify that the composition of those two reflections is the same as a rotation through 180°.

***1.5.2.** Construct two parallel lines ℓ and m. Construct a stick figure. Reflect the figure across ℓ and then reflect across m. Verify that the composition of those two reflections is a translation. How is the translation vector related to the lines ℓ and m?

***1.5.3.** Construct a triangle $\triangle ABC$. Translate the entire triangle using the vector from B to C. Let $A' = T(A)$. Measure the angle $\angle ACA'$ and compare with the measure of $\angle BAC$.

1.5.4. Explain the relationship between the angle measures in the preceding exercise in terms of theorems from Chapter 0. Use this relationship to give a transformational proof of the angle sum theorem.

***1.5.5.** Construct a triangle $\triangle ABC$ and a segment $\overline{DE}$ such that $\overline{DE}$ is longer than $\overline{AB}$ and not parallel to it. Construct a triangle $\triangle DEF$ such that $\triangle DEF \sim \triangle ABC$ as follows: First dilate $\triangle ABC$ to make a triangle $\triangle A'B'C'$ such that $A'B' = DE$. Then translate A' to D. Finally, rotate the triangle to make the sides line up.

1.6 ENHANCING THE SKETCH

As you become a more experienced user of GSP you should begin to pay more attention to the subjective aspects of your sketches. If you take the time to add details that make your diagram easy to comprehend and include appropriate explanatory text, you will make your sketch much more user friendly. You should strive for a document that the reader can understand without undue effort as well as one that is pleasing to look at.

A good way to avoid clutter in the sketch is to hide unnecessary objects. Any objects that are needed as part of the construction but are not important to the final product should be hidden. In order to hide objects, select them with the selection tool and then choose Hide Objects under the Display menu. The objects can be brought back out of hiding with the Show All Hidden command.

Another way to improve the organization of your sketches is to divide them into pages. In order to do so, go to Document Options under the File menu. You can add pages to your document and name them individually. When you do this, tabs will appear along the bottom of your document showing the names you have chosen for the pages. By clicking on these tabs you can move from one page to another.

You should make use of the Line Width, Color, and Text commands under the Display menu. Good use of colors to distinguish the various objects in your sketches can make the sketches much easier to comprehend.

You should also add text boxes in which you give the reader any information he or she needs in order to understand your sketch. You should explain the construction, what it illustrates, and what conclusions you drew from it. In the sketches you produce as solutions to exercises, you should include text boxes in which you record any observations you made or conclusions you drew from your observations.

Finally, you should make sure that your sketch is robust. The diagram should not fall apart when points and other objects are moved. In order to accomplish this, you must make certain that the appropriate parent-child relationships hold. A tangent line should remain tangent to the circle when the circle is moved and the midpoint of a segment should remain the midpoint when the segment is moved, etc.

EXERCISES

***1.6.1.** Go back to the sketch you produced for Exercise 1.3.2. Hide everything except the circle, a point on the circle, and the tangent line at that point. Use color and line width to clearly distinguish the line from the circle. Add a text box in which you explain how you constructed the tangent line.

***1.6.2.** Go back to the sketch you made for Exercise 1.4.5. Add a text box in which you explain what the diagram illustrates and what exploration the reader can do to see this for herself. Now add a second page to the sketch. On the second page, include text boxes containing a statement of the theorem from Exercise 1.4.6 and your proof.

***1.6.3.** Go back to the sketch you made for Exercise 1.4.8. Make use of color and line width to distinguish the original quadrilateral from the associated midpoint quadrilateral. Add a text box that tells the reader what the sketch illustrates and what she can do with the sketch to explore this relationship for herself.

For the remainder of the course you should try to include the kinds of enhancements discussed in this section in all of your GSP sketches.

CHAPTER 2

The classical triangle centers

This chapter studies several points associated with a triangle. These points are all *triangle centers* in the sense that each of them can claim to be at the center of the triangle in a certain sense. They are *classical* in that they were known to the ancient Greeks. The classical triangle centers form a bridge between elementary and advanced Euclidean geometry. They also provide an excellent setting in which to develop proficiency with GSP.

While the three triangle centers were known to the ancient Greeks, the ancients somehow missed a simple relationship between the three centers. This relationship was discovered by Leonhard Euler in the eighteenth century. Euler's theorem serves as a fitting introduction to advanced Euclidean geometry because Euler's discovery can be seen as the beginning of a revival of interest in Euclidean geometry and most of the theorems we will study in the course were discovered in the century after Euler lived.[1] Euler's original proof of his theorem was a complicated analytic argument, but it is simple to discover (the statement of) the theorem with GSP.

2.1 CONCURRENT LINES

Definition. Three lines are *concurrent* if there is a point P such that P lies on all three of the lines. The point P is called the *point of concurrency*. Three segments are concurrent if they have an interior point in common.

Two arbitrary lines will intersect in a point—unless the lines happen to be parallel, which is unusual. Thus concurrency is an expected property of two lines. But it is rare that three lines should have a point in common. One of the surprising and beautiful aspects of advanced Euclidean geometry is the fact that so many triples of lines determined by triangles are concurrent. Each of the triangle centers in this chapter is an example of that phenomenon.

[1] See Chapter 7 of [7] for a nice discussion of Euler's contributions to geometry.

2.2 THE CENTROID

Definition. The segment joining a vertex of a triangle to the midpoint of the opposite side is called a *median* for the triangle.

EXERCISES

***2.2.1.** Construct a triangle $\triangle ABC$. Construct the three midpoints of the sides of $\triangle ABC$. Label the midpoints D, E, and F in such a way that D lies on the side opposite A, E lies on the side opposite B, and F lies on the side opposite C. Construct the three medians for $\triangle ABC$. What do the three medians have in common? Verify that this continues to be true when the vertices of the triangle are moved around in the plane.

***2.2.2.** In the preceding exercise you should have discovered that the three medians are concurrent (they have a point in common). Mark the point of intersection and label it G. Measure AG and GD, and then calculate AG/GD. Make an observation about the ratio. Now measure BG, GE, CG, and GF, and then calculate BG/GE and CG/GF. Leave the calculations displayed on the screen while you move the vertices of the triangle. Make an observation about the ratios.

The exercises you just completed should have led you to discover the following theorem.

Median Concurrence Theorem. *The three medians of any triangle are concurrent; that is, if $\triangle ABC$ is a triangle and D, E, and F are the midpoints of the sides opposite A, B, and C, respectively, then $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ all intersect in a common point G. Moreover, $AG = 2GD$, $BG = 2GE$, and $CG = 2GF$.*

Definition. The point of concurrency of the three medians is called the *centroid* of the triangle. The centroid is usually denoted by G.

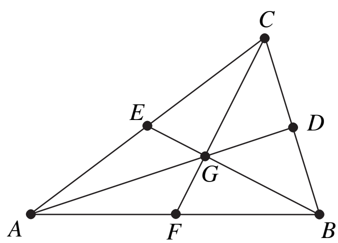

FIGURE 2.1: The three medians and the centroid

We will not prove the concurrency part of the theorem at this time. That proof will be postponed until after we have developed a general principle, due to Ceva, that allows us to give a unified proof of all the concurrence theorems at the same time. For now we will assume the concurrency part of the theorem and prove that the centroid divides each median in a 2:1 ratio. That proof is outlined in the next four exercises.

EXERCISES

***2.2.3.** Construct a triangle $\triangle ABC$, then construct the midpoints of the three sides and label them as in Exercise 2.2.1. Construct the two medians $\overline{AD}$ and $\overline{BE}$ and the segment $\overline{DE}$. Check, by measuring angles, that $\triangle ABC \sim \triangle EDC$. Then check that $\triangle ABG \sim \triangle DEG$.

Note. The notation of Exercise 2.2.3 is assumed in the next three exercises.

2.2.4. Use theorems from Chapter 0 to prove that $\triangle ABC \sim \triangle EDC$ and that $AB = 2ED$.

2.2.5. Use Theorems from Chapter 0 to prove that $\triangle ABG \sim \triangle DEG$.

2.2.6. Use the two preceding exercises to prove that $AG = 2GD$ and $BG = 2GE$. Explain how this allows you to conclude that $CG = 2GF$ as well.

2.2.7. Explain why the centroid is the center of mass of the triangle. In other words, explain why a triangle made of a rigid, uniformly dense material would balance at the centroid.

***2.2.8.** The three medians subdivide the triangle $\triangle ABC$ into six smaller triangles. Determine by experimentation the shape of $\triangle ABC$ for which these six subtriangles are congruent. Measure the areas of the six subtriangles and verify that the areas are always equal (regardless of the shape of the original triangle).

2.2.9. Supply a proof that the six subtriangles in the preceding exercise have equal areas.

2.3 THE ORTHOCENTER

Definition. The line determined by two vertices of a triangle is called a *sideline* for the triangle.

Definition. An *altitude* for a triangle is a line through one vertex that is perpendicular to the opposite sideline. The point at which an altitude intersects the opposite sideline of a triangle is called the *foot* of the altitude.

EXERCISES

***2.3.1.** Construct a triangle and construct the three altitudes for the triangle. Observe that no matter how the vertices of the triangle are moved around in the plane, the three altitudes continue to be concurrent.

The exercise should have confirmed the following theorem. It will be proved in Chapter 8.

Altitude Concurrence Theorem. *The three altitudes of any triangle are concurrent.*

Definition. The point of concurrency of the three altitudes is called the *orthocenter* of the triangle. It is usually denoted by H.

EXERCISES

***2.3.2.** Construct another triangle. Mark the orthocenter of your triangle and label it H. Move one vertex and watch what happens to H. Add the centroid G to your sketch and again move the vertices of the triangle. Observe that the centroid always stays inside the triangle, but that the orthocenter can be outside the triangle or even on the triangle.

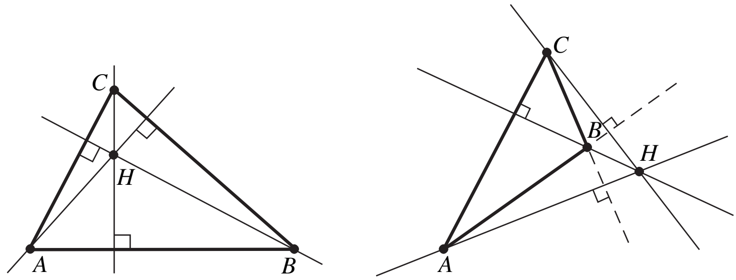

FIGURE 2.2: Two triangles with their three altitudes and orthocenter

***2.3.3.** Determine by experimentation the shape of triangles for which the orthocenter is outside the triangle. Find a shape for the triangle so that the orthocenter is equal to one of the vertices of the triangle. Observe what happens to the orthocenter when one vertex crosses over the line determined by the other two vertices. Make notes on your observations.

2.3.4. Prove that the centroid is always inside the triangle.
[Hint: The inside of the triangle is the intersection of the interiors of the three interior angles of the triangle.]

***2.3.5.** Determine by experimentation whether or not it is possible for the centroid and the orthocenter to be the same point. If it is possible, for which triangles does this happen?

2.4 THE CIRCUMCENTER

In this section you will verify that the perpendicular bisectors of the three sides of any triangle are concurrent.

Definition. The point of concurrency of the three perpendicular bisectors of the sides of a triangle is called the *circumcenter* of the triangle. The circumcenter is usually denoted by O.

The reason for the name "circumcenter" will become clear later when we study circles associated with triangles. The fact that the three perpendicular bisectors of any triangle are concurrent will be proved in Chapter 8.

EXERCISES

***2.4.1.** Construct a triangle $\triangle ABC$ and construct the three perpendicular bisectors for the sides of your triangle. These perpendicular bisectors should be concurrent. Mark the point of concurrency and label it O.

***2.4.2.** Move one vertex of the triangle around and observe how the circumcenter changes. Note what happens when one vertex crosses over the line determined by the other two vertices. Find example triangles which show that the circumcenter may be inside, on, or outside the triangle, depending on the shape of the triangle. Make notes on your findings.

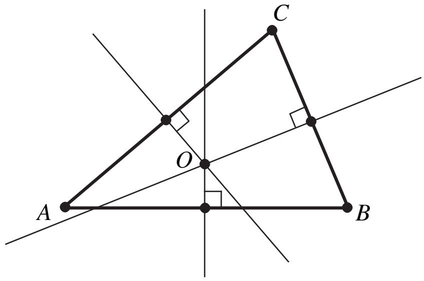

FIGURE 2.3: The three perpendicular bisectors and the circumcenter

*__2.4.3.__ Measure the distances OA, OB, and OC and make an observation about them.
2.4.4. Prove that the circumcenter is equidistant from the vertices of the triangle.
*__2.4.5.__ Determine by experimentation whether or not it is possible for the circumcenter and the centroid to be the same point. If it is possible, for which triangles does this happen?

2.5 THE EULER LINE

In this section we will investigate the relationship between the three triangle centers that was mentioned at the beginning of the chapter. The theorem was discovered by the German mathematician Leonhard Euler (1707–1783).

EXERCISES

*__2.5.1.__ Construct a triangle $\triangle ABC$ and construct all three centers G, H and O. Hide any lines that were used in the construction so that only the triangle and the three centers are visible. Put a line through two of the centers and observe that the third also lies on that line. Verify that G, H, and O continue to be collinear even when the shape of the triangle is changed.
*__2.5.2.__ Measure the distances HG and GO. Calculate HG/GO. Leave the calculation visible on the screen as you change the shape of your triangle. Observe what happens to HG/GO as the triangle changes.

In the exercises above you should have discovered the following theorem.

Euler Line Theorem. *The orthocenter H, the circumcenter O, and the centroid G of any triangle are collinear. Furthermore, G is between H and O (unless the triangle is equilateral, in which case the three points coincide) and $HG = 2GO$.*

Definition. The line through H, O, and G is called the *Euler line* of the triangle.

The proof of the Euler Line Theorem is outlined in the following exercises. Since the existence of the three triangle centers depends on the concurrency theorems stated earlier in the chapter, those results are implicitly assumed in the proof.

EXERCISES

2.5.3. Prove that a triangle is equilateral if and only if its centroid and circumcenter are the same point. In case the triangle is equilateral, the centroid, the orthocenter, and the circumcenter all coincide.

2.5.4. Fill in the details in the following proof of the Euler Line Theorem. Let $\triangle ABC$ be a triangle with centroid G, orthocenter H, and circumcenter O. By the previous exercise, it may be assumed that $G \neq O$ (explain why). Choose a point H' on $\overrightarrow{OG}$ such that G is between O and H' and $GH' = 2OG$. The proof can be completed by showing that $H' = H$ (explain). It suffices to show that H' is on the altitude through C (explain why this is sufficient). Let F be the midpoint of $\overline{AB}$. Use Exercise 2.2.6 and the SAS Similarity Criterion to prove that $\triangle GOF \sim \triangle GH'C$. Conclude that $\overleftrightarrow{CH'} \parallel \overleftrightarrow{OF}$ and thus $\overleftrightarrow{CH'} \perp \overleftrightarrow{AB}$.

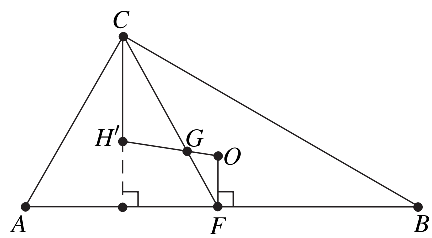

FIGURE 2.4: Proof of Euler Line Theorem

***2.5.5.** Figure 2.4 shows a diagram of the proof of the Euler Line Theorem in case the original triangle is acute. Use GSP to experiment with triangles of other shapes to determine what the diagram looks like in case $\triangle ABC$ has some other shape.

2.5.6. Prove that a triangle is isosceles if and only if two medians are congruent (i.e., have the same length).

2.5.7. Prove that a triangle is isosceles if and only if two altitudes are congruent.

***2.5.8.** Construct a triangle $\triangle ABC$ and the bisectors of angles at A and B. Mark the points at which the angle bisectors intersect the opposite sides of the triangle and label the points D and E, respectively. The two segments $\overline{AD}$ and $\overline{BE}$ are called *internal angle bisectors*. Measure the lengths of the two internal angle bisectors and then measure the lengths of the sides $\overline{BC}$ and $\overline{AC}$. Use your measurements to verify the following theorem: *A triangle is isosceles if and only if two internal angle bisectors are congruent.*

The theorem in the last exercise is known as the *Steiner-Lehmus Theorem* because the question was proposed by C. L. Lehmus in 1840 and the theorem was proved by the Swiss mathematician Jacob Steiner in 1842. A proof of the theorem and discussion of the interesting history of the proof may be found in §1.5 of [3].

C H A P T E R 3

Advanced techniques in GSP

This chapter introduces two features of GSP that greatly enhance its usefulness. Custom tools allow you to quickly accomplish routine constructions and action buttons allow you to make sketches that illustrate a process, not just a finished product.

3.1 CUSTOM TOOLS

One of the most effective ways to tap the power of GSP is to create custom tools to perform constructions that must be repeated many times. Once you learn how to make custom tools of your own, you should begin to accumulate a collection of specialized tools that you can use when you need them.

To create a custom tool, you begin by making a GSP sketch that includes the construction you want to make into a tool. Once you have made the sketch, you use the selection arrow to select the the objects in the construction and then choose Create New Tool under the Custom Tool icon in the tool bar. You must be careful to select a set of objects that are related in such a way that at least one of them is completely determined by the others. The objects you selected that are to be produced by the tool are called the *results* and the objects that don't depend on any others are called *givens*. Any object that is part of the construction but is neither a given nor a result is called an *intermediate object*.

Here is a specific example. Suppose we wish to create a tool that constructs the perpendicular bisector of a segment. The first step is to construct two points A and B, the segment $\overline{AB}$ joining them, the midpoint M of the segment, and finally the line ℓ that is perpendicular to $\overline{AB}$ and passes through M. The perpendicular bisector of a segment is completely determined by the endpoints of the segment, so you can create the tool by selecting the two points A and B and the line ℓ. Choose Create New Tool and a dialog box will pop up. Type in the name you want to give your tool and click OK. Now your tool is ready to use and its name should appear in the menu that pops up when you press on the Custom Tool icon. To use it, simply choose it by name from the pop-up menu and then click in your sketch. The first time you click, the point A should be constructed; the second time you click, both the point B and the line ℓ will be constructed.

In the example above, the points A and B are the givens and the line ℓ is the result. The segment $\overline{AB}$ and the midpoint M are intermediate objects. When you select the objects that are to be part of your tool, you can control which of them are to be shown in the result and which are to be hidden. It is usually best to make tools that have only points as givens, but it is also possible to use other objects as the givens. For example, you could use the segment $\overline{AB}$ as the given in your perpendicular bisector tool instead of the endpoints A and B. To accomplish this, you simply select the segment $\overline{AB}$ and the line ℓ when you create the tool. On the other hand, you could include the segment $\overline{AB}$ and midpoint M as results. To accomplish that, you should select all of the objects you have constructed (the points A and B, the segment $\overline{AB}$, the midpoint M and the line ℓ) when you create the tool. If you make the tool that way, the givens are A and B since they are the parents of everything else in the construction.

EXERCISES

***3.1.1.** Create a Perpendicular Bisector tool exactly as in the example above. Your tool should accept two points as givens and should produce only the perpendicular bisector as result.

***3.1.2.** Create a second version, Perpendicular Bisector 2, of the perpendicular bisector tool. This one should again have the two points as givens, but should produce the segment, the midpoint, and the perpendicular bisector as results.

***3.1.3.** Make a third version of the perpendicular bisector tool that accepts a segment as the only given and produces both the midpoint and the perpendicular bisector as results.

Any tool you make becomes part of the sketch in which it is created. The tool will be available for use any time that particular sketch is open. The Sketchpad directory that is installed on your computer includes a Tool Folder. Any custom tool that is part of a document stored in this folder is automatically loaded when you start Sketchpad. It is best to put your tools in that folder, if possible, so that they are always available for use when you need them. The alternative is to open the sketch containing the tool and leave it open as long as you might want to use the tool; there is no limit to the number of Sketchpad documents you can have open at one time. It is usually best to include each tool in a separate Sketchpad document. The sketch and the custom tool contained in the sketch can have the same name.

If you choose Show Script View from the Custom Tool pop-up menu you will see a written description of the tool. It is sometimes helpful to look at that window in order to determine the object requirements of a given tool. You might also want to consult the script view if you are trying to determine what a tool does and how it was originally defined. There is a box at the top of the window where you can type comments; it is a good idea to include comments, both for your own later use and for the benefit of anyone else who might use the tool. You can also include explanatory text in the sketch itself.

EXERCISES

***3.1.4.** Make a custom tool that accepts two points as givens and whose result is an equilateral triangle with the given points as the endpoints of one side. Save your tool for future use. [Compare Exercise 1.3.3.]

***3.1.5.** Make a custom tool that accepts two points as givens and whose result is a square with the given points as the endpoints of one side. Save your tool for future use. [Compare Exercise 1.3.4.]

***3.1.6.** Make a custom tool that accepts a segment as given and whose result is a square region with the given segment as one side. Save your tool for future use.

***3.1.7.** Create a tool that accepts a segment as its given and constructs a circle with the given segment as diameter. Save your tool for later use.

3.2 ACTION BUTTONS

As the name suggests, action buttons allow you to add action to your GSP sketches. They also allow you to make more interesting presentations with GSP. In this section you will learn how to create action buttons; for the remainder of the course you should make use of them to improve your GSP sketches.

To create an action button, use the selection arrow to select some objects and then choose the appropriate type of action button from the Action Buttons submenu that is found under the Edit menu in GSP. The action button itself is considered to be a GSP object and the objects you selected when you created the button are its parents.

Here is a description of the various kinds of action buttons.

- Hide/Show. This button allows you to alternately hide or show an object or collection of objects. To use it, you simply select the objects you want to hide or show and then choose Hide/Show from the action button submenu. A button will appear in your document. When you click on it with the selection arrow, the objects will disappear; when you click on it a second time they will reappear.
- Animation. You can add a button that turns animation on or off. Simply select the objects you want to animate and then choose Animation from action button submenu. A window will appear that can can be used to control some of the characteristics of the animation if you wish. Click OK to make the window go away. An animate button will appear in the sketch. The first time you click on it with the selection arrow the animation will be turned on and the next time you click on the button the animation will be turned off.
- Movement. To use this button, select a pair of points in your sketch and then choose Movement in the action buttons submenu. Again a window will appear that can can be used to control some of the characteristics of the movement if you wish. Click OK to make the window go away. A button will appear in your sketch. When you click on it, the first point will move to the second. You also have the option of selecting several pairs of points at once;

when you do so, the Movement button will cause the first point in each pair to move to the second point in the pair.

- Presentation. A presentation button automatically activates a group of other buttons. The buttons can be activated either simultaneously or in sequence. Use a presentation button to choreograph a complex set of motions or to present a Sketchpad slide show. To create a presentation button, select a group of buttons by clicking on the black rectangle at the left edge of the buttons. Then choose Presentation under the action button submenu. A window will pop up in which you can control some of the characteristics of the presentation. Click OK to make the window go away.
- Link. A Link button allows you to link to another page of your document or to an internet URL.
- Scroll A scroll button scrolls the sketch window so that a selected point is located either in the center of the window or at the top left corner of the window. To create a scroll button, select a point and then choose Scroll under the action button submenu. Use a scroll button in large sketches to position the window or to move to a different part of your sketch.

3.3 THE EULER LINE REVISITED

The sketches you created in the last chapter can be greatly improved with the use of custom tools and action buttons.

EXERCISES

***3.3.1.** Make a GSP tool that constructs the centroid of a triangle. Your tool should accept three vertices of a triangle as the givens and produce the triangle and the centroid G as results. The tool should label the centroid G. The tool should not display the intermediate objects that were used in the construction of G.
[Hint: As you are constructing the centroid and before you create the custom tool, choose the Text tool and double click on the centroid. A Properties box will pop up. Type the label G in the box and click on Use Label in Custom Tools. Then proceed to create the tool. It will consistently name the centroid G.]

***3.3.2.** Make a GSP tool that constructs the orthocenter of a triangle. Your tool should accept three vertices of a triangle as givens and produce the triangle and the orthocenter H as results. The tool should label the orthocenter H. The tool should not display the intermediate objects that were used in the construction of H.

***3.3.3.** Make a GSP tool that constructs the circumcenter of a triangle. Your tool should accept three vertices of a triangle as givens and produce the triangle and the circumcenter O as results.

***3.3.4.** Make a new sketch that illustrates the Euler Line Theorem. First construct a triangle and then use the tools from the preceding exercises to create the three triangle centers. Hide any intermediate objects so that only the triangle itself and the three triangle centers are visible. Construct a line through two of the points and observe that it also passes through the third. Add hide/show buttons for the triangle, the three centers, and the Euler line. Add a presentation button that turns the sketch into a slide show that illustrates the Euler Line Theorem. Make sure that you have made good use of color and text boxes to make your sketch user friendly.

3.4 THE PYTHAGOREAN THEOREM REVISITED

The Pythagorean theorem is part of elementary Euclidean geometry, but it is worth studying again because of its importance to geometry generally. In the exercises below you will first explore the statement of the theorem and then look at Euclid's proof. Even though there are literally hundreds of different proofs of the Pythagorean theorem, Euclid's proof remains one of the most beautiful and elegant.

EXERCISES

*__3.4.1.__ Construct a right triangle. Use your tool from Exercise 3.1.6 to construct the square region on each side of the triangle. Measure the areas of the squares and verify the Pythagorean relationship.
[Hint: Be careful about the order in which the givens in your square tool are selected so that all three of the squares appear on the outside of the triangle.]

*__3.4.2.__ Construct a right triangle. Use your tool from Exercise 3.1.4 to construct an equilateral triangle on each side of the right triangle. Measure the areas of the associated triangular regions. Find a relationship between the areas.

*__3.4.3.__ Construct a right triangle. Use the tool from Exercise 3.1.7 to construct three circles whose diameters are the three sides of the triangle. (These circles will overlap.) Measure the areas of the circles and find a relationship between them. Explain how this relationship can be used to determine which is larger: a large pizza or the combination of one small and one medium pizza.[1]

3.4.4. State a theorem that summarizes the results of the last three exercises.
[Hint: Check Euclid's Proposition VI.31, if necessary.]

3.4.5. Find Euclid's proof of the Pythagorean theorem on the world wide web. Make sure you understand the proof.
[Hint: The Pythagorean theorem is Euclid's Proposition I.47. You can also find the proof on page 201 of [16]. The proof goes by the name of *bride's chair* for historical reasons that are quite obscure.]

*__3.4.6.__ Create a GSP sketch that illustrates Euclid's proof. The sketch should include action buttons that show objects as they are needed. It should also have buttons that show how triangular regions are related. There are two kinds of relationships: the first is a shear in which one vertex of a triangle moves along a line that is parallel to the base of the triangle and the second type of movement is a rotation about a point.

*__3.4.7.__ **Challenge Exercise.** Make a GSP sketch that animates Euclid's proof. Your sketch should show the triangles moving, first by a shear, then by a rotation, and finally by a second shear.

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[1]This is the origin of the familiar saying, "The pizza on the hypotenuse is equal to the sum of the pizzas on the legs."

C H A P T E R 4

Circumscribed circles, inscribed circles, and escribed circles

In this chapter we explore the construction of several circles that are associated with a triangle and also introduce more triangle centers. Again the emphasis is mainly on GSP exploration, with formal proofs of most results postponed until later. The chapter ends with a proof of Heron's formula for the area of a triangle.

4.1 THE CIRCUMSCRIBED CIRCLE AND THE CIRCUMCENTER

The first circle we will study is the circumscribed circle. It can be thought of as the smallest circle that contains a given triangle.

Definition. A circle that contains all three vertices of the triangle $\triangle ABC$ is said to *circumscribe* the triangle. The circle is called the *circumscribed circle* or simply the *circumcircle* of the triangle. The radius of the circumscribed circle is called the *circumradius*.

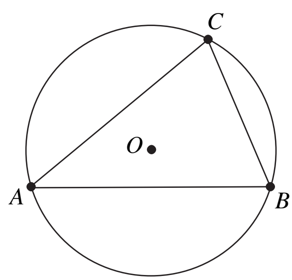

FIGURE 4.1: The circumcenter and the circumcircle

It is, of course, no accident that the term circumcenter was used as the name of one of the triangle centers that was introduced in Chapter 2. In fact, Exercise 2.4.4

shows that the three vertices of a triangle are equidistant from the circumcenter of the triangle, so all three vertices lie on a circle centered at the circumcenter. It follows that every triangle has a circumcircle and the circumcenter is the center of the circumcircle.

It should be noted that part of the definition of triangle is the assumption that the vertices are noncollinear. Thus the real assertion is that three noncollinear points determine a circle. Furthermore, the fact that we know how to construct the center of the circumcircle means we can construct the circumscribed circle itself.

EXERCISES

***4.1.1.** Make a tool that constructs the circumcircle of a triangle. The tool should accept three noncollinear points as given and produce a circle containing the three points as result. Make sure your tool is robust enough so that the circle you construct continues to go through all the vertices of the triangle even when the vertices of the triangle are moved.

***4.1.2.** Use the tool you made in the previous exercise to explore the circumscribed circle of various triangles. For which triangles is the circumcenter inside the triangle? For which triangles is the circumcenter on the edge of the triangle? What happens to the circumcircle when one vertex of the triangle is moved across the sideline determined by the other two vertices? Make notes on your observations.

4.1.3. Prove that the circumcircle is unique. In other words, prove that there can be at most one circle that passes through three given noncollinear points. How many circles pass through two given points?
[Hint: Begin by proving that any circle that passes through A, B, and C must have the circumcenter as its center.]

***4.1.4.** Construct a triangle $\triangle ABC$, its circumcircle γ, and measure the circumradius R of $\triangle ABC$. Verify the following result, which extends the usual Law of Sines that you learned in high school.
Extended Law of Sines. *If $\triangle ABC$ is a triangle with circumradius R, then*

$$\frac{BC}{\sin(\angle BAC)} = \frac{AC}{\sin(\angle ABC)} = \frac{AB}{\sin(\angle ACB)} = 2R.$$

4.1.5. Prove the extended law of sines.
[Hint: Let γ be the circumscribed circle of $\triangle ABC$ and let D be the point on γ such that $\overline{DB}$ is a diameter of γ. Prove that $\angle BAC \cong \angle BDC$. Use that result to prove that $\sin(\angle BAC) = BC/2R$. The other proofs are similar.]

4.2 THE INSCRIBED CIRCLE AND THE INCENTER

The second circle we study is called the *inscribed circle*, or simply the *incircle*. It is the opposite of the circumcircle in the sense that it is the largest circle that is contained in the triangle.

EXERCISES

***4.2.1.** Construct a triangle and the three bisectors of the interior angles of the triangle. Note that the three angle bisectors are concurrent. The point of concurrency is called the *incenter* of the triangle. Experiment with triangles of different shapes to determine whether the incenter can ever be on the triangle or outside the triangle. For which triangles is the incenter on the triangle? For which triangles is the incenter outside the triangle?

***4.2.2.** Construct another triangle and its incenter. Label the incenter I; keep it visible but hide the angle bisectors. Experiment with the triangle and the incenter in order to answer the following questions.

- **(a)** Are there triangles for which the incenter equals the circumcenter? What shape are they?
- **(b)** Are there triangles for which the incenter equals the centroid? What shape are they?

***4.2.3.** Construct another triangle and its incenter. Label the incenter I; keep it visible but hide the angle bisectors. For each side of the triangle, construct a line that passes through the incenter and is perpendicular to the sideline. Mark the feet of these perpendiculars and label them X, Y and Z as indicated in Figure 4.2. Measure the distances IX, IY, and IZ and observe that they are equal. This number is called the *inradius* of the triangle.

***4.2.4.** Hide the perpendiculars in your sketch from the last exercise, but keep the points I, X, Y, and Z visible. Construct the circle with center I and radius equal to the inradius. Observe that this circle is tangent to each of the three sides of the triangle. This circle is the inscribed circle for the triangle.

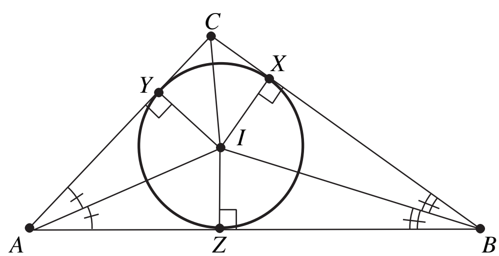

FIGURE 4.2: The incenter and the incircle

We now state a theorem and definition that formalize some of the observations you made in the exercises above.

Angle Bisector Concurrence Theorem. *If $\triangle ABC$ is any triangle, the three bisectors of the interior angles of $\triangle ABC$ are concurrent. The point of concurrency is equidistant from the sides of the triangle.*

Definition. The point of concurrency of the bisectors is called the *incenter* of the triangle. The distance from the incenter to sides of the triangle is the *inradius*. The circle that has its center at the incenter and is tangent to each of the sides of the triangle is called the *inscribed circle*, or simply the *incircle* of the triangle.

The concurrency part of the theorem will be assumed for now, but it will be proved in Chapter 8. You will prove the rest of the theorem in the exercises below.

EXERCISES

4.2.5. Prove that the point at which any two interior angle bisectors intersect is equidistant from all three sidelines of the triangle.

4.2.6. Use the definition of triangle interior to prove that the incenter is inside the triangle.

***4.2.7.** Make a tool that constructs the incenter of a given triangle.

***4.2.8.** Make a tool that constructs the inscribed circle of a given triangle.

4.3 THE ESCRIBED CIRCLES AND THE EXCENTERS

The incircle is not the only circle that is tangent to all three sidelines of the triangle. Any circle that is tangent to all three sidelines of a triangle is called an *equicircle* (or *tritangent circle*) for the triangle. In this section we will construct and study the remaining equicircles.

Definition. A circle that is outside the triangle and is tangent to all three sidelines of the triangle is called an *escribed circle* or an *excircle*. The center of an excircle is called an *excenter* for the triangle.

There are three excircles, one opposite each vertex of the triangle. The excircle opposite vertex A is shown in Figure 4.3; it is called the A*-excircle* and is usually denoted γ_A. There is also a B-excircle γ_B and a C-excircle γ_C.

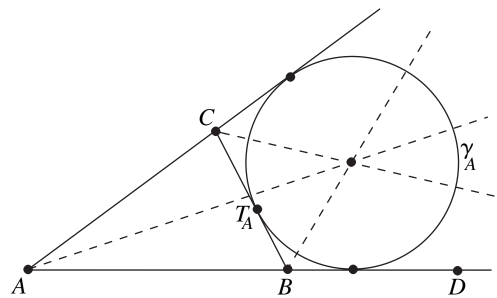

FIGURE 4.3: The A-excircle

Figure 4.3 indicates how the A-excircle is constructed. The A-excenter is the point at which the bisector of the interior angle at A and the bisectors of the exterior angles at B and C concur. (There are two exterior angles at B; when constructing the A-excenter use the exterior angle formed by extending $\overline{AB}$. At vertex C use the exterior angle formed by extending $\overline{AC}$.) Once the excenter has been located, a radius of the excenter is obtained by dropping a perpendicular from the excenter to one of the sidelines and marking the foot of the perpendicular.

EXERCISES

***4.3.1.** Construct a triangle $\triangle ABC$, the rays $\overrightarrow{AC}$ and $\overrightarrow{AB}$ and the three angle bisectors shown in Figure 4.3. Experiment with triangles of different shape to verify that the three angle bisectors are always concurrent.

***4.3.2.** Make a tool that constructs an excircle. Your tool should accept the three vertices of the triangle as givens and should construct the excircle opposite the first vertex selected as its result.
[Hint: Be careful with the construction of this tool. Start with the three vertices A, B, and C. Then choose a point D as shown in Figure 4.3. If you simply choose D to be a movable point on the sideline, it is not a well defined descendant of the vertices and GSP will not consistently place it where you want it. The result will be a tool that works correctly most of the time, but occasionally produces the incircle rather than the excircle. In order to specify the point D uniquely, you might define it to be the reflection of A across a line through B. Once D is defined, the excenter may be defined as the intersection the bisectors of $\angle CAB$ and $\angle CBD$.]

***4.3.3.** Construct a triangle, the three sidelines of the triangle, and the three excircles for the triangle. Explore triangles of different shapes to learn what the configuration of excircles looks like in different cases. What happens to the excircles when one vertex passes over the sideline determined by the other two vertices?

The fact that the three angle bisectors described above are always concurrent will be proved in Chapter 8. Assuming that result, the following exercise shows that the point of concurrency is an excenter.

EXERCISES

4.3.4. Prove that the point at which any two of the three angle bisectors shown in shown in Figure 4.3 intersect is equidistant from all three sidelines of the triangle.

4.4 THE GERGONNE POINT AND THE NAGEL POINT

There are two additional triangle centers that are associated with the inscribed and escribed circles.

EXERCISES

***4.4.1.** Construct a triangle $\triangle ABC$ and its incircle. Mark the three points at which the incircle is tangent to the triangle. The point of tangency opposite to vertex A should be labeled X, the point of tangency opposite B should be labeled Y, and the one opposite C should be labeled Z. Construct the segments $\overline{AX}$, $\overline{BY}$, and $\overline{CZ}$. Observe that the three segments are always concurrent, regardless of the shape of the triangle. The point of concurrency is call the *Gergonne point* of the triangle. It is denoted Ge. (Don't confuse it with the centroid G.)

***4.4.2.** Construct a triangle, the incenter I, and the Gergonne point Ge. For which triangles is $I = Ge$?

***4.4.3.** Construct a triangle and its three excircles. Mark the three points at which the three excircles are tangent to the triangle. Use the label T_A for the point at which the A-excircle is tangent to $\overline{BC}$, T_B for the point at which the B-excircle is tangent to $\overline{AC}$, and T_C for the point at which the C-excircle is tangent to $\overline{AB}$. (In GSP you can use T−A as a substitute for T_A.) Construct the segments $\overline{AT_A}$, $\overline{BT_B}$, and $\overline{CT_C}$. Observe that the three segments are always concurrent, regardless of the shape of the triangle. The point of concurrency is call the *Nagel point* of the triangle. It is denoted Na.

***4.4.4.** Is the Nagel point ever equal to the Gergonne point? If so, for which triangles?

***4.4.5.** Which of the three points I, Ge, or Na lies on the Euler line? Do the points you have identified lie on the Euler line for every triangle, or only for some triangles?

The Nagel point is named for the German high school teacher and geometer Christian Heinrich von Nagel (1803–1882) while the Gergonne point is named for the French mathematician Joseph Diaz Gergonne (1771–1859). Each of the Nagel and Gergonne points is defined as the point at which three segments are concurrent. You have accumulated GSP evidence that these segments do concur, but we have not actually proved that. The two concurrence theorems that allow us to define the Gergonne and Nagel points will be proved in Chapter 8. In that chapter we will also show that the two points are "isotomic conjugates" of one another.

4.5 HERON'S FORMULA

In this section we use the geometry of the incircle and the excircle to derive a famous formula that expresses the area of a triangle in terms to the lengths of the sides of the triangle. This formula is named for the ancient Greek geometer Heron of Alexandria who lived from approximately AD 10 until about AD 75. Even though the formula is commonly attributed to Heron, it was probably already known to Archimedes (287–212 BC).

Let us begin with some notation that will be assumed for the remainder of this section. Fix $\triangle ABC$. Let I be the center of the incircle and let E be the center of the A-excircle. The feet of the perpendiculars from I and E to the sidelines of $\triangle ABC$ are labeled X, Y, Z, T, U, and V, as indicated in Figure 4.4.

Define $a = BC$, $b = AC$, and $c = AB$. We will use r to denote the inradius of $\triangle ABC$ and r_a to denote the radius of the A-excircle.

Definition. The *semiperimeter* of $\triangle ABC$ is $s = (1/2)(a + b + c)$.

EXERCISES

4.5.1. Prove that the area of $\triangle ABC$ satisfies $\alpha(\triangle ABC) = sr$.

4.5.2. By the External Tangents Theorem, $AZ = AY$, $BZ = BX$, $CX = CY$, $BV = BT$, and $CT = CU$. In order to simplify notation, let $z = AZ$, $x = BX$, $y = CY$, $u = CU$, and $v = BV$.

(a) Prove that $x + y + z = s$, $x + y = a$, $y + z = b$, and $x + z = c$.
(b) Prove that $x = s - b$, $y = s - c$, and $z = s - a$.
(c) Prove that $u + v = x + y$ and $x + z + v = y + z + u$.
(d) Prove that $u = x$ and $v = y$.

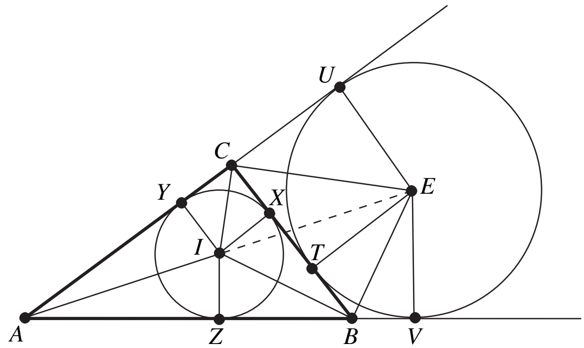

FIGURE 4.4: Notation for proof of Heron's formula

4.5.3. Prove that $AV = s$.
4.5.4. Use similar triangles to prove that $r_a/s = r/(s - a)$.
4.5.5. Prove that $\mu(\angle ZBI) + \mu(\angle EBV) = 90^\circ$. Conclude that $\triangle ZBI \sim \triangle VEB$.
4.5.6. Prove that $(s - b)/r = r_a/(s - c)$.
4.5.7. Combine Exercises 4.5.1, 4.5.4, and 4.5.6 to prove Heron's Formula:

$$\alpha(\triangle ABC) = \sqrt{s(s - a)(s - b)(s - c)}.$$

CHAPTER 5

The medial and orthic triangles

In this chapter we investigate constructions of new triangles from old. We begin by studying two specific examples of such triangles, the medial triangle and the orthic triangle, and then generalize the construction in two different ways.

Note on terminology. A median of a triangle is defined to be the segment from a vertex of the triangle to the midpoint of the opposite side. While this is usually the appropriate definition, there are occasions in this chapter when it is more convenient to define the median to be the line determined by the vertex and midpoint rather than the segment joining them. The reader should use whichever definition fits the context. In the same way an altitude of a triangle is usually to be interpreted as a line, but may occasionally be thought of as the segment joining a vertex to a point on the opposite sideline.

5.1 THE MEDIAL TRIANGLE

Throughout this section $\triangle ABC$ is a triangle, D is the midpoint of $\overline{BC}$, E is the midpoint of the segment $\overline{AC}$, and F is the midpoint of $\overline{AB}$.

Definition. The triangle $\triangle DEF$ is the *medial triangle* of $\triangle ABC$.

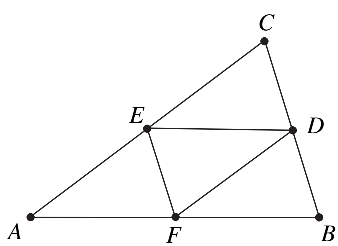

FIGURE 5.1: The medial triangle

EXERCISES

***5.1.1.** Make a tool that constructs the medial triangle of a given triangle.

5.1.2. Prove that the medial triangle subdivides the original triangle into four congruent triangles.

5.1.3. Prove that the original triangle $\triangle ABC$ and medial triangle $\triangle DEF$ have the same medians.

5.1.4. Prove that the perpendicular bisectors of the sides of the original triangle $\triangle ABC$ are the same as the altitudes of the medial triangle $\triangle DEF$.

***5.1.5.** Start with a triangle $\triangle ABC$ and construct a line through each vertex that is parallel to the opposite side of the triangle. Let A'', B'', and C'' be the points at which these three lines intersect. (See Figure 5.2.) The triangle $\triangle A''B''C''$ is called the *anticomplementary triangle* of $\triangle ABC$.[1]

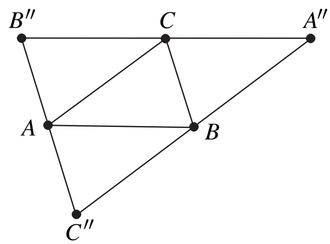

FIGURE 5.2: $\triangle A''B''C''$ is the anticomplementary triangle of $\triangle ABC$

5.1.6. Prove that $\triangle ABC$ is the medial triangle of $\triangle A''B''C''$.

***5.1.7.** Make a tool that constructs the anticomplementary triangle of a given triangle.

***5.1.8.** Let L be the orthocenter of the anticomplementary triangle $\triangle A''B''C''$. Use GSP to verify that L lies on the Euler line of the original triangle $\triangle ABC$. The point L is another triangle center for $\triangle ABC$; it is known as the *de Longchamps point* of $\triangle ABC$ for G. de Longchamps (1842-1906).

***5.1.9.** Let H be the orthocenter, O the circumcenter, and L the de Longchamps point of $\triangle ABC$. Use measurement to verify that O is the midpoint of the segment $\overline{HL}$.

5.2 THE ORTHIC TRIANGLE

Throughout this section $\triangle ABC$ is a triangle, A' is the foot of the altitude through A, B' is the foot of the altitude through B, and C' is the foot of the altitude through C. Note that A' is a point on the sideline $\overleftrightarrow{BC}$ and is not necessarily on the side $\overline{BC}$. Similar comments apply to B' and C'.

Definition. The triangle $\triangle A'B'C'$ is the *orthic triangle* of $\triangle ABC$.

[1]The term *antimedial* would be more appropriate since the construction is the opposite of the construction of the medial triangle. But the term anticomplementary is firmly established in the literature.

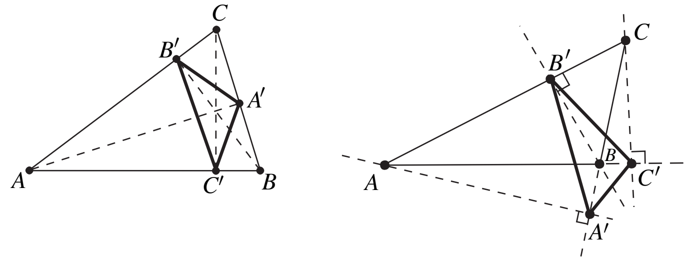

FIGURE 5.3: Two orthic triangles

EXERCISES

***5.2.1.** Make a tool that constructs the orthic triangle of a given triangle.

***5.2.2.** Construct a triangle $\triangle ABC$ and use the tool you just created to construct the orthic triangle $\triangle A'B'C'$. Move the vertices of $\triangle ABC$ and observe what happens to the orthic triangle.

(a) Under what conditions is $\triangle A'B'C'$ contained in $\triangle ABC$?

(b) Is it possible for the orthic triangle to be completely outside the original triangle?

(c) Observe that in a strict sense the orthic triangle may not be a triangle at all since the vertices can be collinear. Under what conditions does the orthic triangle degenerate into a line segment?

(d) Under what conditions is one of the vertices of the orthic triangle equal to one of the vertices of the original triangle?

(e) Under what conditions will two of the vertices of the orthic triangle equal the same vertex of the original triangle?

Make notes on your observations.

5.2.3. Let $\triangle ABC$ be a triangle.

(a) Prove that if $\angle BAC$ and $\angle ACB$ are acute, then B' lies in the interior of $\overline{AC}$.

(b) Prove that every triangle must have at least one altitude with its foot on the triangle.

(c) Prove that if $\angle ABC$ is obtuse, then both A' and C' lie outside $\triangle ABC$.

***5.2.4.** Use the Measure command to investigate the relationship between the measures of the angles $\angle ABC$, $\angle AB'C'$, and $\angle A'B'C$. Find two other triples of angles in the diagram whose measures are related in the same way.

5.2.5. Prove that sides of the orthic triangle cut off three triangles from $\triangle ABC$ that are all similar to $\triangle ABC$. Specifically, if $\triangle A'B'C'$ is the orthic triangle for $\triangle ABC$, then $\triangle ABC \sim \triangle AB'C' \sim \triangle A'BC' \sim \triangle A'B'C$.
[Hint: First prove that $\triangle AB'B \sim \triangle AC'C$ and then apply the SAS similarity criterion to prove $\triangle ABC \sim \triangle AB'C'$. Use similar proofs to show that $\triangle ABC \sim \triangle A'BC'$ and $\triangle ABC \sim \triangle A'B'C$.]

***5.2.6.** Construct a triangle $\triangle ABC$, the three altitudes, and the orthic triangle $\triangle A'B'C'$. Now add the incenter and incircle of $\triangle A'B'C'$ to your sketch Verify that the orthocenter of $\triangle ABC$ is the same as the incenter of $\triangle A'B'C'$ provided all the angles in $\triangle ABC$ are acute. What is the incenter of $\triangle A'B'C'$ in case $\angle BAC$ is obtuse? Try to determine how the orthocenter of $\triangle ABC$ is related to $\triangle A'B'C'$ in case $\angle BAC$ is obtuse.

5.2.7. Let $\triangle ABC$ be a triangle in which all three interior angles are acute and let $\triangle A'B'C'$ be the orthic triangle.
(a) Prove that the altitudes of $\triangle ABC$ are the angle bisectors of $\triangle A'B'C'$.
(b) Prove that the orthocenter of $\triangle ABC$ is the incenter of $\triangle A'B'C'$.
[Hint: Use Exercise 5.2.5.]

5.2.8. Let $\triangle ABC$ be a triangle such that $\angle BAC$ is obtuse and let $\triangle A'B'C'$ be the orthic triangle.
(a) Prove that A is the incenter of $\triangle A'B'C'$.
(b) Prove that the orthocenter of $\triangle ABC$ is the A'-excenter of $\triangle A'B'C'$.

5.3 CEVIAN TRIANGLES

The construction of the medial triangle and the orthic triangle are both examples of a more general construction. Start with a triangle $\triangle ABC$ and a point P that is not on any of the sidelines of $\triangle ABC$. Let L be the point at which $\overleftrightarrow{AP}$ intersects $\overleftrightarrow{BC}$, let M be the point at which $\overleftrightarrow{BP}$ intersects $\overleftrightarrow{AC}$, and let N be the point at which $\overleftrightarrow{CP}$ intersects $\overleftrightarrow{AB}$.

Definition. The triangle $\triangle LMN$ is the a *Cevian triangle* for $\triangle ABC$ associated with the point P.

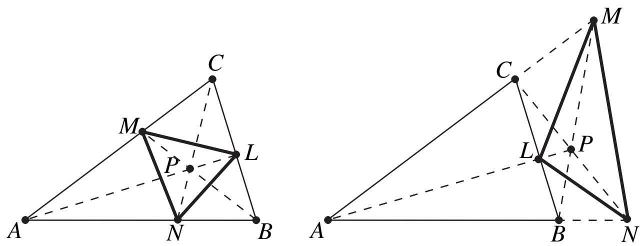

FIGURE 5.4: Two Cevian triangles associated with different points P

The medial triangle is the special case of a Cevian triangle in which P is the centroid of $\triangle ABC$ and the orthic triangle is the special case in which P it the orthocenter of $\triangle ABC$. Cevian triangles are named for Giovanni Ceva who studied the question of when three lines through the vertices of a triangle would be concurrent. Because of Ceva's work, such lines are called *Cevian lines* (or simply *cevians*) for the triangle. We will investigate Ceva's Theorem in Chapter 8.

EXERCISES

***5.3.1.** Make a tool that constructs the Cevian triangle associated with a given triangle $\triangle ABC$ and point P.

5.3.2. What happens if you attempt to construct a Cevian triangle associated with a point P that is on a sideline?

***5.3.3.** **Preview of Ceva's Theorem.** Construct a triangle $\triangle ABC$ and the Cevian triangle associated with the point P. Compute the quantity

$$d = \frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA}.$$

Observe that d has the same value regardless of the positions of A, B, C, and P (provided that none of the distances in the expression is 0).

***5.3.4.** **Preview of Menelaus's Theorem.** Construct a triangle $\triangle ABC$ and the Cevian triangle associated with the point P. Let A'' be the point of intersection of the lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{MN}$, let B'' be the point of intersection of $\overleftrightarrow{AC}$ and $\overleftrightarrow{LN}$, and let C'' be the point of intersection of the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{LM}$. (These points may not exist if any two of the lines are parallel, but you can always move the vertices of $\triangle ABC$ slightly so that the lines do intersect.) Verify that A'', B'', and C'' are collinear.

5.4 PEDAL TRIANGLES

There is a second way in which to generalize the construction of the medial and orthic triangles that leads to another useful class of triangles. Again start with a triangle $\triangle ABC$ and point P that is not on any of the sidelines of the triangles. Drop perpendiculars from P to each of the sidelines of the triangle. Let A' be the foot of the perpendicular to the sideline opposite A, let B' be the foot of the perpendicular to the sideline opposite B, and let C' be the foot of the perpendicular to the sideline opposite C.

Definition. The triangle $\triangle A'B'C'$ is the *pedal triangle* associated with $\triangle ABC$ and P.

It is obvious that the orthic triangle is the pedal triangle associated with the orthocenter. Since the midpoints of the sides of the triangle are the feet of the perpendiculars from the circumcenter, we see that the medial triangle is also a pedal triangle; it is the pedal triangle associated with the circumcenter.

EXERCISES

***5.4.1.** Make a tool that constructs the pedal triangle associated with a given triangle $\triangle ABC$ and point P.

***5.4.2.** Use your incenter tool and the pedal triangle tool to construct the pedal triangle associated with the incenter of the triangle $\triangle ABC$. Verify that the inscribed circle for the original triangle is the same as the circumscribed circle for the pedal triangle.

***5.4.3.** Construct a triangle and choose a point P in the interior of the triangle. Use your pedal triangle tool to construct the pedal triangle with respect to P. Now construct the pedal triangle of the pedal triangle with respect to the same point P. This new triangle is called the *second pedal* triangle. There is also a *third pedal triangle*, which is the pedal triangle of the second pedal triangle (with respect to the same point P). Verify that the third pedal triangle is similar to the original triangle.
[Hint: This construction will work best if your pedal triangle returns the vertices of the pedal triangle as results, not just the triangle itself.]

*5.4.4. **Preview of Simson's Theorem.** Construct a triangle $\triangle ABC$, the circumscribed circle γ for $\triangle ABC$, and a point P that is not on any of the sidelines of the triangle. Verify that the vertices of the pedal triangle are collinear if and only if P lies on γ.

5.4.5. Let $\triangle ABC$ be a triangle, let P be a point, and let A', B', and C' be the feet of the perpendiculars from P to the sidelines opposite A, B, and C, respectively. Prove that

$$B'C' = \frac{BC \cdot AP}{2R}, \quad A'C' = \frac{AC \cdot BP}{2R}, \quad \text{and} \quad A'B' = \frac{AB \cdot CP}{2R}.$$

[Hint: Use the converse to Thales' Theorem to prove that B' and C' lie on the circle with diameter $\overline{AP}$. Conclude that P is on the circumcircle of $\triangle AB'C'$. Apply the extended law of sines (page 31) to the triangles $\triangle ABC$ and $\triangle AB'C'$ to get $BC/\sin(\angle BAC) = 2R$ and $B'C'/\sin(\angle BAC) = AP$. Solve for $B'C'$ to obtain the first equation. The other two equations are proved similarly.]

The result in Exercise 5.4.3 was discovered by J. Neuberg in 1892. The line containing the three collinear feet of the perpendiculars in the Exercise 5.4.4 is called a *Simson line* for the triangle. It is named for Robert Simson (1687–1768). We will prove Simson's Theorem in Chapter 11. Exercise 5.4.5 is a technical fact that will be needed in the proof of Ptolemy's Theorem in that same chapter.

CHAPTER 6

Quadrilaterals

The objects we have studied until now have been remarkably simple: just lines, triangles, and circles. In the following chapters we will need to use polygons with more sides; specifically, we will study four-sided and six-sided figures. This chapter contains the necessary information about four-sided figures.

6.1 BASIC DEFINITIONS

Let us begin by repeating the definitions from Chapter 0. Four points A, B, C, and D such that no three of the points are collinear determine a *quadrilateral*, which we will denote by $\square ABCD$. Specifically,

$$\square ABCD = \overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}.$$

The four segments are the *sides* of the quadrilateral and the points A, B, C, and D are the *vertices* of the quadrilateral. The sides $\overline{AB}$ and $\overline{CD}$ are called *opposite sides* of the quadrilateral as are the sides $\overline{BC}$ and $\overline{AD}$. Two quadrilaterals are *congruent* if there is a correspondence between their vertices so that all four corresponding sides are congruent and all four corresponding angles are congruent.

In the past, the term *quadrangle* was frequently used for what we now call a quadrilateral. Obviously the difference between the two terms is that one emphasizes the fact that the figure has four sides while the other emphasizes the fact that the figure contains four angles. In the same way either of the terms *triangle* or *trilateral* can be used to name a three-sided or three-angled figure. We will follow current practice in using the terms triangle and quadrilateral even though it could be argued that this is inconsistent terminology.[1] The duality between sides and vertices will be discussed again in Chapter 9.

Each quadrilateral separates the plane into an inside and an outside. A first course on the foundations of geometry will often avoid defining the interior of a

[1]It has been argued that the term *quadrangle* is preferable to *quadrilateral* and that we should resist the use of the term quadrilateral [3, page 52].

general quadrilateral because defining it precisely requires care.[2] In fact, a rigorous definition of interior belongs to the branch of mathematics called *topology* since the definition uses the famous Jordan Curve Theorem, which asserts that a closed curve separates the plane into an inside and an outside. In this course we will not worry about such foundational issues, but will accept the intuitively obvious fact that a quadrilateral in the plane has an interior. GSP also has no qualms about this either and will allow you to define the interior of a general quadrilateral.

6.2 CONVEX AND CROSSED QUADRILATERALS

We will distinguish three classes of quadrilaterals: convex, concave, and crossed. There are several equivalent ways to define the most important of these, the convex quadrilaterals.

A quadrilateral is *convex* if each vertex lies in the interior of the opposite angle. Specifically, the quadrilateral $\square ABCD$ is convex if A is in the interior of $\angle BCD$, B is in the interior of $\angle CDA$, C is in the interior of $\angle DAB$, and D is in the interior of $\angle ABC$. This is significant because it allows us to use additivity of angle measure. For example, if $\square ABCD$ is convex, then $\mu(\angle ABC) = \mu(\angle ABD) + \mu(\angle DBC)$. Equivalently, a quadrilateral is convex if the region associated with the quadrilateral is convex in the sense that the line segment joining any two points in the region is completely contained in the region. It can also be shown that a quadrilateral is convex if and only if the two diagonals intersect at a point that is in the interior of both diagonals [16, Theorem 6.7.9].

Our definition of quadrilateral says nothing about how the sides intersect. The requirement that no three vertices are collinear prevents adjacent sides from intersecting at any point other than their common endpoint, but opposite sides can intersect. We will call a quadrilateral whose sides intersect at an interior point a *crossed* quadrilateral. It is easy to see that a convex quadrilateral cannot be crossed, so we have defined two disjoint collections of quadrilaterals. Any quadrilateral that is neither convex nor crossed will be called a *concave* quadrilateral.

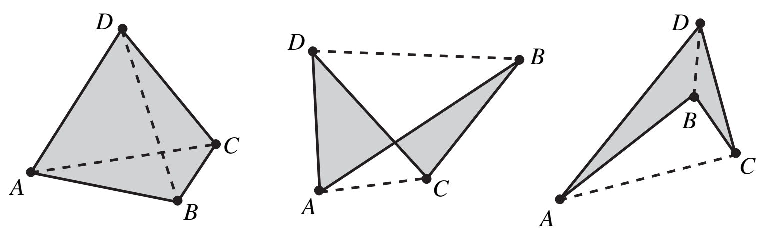

FIGURE 6.1: Convex, crossed, and concave

Figure 6.1 shows three quadrilaterals; the first is convex, the second is crossed, and the third is concave. In each case the interior is shaded and the two diagonals

[2]It is customary in such a course to define the interior of a quadrilateral only if the quadrilateral is convex—see §6.2 for the definition of convex.

are shown as dashed lines. Observe that the three kinds of quadrilaterals can be distinguished by their diagonals. The diagonals of a convex quadrilateral are both inside the quadrilateral, the diagonals of a crossed quadrilateral are both outside the quadrilateral, while one diagonal of a concave quadrilateral is inside and the other is outside.

One fact that will be assumed is that every trapezoid is convex [16, Exercise 6.28]. In particular, every parallelogram is convex.

EXERCISES

***6.2.1.** Construct three quadrilaterals, one of each kind. Construct the interior of each. Observe that GSP determines the interior of a quadrilateral by the order in which you select the vertices and that the interior may not correspond with the segments you have constructed between those vertices.

6.2.2. Review Exercise 1.4.8. Prove the following theorem.
Varignon's Theorem. *The midpoint quadrilateral of any quadrilateral is a parallelogram.*
[Hint: Use Euclid's Proposition VI.2 (page 7).]

***6.2.3.** Construct a convex quadrilateral and its associated midpoint quadrilateral. Calculate the area of each. What is the relationship between the areas? Find analogous relationships for concave and crossed quadrilaterals.

The theorem above is named for the French mathematician Pierre Varignon (1654–1722). The area relationship in the last exercise is often stated as part of Varignon's theorem.

6.3 CYCLIC QUADRILATERALS

We have seen that every triangle can be circumscribed. By contrast, the vertices of a quadrilateral do not necessarily lie on a circle. To see this, simply observe that the first three vertices determine a unique circle and that the fourth vertex may or may not lie on that circle. A quadrilateral whose vertices lie on a circle is quite special and such quadrilaterals will be useful in the next chapter.

Definition. A *cyclic quadrilateral* is a quadrilateral that is convex and whose vertices lie on a circle.

A quadrilateral $\square ABCD$ is *inscribed* in the circle γ if all the vertices of $\square ABCD$ lie on γ.

EXERCISES

***6.3.1.** Construct a circle γ and construct four points A, B, C, and D (in cyclic order) on γ. Measure angles $\angle ABC$ and $\angle CDA$ and calculate the sum of the measures. Do the same with angles $\angle BCD$ and $\angle DAB$. What relationship do you observe?

6.3.2. Prove the following theorem.
Euclid's Proposition III.22. *If $\square ABCD$ is a convex quadrilateral inscribed in the circle γ, then the opposite angles are supplements; i.e.,*

$$\mu(\angle ABC) + \mu(\angle CDA) = 180^\circ = \mu(\angle BCD) + \mu(\angle DAB).$$

[Hint: Divide the angles using diagonals of the quadrilateral and apply the Inscribed Angle Theorem.]

6.3.3. Prove the following theorem.
Converse to Euclid's Proposition III.22. *If $\square ABCD$ is a convex quadrilateral such that the opposite angles are supplements, then $\square ABCD$ is a cyclic quadrilateral.*
[Hint: The idea is very much like that in the proof of Exercise 0.10.4. Let γ be the circumscribed circle for $\triangle ABC$. Use the fact that $\square ABCD$ is convex to prove that there is a point D' such that D' lies on $\overrightarrow{AD}$ and γ. Show that the assumption $D \neq D'$ leads to a contradiction.]

***6.3.4.** Construct a circle γ and construct four points A, B, C, and D (in cyclic order) on γ. Let a, b, c, and d denote the lengths of the sides of $\square ABCD$. Define $s = (1/2)(a + b + c + d)$. Verify that

$$\alpha(\square ABCD) = \sqrt{(s - a)(s - b)(s - c)(s - d)}.$$

The formula in the last exercise is known as *Brahmagupta's formula*. It is named for the Indian mathematician Brahmagupta who discovered the formula in the seventh century. Heron's formula can be viewed as the special case in which $d = 0$.

6.4 DIAGONALS

Before you work the following exercises, you should review Exercise 1.4.9.

EXERCISES

6.4.1. Prove that a quadrilateral is a parallelogram if and only if the diagonals bisect each other.

6.4.2. Prove that a quadrilateral is a rhombus if and only if the diagonals are perpendicular and bisect each other.

6.4.3. Prove that a quadrilateral is a rectangle if and only if the diagonals are congruent and bisect each other.

6.4.4. Complete the following sentence: A quadrilateral is a square if and only if the diagonals ....

CHAPTER 7

The Nine-Point Circle

One of the most remarkable discoveries of nineteenth century Euclidean geometry is the fact that there is one circle that contains nine important points associated with a triangle. In 1765 Euler proved that the medians of the sides and the feet of the altitudes of a triangle lie on a single circle. In other words, the medial and orthic triangles always share the same circumcircle. Furthermore, the center of this common circumcircle lies on the Euler line of the original triangle.

It was not until 1820 that Charles-Julien Brianchon (1783–1864) and Jean-Victor Poncelet (1788–1867) proved that the midpoints of the segments joining the orthocenter to the vertices lie on the same circle. As a result this circle became known as the nine-point circle. Later, Karl Wilhelm Feuerbach (1800–1834) proved that the nine-point circle has the additional property that it is tangent to all four of the equicircles; for this reason Feuerbach's name is often associated with the nine-point circle.

7.1 THE NINE-POINT CIRCLE

Let us begin with a statement of the theorem.

Nine-point Circle Theorem. *If $\triangle ABC$ is any triangle, then the midpoints of the sides of $\triangle ABC$, the feet of the altitudes of $\triangle ABC$, and the midpoints of the segments joining the orthocenter of $\triangle ABC$ to the three vertices of $\triangle ABC$ all lie on a single circle.*

EXERCISES

***7.1.1.** Construct a triangle and the nine points indicated in the theorem. Verify that they all lie on a circle, regardless of the shape of the triangle.

***7.1.2.** Make a tool that constructs the nine-point circle for a triangle. The tool should accept the vertices of the triangle as givens and should return both the circle and the nine points as results. Label all the points as in Figure 7.1. In particular, the vertices of the triangle are A, B, and C, the midpoints of the sides are D, E, and F as before, the feet of the altitudes are A', B', and C', the orthocenter is H, and the midpoints of the segments from H to the vertices of the triangle are K, L, and M. Experiment with triangles of different shapes to get a feel for the nine-point circle. Is it possible for the nine-point circle to be completely contained inside $\triangle ABC$? What is true of $\triangle ABC$ in that case?

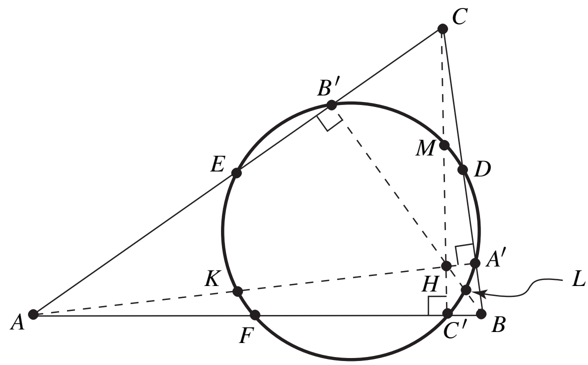

FIGURE 7.1: The nine-point circle

Three points determine a circle, so the strategy for proving the nine-point circle theorem is to start with the circle determined by three of the points and then prove that the other six points also lie on that circle. Specifically, let $\triangle ABC$ be a triangle and let D, E, and F be the midpoints of the sides of $\triangle ABC$. Define γ to be the circumcircle for the medial triangle $\triangle DEF$. Then the midpoints of the sides of the triangle obviously lie on γ. We will first show that the feet of the altitudes lie on γ and then we will show that the midpoints of the segments joining the orthocenter to the three vertices lie on γ.

EXERCISES

***7.1.3.** Go back to your GSP sketch from Exercise 7.1.2. Move the vertices A, B and C so that your diagram looks as much like Figure 7.1 as possible. Use angle measurement and calculation to verify that each of the quadrilaterals $\square DEFA'$, $\square EFDB'$, and $\square DEFC'$ satisfies the following conditions:
- **(a)** one pair of opposite sides is parallel,
- **(b)** the other pair of opposite sides is congruent, and
- **(c)** both pairs of opposite angles are supplements.

***7.1.4.** When the vertices of $\triangle ABC$ are moved so that the diagram changes, there continue to be three trapezoids that satisfy the conditions of the previous exercise. But the order of the vertices must be adjusted. Try to understand the pattern before proceeding.

7.1.5. Assume $\triangle ABC$ is a triangle such that C' lies between F and B (see Figure 7.1). Prove that $\square DEFB$ is a parallelogram. Use the converse to Thales's theorem to prove that $DC' = DB$. Prove that $\square DEFC'$ is a cyclic quadrilateral. Conclude that C' always lies on the circle determined by D, E, and F.

***7.1.6.** Go back to your GSP sketch of the nine-point circle and move the vertices of the triangle $\triangle ABC$ to determine the other possible locations for C' on the line $\overleftrightarrow{AB}$. Modify the argument in the preceding exercise as necessary in order to conclude that in every case C' lies on the circle determined by D, E, and F.

The exercises above show that C' lies on the circumcircle of $\triangle DEF$. We could use the same argument (with the labels of points changed appropriately) to conclude that A' and B' lie on this circle as well. We can therefore conclude that the feet of all three of the altitudes lie on the circle γ.

The next set of exercises will complete the proof of the nine-point circle theorem by showing that the remaining three points lie on γ as well. We continue to use the notation of Figure 7.1.

EXERCISES

7.1.7. Prove that $\overleftrightarrow{EM} \parallel \overleftrightarrow{AH}$ and conclude that $\overleftrightarrow{EM} \perp \overleftrightarrow{BC}$. Prove that $\overleftrightarrow{EF} \parallel \overleftrightarrow{BC}$ and conclude that $\overleftrightarrow{EM} \perp \overleftrightarrow{EF}$. Prove that E lies on the circle with diameter $\overline{MF}$. [Hint: Use Exercise 0.10.4.]

7.1.8. Prove that C' also lies on the circle with diameter $\overline{MF}$.

7.1.9. Use the two preceding exercises to prove that M lies on the circle determined by E, F, and C'.

By uniqueness of the circumcircle, there is only one circle that contains the three points E, F, and C'. Thus the exercises above show that M lies on γ. Similar arguments show that K and L lie on γ, so the proof of the nine-point circle theorem is complete.

The last few exercises actually prove more than is stated in the theorem—not only do the points K, L, and M lie on the nine-point circle, but $\overline{KD}$, $\overline{LE}$, and $\overline{MF}$ are diameters of the circle.

7.2 THE NINE-POINT CENTER

It should be clear from Figure 7.1 that the orthocenter H is not the center of the nine-point circle. In fact the center of the nine-point circle is a new triangle center that we have not encountered before.

Definition. The center of the nine-point circle is the *nine-point center* of $\triangle ABC$. It is denoted by N.

EXERCISES

***7.2.1.** Make a tool that constructs the nine-point center of a triangle. [Hint: the nine-point center is the circumcenter of the medial triangle, so you should be able to make this tool by combining two others.]

***7.2.2.** Construct a triangle, its circumcenter, its orthocenter, and its nine-point center. Verify that the nine-point center is the midpoint of the segment joining the circumcenter to the orthocenter.

Here is a statement of the theorem you verified in the last exercise.

Nine-Point Center Theorem. *The nine-point center is the midpoint of the line segment from the circumcenter to the orthocenter.*

In particular, the nine-point center lies on the Euler line. We now know a total of five points on the Euler line: the centroid, the circumcenter, the orthocenter, the de Longchamps point, and the nine-point center.

EXERCISES

7.2.3. Prove the Nine-Point Center Theorem.
[Hint: Use Figure 7.2 and the Secant Line Theorem.]

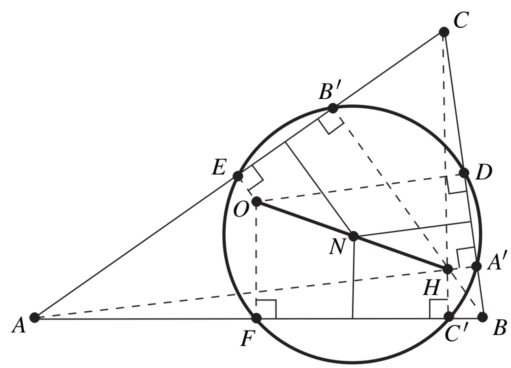

FIGURE 7.2: The orthocenter, the circumcenter, and the nine-point center

7.3 FEUERBACH'S THEOREM

Well after the nine-point circle had been discovered, Feuerbach proved the amazing result that the nine-point circle is tangent to each of the four equicircles.

Feuerbach's Theorem. *The nine-point circle is tangent to each of the four equicircles.*

EXERCISES

***7.3.1.** Construct a triangle $\triangle ABC$ and use tools from your toolbox to construct the incircle, the three excircles, and the nine-point circle. Verify that the nine-point circle is tangent to each of the others. Explore different shapes for $\triangle ABC$ to determine possible configurations for the five circles.

***7.3.2.** The *Feuerbach point* of the triangle is the point of tangency of the nine-point circle and the incircle. Construct the Feuerbach point for your triangle.

***7.3.3.** The *Feuerbach triangle* is the triangle whose vertices are the three points of tangency of the nine-point circle with the three excircles. Construct the Feuerbach triangle for your triangle.

We will not attempt a proof of Feuerbach's Theorem, but a proof may be found in §5.6 of [3].

C H A P T E R 8

Ceva's Theorem

In this chapter we finally complete the proofs of the various concurrency theorems that were investigated in earlier chapters. We will prove a theorem due to Giovanni Ceva (1647–1734) that gives a necessary and sufficient condition for three lines through the vertices of a triangle to be concurrent and then we will derive all of the concurrency theorems as corollaries of that one result. We could have given *ad hoc* proofs of each of the concurrency results separately, but it seems better to expose the unifying principle behind the various results. One nice aspect of doing things this way is that the proof of Ceva's general theorem is not much more difficult than the proof of any one of the special concurrency results.

8.1 EXPLORING CEVA'S THEOREM

All of the concurrency theorems studied so far have a common element. In each one there is a triangle and there is one line that passes through each of the vertices of the triangle. They are all special cases of the following general problem.

Concurrency Problem. *Let $\triangle ABC$ be a triangle and let ℓ, m, and n be three lines such that A lies on ℓ, B lies on m, and C lies on n. Find necessary and sufficient conditions under which ℓ, m, and n are concurrent.*

The lines through the vertices will either be parallel to their opposite sidelines or they will intersect the opposite sidelines. We begin by looking at the two possibilities separately. Those lines that intersect the opposite sideline are more common and have a special name.

Definition. Let $\triangle ABC$ be a triangle and let L, M, and N be points on the sidelines $\overleftrightarrow{BC}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{AB}$, respectively. The lines $\ell = \overleftrightarrow{AL}$, $m = \overleftrightarrow{BM}$, and $n = \overleftrightarrow{CN}$ are called *Cevian lines* (or simply *cevians*) for the triangle $\triangle ABC$. A Cevian line is *proper* if it passes through exactly one vertex of the triangle $\triangle ABC$.

A Cevian line is specified by naming the vertex it passes through along with the point at which it intersects the opposite sideline. Thus the assertion that $\overleftrightarrow{AL}$ is a Cevian line for $\triangle ABC$ is understood to mean that L is a point on $\overleftrightarrow{BC}$.

Exercise 5.3.3 provides a hint about what happens when three proper Cevian lines are concurrent. As you saw there, the following quantity is important:

$$d = \frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA}.$$

This quantity is defined and is positive provided all three Cevian lines are proper.

In the following exercises you will do some GSP exploration that will help to clarify the statement of the theorem we seek. It is assumed in all of the exercises that $\triangle ABC$ and ℓ, m, and n are as in the statement of the Concurrency Problem.

EXERCISES

***8.1.1.** Construct three noncollinear points A, B, and C, the three sidelines of $\triangle ABC$, and movable points L, M, and N on the sidelines $\overleftrightarrow{BC}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{AB}$, respectively. Explore various possible positions for the vertices of the triangle and the points L, M, and N to verify the following result: *If three proper Cevian lines are concurrent, then $d = 1$.* Find examples of concurrent Cevian lines in which all three of the points L, M, and N lie on the triangle $\triangle ABC$ and other examples in which at least one of the three points is not on $\triangle ABC$. Verify that $d = 1$ in all cases in which the proper Cevians are concurrent.

***8.1.2.** Find an example in which $d = 1$ but the three Cevian lines are not concurrent.

***8.1.3.** If the three Cevians are concurrent, how many of the points L, M, and N must lie on the triangle $\triangle ABC$? In your example for the preceding exercise, how many of the points were on the triangle?

***8.1.4.** Find points L, M, and N such that the three Cevian lines are mutually parallel. (Use the Parallel Lines command under the Construct menu to make sure the lines are exactly parallel, not just approximately parallel.) What is d equal to in this case? How many of the points L, M, and N lie on the triangle $\triangle ABC$?

***8.1.5.** Let ℓ be the line through A that is parallel to the sideline $\overleftrightarrow{BC}$. Find points M and N so that ℓ is concurrent with the Cevian lines $\overleftrightarrow{BM}$ and $\overleftrightarrow{CN}$. When the three lines are concurrent, how many of the points M and N lie on the triangle? There is no point L in this case, but it is still possible to compute the other two factors in d. Calculate

$$d' = \frac{AN}{NB} \cdot \frac{CM}{MA}$$

when the three lines are concurrent.

***8.1.6.** Now let ℓ be the line through A that is parallel to the sideline $\overleftrightarrow{BC}$ and let m be the line through B that is parallel to the sideline $\overleftrightarrow{AC}$. Is there a Cevian line $n = \overleftrightarrow{CN}$ such that the three lines ℓ, m, and n are concurrent? If so, calculate AN/NB. Is N on the triangle?

***8.1.7.** Is it possible for all three of the lines ℓ, m, and n to be parallel to the opposite sidelines and also concurrent?

***8.1.8.** Suppose $\ell = \overleftrightarrow{AB}$. What must be true of m and n if the three lines are concurrent?

In the exercises above you probably discovered that $d = 1$ if three proper Cevian lines are concurrent or if they are all mutually parallel. But the converse does not hold: it is possible that $d = 1$ even if the lines are not concurrent or mutually parallel. It appears that the converse depends on how many of the points L, M, and N lie on the triangle $\triangle ABC$. You also saw that there are various exceptional cases in which the lines ℓ, m, and n are not Cevian lines but are still concurrent. It would appear that the statement of a theorem that covers all the possibilities and all the special cases we have encountered would have to be quite complicated. The next order of business is to extend the Euclidean plane and distance measurements in that plane in a way that allows us to give one succinct statement that summarizes all of these observations.

8.2 SENSED RATIOS AND IDEAL POINTS

This section contains several new ideas and definitions that will allow us to give a concise statement of the most general form of Ceva's Theorem.

The first issue we address is how to include in d information about how many of the points L, M, and N are on the triangle and how many are not. We will accomplish that by adding a sign to each of the three ratios in the formula for d. Start with three distinct collinear points P, Q, and R. Since both distances involved are positive, the ratio PQ/QR is always positive. We will introduce a sensed ratio, which is sometimes positive and sometimes negative.

Definition. Assume P, Q, and R are three distinct collinear points. Define the *sensed ratio* $\mathbf{PQ}/\mathbf{QR}$ by

$$\frac{\mathbf{PQ}}{\mathbf{QR}} = \begin{cases} \dfrac{PQ}{QR} & \text{if } Q \text{ is between } P \text{ and } R\text{, and} \\[2ex] -\dfrac{PQ}{QR} & \text{if } Q \text{ is not between } P \text{ and } R. \end{cases}$$

Note that boldface is used to denote the sensed ratio in order to distinguish it from the unsensed ratio.

Given a line it is possible to assign a direction to it. Once a direction is assigned, it makes sense to speak of *directed distances*. The directed distance $\mathbf{AB}$ between two points A and B on the line is considered to be positive if the direction from A to B agrees with the direction assigned to the line and is negative otherwise. Directed distances such as this are not well defined because there are two ways to assign a direction to a given line. But reversing the direction of the line changes the sign of all the directed distances between points on the line. Thus the ratio of two directed distances is well defined in the sense that the same answer is obtained regardless of which direction is assigned to the line. The sensed ratio in the previous paragraph should be thought of as the quotient of two such directed distances.

When we state Ceva's Theorem, we will replace the positive quantity d (defined in the last section) by a real number s, where s is the product of the sensed ratios.

Specifically, we will define

$$s = \frac{\mathbf{AN}}{\mathbf{NB}} \cdot \frac{\mathbf{BL}}{\mathbf{LC}} \cdot \frac{\mathbf{CM}}{\mathbf{MA}}.$$

Notice that $s = d$ in case all three of the points L, M, and N lie on the triangle $\triangle ABC$ and $s = d$ in case exactly one of the points lies on the triangle. If either two of the points or none of the points L, M, and N lie on the triangle $\triangle ABC$, then s is negative and $s = -d$. Using s in the statement of the theorem will allow us to specify the numerical value of d and impose restrictions on the number of points that lie on the triangle at the same time.

Two other reasons Ceva's Theorem is complicated to state in complete generality are the fact that mutually parallel lines behave like concurrent lines and the fact that a line through a vertex might be parallel to the opposite sideline and hence not a Cevian line. The correct setting for Ceva's Theorem is actually the extended Euclidean plane, in which any two lines intersect. We will define the extended Euclidean plane to consist of all points in the Euclidean plane together with some additional ideal points, called "points at infinity." These ideal points are added to the plane in such a way that two parallel lines intersect at one of these ideal points.

Definition. The *extended Euclidean plane* consists of all the points in the ordinary Euclidean plane together with one additional point for each collection of mutually parallel lines in the plane. The points in the Euclidean plane will be called *ordinary points* and the new points that are added are called *ideal points* or *points at infinity*. The set of all ideal points is called the *line at infinity*.

Intuitively what we have done is we have added one point to each line in the plane. This point is "at infinity" in the sense that it is infinitely far away from any of the ordinary points. We can think of it as being a point out at the end of the line, although there is only one ideal point on each line so we approach the same ideal point on a line by traveling towards either of the two ends of the line. We will not attempt to picture the ideal points in our diagrams since they are infinitely far away and thus beyond what we can see. Their role is to complete the ordinary plane in such a way that it is not necessary to mention exceptional special cases in the statements of theorems. For example, every pair of distinct lines in the extended Euclidean plane intersect in exactly one point (with no exceptions). If the two lines are parallel in the ordinary plane, they share an ideal point and that is their point of intersection. If the two lines are not parallel in the ordinary plane, then they have different ideal points but intersect at an ordinary point.

Here is a summary of the important properties of the ideal points.

- Each ordinary line contains exactly one ideal point.
- Parallel ordinary lines share a common ideal point.
- Ordinary lines that are distinct and nonparallel have distinct ideal points.

In the extended plane every line through a vertex of an ordinary triangle is a Cevian line. To see this, consider an ordinary triangle $\triangle ABC$ and a line ℓ that passes through vertex A. Either ℓ intersects the sideline $\overleftrightarrow{BC}$ at a point L or $\ell \parallel \overleftrightarrow{BC}$. In the first case ℓ is obviously a Cevian line, but in the second case ℓ is also a Cevian line with L equal to the ideal point on $\overleftrightarrow{BC}$.

Now that we have added additional points to the plane, we must extend our definition of sensed ratio to include them. In order to understand the next definition, think of two fixed points A and B and a movable point C on the line $\overleftrightarrow{AB}$. As C approaches either end of the line, C is not between A and B so the sensed ratio $\mathbf{AC}/\mathbf{CB}$ is negative. Furthermore, the distances AC and CB both approach infinity. But the difference between AC and CB is constant, so they approach infinity at the same rate and their ratio has a limit of 1. In other words, if I is the ideal point on $\overleftrightarrow{AB}$, then

$$\lim_{C\to I} \frac{\mathbf{AC}}{\mathbf{CB}} = -1.$$

This reasoning justifies the following definition.

Definition. Let A and B be distinct ordinary points and let I be the ideal point on $\overleftrightarrow{AB}$. Define

$$\frac{\mathbf{AI}}{\mathbf{IB}} = -1.$$

Finally, we must extend ordinary arithmetic of fractions to cover the possibility that a Cevian line might not be proper.[1] We will adopt the convention that $a/b = c/d$ provided $ad = bc$. Thus it is possible for b to be zero in the fraction a/b, but if $b = 0$ and $a/b = c/d$, then either $a = 0$ or $d = 0$. In particular, if we write

$$\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = 1,$$

this does not necessarily imply that $LC \neq 0$. Instead it means that if $LC = 0$, then at least one of AN, BL, or CM must also be zero.

EXERCISES

***8.2.1.** Construct two lines ℓ and m. Mark the point of intersection of the two lines. Now rotate one line and keep the other fixed. Watch what happens to the point of intersection as the lines pass the position at which they are parallel. Explain how what you observe justifies the choice of just one ideal point on each line even though the line has two ends.
[Hint: There is a special version of the selection tool that can be used to rotate objects.]

***8.2.2.** Construct a line and points A, B, and X on the line. Calculate the ratio AX/XB. (GSP will only calculate an unsensed ratio so you will have to supply the sign.)

[1] This can happen, for example, in the altitude concurrence theorem.

For which points X on $\overleftrightarrow{AB}$ is this ratio defined? Watch what happens to the ratio as X moves along the line. Use your observations to draw a graph of the function

$$f(X) = \frac{\mathbf{AX}}{\mathbf{XB}}.$$

[Hint: f is a function whose domain consists of points on $\overleftrightarrow{AB}$ and whose range consists of real numbers, so your graph should have the line $\overleftrightarrow{AB}$ as its horizontal axis and the real line as the vertical axis.]

8.2.3. Let A and B be two distinct points. Prove that for each real number r there is exactly one point on the extended line $\overleftrightarrow{AB}$ such that $\mathbf{AX}/\mathbf{XB} = r$.

8.2.4. Draw an example of a triangle in the extended Euclidean plane that has one ideal vertex. Is there a triangle in the extended plane that has two ideal vertices? Could there be a triangle with three ideal vertices?

8.3 THE STANDARD FORM OF CEVA'S THEOREM

We are finally ready for the statement of Ceva's theorem.

Ceva's Theorem. *Let $\triangle ABC$ be an ordinary triangle. The Cevian lines $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ are concurrent if and only if*

$$\frac{\mathbf{AN}}{\mathbf{NB}} \cdot \frac{\mathbf{BL}}{\mathbf{LC}} \cdot \frac{\mathbf{CM}}{\mathbf{MA}} = +1.$$

The statement of the theorem assumes that the triangle is an ordinary triangle, so A, B, and C are all ordinary points. There is no assumption that the points L, M, and N are ordinary; nor is there there an assumption that the point of concurrency is ordinary. In addition, L, M, and N may be equal to vertices of the triangle. Thus the theorem covers all possible lines through the vertices of the triangle and gives a complete solution to the Concurrency Problem stated at the beginning of the chapter.

Because we have stated Ceva's theorem in its ultimate generality, a complete proof will involve looking at a number of special cases and this might make the proof appear to be complicated. In order to make certain that the simplicity of the basic theorem is not lost, we will begin with a proof of the special case in which the point of concurrency is inside the triangle; this case covers most of the applications. It is the special case in which all three of the points L, M, and N are in the interior of the segments that form the sides of $\triangle ABC$.[2]

[2]It follows from the crossbar theorem and related foundational results that the point of concurrency of the lines $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ is inside the triangle $\triangle ABC$ if and only if all three of the points L, M, and N are in the interiors of the segments that form the sides of $\triangle ABC$. That fact will be assumed in this section.

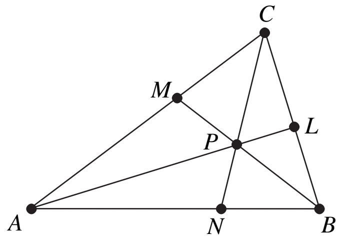

FIGURE 8.1: A special case of Ceva's theorem

EXERCISES

8.3.1. Let $\triangle ABC$ be a triangle and let L, M, and N be points in the interiors of the sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively. Prove that if the three Cevian lines $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ are concurrent, then

$$\frac{\mathbf{AN}}{\mathbf{NB}} \cdot \frac{\mathbf{BL}}{\mathbf{LC}} \cdot \frac{\mathbf{CM}}{\mathbf{MA}} = 1.$$

[Hint: Let k be the line through A that is parallel to $\overleftrightarrow{BC}$, let R be the point at which $\overleftrightarrow{BM}$ intersects k, and let S be the point at which $\overleftrightarrow{CN}$ intersects k. Use similar triangles to show that $AN/NB = SA/CB$, $CM/MA = CB/AR$, and $BL/LC = AR/SA$. The result follows by algebra.]

8.3.2. Let $\triangle ABC$ be a triangle and let L, M, and N be points in the interiors of the sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively. Prove that if

$$\frac{\mathbf{AN}}{\mathbf{NB}} \cdot \frac{\mathbf{BL}}{\mathbf{LC}} \cdot \frac{\mathbf{CM}}{\mathbf{MA}} = 1,$$

then the three Cevian lines $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ are concurrent.
[Hint: By the Crossbar Theorem there is a point P at which $\overline{AL}$ intersects $\overline{BM}$. Choose N' to be the point at which $\overrightarrow{CP}$ intersects $\overline{AB}$. Note that $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN'}$ are concurrent. Use the part of the theorem you have already proved and Exercise 8.2.3 to prove that $N = N'$.]

Now we proceed to prove Ceva's Theorem in general. We will begin by proving that if the three lines are concurrent, then $s = 1$. Once that has been established, the proof used in the last exercise (above) can easily be modified to prove the converse.

Assume in the following exercises that $\triangle ABC$ is triangle, the lines $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ are Cevian lines for $\triangle ABC$, and

$$s = \frac{\mathbf{AN}}{\mathbf{NB}} \cdot \frac{\mathbf{BL}}{\mathbf{LC}} \cdot \frac{\mathbf{CM}}{\mathbf{MA}}.$$

It is also assumed that A, B, and C are ordinary points in the Euclidean plane, but L, M, and N are points in the extended plane. Thus one or more of the points L,

M, and N may be ideal, in which case the corresponding factors in s are -1. It is also possible that one or more of L, M, and N may be a vertex of $\triangle ABC$. In that case there might be a 0 in the denominator of s. When that happens the factors of s should not be considered separately, but the entire numerator and the entire denominator should be considered. If one of the factors in the denominator of s is 0, then $s = -1$ simply means that there is also a factor in the numerator that is equal to 0.

EXERCISES

8.3.3. Prove that if $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ are proper Cevian lines that are concurrent at an ordinary point P, then either all three of the points L, M, and N lie on $\triangle ABC$ or exactly one of them does.
[Hint: The three sidelines divide the exterior of $\triangle ABC$ into six regions. Consider the possibility that P lies in each of them separately. Don't forget that one or more of the points L, M, and N might be ideal.]

8.3.4. Prove that if $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ are proper Cevian lines that are concurrent at an ideal point, then exactly one of the three of the points L, M, and N lies on $\triangle ABC$.

8.3.5. Prove that if $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ are proper Cevian lines that are concurrent at an ordinary point P and all three of L, M, and N are ordinary, then $s = 1$.
[Hint: The case in which all three of the points L, M, and N lie on $\triangle ABC$ was covered in Exercise 8.3.1, so you may assume that only M lies on $\triangle ABC$. Proceed as in the proof of Exercise 8.3.1; there are two possible diagrams, both different from that in Exercise 8.3.1, but you should still be able to find the similar triangles you need.]

8.3.6. Prove that if $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ are proper Cevian lines that are concurrent at an ideal point and all three of L, M, and N are ordinary, then $s = 1$.
[Hint: Concurrent at an ideal point means parallel. Essentially the same proof works again.]

8.3.7. Prove that if $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ are proper Cevian lines that are concurrent at an ordinary point P and L is ideal but M and N are ordinary, then $s = 1$.
[Hint: Take another look at your sketch and your conclusions in Exercise 8.1.5.]

8.3.8. Prove that if $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ are proper Cevian lines that are concurrent at an ordinary point P and L and M are ideal but N is ordinary, then $s = 1$.
[Hint: Take another look at your sketch and your conclusions in Exercise 8.1.6.]

8.3.9. Prove that at least one of L, M, and N must be ordinary if the proper Cevian lines $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ are concurrent.

8.3.10. Prove that if $L = B$ (so that $\overleftrightarrow{AL}$ is the sideline $\overleftrightarrow{AB}$ of the triangle) and the three Cevians $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ are concurrent, then either $\overleftrightarrow{BM} = \overleftrightarrow{AB}$ or $\overleftrightarrow{CN} = \overleftrightarrow{BC}$. Prove that $s = 1$ in either case.

8.3.11. Prove that if $s = 1$ then the Cevian lines $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ are concurrent.
[Hint: Proceed as in Exercise 8.3.2. Be sure to check that this proof works in every case.]

8.4 THE TRIGONOMETRIC FORM OF CEVA'S THEOREM

There is second form of Ceva's Theorem that is sometimes more convenient to apply than the standard form. This second version of the theorem expresses the concurrence criterion in terms of sines of angles rather than distances, so it will be preferred for applications in which it is simpler to measure the angles than the lengths.

As in the standard form, sense must to taken into account in the concurrence condition. For the trigonometric form this is simply a matter of using directed measures for the angles. The measure of an angle $\angle BAC$ is considered to be positive if it is measured in the counterclockwise direction from $\overrightarrow{AB}$ to $\overrightarrow{AC}$ and it is negative if the rotation from $\overrightarrow{AB}$ to $\overrightarrow{AC}$ is clockwise. (This is the same way angles are measured in calculus.) We will use boldface for the angle when it is to be considered a directed angle. Thus $\angle\mathbf{BAC}$ denotes the directed angle from $\overrightarrow{AB}$ to $\overrightarrow{AC}$.

Trigonometric Form of Ceva's Theorem. *Let $\triangle ABC$ be an ordinary triangle. The Cevian lines $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ are concurrent if and only if*

$$\frac{\sin(\angle\mathbf{BAL})}{\sin(\angle\mathbf{LAC})}\cdot\frac{\sin(\angle\mathbf{CBM})}{\sin(\angle\mathbf{MBA})}\cdot\frac{\sin(\angle\mathbf{ACN})}{\sin(\angle\mathbf{NCB})}=+1.$$

The proof is based on the following simple lemma.

Lemma. *If $\triangle ABC$ is any ordinary triangle and L is a point on $\overleftrightarrow{BC}$, then*

$$\frac{\mathbf{BL}}{\mathbf{LC}}=\frac{AB\cdot\sin(\angle\mathbf{BAL})}{AC\cdot\sin(\angle\mathbf{LAC})}.$$

Proof. We will prove the theorem in case L is between B and C (see Figure 8.2) and leave the general case as an exercise. Let h denote the height of triangle $\triangle ABC$. Then

$$\frac{\mathbf{BL}}{\mathbf{LC}}=\frac{(1/2)\,h\cdot BL}{(1/2)\,h\cdot LC}=\frac{(1/2)\,AB\cdot AL\cdot\sin(\angle BAL)}{(1/2)\,AL\cdot AC\cdot\sin(\angle LAC)}=\frac{AB\cdot\sin(\angle\mathbf{BAL})}{AC\cdot\sin(\angle\mathbf{LAC})}.$$

The first equation is just algebra, the second equation is based on two applications of Exercise 0.11.1, and the third equation uses the fact that L is between B and C so the two angles have the same direction. □

EXERCISES

8.4.1. Modify the proof above to cover the remaining cases of the lemma. [Hint: L might equal B or C, or L might be outside $\overline{BC}$.]

8.4.2. Use the lemma to prove the trigonometric form of Ceva's theorem.

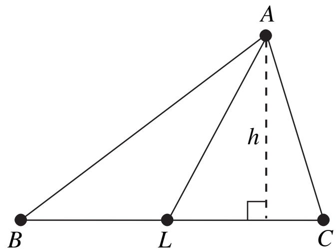

FIGURE 8.2: Proof of the lemma

8.5 THE CONCURRENCE THEOREMS

In this section we finally apply Ceva's Theorem to prove the concurrence results that we have encountered thus far in the course. Before getting to those proofs, however, we should make one observation about applying Ceva's theorem.

When we want to apply Ceva's theorem we will usually not find the triangle and the Cevian lines conveniently labeled with exactly the letters that were used in the statement of Ceva's Theorem. The pattern that you should see in the expression

$$\frac{\mathbf{AN}}{\mathbf{NB}} \cdot \frac{\mathbf{BL}}{\mathbf{LC}} \cdot \frac{\mathbf{CM}}{\mathbf{MA}}$$

is that the vertices of the triangle are listed in cyclic order around the triangle. The first fraction involves the two endpoints of the first side of the triangle and the point on the corresponding sideline. The second fraction involves the second side of the triangle, oriented so that the second side begins at the point where the first segment ended, and so on. When you apply the theorem you may begin at any vertex of the triangle and you may go around the triangle in either direction. But once you select a starting vertex and a direction around the triangle, you must follow the rest of the pattern.

EXERCISES

8.5.1. Existence of the centroid. Prove that the medians of a triangle are concurrent.

8.5.2. Existence of the orthocenter. Prove that the altitudes of any triangle are concurrent.
[Hint: Use the angle sum theorem and the trigonometric form of Ceva's theorem. Be careful with the special case of a right triangle.]

8.5.3. Existence of the circumcenter. Prove that the perpendicular bisectors of the sides of any triangle are concurrent.
[Hint: Be sure to notice that the circumcenter breaks the pattern: it is not the point of concurrence of Cevian lines. Use Exercise 5.1.4.]

8.5.4. Existence of the incenter. Prove that the interior angle bisectors of any triangle are concurrent.

8.5.5. Existence of the Gergonne point. Let L, M, and N be the points at which the incircle touches the sides of $\triangle ABC$. Prove that $\overline{AL}$, $\overline{BM}$, and $\overline{CN}$ are concurrent.
[Hint: Use the external tangents theorem.]

8.5.6. Existence of excenters. Prove that the bisector of an interior angle of a triangle and the bisectors of the remote exterior angles are concurrent.

8.5.7. Existence of the Nagel point. Let $\triangle ABC$ be a triangle, let T_A be the point at which the A-excircle is tangent to $\overline{BC}$, let T_B be the point at which the B-excircle is tangent to $\overline{AC}$, and let T_C be the point at which the C-excircle is tangent to $\overline{AB}$. Prove that the segments $\overline{AT_A}$, $\overline{BT_B}$, and $\overline{CT_C}$ are concurrent.
[Hint: Use Exercise 4.5.2 to prove that the ratios in this problem are the reciprocals of the ratios in Exercise 8.5.5.]

8.6 ISOTOMIC AND ISOGONAL CONJUGATES AND THE SYMMEDIAN POINT

Ceva's Theorem allows us to define two very interesting transformations of the plane associated with a triangle. Both these transformations are called *conjugates* because applying them twice results in the identity transformation. This is the same sense in which the word "conjugate" is used in connection with complex numbers. In other contexts a transformation with this property would be called an *involution*.

Let $\triangle ABC$ be an ordinary triangle and let P be a point in the extended plane. For simplicity we will assume that P does not lie on any of the sidelines of the triangle. In that case P is the point of concurrence of three proper Cevian lines $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$.

Definition. The *isotomic conjugate* of P is the point of concurrence of the lines $\overleftrightarrow{AL'}$, $\overleftrightarrow{BM'}$, and $\overleftrightarrow{CN'}$, where L' is the reflection of L across the perpendicular bisector of $\overline{BC}$, M' is the reflection of M across the perpendicular bisector of $\overline{AC}$, and N' is the reflection of N across the perpendicular bisector of $\overline{AB}$.

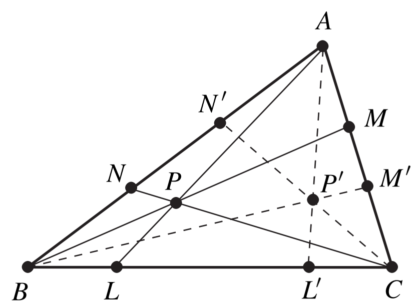

FIGURE 8.3: P' is the isotomic conjugate of P

Definition. The *isogonal* of a Cevian line $\overleftrightarrow{AL}$ is the reflection of $\overleftrightarrow{AL}$ through the angle bisector of $\angle CAB$. Similarly the isogonal of a Cevian line $\overleftrightarrow{BM}$ is the reflection of $\overleftrightarrow{BM}$ through the bisector of $\angle ABC$ and the isogonal of a Cevian line $\overleftrightarrow{CN}$ is the reflection of $\overleftrightarrow{CN}$ through the bisector of $\angle BCA$. The *isogonal conjugate* of P is the point of concurrence of the isogonals of $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$.

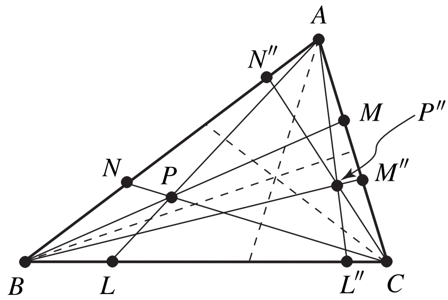

FIGURE 8.4: P'' is the isogonal conjugate of P

Both definitions presume that the new Cevian lines intersect; that is where Ceva's theorem comes in. The next two exercises show that the isotomic conjugate and the isogonal conjugate exist.

EXERCISES

8.6.1. Use Ceva's theorem to prove that the three Cevian lines in the definition of isotomic conjugate are concurrent.

8.6.2. Use the trigonometric form of Ceva's theorem to prove that the three Cevian lines in the definition of isogonal conjugate are concurrent.

***8.6.3.** Let $\triangle ABC$ be a triangle. Make a tool that constructs the isotomic conjugate of a point P that is not on any of the sidelines of the triangle.

***8.6.4.** Let $\triangle ABC$ be a triangle. Make a tool that constructs the isogonal conjugate of a point P that is not on any of the sidelines of the triangle.

***8.6.5.** Verify that the Gergonne point and the Nagel point are isotomic conjugates.

8.6.6. Prove that the Gergonne point and the Nagel point are isotomic conjugates. [Hint: Use Exercise 4.5.2.]

***8.6.7.** Verify that the incenter is, in general, the only point in the interior of a triangle that is its own isogonal conjugate. Find all points in the exterior of the triangle that are isogonal conjugates of themselves.

8.6.8. Prove that the incenter is its own isogonal conjugate.

8.6.9. Prove that the orthocenter and the circumcenter are isogonal conjugates. [Hint: Let γ be the circumcircle of $\triangle ABC$. Extend $\overline{AO}$ and $\overline{AH}$ until they intersect γ at points R and S as indicated in Figure 8.5. Let $\overline{PQ}$ be a diameter of γ that is parallel to $\overline{BC}$ and let T be a point on γ such that $\overline{ST}$ is a diameter (see Figure 8.5).

Prove that $\overleftrightarrow{RT} \perp \overleftrightarrow{PQ}$. Prove that $\angle ROP \cong \angle POT \cong \angle QOS$ and prove that $\angle BOP \cong \angle COQ$. Conclude that $\angle BOR \cong \angle COS$. Apply Exercise 0.10.5 twice to conclude that $\angle BAR \cong \angle CTS$. Finally, use the inscribed angle theorem to conclude that $\angle BAO \cong \angle HAC$.]

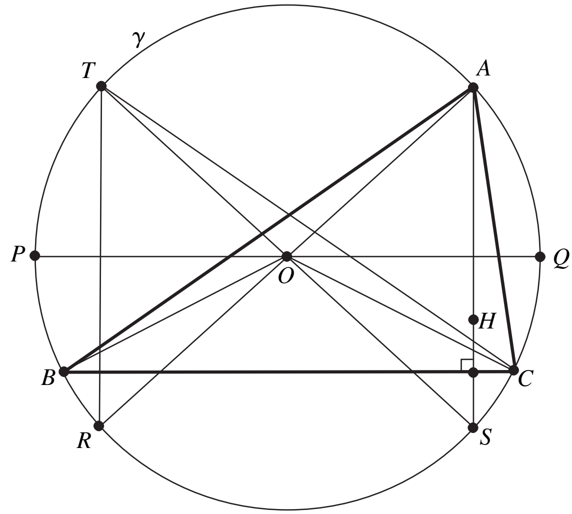

FIGURE 8.5: Proof that the circumcenter and the orthocenter are isogonal conjugates

***8.6.10.** What happens to the isogonals of the Cevians through P as P approaches the circumcircle of $\triangle ABC$? What is the isogonal conjugate of a point on the circumcircle?

8.6.11. Prove that the isogonal complement of a point on the circumcircle is an ideal point.
[Hint: Observe that the isogonal complement of P is ideal if and only if the isogonal Cevians are parallel. Since the isogonal Cevians have already been proved concurrent, it is enough to show that two of them are parallel. Figure 8.6 shows one possible diagram. Apply Euclid's Proposition III.22 and the angle sum theorem.]

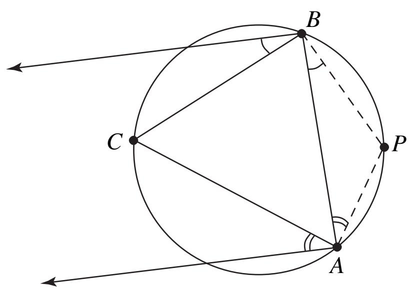

FIGURE 8.6: Proof that the isogonal complement of a point on the circumcircle is ideal

Definition. A *symmedian* for $\triangle ABC$ is the isogonal of a median of the triangle. The *symmedian point* of $\triangle ABC$ is the point of concurrence of the three symmedians; i.e., the symmedian is the isogonal conjugate of the centroid.

The symmedian point is another triangle center. It is usually denoted by K.

EXERCISES

***8.6.12.** Verify that the isotomic conjugate of the orthocenter is the symmedian point of the anticomplementary triangle.

CHAPTER 9

The Theorem of Menelaus

The theorem we will study in this chapter is ancient, dating from about the year AD 100. It was originally discovered by Menelaus of Alexandria (70–130), but it did not become well known until it was rediscovered by Ceva in the seventeenth century. The theorem is powerful and has many interesting applications, some of which will be explored in later chapters.

9.1 DUALITY

In the last chapter we studied the problem of determining when three lines through the vertices of a triangle are concurrent; in this chapter we study the problem of determining when three points on the sidelines of a triangle are collinear. The relationship between these two problems is an example of *duality*. Before studying the main result of the chapter we will pause in this section to consider the principle of duality because discerning larger patterns such as duality can lead to a much deeper understanding of the theorems of geometry than just studying each geometric result in isolation.

Roughly speaking, the principle of duality asserts that any true statement in geometry should remain true when the words point and line are interchanged. Just as two points lie on exactly one line, so two lines intersect in exactly one point; just as three points may be collinear, so three lines may be concurrent. The use of the term *incident* makes these statements sound even more symmetric. For example, two distinct points are incident with exactly one line and two distinct lines are (usually) incident with exactly one point.

Passing to the extended Euclidean plane allows us to eliminate the exceptions to these rules, so the extended plane is the natural setting for duality. Specifically, the statement that two distinct lines are incident with exactly one point is true without exception in the extended plane. This makes it completely dual to the statement that two distinct points are incident with exactly one line (Euclid's first postulate). Girard Desargues (1591–1661) was the first to make systematic use of ideal points and his work lead eventually to the development of a whole new branch of geometry known as *projective geometry*. A later chapter will include a study of the theorem of Desargues, which is a beautiful result that relates two dual properties of triangles.

Desargues's theorem is foundational in projective geometry, but we will not pursue that branch of geometry any further.

The triangle itself is self dual. It can be thought of as determined by three noncollinear points (the vertices) or by three nonconcurrent lines (the sidelines). When this latter view of the triangle is being emphasized, the triangle is referred to by its dual name *trilateral*. Vertices and sidelines (*not* sides) are dual aspects of the triangle, so they should be interchanged when theorems regarding triangles are being dualized. This is the reason it is natural to use the sidelines of a triangle, rather than its sides, in so many of the theorems we have encountered in this course.

The theorems of Ceva and Menelaus will be our primary examples of dual theorems. The theorem of Ceva gives a criterion that tells us when three lines through the vertices of a triangle are concurrent; the theorem of Menelaus tells us that essentially the same criterion can be used to determine when three points on the sidelines of a triangle are collinear. Apparently what happened historically is that Ceva rediscovered the theorem of Menelaus and then discovered his own theorem by applying the principle of duality.

The principle of duality was first formalized by Charles Julien Brianchon (1785–1864) when he applied it to a theorem of Blaise Pascal (1623–1663). The theorems of Pascal and Brianchon serve as a beautiful illustration of a pair of dual theorems. The theorem of Pascal asserts that if a hexagon is inscribed in a circle, then the three points determined by pairs of opposite sidelines are collinear. Brianchon's theorem asserts that if a hexagon is circumscribed about a circle, then the lines determined by pairs of opposite vertices are concurrent. We will study those theorems later in the chapter.

9.2 THE THEOREM OF MENELAUS

In order to simplify the statements in this section, let us make a definition.

Definition. Let $\triangle ABC$ be a triangle. Three points L, M, and N such that L lies on $\overleftrightarrow{BC}$, M lies on $\overleftrightarrow{AC}$, and N lies on $\overleftrightarrow{AB}$ are called *Menelaus points* for the triangle. We say that a Menelaus point is *proper* if it is not equal to any of the vertices of the triangle.

It will be assumed that the vertices of the triangle are ordinary points, but one or more of the Menelaus points may be ideal points. Let us begin with some GSP exploration of when three Menelaus points are collinear.

EXERCISES

***9.2.1.** Construct a triangle $\triangle ABC$ and proper Menelaus points L, M, and N for $\triangle ABC$. Calculate the quantity
$$d = \frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA}.$$
Move the vertices of the triangle and the Menelaus points to verify that if the Menelaus points are collinear, then $d = 1$.

***9.2.2.** Verify that $d = 1$ if L is an ideal point and M and N lie on a line parallel to $\overleftrightarrow{BC}$.

***9.2.3.** Find an example of a triangle and Menelaus points such that $d = 1$ even though the Menelaus points are not collinear.

***9.2.4.** Construct a triangle $\triangle ABC$ and a line ℓ that does not pass through any of the vertices of the triangle. Determine the answer to the following question: *What are the possible numbers of points of intersection of ℓ with the triangle?*

The exercises above indicate that if the Menelaus points are collinear, then $d = 1$. But they also show that the converse does not hold. Again the number of Menelaus points that that lie on the triangle itself is important, so we must use

$$s = \frac{\mathbf{AN}}{\mathbf{NB}} \cdot \frac{\mathbf{BL}}{\mathbf{LC}} \cdot \frac{\mathbf{CM}}{\mathbf{MA}},$$

the product of the sensed ratios, rather than d. All of this is probably expected since it is exactly analogous to what happened in the case of Ceva's theorem.

One difference between this situation and that of Ceva's theorem is that the number of Menelaus points that are expected to lie on the triangle is even, so the number of factors in s that are negative is even rather than odd. This is because any line in the plane that does not contain any of the vertices of the triangle $\triangle ABC$ will either miss the triangle entirely or will intersect it in exactly two points. You should have convinced yourself of that fact when you did your GSP exploration in Exercise 9.2.4. Pasch's Axiom, which was assumed in Chapter 0 makes this precise. We can conclude that if the three Menelaus points are all proper, then either zero or two of them lie on the triangle. In that case either three or one of the sensed ratios in s will be negative, so s itself is negative.

We can now state Menelaus's theorem in full generality. Again we assume that the vertices of the triangle are ordinary points in the Euclidean plane, but we allow the possibility that one or more of the Menelaus points is ideal.

Theorem of Menelaus. *Let $\triangle ABC$ be an ordinary triangle. The Menelaus points L, M, and N for $\triangle ABC$ are collinear if and only if*

$$\frac{\mathbf{AN}}{\mathbf{NB}} \cdot \frac{\mathbf{BL}}{\mathbf{LC}} \cdot \frac{\mathbf{CM}}{\mathbf{MA}} = -1.$$

The next few exercises outline a proof of Menelaus's theorem. It would be possible to prove the theorem as a corollary of Ceva's theorem, but that proof is no simpler than a proof based on similar triangles. Since it is no more difficult, we give a proof based only on elementary Euclidean geometry and do not use Ceva's theorem.

EXERCISES

9.2.5. Prove the theorem in case one or more of the Menelaus points is improper. Specifically, prove each of the following statements and then explain why this suffices to prove the theorem in every case in which at least one of the Menelaus points is improper.

(a) If $L = C$ and the three Menelaus points are collinear, then either $N = A$ or $M = C$ and (in either case) $s = -1$.

(b) If $L = C$ and $s = -1$, then either $N = A$ or $M = C$.

9.2.6. Assume that all three Menelaus points are proper and ordinary. Prove that if the three Menelaus points are collinear, then $s = -1$.
[Hint: Assume, first, that L and M lie on the triangle and N does not (see Figure 9.1). Drop perpendiculars from A, B, and C to the line $\overleftrightarrow{LM}$ and call the feet R, S, and T, respectively. Let $r = AR$, $s = BS$, and $t = CT$. Use similar triangles to express each of the sensed ratios AN/NB, BL/LC, and CM/MA in terms of r, s, and t and then use algebra to derive the Menelaus formula. Use GSP to determine what the other possible figures look like and modify the argument to fit them as well.]

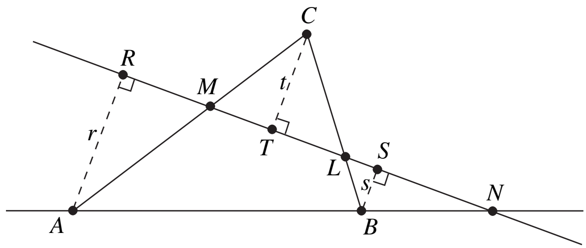

FIGURE 9.1: Proof of Menelaus's theorem in case two points lie on the triangle

9.2.7. Assume that all three Menelaus points are proper and that exactly one of them is ideal. Prove that $s = -1$ if the three Menelaus points are collinear.

9.2.8. Prove that it is impossible for the Menelaus points to be collinear if exactly two of them are ideal points. Prove that $s = -1$ in case all three Menelaus points are ideal. [In that case they are collinear because they all lie on the line at infinity.]

9.2.9. Prove the converse of Menelaus's theorem; i.e., prove that if $s = -1$, then the three Menelaus points are collinear.

There is also a trigonometric form of Menelaus's theorem. The proof is much like the proof of the trigonometric form of Ceva's theorem.

Trigonometric Form of the Theorem of Menelaus. *Let $\triangle ABC$ be an ordinary triangle. The Menelaus points L, M, and N for $\triangle ABC$ are collinear if and only if*

$$\frac{\sin(\angle \mathbf{BAL})}{\sin(\angle \mathbf{LAC})} \cdot \frac{\sin(\angle \mathbf{CBM})}{\sin(\angle \mathbf{MBA})} \cdot \frac{\sin(\angle \mathbf{ACN})}{\sin(\angle \mathbf{NCB})} = -1.$$

EXERCISES

9.2.10. Prove the trigonometric form of the theorem of Menelaus.

CHAPTER 10

Circles and lines

In this short chapter we develop some technical results that will be needed in the proofs of the applications of Menelaus's theorem in the next chapter.

10.1 THE POWER OF A POINT

We begin by defining the power of a point with respect to a circle.

Definition. Let β be a circle and let O be a point. Choose a line ℓ such that O lies on ℓ and ℓ intersects β. Define the *power of O with respect to β* to be

$$p(O,\beta) = \begin{cases} (OP)^2 & \text{if } \ell \text{ is tangent to } \beta \text{ at } P\text{, or} \\ (OQ)(OR) & \text{if } \ell \text{ intersects } \beta \text{ at two points } Q \text{ and } R. \end{cases}$$

In this section we will prove two of Euclid's most famous propositions. The first asserts that the power of a point is well defined in the sense that it does not depend on which line ℓ is used in the definition (provided O lies on the line and the line intersects β). The second gives a relationship between the angles of a triangle and the angles made by a line that is tangent to the circumcircle at a vertex. Both results will prove to be useful in the next chapter.

Euclid's Proposition III.36. *The power of O with respect to β is well defined; i.e., the same numerical value for the power is obtained regardless of which line ℓ is used in the definition (provided O lies on ℓ and ℓ intersects β).*

Euclid's Proposition III.32. *Let $\triangle ABC$ be a triangle inscribed in the circle γ and let t be the line that is tangent to γ at A. If D and E are points on t such that A is between D and E and D is on the same side of $\overleftrightarrow{AC}$ as B, then $\angle DAB \cong \angle ACB$ and $\angle EAC \cong \angle ABC$.*

EXERCISES

***10.1.1.** Construct two points O and S and construct a circle β. Keep O fixed and move S so that $\overleftrightarrow{OS}$ that intersects β in two points. Mark the two points of intersection and label them Q and R. Measure OQ and OR and calculate $(OQ)(OR)$. Now move S (keeping O and β fixed) and observe that $(OQ)(OR)$ remains constant. Verify that this is true regardless of whether O is inside β or outside β.

10.1.2. Prove Euclid's Proposition III.36.

[Hint: Let B be the center of β. First check the case in which $O = B$. Then assume $O \neq B$. Let S and T be the two points at which $\overleftrightarrow{OB}$ intersects β; see Figure 10.1. If ℓ intersects β in two points Q and R, use similar triangles to prove that $(OQ)(OR) = (OS)(OT)$. If ℓ is tangent to β at P, use the Pythagorean theorem to prove that $(OP)^2 = (OS)(OT)$. Explain why this is enough to prove the theorem.

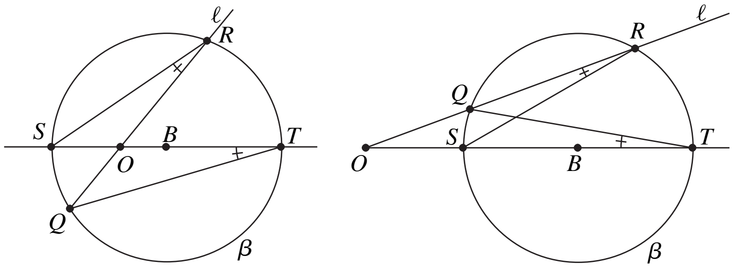

FIGURE 10.1: The proof of Euclid's Proposition III.36

***10.1.3.** Construct a circle γ and three points A, B, and C on γ. Then construct the tangent line t at A. Measure the angles made by t with $\overrightarrow{AB}$ and $\overrightarrow{AC}$ as well as the interior angles of $\triangle ABC$. Find a relationship between the measures of the angles of $\triangle ABC$ and those made by t and the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

10.1.4. Prove Euclid's Proposition III.32.

[Hint: First consider the case in which $\overleftrightarrow{BC}$ is parallel to t. Show that $\triangle ABC$ is an isosceles triangle in this case. In the other case, let O be the point at which $\overleftrightarrow{BC}$ intersects t. It may be assumed that C is between B and O (explain). Use Euclid's Proposition III.36 and the SAS Similarity Criterion to show that $\triangle OAC \sim \triangle OBA$.]

10.2 THE RADICAL AXIS

In this section we investigate the locus of points that have the same power with respect to two circles.

EXERCISES

***10.2.1.** Construct two circles α and β that have different centers. Construct a point P and calculate both $p(P, \alpha)$ and $p(P, \beta)$. Move P and determine by experimentation all points P for which $p(P, \alpha) = p(P, \beta)$. Make a conjecture about the shape of the set of points P for which $p(P, \alpha) = p(P, \beta)$. Do this in case the circles are disjoint and in case the circles intersect. Compare your results in the two cases.

Definition. The *radical axis* $r(\alpha, \beta)$ of two circles α and β is the locus of points P such that $p(P, \alpha) = p(P, \beta)$; i.e.,

$$r(\alpha, \beta) = \{P \mid p(P, \alpha) = p(P, \beta)\}.$$

The next few exercises will prove that the radical axis of two circles is always a line. Note that Exercise 10.2.3 can be viewed as a generalization of the pointwise characterization of perpendicular bisector.

EXERCISES

10.2.2. Let A and B be two distinct points. Prove that if E and F lie on $\overleftrightarrow{AB}$ and $(EA)^2 - (EB)^2 = (FA)^2 - (FB)^2$, then $E = F$.
[Hint: Introduce coordinates on the line so that A corresponds to the real number a, B corresponds to b, E corresponds to e, and F corresponds to f. Translate the given equation into an equation relating a, b, e, and f (for example, $EA = |e - a|$), and show that $e = f$.]

10.2.3. Let A and B be two distinct points, let E be a point on $\overleftrightarrow{AB}$, and let ℓ be the line that is perpendicular to $\overleftrightarrow{AB}$ at E. Prove that a point P lies on ℓ if and only if $(PA)^2 - (PB)^2 = (EA)^2 - (EB)^2$.
[Hint: The forward implication is an application of the Pythagorean theorem. For the converse, drop a perpendicular from P to $\overleftrightarrow{AB}$ and call the foot F. Use the hypothesis and the previous exercise to show that $E = F$. Conclude that P lies on ℓ.]

10.2.4. Let A and B be two distinct points. Prove that for every real number c there exists a unique point X on $\overleftrightarrow{AB}$ such that $(AX)^2 - (BX)^2 = c$.
[Hint: Again introduce coordinates on the line so that A corresponds to the real number a, B corresponds to b, and X corresponds to x. Translate the given equation into an equation relating a, b, x, and c. Check that there is a unique solution for x in terms of a, b, and c.]

10.2.5. Let α be a circle with center A and let β be a circle with center B, $A \neq B$. Prove that $r(\alpha, \beta)$ is a line that is perpendicular to $\overleftrightarrow{AB}$.
[Hint: Prove that $p(P, \alpha) = (PA)^2 - r^2$, where r is the radius of α. There is a similar equation for $p(P, \beta)$. Apply the last two exercises.]

Here is a statement of the theorem that was proved in the last exercise.

Radical Axis Theorem. *If α and β are two circles with different centers, then $r(\alpha, \beta)$ is a line that is perpendicular to the line through the centers of the circles.*

EXERCISES

10.2.6. Prove that the radical axis of a pair of tangent circles is the common tangent line.

10.2.7. Prove that the radical axis of a pair of circles that intersect at two points is the common secant line.

10.3 THE RADICAL CENTER

Three circles can be taken in pairs to determine three radical axes. The next theorem asserts that these three radical axes are always concurrent. That result is important in the proof of Brianchon's theorem.

Radical Center Theorem. *If α, β, and γ are three circles with distinct centers, then the three radical axes determined by the circles are concurrent (in the extended plane).*

EXERCISES

10.3.1. Prove the radical center theorem.
[Hint: Prove that the radical axes intersect at an ideal point in case the centers of the three circles are collinear. In the other case, prove that any two radical axes intersect and that the third radical axis must pass through the point at which the first two intersect.]

Definition. The point of concurrence of the three radical axes of the three circles α, β, and γ is called the *radical center* of the three circles.

C H A P T E R 11

Applications of the theorem of Menelaus

This chapter contains a sampling of the many applications of the theorem of Menelaus.

11.1 TANGENT LINES AND ANGLE BISECTORS

The first applications are simple results about how tangent lines and angle bisectors intersect the sidelines of the triangle. All applications in this section use the trigonometric form of Menelaus's theorem.

EXERCISES

***11.1.1.** Construct a triangle and its circumscribed circle. For each vertex of the triangle, construct the line that is tangent to the circumcircle at that point. Mark the point at which the line that is tangent at a vertex intersects the opposite sideline of the triangle. Verify that the three points you have marked are always collinear. Under what conditions is one or more of the intersection points an ideal point?

11.1.2. Let $\triangle ABC$ be a triangle. Prove that the lines that are tangent to the circumcircle of $\triangle ABC$ at the vertices of the triangle cut the opposite sidelines at three collinear points.
[Hint: Use Euclid's Proposition III.32.]

***11.1.3.** Construct a triangle $\triangle ABC$. At each vertex of the triangle, construct the line that bisects the two exterior angles at that vertex. Mark the point at which the bisector intersects the opposite sideline. Verify that the three points you have marked are always collinear. Under what conditions is one of more of the intersection points an ideal point?

11.1.4. Let $\triangle ABC$ be a triangle. Prove that the external angle bisectors of $\triangle ABC$ meet the opposite sidelines of the triangle in three collinear points.

***11.1.5.** Construct a triangle $\triangle ABC$. At each of vertices A and B construct the line that bisects the interior angle at that vertex and mark the point at which the bisector intersects the opposite sideline. At vertex C, construct the line that bisects the exterior angles and mark the point at which the bisector intersects the opposite sideline. Verify that the three points you have marked are always collinear. Under what conditions is one or more of the intersection points an ideal point?

11.1.6. Let $\triangle ABC$ be a triangle. Prove that internal angle bisectors at A and B and the external angle bisector at C meet the opposite sidelines of the triangle in three collinear points.

11.2 DESARGUES'S THEOREM

The theorem in this section is due to Girard Desargues (1591–1661). It has important applications to the theory of perspective drawing in art.

Definition. Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are said to be *perspective from the point O* if the three lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$ joining corresponding vertices are concurrent at O. The point O is called the *perspector* or the *point of perspective* (see Figure 11.1).

Some authors use the term *copolar* to describe triangles that are perspective from a point.

Definition. Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are said to be *perspective from a line* if the three points L, M, and N at which corresponding sidelines $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$, $\overleftrightarrow{AC}$ and $\overleftrightarrow{A'C'}$, and $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ intersect are collinear. The line containing L, M, and N is called the *perspectrix* (see Figure 11.1).

Some authors use the term *coaxial* to describe triangles that are perspective from a line.

Desargues's Theorem. *Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are perspective from a point if and only if they are perspective from a line.*

EXERCISES

***11.2.1.** Make a GSP sketch showing two triangles that are perspective from a point. Verify that they are perspective from a line. What happens when the perspector is an ideal point? What happens when one or more of the pairs of corresponding sidelines of the triangles are parallel? How many pairs of corresponding sides can be parallel?

***11.2.2.** Make a GSP sketch showing two triangles that are perspective from a line. Verify that they are perspective from a point.

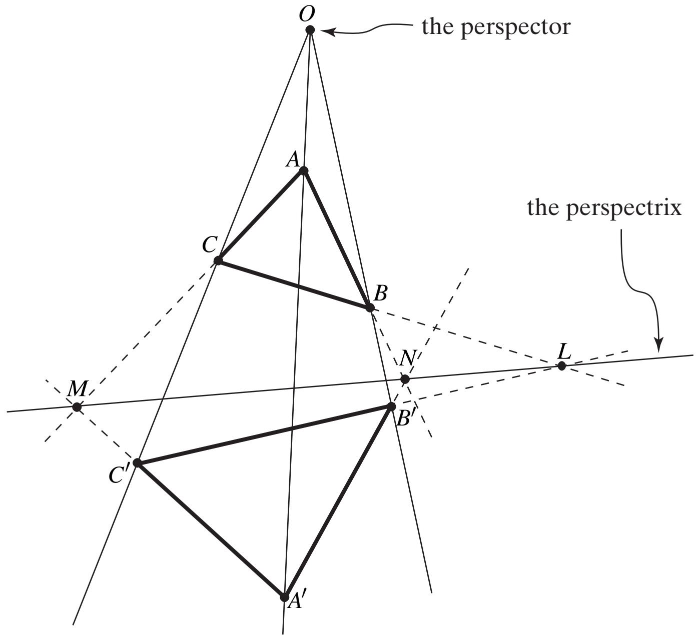

FIGURE 11.1: Desargues's Theorem

11.2.3. Prove that if the triangles $\triangle ABC$ and $\triangle A'B'C'$ are perspective from a point, then they are perspective from a line.
[Hint: Assume that the two triangles are perspective from O. The points L, M, and N may be defined as the intersections of corresponding sidelines of the two triangles. Explain why it is enough to prove that

$$\frac{\mathbf{AN}}{\mathbf{NB}} \cdot \frac{\mathbf{BL}}{\mathbf{LC}} \cdot \frac{\mathbf{CM}}{\mathbf{MA}} = -1.$$

Apply Menelaus's theorem to the triangle $\triangle OAB$ with collinear Menelaus points N, A', and B'. Apply it in a similar way to $\triangle OAC$ and $\triangle OBC$, and then multiply the resulting equations together.]

11.2.4. Prove that if the triangles $\triangle ABC$ and $\triangle A'B'C'$ are perspective from a line, then they are perspective from a point.
[Hint: Assume that the points L, M, and N are collinear. Define O to be the point of intersection of $\overleftrightarrow{BB'}$ and $\overleftrightarrow{CC'}$. Explain why it is enough to prove that A, A', and O are collinear. Note that the triangles $\triangle MCC'$ and $\triangle NBB'$ are perspective from the point L, so you can apply the part of the theorem that you have already proved to them.]

11.3 PASCAL'S MYSTIC HEXAGRAM

Polygons inscribed in circles have some surprising properties. One of the most interesting was discovered by Blaise Pascal (1623–1662). Pascal discovered the next theorem when he was only sixteen years old and he gave it the colorful Latin title *mysterium hexagrammicum*. For that reason the theorem is still referred to as *Pascal's mystic hexagram*.

Definition. A *hexagon* is a polygon with six vertices $ABCDEF$. It is required that no three consecutive vertices (in cyclic order) are collinear. Just as is the case with quadrilaterals, we allow the sides to cross. A hexagon is *inscribed* in the circle γ if all the vertices lie on γ.

Pascal's Mystic Hexagram. *If a hexagon is inscribed in a circle, then the three points at which opposite sidelines intersect are collinear.*

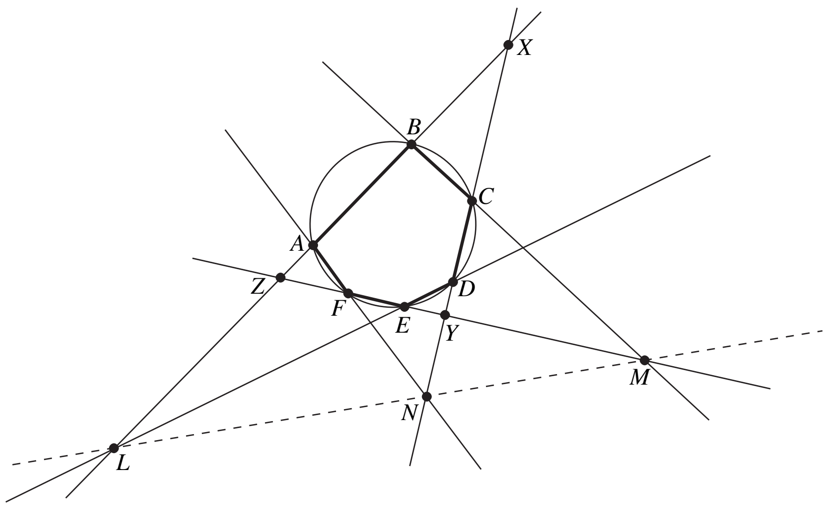

FIGURE 11.2: Pascal's Mystic Hexagram

EXERCISES

*__11.3.1.__ Construct a circle and six points A, B, C, D, E, and F, cyclically ordered around the circle. Draw the lines through the sides and mark the points at which the opposite sidelines intersect. Verify that these three points are collinear. Does the theorem hold for crossed hexagons whose vertices lie on a circle?

*__11.3.2.__ Draw examples of inscribed hexagons for which 0, 1, and 3 of the points of intersection are ideal points. Is it possible for exactly two of the points to be ideal?

11.3.3. Explain how the result of Exercise 11.1.1 can be viewed as a limiting case of Pascal's theorem.
[Hint: Take the limit as pairs of adjacent vertices of the hexagon converge.]

Notation. Let γ be a circle, and let A, B, C, D, E, and F be six points on γ, cyclically ordered around the circle. Define six additional points as follows.

L is the point at which $\overleftrightarrow{AB}$ meets $\overleftrightarrow{DE}$
M is the point at which $\overleftrightarrow{BC}$ meets $\overleftrightarrow{EF}$
N is the point at which $\overleftrightarrow{CD}$ meets $\overleftrightarrow{AF}$
X is the point at which $\overleftrightarrow{AB}$ meets $\overleftrightarrow{CD}$
Y is the point at which $\overleftrightarrow{EF}$ meets $\overleftrightarrow{CD}$
Z is the point at which $\overleftrightarrow{AB}$ meets $\overleftrightarrow{EF}$

EXERCISES

***11.3.4.** Use GSP to make a sketch showing the circle γ and the twelve points listed above. Observe that L, M, and N are Menelaus points for $\triangle XYZ$.

11.3.5. Prove Pascal's theorem in case all three of the points X, Y, and Z are ordinary points.
[Hint: The proof is accomplished by using Menelaus's theorem to show that L, M, and N are collinear Menelaus points for the triangle $\triangle XYZ$. Apply Menelaus's theorem three times to $\triangle XYZ$ with collinear Menelaus points $\{B, C, M\}$, $\{A, F, N\}$, and $\{D, E, L\}$. Multiply the resulting equations together and apply Euclid's Proposition III.36 to reach the desired conclusion.]

11.3.6. Prove Pascal's theorem in case one of the points X, Y, Z is an ideal point.
[Hint: Instead of using X, Y, and Z, use the points of intersection of $\overleftrightarrow{BC}$ with $\overleftrightarrow{DE}$, $\overleftrightarrow{DE}$ with $\overleftrightarrow{FA}$ and $\overleftrightarrow{FA}$ with $\overleftrightarrow{BC}$. First check that if one of X, Y, or Z is ideal, then all three of these points are ordinary. Then proceed as in the previous proof.]

11.4 BRIANCHON'S THEOREM

As mentioned at beginning of chapter, Brianchon's theorem is dual to Pascal's theorem. It illustrates a new aspect of duality: a point on circle is dual to a line that is tangent to the circle. The theorem is included in this chapter because it is such a beautiful example of duality, even though the proof does not use Menelaus's theorem. Instead the proof is based on the radical center theorem. Brianchon's theorem is named for its discoverer, Charles Julien Brianchon (1783–1864).

Brianchon's Theorem. *If a hexagon is circumscribed about a circle, then the lines determined by pairs of opposite vertices are concurrent.*

EXERCISES

***11.4.1.** Make a GSP sketch that can be used to verify and illustrate Brianchon's theorem.

The proof relies on a construction that is relatively easy to reproduce with GSP, but which few people would think of for themselves. We will describe the construction first, and then look at the proof itself. As you read the next two paragraphs you should make a GSP sketch that reproduces Figure 11.3.

Let $\mathcal{C}$ be a circle and let $ABCDEF$ be a hexagon that is circumscribed about $\mathcal{C}$. More specifically, this means that all the vertices of $ABCDEF$ lie outside $\mathcal{C}$ and each side of $ABCDEF$ is tangent to $\mathcal{C}$ at an interior point of that side. Let P, Q, R, S, T, and U be the points of tangency, labeled as in Figure 11.3.

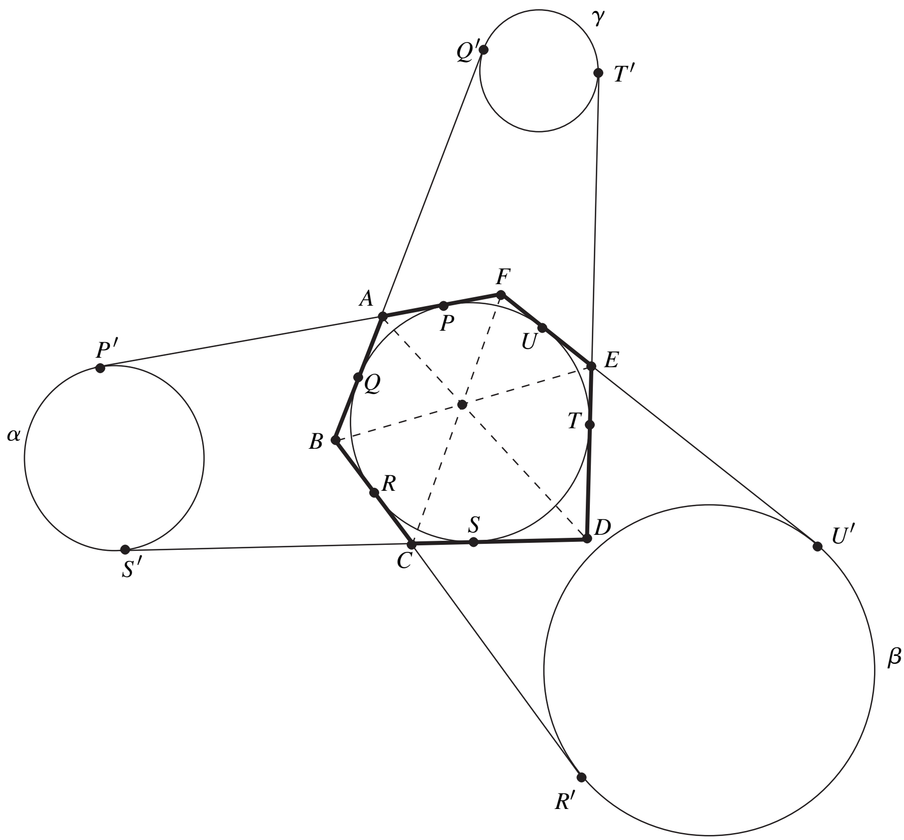

FIGURE 11.3: Construction for the proof of Brianchon's theorem

Construct points P' on $\overrightarrow{FA}$, Q' on $\overrightarrow{BA}$, R' on $\overrightarrow{BC}$, S' on $\overrightarrow{DC}$, T' on $\overrightarrow{DE}$, and U' on $\overrightarrow{FE}$ such that $PP' = QQ' = RR' = SS' = TT' = UU'$. (The precise value of these distances is not important, provided all six of the distances are equal.) Construct three circles α, β, and γ such that α is tangent to $\overleftrightarrow{AF}$ at P' and tangent to $\overleftrightarrow{CD}$ at S', β is tangent to $\overleftrightarrow{BC}$ at R' and tangent to $\overleftrightarrow{FE}$ at U', and γ is tangent to $\overleftrightarrow{AB}$ at Q' and tangent to $\overleftrightarrow{DE}$ at T'. These three circles exist by Exercise 11.4.2.

EXERCISES

11.4.2. Let $\mathcal{C}$ be a circle and let t and s be two lines that are tangent to $\mathcal{C}$ at points P and S, respectively. If P' is a point on t and $S' \neq P'$ is a point on s such that P'

and S' are on the same side of $\overleftrightarrow{PS}$ and $PP' = SS'$, then there exists a circle α that is tangent to t and s at P' and S'.
[Hint: The circles $\mathcal{C}$ and α in Figure 11.3 illustrate the exercise. First prove the theorem in case $\overleftrightarrow{PP'} \parallel \overleftrightarrow{SS'}$ and then use similar triangles to prove the other case.]

11.4.3. Prove that $FP' = FU'$, $CS' = CR'$, $BQ' = BR'$, $ET' = EU'$, $AQ' = AP'$, and $DT' = DS'$ in Figure 11.3.

11.4.4. Prove that $r(\alpha, \beta) = \overleftrightarrow{CF}$, $r(\beta, \gamma) = \overleftrightarrow{BE}$, and $r(\alpha, \gamma) = \overleftrightarrow{AD}$.

11.4.5. Use the preceding exercise and the radical center theorem to prove Brianchon's theorem.

11.5 PAPPUS'S THEOREM

The next theorem was discovered by Pappus of Alexandria. Pappus, who live from approximately AD 290 until about 350, was one of the last great geometers of antiquity. Much later his theorem became an important result in the foundations of projective geometry. Both the statement and proof of Pappus's theorem are reminiscent of Pascal's theorem. Again there is a hexagon involved. In this case the vertices of the hexagon lie on two lines rather than on a circle. The hexagon is necessarily a crossed hexagon.

Pappus's Theorem. *Let A, B, C, D, E, and F be six points. Define L to be the point at which* $\overleftrightarrow{AB}$ *meets* $\overleftrightarrow{DE}$, *define M to be the point at which* $\overleftrightarrow{BC}$ *meets* $\overleftrightarrow{EF}$, *and define N to be the point at which* $\overleftrightarrow{CD}$ *meets* $\overleftrightarrow{AF}$. *If A, C, E lie on one line and B, D, F lie on another line, then the points L, M, and N are collinear.*

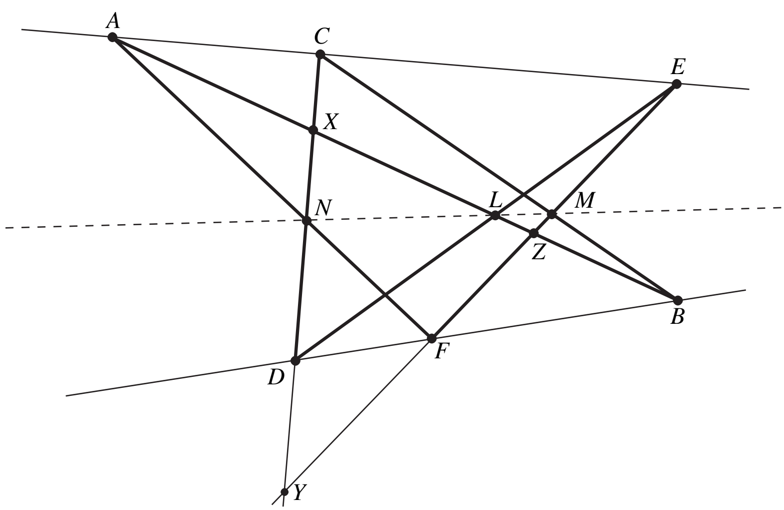

FIGURE 11.4: Pappus's Theorem

EXERCISES

***11.5.1.** Make a sketch that can be used to verify and illustrate Pappus's theorem.

Before we begin the proof of the theorem, let us establish some notation (see Figure 11.4).

Notation. Start with six points A, B, C, D, E, and F. Assume A, C, and E lie on the line ℓ and that B, D, and F lie on a line m. Define six additional points as follows.

L is the point at which $\overleftrightarrow{AB}$ meets $\overleftrightarrow{DE}$
M is the point at which $\overleftrightarrow{BC}$ meets $\overleftrightarrow{EF}$
N is the point at which $\overleftrightarrow{CD}$ meets $\overleftrightarrow{AF}$
X is the point at which $\overleftrightarrow{AB}$ meets $\overleftrightarrow{CD}$
Y is the point at which $\overleftrightarrow{EF}$ meets $\overleftrightarrow{CD}$
Z is the point at which $\overleftrightarrow{AB}$ meets $\overleftrightarrow{EF}$

In order to keep the proof relatively simple, we will only prove Pappus's theorem in case the three points X, Y, and Z are ordinary points.

EXERCISES

***11.5.2.** Use GSP to make a sketch showing the ℓ and m along with the twelve points listed above. Observe that L, M, and N are Menelaus points for $\triangle XYZ$.

11.5.3. Prove Pappus's theorem in case the three points X, Y, and Z defined above are ordinary points.
[Hint: The proof is accomplished by using Menelaus's theorem to show that L, M, and N are collinear Menelaus points for the triangle $\triangle XYZ$. Apply Menelaus's theorem five times to $\triangle XYZ$ with collinear Menelaus points $\{B, C, M\}$, $\{A, F, N\}$, $\{D, E, L\}$, $\{A, C, E\}$, and $\{B, D, F\}$. Multiply the first three equations, divide by the product of the last two, and cancel lots of terms.]

11.6 SIMSON'S THEOREM

This theorem is usually attributed to the Scottish mathematician Robert Simson (1687–1768), but there is no record that he ever published it. The theorem was published in 1799 by William Wallace (1768–1843). The result was discussed earlier in the context of pedal triangles.

Simson's Theorem. *A point P is on the circumscribed circle of triangle $\triangle ABC$ if and only if the feet of the perpendiculars from P to the sidelines of $\triangle ABC$ are collinear.*

Definition. A line that contains the feet of three perpendiculars from a point P to the triangle $\triangle ABC$ is called a *Simson line* for $\triangle ABC$. The point P is called the *pole* of the Simson line.

We will prove Simson's theorem below, but first we explore several interesting properties of Simson lines.

EXERCISES

***11.6.1.** Construct a triangle $\triangle ABC$ and its circumcircle. Choose a point P and construct the feet of the three perpendiculars from P to the sidelines of the triangle. Put a line through two of the feet and verify that the third foot is on that line if and only if P is on the circumcircle.
[Hint: In order to make sure that you consider all possible shapes for $\triangle ABC$, it is best to construct the circumcircle first and then construct A, B, and C to be three movable points on the circle.]

***11.6.2.** Find triangles $\triangle ABC$ and points P on the circumcircle of $\triangle ABC$ such that the Simson line with pole P intersects $\triangle ABC$ in exactly two points. Now find examples for which the Simson line is disjoint from the triangle. Make observations about the shape of the triangles and the location of P for which the latter possibility occurs.

***11.6.3.** Let Q be the point at which the altitude through A meets the circumcircle. Verify that the Simson line with pole Q is parallel to the tangent to the circle at A.

***11.6.4.** Let P and Q be two points on the circumcircle of $\triangle ABC$. Verify that the measure of the angle between the Simson lines with poles P and Q is half the measure of $\angle POQ$ (where O is the circumcenter).

***11.6.5.** Verify that if $\overline{PQ}$ is a diameter for the circumcircle, then the Simson lines with poles P and Q intersect at a point on the nine-point circle.

***11.6.6.** Let H be the orthocenter of $\triangle ABC$. Verify that if P is any point on the circumcircle, then the midpoint of $\overline{HP}$ lies on the Simson line with pole P.

11.6.7. Prove that the feet of the perpendiculars are collinear if and only if P lies on the circumcircle.
[Hint: Apply Menelaus's theorem along with similar triangles and Euclid's Proposition III.22.]

***11.6.8.** Construct a circle γ and three movable points A, B, and C on γ. Construct an additional point P that lies on the arc of γ from A to C that does not contain B. Verify that B' is always between A' and C', whether the Simson line intersects $\triangle ABC$ or not.

11.6.9. Prove the result you verified in the last exercise.

11.7 PTOLEMY'S THEOREM

Ptolemy's theorem is a very useful result regarding cyclic quadrilaterals that is attributed to Claudius Ptolemaeus of Alexandria (85–165). It asserts that the product of the lengths of the two diagonals of a cyclic quadrilateral is equal to the sum of the product of the lengths of the opposite sides. Here is a precise statement of the theorem.

Ptolemy's Theorem. *If $\square ABCD$ is a cyclic quadrilateral, then*

$$AB \cdot CD + BC \cdot AD = AC \cdot BD.$$

EXERCISES

***11.7.1.** Make a GSP sketch that illustrates and verifies Ptolemy's theorem. Part of the definition of cyclic quadrilateral requires that the quadrilateral be convex; is the theorem valid for crossed quadrilaterals whose vertices lie on a circle?

11.7.2. Prove Ptolemy's theorem.
[Hint: Consider the Simson line for $\triangle ABC$ with pole D. By Exercise 11.6.9, B' is between A' and C', so $A'B' + B'C' = A'C'$. Apply Exercise 5.4.5 to reach the desired conclusion.]

11.8 THE BUTTERFLY THEOREM

The next theorem is known as the butterfly theorem. The reason for the name is evident from Figure 11.5.

Butterfly Theorem. *Assume γ is a circle, $\overline{PQ}$ is a chord of γ, and M is the midpoint of $\overline{PQ}$. Let $\overline{AB}$ and $\overline{CD}$ be two chords of γ such that A and C are on the same side of $\overleftrightarrow{PQ}$ and both $\overline{AB}$ and $\overline{CD}$ intersect $\overline{PQ}$ at M. If $\overline{AD}$ intersects $\overline{PQ}$ at X and $\overline{BC}$ intersects $\overline{PQ}$ at Y, then M is the midpoint of $\overline{XY}$.*

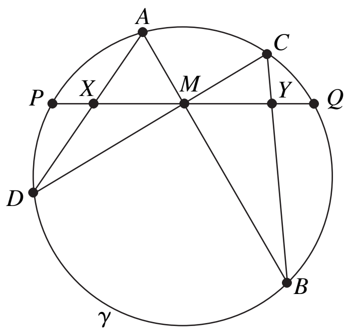

FIGURE 11.5: The Butterfly Theorem

EXERCISES

***11.8.1.** Make a GSP sketch that illustrates and verifies the butterfly theorem.

***11.8.2.** Verify that the butterfly theorem remains true even when the assumption that A and C are on the same side of $\overleftrightarrow{PQ}$ is dropped. In that case the segment $\overline{PQ}$ must be extended to the line $\overleftrightarrow{PQ}$ and the points X and Y lie outside γ. Can you find a butterfly in this diagram? While the result is true in this generality, we will only prove the theorem as stated above.

11.8.3. Prove the butterfly theorem in the special case in which $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$.
[Hint: Prove that M is the center of γ (in this case) and then use ASA to prove that $\triangle DMX \cong \triangle CMY$.]

11.8.4. Prove the following simple algebra result that that is needed in the proof of the butterfly theorem: *If x, y, and m are three positive numbers such that*

$$\frac{x^2(m^2 - y^2)}{y^2(m^2 - x^2)} = 1,$$

then $x = y$.

11.8.5. Prove the butterfly theorem in case $\overleftrightarrow{AD}$ intersects $\overleftrightarrow{BC}$.

[Hint: Let R be the point at which $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ intersect. To simplify the notation, let $x = MX$, $y = MY$, and $m = PM$. Apply Menelaus's theorem twice to $\triangle RXY$, first with collinear Menelaus points $\{A, M, B\}$ and then with collinear Menelaus points $\{C, M, D\}$. Multiply the resulting equations. Apply Euclid's Proposition III.36 three times to conclude that $RA \cdot RD = RB \cdot RC$, $XA \cdot XD = XP \cdot XQ$, and $YB \cdot YC = YP \cdot YQ$. All of that should result in the equation

$$\frac{x^2(m^2 - y^2)}{y^2(m^2 - x^2)} = 1.$$

Use Exercise 11.8.4 to complete the proof.]

CHAPTER 12

More topics in triangle geometry

This chapter examines a number of loosely connected topics regarding the geometry of the triangle. We will explore the the statements of the theorems using GSP, but will not prove them.

12.1 NAPOLEON'S THEOREM AND THE NAPOLEON POINT

The theorem in this section is commonly attributed to the French emperor Napoleon Bonaparte (1769-1821). Napoleon was an amateur mathematician, with a particular interest in geometry, who took pride in his mathematical talents. Thus this attribution may be based at least partially on historical fact. On the other hand, Coxeter and Greitzer [3, page 63] make the following comment regarding the possibility that the theorem might in fact be due to Napoleon: "...the possibility of his knowing enough geometry for this feat is as questionable as the possibility of his knowing enough English to compose the famous palindrome ABLE WAS I ERE I SAW ELBA."

EXERCISES

***12.1.1.** Construct a triangle $\triangle ABC$ and use the tool you made in Exercise 3.1.4 to construct an equilateral triangle on each side of $\triangle ABC$. Make sure that the three new triangles are on the outside of the original triangle. Label vertices so that the three new triangles are $\triangle A'BC$, $\triangle AB'C$, and $\triangle ABC'$. Construct the three centroids of triangles $\triangle A'BC$, $\triangle AB'C$, and $\triangle ABC'$ and label them U, V, and W, respectively. Verify the following.

(a) $\triangle UVW$ is equilateral.

(b) Lines $\overleftrightarrow{AU}$, $\overleftrightarrow{BV}$, and $\overleftrightarrow{CW}$ are concurrent.

***12.1.2.** Do the same construction as in the previous exercise, but this time construct the equilateral triangles so that they are oriented towards the inside of $\triangle ABC$. (The equilateral triangles will overlap and may even stick out of $\triangle ABC$.) Do the same two conclusions hold in this case?

Definition. The triangle $\triangle UVW$ in Exercise 12.1.1 is called the *Napoleon triangle* associated with $\triangle ABC$. *Napoleon's theorem* is the assertion that the Napoleon triangle is always equilateral, regardless of the shape of $\triangle ABC$. The point at which lines $\overleftrightarrow{AU}$, $\overleftrightarrow{BV}$, and $\overleftrightarrow{CW}$ concur is called the *Napoleon point* of $\triangle ABC$. It is another new triangle center.

12.1.1 The Torricelli point

There is still another triangle center that is closely related to the Napoleon point. This new point is called the Torricelli point of the triangle. Be careful not to confuse it with the Napoleon point. The point is named for the Italian Mathematician Evangelista Torricelli (1608–1647).

EXERCISES

***12.1.3.** Construct a triangle $\triangle ABC$ and construct external equilateral triangles on the three sides of the triangle. Label the vertices as in Exercise 12.1.1. Construct the lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$. Note that they are concurrent. The point at which these three lines concur is called the *Torricelli point* of $\triangle ABC$.

***12.1.4.** For which triangles is the Torricelli point inside the triangle and for which is it outside?

***12.1.5.** For which triangles is the Torricelli point equal to the Napoleon point?

***12.1.6.** Construct the circumcircles for the three triangles $\triangle A'BC$, $\triangle AB'C$, and $\triangle ABC'$. Observe that all three of these circles pass through the Torricelli point. Thus an alternative way to define the Torricelli point would be to say that it is the point at which the three circumcircles of the external equilateral triangles intersect.

12.1.2 van Aubel's theorem

There is also a theorem for quadrilaterals that is closely related to Napoleon's theorem for triangles. The theorem for quadrilaterals is known as *van Aubel's theorem*. The *center of a square* is the point at which the diagonals intersect. The center of a square is obviously equidistant from the vertices.

EXERCISES

***12.1.7.** Construct a convex quadrilateral $\square ABCD$. Construct a square on each side of the quadrilateral. Make sure the fours squares are all on the outside of the quadrilateral. Construct the centers of the four squares and the two segments joining the centers of the squares based on opposite sides of $\square ABCD$. Measure the lengths of these segments. What do you observe? Measure the angles between the two segments. What do you observe?

***12.1.8.** Now try the same thing with the squares constructed towards the inside of the quadrilateral. Is the result still true in this case?

12.2 MIQUEL'S THEOREM AND MIQUEL POINTS

The theorem in this section is attributed to the nineteenth century French mathematician Auguste Miquel. We have proved many theorems regarding concurrent lines; by contrast, this theorem gives a condition under which three circles are concurrent.

EXERCISES

*__12.2.1.__ Construct a triangle $\triangle ABC$ and movable points D, E, and F on the sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively. Now construct the circumcircles for the three triangles $\triangle AEF$, $\triangle BDF$, and $\triangle CDE$. Observe that the three circles are concurrent regardless of the shape of $\triangle ABC$ and regardless of how the points D, E and F are chosen. Find examples for which the point of concurrence is inside the triangle as well as examples for which it is outside.

*__12.2.2.__ What happens if the points D, E, and F are chosen to lie on the sidelines of $\triangle ABC$ rather than the sides?

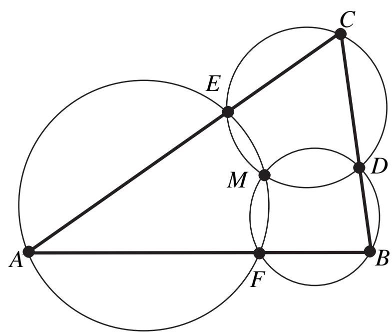

FIGURE 12.1: Miquel's Theorem

The assertion that the three circles in Exercise 12.2.1 are concurrent is known as *Miquel's theorem*. The point of concurrence is called a *Miquel point* for the triangle.

12.3 THE FERMAT POINT

The triangle center studied in this section is named after Pierre de Fermat (1601–1665).

Definition. The *Fermat point* of a triangle is the point F for which the sum of the distances from F to the vertices is as small as possible; i.e., the Fermat point of $\triangle ABC$ is the point F such that $FA + FB + FC$ is minimized.

In the exercises below you will see that for most triangles the Fermat point is the point that makes equal angles with of the vertices. For such triangles the Fermat point is equal to the Torricelli point.

EXERCISES

***12.3.1.** Construct a triangle $\triangle ABC$ such that all three angles in the triangle measure less than $120°$. Construct a point P in the interior of the triangle and calculate $PA + PB + PC$. Move P around until you locate the Fermat point. Once you have found the point P for which $PA + PB + PC$ is minimized, measure the three angles $\angle APB$, $\angle BPC$, and $\angle CPA$. Observe that the sum of the distances is minimized at the same point where the three angles are congruent.

***12.3.2.** Now construct a triangle $\triangle ABC$ in which angle $\angle BAC$ has measure greater than $120°$. Where is the Fermat point for this triangle located?

***12.3.3.** Construct a triangle $\triangle ABC$ and its Torricelli point. Verify that the Torricelli point T and the Fermat point F are the same in case all angles in the triangle have measure less than $120°$. Find examples of triangles for which the Torricelli point and the Fermat point are different.

12.4 MORLEY'S THEOREM

Our final theorem requires that we trisect the angles of a triangle. To *trisect* an angle $\angle BAC$ means to find two rays $\overrightarrow{AD}$ and $\overrightarrow{AE}$ such that $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and $\mu(\angle BAD) = (1/3)\mu(\angle BAC)$ while $\overrightarrow{AE}$ is between $\overrightarrow{AD}$ and $\overrightarrow{AC}$ and $\mu(\angle DAE) = (1/3)\mu(\angle BAC)$. It follows easily that $\mu(\angle EAC) = (1/3)\mu(\angle BAC)$ as well, so the original angle is divided into three congruent angles.

The ancient Greeks were interested in the problem of trisecting an angle using only a straightedge and compass. They never succeeded in trisecting an arbitrary angle, and it has been known since the nineteenth century that it is impossible to do so using only the Euclidean compass and straightedge. But it is quite easy to trisect angles using the measurement and calculation capabilities of GSP.

EXERCISES

***12.4.1.** Make a tool that trisects an angle. The tool should accept three points A, B, and C as givens and should produce as results the two rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ together with the two additional rays in the interior of $\angle BAC$ that trisect the angle.

[Hint: Measure the angle $\angle BAC$ and calculate $(1/3)\mu(\angle BAC)$. Rotate $\overrightarrow{AB}$ through $(1/3)\mu(\angle BAC)$ to construct the first trisecting ray and then rotate again through the same angle to produce the second trisecting ray.]

***12.4.2.** Construct a triangle $\triangle ABC$. Construct the two angle trisectors for each of the three interior angles of $\triangle ABC$. These six rays intersect in a total of twelve points in the interior of $\triangle ABC$. Label the point at which the rays through B and C that are closest to $\overline{BC}$ intersect as A'. Similarly, label the intersection of the two trisectors closest to $\overline{AC}$ as B' and label the intersection of the two trisectors closest to $\overline{AB}$ as C'. The triangle $\triangle A'B'C'$ is called the *Morley triangle* for the triangle $\triangle ABC$. (See Figure 12.2.)

***12.4.3.** Hide all the angle trisectors and concentrate on the Morley triangle $\triangle A'B'C'$. Measure all the sides and the angles of $\triangle A'B'C'$ and verify that it is always equilateral.

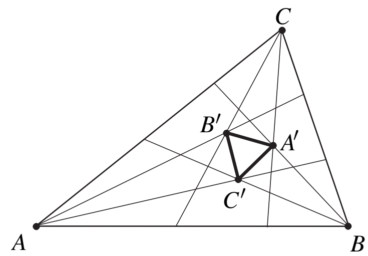

FIGURE 12.2: $\triangle A'B'C'$ is the Morley triangle for $\triangle ABC$

The theorem asserting that the Morley triangle is equilateral regardless of the shape of the original triangle is known as *Morley's theorem*. It was discovered around 1904 by the American mathematician Frank Morley (1869–1937). It is the most recently discovered theorem we have studied.

C H A P T E R 13

Inversions in circles

In this chapter we will investigate a class of transformations of the Euclidean plane called inversions in circles. The study of inversions is a standard part of college geometry courses, so we will leave most of the details of the subject to those courses and will not attempt to give a complete treatment here. Instead we will concentrate on the construction of the GSP tools that will be needed in the study of the Poincaré disk model of hyperbolic geometry in the next chapter. We will state the basic definitions and theorems of the subject, but will not prove the theorems. Proofs may be found in many sources; we will use §12.7 of [16] as our basic reference. In the exercises you will be asked to use the theorems to verify that certain constructions work and then to make GSP tools based on those constructions.

13.1 INVERTING POINTS

Let us begin with the definition of inversion.

Definition. Let $\mathcal{C} = \mathcal{C}(O, r)$ be a circle. The *inverse of P in $\mathcal{C}$* is the point $P' = I_{O,r}(P)$ on $\overrightarrow{OP}$ such that $(OP)(OP') = r^2$.

Observe that $I_{O,r}(P)$ is defined for any $P \neq O$, but that the definition breaks down when $P = O$. As P approaches O, the point $I_{O,r}(P)$ will move farther and farther from O. In order to define $I_{O,r}$ at O, we extend the plane by adding a *point at infinity*. This new point is denoted by the symbol ∞. The plane together with this one additional point at infinity is called the *inversive plane*. We extend $I_{O,r}$ to the inversive plane by defining $I_{O,r}(O) = \infty$ and $I_{O,r}(\infty) = O$.

The inversive plane is quite different from the extended Euclidean plane that was introduced in Chapter 8. In particular, the extended Euclidean plane includes an infinite number of different ideal points, while the inversive plane includes just one point at infinity. The inversion $I_{O,r}$ is a transformation of the inversive plane.[1] There is just one inversive plane on which every inversion is defined (not a different inversive plane for each inversion).

[1] A *transformation* is a function of a space to itself that is one-to-one and onto.

Construction. Let P be a point inside $\mathcal{C}$. Construct the perpendicular to $\overleftrightarrow{OP}$ at P, define T and U to be the points at which the perpendicular intersects $\mathcal{C}$, and then construct P' as shown in Figure 13.1.

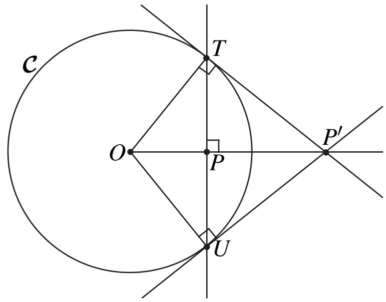

FIGURE 13.1: Contruction of P' in case P is inside $\mathcal{C}$

EXERCISES

***13.1.1.** Carry out the construction in GSP. Measure OP, OP', and OT and verify that $(OP)(OP') = (OT)^2$.

13.1.2. Prove that the point P' constructed above is the inverse of P.

***13.1.3.** Use the construction above to make a tool that finds $P' = I_{O,r}(P)$ for each point P inside $\mathcal{C}$. Your tool should accept three points as givens (the center O, a point on $\mathcal{C}$, and the point P) and it should return the point P' as its result.

Construction. Let P be a point outside $\mathcal{C}$. Find the midpoint M of $\overline{OP}$, construct a circle α with center M and radius MP, and let T and U be the two points of $\alpha \cap \mathcal{C}$. Then construct P' as shown in Figure 13.2.

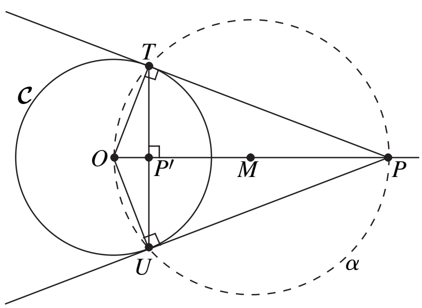

FIGURE 13.2: Contruction of P' in case P is outside of $\mathcal{C}$

EXERCISES

***13.1.4.** Carry out the construction in GSP. Measure OP, OP', and OT and verify that $(OP)(OP') = (OT)^2$.

13.1.5. Prove that the point P' constructed above is the inverse of P.

***13.1.6.** Use the construction above to make a tool that returns $P' = I_{O,r}(P)$ for each point P outside $\mathcal{C}$.

The drawback to the tools you made in the last several exercises is that one tool inverts points inside $\mathcal{C}$ but a different tool is required to invert points that are outside $\mathcal{C}$. In the next two exercises you will use a dilation to define one tool that inverts all points, whether they are inside $\mathcal{C}$ or outside $\mathcal{C}$. Note that $P' = I_{O,r}(P)$ is the point on $\overrightarrow{OP}$ that satisfies

$$OP' = \frac{r^2}{(OP^2)}(OP).$$

This does not make $I_{O,r}$ a dilation, because the scale factor relating OP' and OP is not constant, but it does allow us to invert an individual point using the Dilation command in GSP.

EXERCISES

***13.1.7.** Start with a circle $\mathcal{C} = \mathcal{C}(O, r)$ and a point $P \neq O$. Measure OP and r and then calculate $r^2/(OP)^2$. Select this quantity and choose Mark Ratio under the Transform menu. Next select O and then Mark Center. Now Dilate P using the Marked Ratio as the scale factor. Move P around in the plane to see that the resulting point P' is the inverse of P regardless of whether P is inside, on, or outside $\mathcal{C}$.

***13.1.8.** Use the construction in the last exercise to make a tool that inverts any point.

Suggestion. In this section you made several tools that find the inverse of a point in a circle. In subsequent sections you will make other tools that invert other objects, such as lines and circles. The givens for each of these tools consist of the center point O, a point R on $\mathcal{C}$ (which implicitly defines the radius r), and the object to be inverted. Since you will be inverting multiple objects in the same circle, it should not be necessary to reselect O and R every time you use one of your tools.

In order to make the tool easier to use, choose Show Script View under the Custom Tools menu. The Givens should be listed at the top of the script. Double click on the center point and a Properties box will appear. Type in a new label for the center and then click Automatically Match Sketch Object. The tool will still work the same way the first time you use it, but it will put the label you have chosen on the center. The next time you use the tool it will recognize this label and will not expect you to reselect the center. If you choose distinctive names that would never be assigned to other objects by GSP, you will only be required to enter the center and radius once.

Examples of distinctive names are "center of inversion" for O and "circle of inversion" for a point that determines $\mathcal{C}$. It is suggested that you consistently use the same names in all the tools you create in this chapter. If you use the same name for the center in all your tools, then you should only have to enter it once, even if you use several different tools to invert various objects. (Assuming, of course, that you want to invert all of them in the same circle.)

13.2 INVERTING CIRCLES AND LINES

One reason inversions are useful is that they preserve certain features of Euclidean geometry. Inversions are not isometries, and they do not preserve such basic properties as distance and collinearity, but they do preserve other geometric relationships. In this section we will see that inversions preserve circles and lines in the sense that the inverse of a circle is either a circle or a line and the inverse of a line is either a circle or a line. The following theorem makes that statement precise.

Theorem 13.2.1. *Let $I_{O,r}$ be inversion in the circle $\mathcal{C} = \mathcal{C}(O, r)$.*
Part 1. *If α is a circle that does not pass through O, then $I_{O,r}(\alpha)$ is a circle that does not pass through O.*
Part 2. *If α is a circle that passes through O, then $I_{O,r}(\alpha \setminus \{O\})$ is a line that does not pass through O.*
Part 3. *If ℓ is a line that does not pass through O, then $I_{O,r}(\ell \cup \{\infty\})$ is a circle that passes through O.*
Part 4. *If ℓ is a line that passes through O, then $I_{O,r}(\ell \cup \{\infty\}) = \ell \cup \{\infty\}$.*

Proof. See [16], Theorems 12.7.4 through 12.7.7. □

EXERCISES

*13.2.1. Construct a circle $\mathcal{C}(O, r)$ and a circle α such that $O \notin \alpha$. Construct a movable point P on α and its inverse P'. Use the Locus command in the Construct menu to confirm that the locus of points P' is a circle.
- **(a)** Where is the circle $I_{O,r}(\alpha)$ in case α is completely outside $\mathcal{C}$?
- **(b)** Where is the circle $I_{O,r}(\alpha)$ in case α intersects $\mathcal{C}$ in two points?
- **(c)** Where is the circle $I_{O,r}(\alpha)$ in case α is completely inside $\mathcal{C}$?

[Hint: To use the Locus command, select α, P, and P'; then choose Locus.]

*13.2.2. Construct a circle $\mathcal{C}(O, r)$ and a circle α such that $O \in \alpha$. Construct a movable point P on α and its inverse P'. Use the Locus command in the Construct menu to confirm that the locus of points P' is a line.
- **(a)** How is the line $I_{O,r}(\alpha \setminus \{O\})$ related to $\mathcal{C}$ and α in case $\mathcal{C}$ and α intersect in two points? (Draw a sketch.)
- **(b)** How is the line $I_{O,r}(\alpha \setminus \{O\})$ related to $\mathcal{C}$ and α in case $\mathcal{C}$ and α intersect in one point? (Draw a sketch.)
- **(c)** How is the line $I_{O,r}(\alpha \setminus \{O\})$ related to $\mathcal{C}$ and α in case α is contained inside $\mathcal{C}$?.

*13.2.3. Construct a circle $\mathcal{C}(O, r)$ and a line ℓ such that $O \notin \ell$. Construct a movable point P on ℓ and its inverse P'. Use the Locus command to confirm that the locus of points P' is a circle through O.
- **(a)** How is the circle $I_{O,r}(\ell \cup \{\infty\})$ related to $\mathcal{C}$ in case ℓ is outside $\mathcal{C}$?
- **(b)** How is the circle $I_{O,r}(\ell \cup \{\infty\})$ related to $\mathcal{C}$ in case $\mathcal{C}$ and ℓ intersect in two points?

[Hint: The Locus command will not show you the inverse of every point on the (infinitely long) line ℓ, but only the inverses of the points on ℓ that you can see in your sketch. You can see a larger portion of the circle if you make $\mathcal{C}$ relatively small.]

***13.2.4.** Make a tool that inverts a circle α. The tool should accept five points as givens (the center O of inversion, a point R on the circle of inversion, and three points that determine α) and should return the circle α' as its result. You may assume that $O \notin \alpha$ when you make the tool, so that α' is simply the circumcircle of the inverses of three given points on α. What happens to α' when you move α through O?

***13.2.5.** Make a tool that inverts a line ℓ. The tool should accept four points as givens (the center O of inversion, a point R on the circle of inversion, and two points that determine ℓ) and should return the circle ℓ' as its result. You may assume that $O \notin \ell$ when you make the tool, so that ℓ' is simply the circumcircle of O and the inverses of two given points on ℓ. What happens to ℓ' when you move ℓ through O?

13.3 OTHOGONALITY

Another geometric relationship preserved by inversions is orthogonality.

Definition. Two circles are said to be *orthogonal* if they intersect and their tangent lines are perpendicular at the points of intersection.

Theorem 13.3.1. *Let $\mathcal{C} = \mathcal{C}(O, r)$ and α be two circles.*
Part 1. *If α is orthogonal to $\mathcal{C}$, then $I_{O,r}(P) \in \alpha$ for every $P \in \alpha$.*
Part 2. *If there exists a point $P \in \alpha$ such that $I_{O,r}(P) \in \alpha$ and $I_{O,r}(P) \neq P$, then α is orthogonal to $\mathcal{C}$.*

Proof. See [16], Theorems 12.7.9 through 12.7.11. □

The theorem will be used in the exercises below to construct two circles that are orthogonal to $\mathcal{C}$. The first construction allows us to specify two points that are to lie on the circle. The second construction allows us to specify one point on the circle and the tangent line at that point.

Construction. Let $\mathcal{C} = \mathcal{C}(O, r)$ be a circle. Let A and B be two points such that O, A, B are noncollinear and A and B are not on $\mathcal{C}$. Construct $A' = I_{O,r}(A)$ and then construct α, the circumcircle for A, B, and A'.

EXERCISES

***13.3.1.** Carry out the construction above and verify that α is orthogonal to $\mathcal{C}$.

13.3.2. Use Theorem 13.3.1 to prove that α is orthogonal to $\mathcal{C}$.

***13.3.3.** Make a GSP tool that constructs a circle through two points that is orthogonal to $\mathcal{C}$. The tool should accept four points as givens (the center O of inversion, a point R on the circle of inversion, and the two points A and B), and return the circle α as result.

(a) What happens to α when you move the two points so that they lie on a diameter of $\mathcal{C}$?

(b) What happens to α if one of A or B lies on $\mathcal{C}$?

Construction. Let $\mathcal{C} = \mathcal{C}(O, r)$ be a circle. Let t be a line and let P be a point on t. Construct the line ℓ that is perpendicular to t at P. Construct $P' = I_{O,r}(P)$ and $Q = \rho_\ell(P)$, the reflection of P in ℓ. Construct the circumcircle α of P, P', and Q (see Figure 13.3).

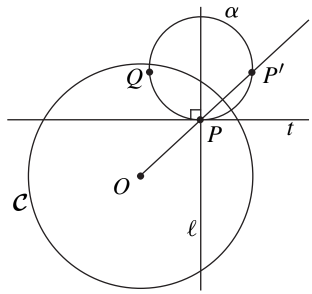

FIGURE 13.3: Construction of circle tangent to t at P

EXERCISES

***13.3.4.** Carry out the construction above and verify that α is a circle that is tangent to t and orthogonal to $\mathcal{C}$.

13.3.5. Prove that α is tangent to t and orthogonal to $\mathcal{C}$.

***13.3.6.** Make a GSP tool that constructs a circle that is tangent to t at P and is orthogonal to $\mathcal{C}$. The tool should accept four points as givens (the center O of inversion, a point R on the circle of inversion, the point P and a second point on t), and return the circle α as result. What happens to α if O lies on ℓ?

13.4 ANGLES AND DISTANCES

Inversions preserve angles in the following sense: Given two intersecting lines in the plane, the inverses of these lines will be two intersecting circles. The angles between the lines are congruent to the angles between the tangent lines of the circles. Inversions do not preserve individual distances, but they do preserve the following combination of distances.

Definition. Let A, B, P, and Q be four distinct points. The *cross-ratio* $[AB, PQ]$ of the four points is defined by

$$[AB, PQ] = \frac{(AP)(BQ)}{(AQ)(BP)}.$$

EXERCISES

***13.4.1.** Construct two intersecting lines ℓ and m. Use the tool of Exercise 13.2.5 to invert the two lines. The result should be two circles that intersect at two points, one of which is O. Construct the two tangent lines at the other point of intersection. Measure the angles between ℓ and m and then measure the angles between the tangent lines to ℓ' and m'. Are the measures equal?

***13.4.2.** Construct four points A, B, P, and Q. Measure the distances and calculate $[AB, PQ]$. Now invert the four points in a circle $\mathcal{C}(O, r)$ and calculate $[A'B', P'Q']$. Verify that $[AB, PQ] = [A'B', P'Q']$.

***13.4.3.** Make a tool that calculates the cross ratio of four points.

CHAPTER 14

The Poincaré disk

In this final chapter we study the Poincaré disk model for hyperbolic geometry. While hyperbolic geometry is a non-Euclidean geometry and the subject matter of these notes is Euclidean geometry, the topic is nonetheless appropriate for inclusion here because the Poincaré disk is built within Euclidean geometry. The main technical tool used in the construction of the Poincaré disk model is inversion in Euclidean circles, so the tools you made in Chapter 13 will be put to use in this chapter to perform hyperbolic constructions. As mentioned in the Preface, many of the constructions in this chapter are based on those in [9].

It was Eugenio Beltrami (1835–1900) who originated the idea of representing hyperbolic geometry within Euclidean geometry. There are many ways in which to construct models of hyperbolic geometry, but the Poincaré disk model is probably the best known. One reason for its popularity is the great beauty of the diagrams that can be constructed in this model. The model is named for the French mathematician Henri Poincaré (1854–1912) who first introduced it.

14.1 THE POINCARÉ DISK MODEL FOR HYPERBOLIC GEOMETRY

A *model* for a geometry is an interpretation of the technical terms of the geometry (such as point, line, distance, angle measure, etc.) that is consistent with the axioms of that geometry. The usual model for Euclidean geometry is $\mathbb{R}^2$, the Cartesian plane, which consists of all ordered pairs of real numbers. That model has been more or less assumed throughout this course, although most of the proofs we have given have been *synthetic*, which means that they are based on the axioms of geometry and not on specific (analytic) properties of any particular model for geometry.

We are about to begin the study of one form of non-Euclidean geometry—-geometry in which the Euclidean parallel postulate is not assumed to hold.[1] We

[1] See Playfair's Postulate on page 6.

will consider two distinct kinds of non-Euclidean geometry: *neutral geometry* is the geometry that is based on all the usual axioms of Euclidean geometry except that no parallel postulate is assumed while *hyperbolic geometry* is the geometry that is based on all the axioms of neutral geometry together with the hyperbolic parallel postulate. The *hyperbolic parallel postulate* asserts that for every line ℓ and for every point P that does not lie on ℓ, there exist multiple lines through P that are parallel to ℓ. Since all the axioms of neutral geometry are also assumed in both Euclidean and hyperbolic geometries, any theorem that can be proved in neutral geometry is a theorem in both the other geometries as well. This is significant because many of the elementary constructions at the beginning of Euclid's elements are neutral, so we can use them in our study of hyperbolic geometry.

Let us now describe the Poincaré disk model for hyperbolic geometry. Fix a circle Γ in the Euclidean plane. A "point" in the Poincaré disk is a Euclidean point that is inside Γ. There are two kinds of "lines" in the Poincaré disk. The first kind of line is a diameter of Γ; more specifically, a Poincaré line of the first kind consists of all the points on a diameter of Γ that lie inside Γ. A second kind of Poincaré line is a Euclidean circle that is orthogonal to Γ; more specifically, a Poincaré line of the second kind consists of all the points of a Euclidean circle orthogonal to Γ that lie inside Γ.

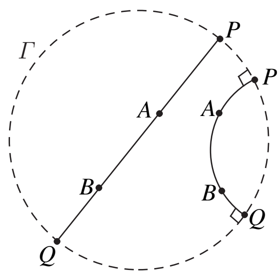

FIGURE 14.1: Two kinds of Poincaré lines and their ideal endpoints

The Poincaré distance between two points A and B in the Poincaré disk is defined by $d(A, B) = |\ln([AB, PQ])|$, where P and Q are the points at which the Poincaré line containing A and B intersects Γ. Observe that the points P and Q lie on Γ so they are not themselves points in the Poincaré disk, but they are still useful in defining the distance between points in the Poincaré disk. The points on Γ are ideal points for the Poincaré disk. (In hyperbolic geometry, each line has two ideal endpoints—see Figure 14.1.) The angle between two Poincaré lines is measured by measuring the Euclidean angle between their tangent lines.

The Poincaré disk is a model for hyperbolic geometry. Proving this assertion means proving that, with the terms point, line, distance, etc. interpreted as above, all the axioms of hyperbolic geometry are satisfied. Since the model is described within Euclidean geometry, those proofs are all Euclidean proofs. For example, we will see in the next section that theorems about Euclidean inversions in circles can be used to construct a unique Poincaré line through any two points in the Poincaré disk.

Thus Euclid's first postulate holds in the model. Note also that Euclidean inversions preserve angles and cross-ratios, so they preserve Poincaré angle measure and Poincaré distance. The Euclidean inversions are therefore isometries of the model and they function as reflections across lines in hyperbolic geometry. Inversions are the main technical tool that are used in proofs of theorems about the Poincaré disk.

In this chapter we will focus on constructions in the Poincaré disk and will not supply proofs that the axioms of hyperbolic geometry are satisfied in the model. Such proofs can be found in most college geometry texts. In particular, §13.3 of [16] contains a more detailed description of the model and proofs that all the usual axioms of geometry are satisfied by this model.

14.2 THE HYPERBOLIC STRAIGHTEDGE

Our first objective is to make a hyperbolic straightedge. In other words, we want to make a GSP tool that constructs the unique Poincaré line through two points in the Poincaré disk. We also want to make variations on the tool that will construct the Poincaré ray or segment determined by two points. The tool you made in Exercise 13.3.3 is almost exactly what is needed in order to construct a Poincaré line; the only refinement required is to trim off the part of the Euclidean circle that lies outside of Γ.

When you make the tools in this and later sections, you may assume that the two given points do not lie on a common diameter of Γ. This means that the tools you make will only construct Poincaré lines of the second kind. As a result, some of the lines may disappear momentarily when you move one point across the diameter determined by another. Making the tools general enough that they can accept any points as givens and produce whichever kind of line is appropriate as result would significantly increase the complication of the constructions, so we will not bother with it and will live with the minor inconvenience that results. One justification for this is that randomly chosen points would not lie on a common diameter, so the case we are omitting is the exceptional case. Another justification is that the simple tools often produce a good approximation to the correct answer anyway since part of a circle of large radius is indistinguishable from a straight line.

EXERCISES

***14.2.1.** Make a tool that constructs the Poincaré line through two points. The tool should accept four points as givens (the center O of Γ, a point R on Γ, and the two points A and B) and return the hyperbolic line ℓ as its result. Assume in your construction that A, B, and O are noncollinear. What happens to ℓ when you move A to make the three points collinear?
[Hints: Use the construction of Exercise 13.3.3 to find a Euclidean circle α that contains A and B and is orthogonal to Γ. Mark the two points P and Q at which α intersects Γ. Hide α and then use the Arc Through 3 Points command in the Construct menu to construct the circular arc from P to Q through A. The suggestions at the end of §13.1 should be helpful in making tools that do not require you to reenter the information about O and Γ every time you use them.]

***14.2.2.** Use the tool from the last exercise to construct a Poincaré line ℓ. Construct a point P that does not lie on ℓ. Construct multiple lines through P that are all parallel to ℓ. Is there any limit to the number of parallel lines you can construct? [Hint: Remember that "parallel" simply means that the line do not intersect.]

***14.2.3.** Make a tool that constructs the Poincaré ray determined by two points. The tool should accept four points as givens (the center O of Γ, a point R on Γ, and the two points A and B) and return the hyperbolic ray $\overrightarrow{AB}$ as its result.
[Hints: The obvious construction is to start with the Poincaré line determined by the two points A and B and then to mark the two ideal points P and Q at which this line meets Γ (labeled as in Figure 14.1). Then hide the line and construct the arc of a circle determined by A, B, and Q. This construction produces a tool that usually returns the correct object, but does not do so reliably unless GSP is given a better rule for determining which of the two points P and Q to use. (You should do the construction and see for yourself what happens when you move A and B around.) One way to make a tool that reliably gives the correct answer is to construct the inverse B' of B in Γ, construct the circular segment determined by A, B, and B', and then to construct Q as the intersection of this arc with Γ. The circular arc determined by A, B, and Q will then be the Poincaré ray $\overrightarrow{AB}$.]

***14.2.4.** Make a tool that constructs the Poincaré segment determined by two points. The tool should accept four points as givens (the center O of Γ, a point R on Γ, and the two points A and B) and return the hyperbolic segment $\overline{AB}$ as its result.
[Hint: Again you must be careful to make sure that your tool reliably returns the correct arc.]

***14.2.5.** Use the segment tool you just made to construct a triangle in the Poincaré disk. Move the vertices around to see what shapes are possible.

14.3 COMMON PERPENDICULARS

In the hyperbolic plane there are two kinds of parallel lines [16, §8.4]. One possibility is that two parallel lines m and n are *asymptotically parallel*, which means that in one direction the two lines get closer and closer together. This is illustrated in the Poincaré disk model by Poincaré lines that limit on a common point of Γ. Figure 14.2 shows three Poincaré lines that are pairwise asymptotically parallel. Even though the three lines converge at Γ, they are parallel because they do not intersect at a point of the Poincaré disk.

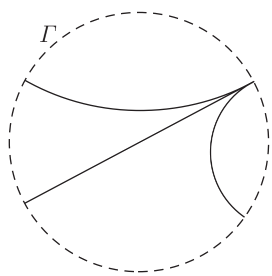

FIGURE 14.2: Three asymptotically parallel lines

The Classification of Parallels Theorem [16, Theorem 8.4.18] asserts that if m and n are two lines that are parallel but not asymptotically parallel, then m and n must admit a common perpendicular. This means that there is a line t that is perpendicular to both m and n (see Figure 14.3). In the exercises below you will construct this common perpendicular line.[2] The written exercises are exercises in Euclidean geometry.

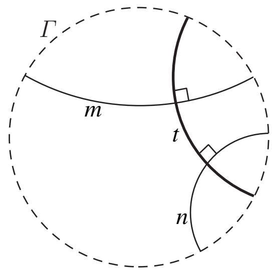

FIGURE 14.3: The Poincaré line t is the common perpendicular for m and n

EXERCISES

14.3.1. Let α be a circle with center A and let C be a point outside α. Construct the circle δ with diameter $\overline{AC}$ and let H be a point of $\alpha \cap \delta$. Define γ to be the circle with center C and radius $\overline{CH}$. Use Thales' Theorem to prove that α and γ are orthogonal.

***14.3.2.** Use Exercise 14.3.1 to make a tool that constructs the circle that is orthogonal to a specified circle α and has a specified point C as center. The tool should accept three points as givens (the center A of α, a point S on α, and a point C outside α) and should return a circle γ that has C as its center and is orthogonal to α as its result.

14.3.3. Let α and β be two orthogonal circles and let ℓ be their common secant line. Let γ be a circle whose center lies on ℓ. Use Theorem 13.3.1 to prove that γ is orthogonal to α if and only if γ is orthogonal to β.

14.3.4. Let α, β, and Γ be three circles such that α and β are both orthogonal to Γ. Let m be the common secant line for α and Γ, let n be the common secant line for β and Γ, and let P be the point at which m and n intersect. By Exercise 14.3.1 there is a circle τ with center P that is orthogonal to Γ. Use Exercise 14.3.3 to prove that τ is also orthogonal to both α and β.

***14.3.5.** Construct two Poincaré lines ℓ and m that are parallel but not asymptotically parallel. Use Exercise 14.3.4 to construct a Poincaré line t such that t is orthogonal to both ℓ and m. Under what conditions will the common perpendicular be a Poincaré line of the first kind (a diameter of Γ)?

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[2]The constructions in this section are based on the ideas in [13].

14.4 THE HYPERBOLIC COMPASS

Our next task is to make a hyperbolic compass tool. This tool should construct the Poincaré circle with a specified point A as center and and a specified point B on the circle. The Poincaré circle with center A and radius r is defined to be the set of all points X in the Poincaré disk such that $d(A, X) = r$. So the obvious way to attempt to construct a Poincaré circle is to make a tool that measures Poincaré distances and then construct the circle as the locus of all points at a fixed Poincaré distance from the center. But this method is not easy to implement and there is a much more elegant way to construct the hyperbolic circle.

The construction is based on two facts. The first is that every Poincaré circle is also a Euclidean circle [16, Exercises 13.7–13.9]. In general the Poincaré center of the circle is different from the Euclidean center and the Poincaré radius is different from the Euclidean radius, but for each Poincaré circle there is a point A' and a number r' such that the Poincaré circle is exactly equal to the Euclidean circle $\mathcal{C}(A', r')$. Thus we need only locate the Euclidean center A' and then construct the Euclidean circle with center A' that passes through B.

The second fact that the construction is based on is the fact that the Tangent Line Theorem (Chapter 0) is a theorem in neutral geometry. As a result, the theorem holds in the Poincaré disk and a Poincaré circle α must be perpendicular to every Poincaré line through the Poincaré center A. In particular, α must be perpendicular to the Euclidean line $\overleftrightarrow{OA}$, where A is the Poincaré center of the circle and O is the center of the Euclidean circle Γ that is used in the definition of the Poincaré disk (since it is a Poincaré line of the first kind). This means that the Euclidean center must lie on $\overleftrightarrow{OA}$. In addition, α must be perpendicular to the Poincaré line determined by A and B. This implies that the Euclidean center must lie on line that is tangent to the Poincaré line at B.

Construction. Assume that the Poincaré disk is defined by the Euclidean circle Γ with center O. Let points A and B inside Γ be given. Assume that A, B, and O are noncollinear. Construct the Euclidean line $\overleftrightarrow{OA}$. Construct the Poincaré line ℓ that contains A and B. Let t be the Euclidean line that is tangent to ℓ at B. Define A' to be the point at which t and $\overleftrightarrow{OA}$ intersect. (See Figure 14.4.) The Euclidean circle with center at A' and passing through B is the Poincaré circle with center A passing through B.

EXERCISES

***14.4.1.** Make a tool that constructs the Poincaré circle determined by two points. The tool should accept four points as givens (the center O of Γ, a point R on Γ, a point A that is to be the center of the Poincaré circle and a point B that is to lie on the Poincaré circle) and return the hyperbolic circle α as its result. Assume in your construction that A, B, and O are noncollinear. What happens to α when you move A to make the three points collinear? What happens when you move the "center" of the circle towards the boundary of the Poincaré disk?

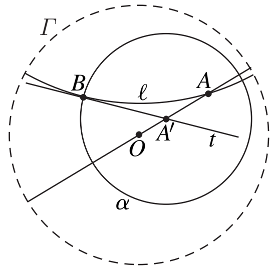

FIGURE 14.4: Construction of the Poincaré circle through B with center A

***14.4.2.** Use your hyperbolic compass to construct a hyperbolic circle and then use your hyperbolic straightedge to construct a triangle inscribed in the circle. Construct an inscribed triangle that has a diameter as one side. Based on your diagram, do you think Thales' Theorem is correct in hyperbolic geometry?

14.5 OTHER HYPERBOLIC TOOLS

Most of the compass and straightedge constructions you learned in high school are part of neutral geometry and are therefore valid in hyperbolic geometry. Since we now have a hyperbolic compass and straightedge, we can employ those tools in connection with the high school constructions to make other hyperbolic tools.

EXERCISES

***14.5.1.** Make a Poincaré perpendicular bisector tool. The tool should accept four points as givens (the center O of Γ, a point R on Γ, and the two endpoints A and B of the Poincaré segment) and it should return a Poincaré line as its result.
[Hint: The Pointwise Characterization of Perpendiculars (Chapter 0) is a neutral theorem. Thus you need only find two points that are equidistant from A and B and then take the Poincaré line through them. We have not yet measured Poincaré distances, but we do know how to construct Poincaré circles.]

***14.5.2.** Make a Poincaré midpoint tool.

***14.5.3.** Make a Poincaré perpendicular line tool; i.e., a tool that drops a perpendicular from a point P to a Poincaré line ℓ. The tool should accept five points as givens (the center O of Γ, a point R on Γ, two points A and B that determine a hyperbolic line ℓ, and a point P that is to lie on the perpendicular) and return a Poincaré line as result.
[Hint: Find two points on ℓ that are equidistant from P (using Poincaré distance). The perpendicular line is the perpendicular bisector of the segment connecting the two points.]

***14.5.4.** Make a Poincaré angle bisector tool. The tool should accept five points as givens (the center O of Γ, a point R on Γ, and three points A, B, and C that define a hyperbolic angle $\angle ABC$) and should return the angle bisector, which is a hyperbolic ray.

[Hint: The Pointwise Characterization of Angle Bisectors (Chapter 0) is a neutral theorem. You should be able to locate a point on the ray as the intersection of two hyperbolic circles.]

***14.5.5.** Use your hyperbolic tools to determine whether or not the Secant Theorem correct in hyperbolic geometry.

***14.5.6.** Make a hyperbolic compass tool that improves on the one you made in Exercise 14.4.1. The new tool should accept three points A, B, and C as givens and construct the Poincaré circle with center A and radius equal to the Poincaré distance from B to C.
[Hint: Find the perpendicular bisector of the Poincaré segment $\overline{AC}$. It is part of a Euclidean circle γ. Reflect B across γ to B'. Convince yourself that the circle with Poincaré center A that passes through B' is the circle you want.]

14.6 TRIANGLE CENTERS IN HYPERBOLIC GEOMETRY

We will use the tools constructed in the last several sections to explore triangle centers in hyperbolic geometry. Some of the classical triangle centers exist in hyperbolic geometry and others do not.

EXERCISES

***14.6.1.** The hyperbolic incenter and incircle.
- **(a)** Construct a Poincaré triangle and the three bisectors of the interior angles of the triangle. Verify that the angle bisectors are always concurrent, regardless of the shape of the triangle.
- **(b)** Construct the hyperbolic incircle for your triangle.

***14.6.2.** Hyperbolic median concurrence.
- **(a)** Construct a Poincaré triangle, the three midpoints of the sides, and the three medians. Verify that the three medians are concurrent regardless of the shape of the triangle.
- **(b)** The point of concurrence is called the hyperbolic centroid of the triangle. Make a tool that constructs the hyperbolic centroid.
- **(c)** Construct the hyperbolic medial triangle.

***14.6.3.** The hyperbolic circumcenter and circumcircle.
- **(a)** Construct a Poincaré triangle and the hyperbolic perpendicular bisectors of the three sides. Move the vertices of the triangle to vary the shape of the triangle. Verify that the perpendicular bisectors are concurrent for some triangles and not for others.
- **(b)** Can you find a triangle for which exactly two of the perpendicular bisectors intersect? If so, describe it.
- **(c)** Can you find a triangle for which all three of the perpendiculars are asymptotically parallel? If so, describe it.
- **(d)** Can you find a triangle for which all three of the perpendiculars admit a common perpendicular? If so, describe it.
- **(e)** Can you find a triangle for which exactly two of the perpendicular bisectors are asymptotically parallel? If so, describe it.
- **(f)** Verify that the triangle has a hyperbolic circumcircle if the three perpendicular bisectors are concurrent.

(g) Make a tool that constructs the hyperbolic circumcenter and circumcircle, provided they exist.

14.6.4. Prove that if two of the hyperbolic perpendicular bisectors of the sides of a hyperbolic triangle intersect, then the third perpendicular bisector also passes through the point of intersection.

***14.6.5.** The hyperbolic orthocenter.

(a) Construct a Poincaré triangle and the three hyperbolic altitudes. Verify that the altitudes are concurrent for some triangles and not for others.

(b) Can you find a triangle for which exactly two of the altitudes intersect? If so, describe it.

(c) Can you find a triangle for which all three of the altitudes are asymptotically parallel? If so, describe it.

(d) Can you find a triangle for which all three of the altitudes admit a common perpendicular? If so, describe it.

(e) Can you find a triangle for which exactly two of the altitudes are asymptotically parallel? If so, describe it.

(f) Make a tool that constructs the hyperbolic orthocenter, provided it exists.

(g) Construct the hyperbolic orthic triangle.

***14.6.6.** The hyperbolic Euler line.

Construct a Poincaré triangle that has both a hyperbolic circumcenter and a hyperbolic orthocenter. Let us agree to call the Poincaré line determined by the circumcenter and the orthocenter the hyperbolic Euler line for the triangle (provided the circumcenter and orthocenter exist). Now construct the hyperbolic centroid of the triangle. Does the hyperbolic centroid always lie on the hyperbolic Euler line?

14.7 MEASURING HYPERBOLIC ANGLES AND DISTANCES

In this section you will make tools that measure hyperbolic angles and hyperbolic distances. We will refer to the tools as the hyperbolic protractor and hyperbolic ruler, since those are the names of the tools we ordinarily use to measure the corresponding Euclidean quantities.

The formula for the Poincaré distance between two points A and B is given on page 97. It is

$$d(A, B) = \left|\ln[AB, PQ]\right| = \left|\ln \frac{(AP)(BQ)}{(AQ)(BP)}\right|,$$

where P and Q are the ideal endpoints of the Poincaré line determined by A and B (see Figure 14.1).

The definition of Poincaré angle measure on page 97 requires a little elaboration. Let A, B, and C be three points in the Poincaré disk that are noncollinear in the hyperbolic sense. The hyperbolic angle $\angle BAC$ is the union of the two hyperbolic rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$. If $A = O$, then the hyperbolic angle is also a Euclidean angle and the hyperbolic measure is defined to be the Euclidean measure.

Another possibility is that $\overrightarrow{AB}$ is part of a Euclidean line and $\overrightarrow{AC}$ is part of a Euclidean circle γ with center G. Let t be the line that is tangent to γ at A and choose a point C^* on t such that C and C^* are on the same side of m, where m is the Euclidean line determined by A and G (see the left half of Figure 14.5).[3] The Poincaré measure of the hyperbolic angle $\angle BAC$ is defined to be the ordinary measure of the Euclidean angle $\angle BAC^*$.

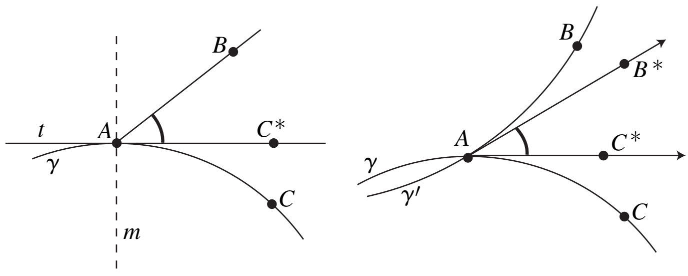

FIGURE 14.5: Definition of Poincaré angle measure

The final possibility is that $\overrightarrow{AC}$ is part of a Euclidean circle γ and $\overrightarrow{AB}$ is part of a Euclidean circle γ'. Choose points B^* and C^* on the tangent lines to γ' and γ as indicated in the right half of Figure 14.5. The Poincaré measure of the hyperbolic angle $\angle BAC$ is defined to be the ordinary measure of the Euclidean angle $\angle B^*AC^*$.

GSP allows the Euclidean angle to be measured in either radians or degrees, whichever is preferred.

EXERCISES

***14.7.1.** Make a tool that measures Poincaré distance between two points (a hyperbolic ruler). You may assume, when you make the tool, that the two points do not lie on a diameter of Γ.

(a) How does your tool respond when you move the two points so that they do lie on a diameter?

(b) What happens when you move one of the points so that it approaches Γ? Did you expect the Poincaré distance to approach infinity? Can you explain why it does not?

(c) Change the size of the Poincaré disk you are using. Does this change the limiting value for $d(A, B)$ as B approaches the boundary of the disk?

***14.7.2.** Construct a Poincaré segment $\overline{AB}$, its perpendicular bisector ℓ, and a movable point C on ℓ. Now measure the distances $d(A, C)$ and $d(B, C)$ and compare the two. What happens when C is moved along ℓ?

***14.7.3.** Construct two Poincaré lines that meet at right angles. Let C be the point at which the two lines intersect, and choose points A and B on the lines so that $\triangle ABC$ is a Poincaré right triangle. Measure the lengths of the sides of the triangle and determine whether or not the Pythagorean Theorem is valid in hyperbolic geometry.

***14.7.4.** Construct a hyperbolic triangle $\triangle ABC$, the midpoints of the sides, and the three medians. The Euclidean Median Concurrence Theorem (Chapter 2) asserts

[3] It is easy to check that the entire hyperbolic ray $\overrightarrow{AC}$ is on one side of m.

that the three medians are concurrent and that the point of concurrence divides each median into two segments, one exactly twice as long as the other. In Exercise 14.6.2 you verified that the concurrence part of the theorem is valid in hyperbolic geometry. Measure distances and determine whether or not the second part of the theorem (the part regarding the distances) is valid in hyperbolic geometry.

***14.7.5.** Make a tool that constructs the tangent Euclidean ray $\overrightarrow{AB^*}$ associated with a hyperbolic ray $\overrightarrow{AB}$ as in Figure 14.5.
[Hints: Again you must be careful when you make this tool to be certain that it always constructs the correct ray; if the point B^* is not specified carefully enough, the tool will sometimes give the opposite ray. The hyperbolic ray is part of a Euclidean circle γ. Locate the center G of γ and construct the tangent line t at A. Let D be the point at which ray $\overrightarrow{GB}$ intersects t. The point D could serve as B^* except for the fact that it might not be a point in the Poincaré disk. A more pleasing tool results if you trim the ray $\overrightarrow{AD}$ so that it lies inside the Poincaré disk and then choose a point B^* on the trimmed ray.]

***14.7.6.** Use the tool from the last exercise to make another tool that measures Poincaré angles (a hyperbolic protractor). The tool should accept five points as givens (the center O of Γ, a point R on Γ, and three points A, B, and C that define a hyperbolic angle $\angle ABC$) and should return the measure of the angle, which is a number. You may assume, when you make the tool, that A and B do not lie on a diameter of Γ and that A and C do not lie on a diameter of Γ.

- **(a)** How does your tool respond when you move A and B so that they do lie on a diameter?
- **(b)** What happens if you move A so that it is at the center of Γ?

***14.7.7.** Construct a Poincaré triangle and calculate its hyperbolic angle sum. How large can you make the angle sum? How small can you make it?

***14.7.8.** Construct a Poincaré circle α with Poincaré center A and construct two movable points B and C on α. Then $\triangle ABC$ is a hyperbolic isosceles triangle. Measure the base angles. Does the Isosceles Triangle Theorem hold in hyperbolic geometry?

***14.7.9.** Use Euclid's construction (Exercise 1.3.3) to construct an equilateral Poincaré triangle. Measure the perimeter of the triangle and the angle sum. Now change the lengths. How are the angle sum and perimeter of the triangle related?

***14.7.10.** Construct a quadrilateral $\square ABCD$ that has three right angles. (Such a quadrilateral is called a *Lambert quadrilateral*.) Measure the fourth angle. Can you adjust the vertices so that the fourth angle is right? How do the lengths of $\overline{AB}$ and $\overline{CD}$ compare?

***14.7.11.** Construct a quadrilateral $\square ABCD$ that has right angles at A and B and such that $d(B, C) = d(A, D)$. (Such a quadrilateral is called a *Saccheri quadrilateral*.) Measure the angles at C and D. Can you adjust the vertices so that all the angles are right? How does the length of $\overline{AB}$ compare with the length of $\overline{CD}$?

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Index