CHAPTER 2

The classical triangle centers

This chapter studies several points associated with a triangle. These points are all *triangle centers* in the sense that each of them can claim to be at the center of the triangle in a certain sense. They are *classical* in that they were known to the ancient Greeks. The classical triangle centers form a bridge between elementary and advanced Euclidean geometry. They also provide an excellent setting in which to develop proficiency with GSP.

While the three triangle centers were known to the ancient Greeks, the ancients somehow missed a simple relationship between the three centers. This relationship was discovered by Leonhard Euler in the eighteenth century. Euler's theorem serves as a fitting introduction to advanced Euclidean geometry because Euler's discovery can be seen as the beginning of a revival of interest in Euclidean geometry and most of the theorems we will study in the course were discovered in the century after Euler lived.[1] Euler's original proof of his theorem was a complicated analytic argument, but it is simple to discover (the statement of) the theorem with GSP.

2.1 CONCURRENT LINES

Definition. Three lines are *concurrent* if there is a point P such that P lies on all three of the lines. The point P is called the *point of concurrency*. Three segments are concurrent if they have an interior point in common.

Two arbitrary lines will intersect in a point—unless the lines happen to be parallel, which is unusual. Thus concurrency is an expected property of two lines. But it is rare that three lines should have a point in common. One of the surprising and beautiful aspects of advanced Euclidean geometry is the fact that so many triples of lines determined by triangles are concurrent. Each of the triangle centers in this chapter is an example of that phenomenon.

[1] See Chapter 7 of [7] for a nice discussion of Euler's contributions to geometry.

2.2 THE CENTROID

Definition. The segment joining a vertex of a triangle to the midpoint of the opposite side is called a *median* for the triangle.

EXERCISES

***2.2.1.** Construct a triangle $\triangle ABC$. Construct the three midpoints of the sides of $\triangle ABC$. Label the midpoints D, E, and F in such a way that D lies on the side opposite A, E lies on the side opposite B, and F lies on the side opposite C. Construct the three medians for $\triangle ABC$. What do the three medians have in common? Verify that this continues to be true when the vertices of the triangle are moved around in the plane.

***2.2.2.** In the preceding exercise you should have discovered that the three medians are concurrent (they have a point in common). Mark the point of intersection and label it G. Measure AG and GD, and then calculate AG/GD. Make an observation about the ratio. Now measure BG, GE, CG, and GF, and then calculate BG/GE and CG/GF. Leave the calculations displayed on the screen while you move the vertices of the triangle. Make an observation about the ratios.

The exercises you just completed should have led you to discover the following theorem.

Median Concurrence Theorem. *The three medians of any triangle are concurrent; that is, if $\triangle ABC$ is a triangle and D, E, and F are the midpoints of the sides opposite A, B, and C, respectively, then $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ all intersect in a common point G. Moreover, $AG = 2GD$, $BG = 2GE$, and $CG = 2GF$.*

Definition. The point of concurrency of the three medians is called the *centroid* of the triangle. The centroid is usually denoted by G.

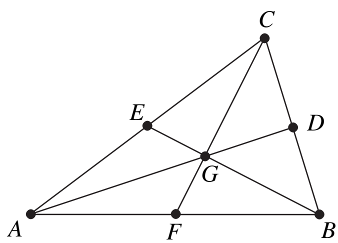

FIGURE 2.1: The three medians and the centroid

We will not prove the concurrency part of the theorem at this time. That proof will be postponed until after we have developed a general principle, due to Ceva, that allows us to give a unified proof of all the concurrence theorems at the same time. For now we will assume the concurrency part of the theorem and prove that the centroid divides each median in a 2:1 ratio. That proof is outlined in the next four exercises.

EXERCISES

***2.2.3.** Construct a triangle $\triangle ABC$, then construct the midpoints of the three sides and label them as in Exercise 2.2.1. Construct the two medians $\overline{AD}$ and $\overline{BE}$ and the segment $\overline{DE}$. Check, by measuring angles, that $\triangle ABC \sim \triangle EDC$. Then check that $\triangle ABG \sim \triangle DEG$.

Note. The notation of Exercise 2.2.3 is assumed in the next three exercises.

2.2.4. Use theorems from Chapter 0 to prove that $\triangle ABC \sim \triangle EDC$ and that $AB = 2ED$.

2.2.5. Use Theorems from Chapter 0 to prove that $\triangle ABG \sim \triangle DEG$.

2.2.6. Use the two preceding exercises to prove that $AG = 2GD$ and $BG = 2GE$. Explain how this allows you to conclude that $CG = 2GF$ as well.

2.2.7. Explain why the centroid is the center of mass of the triangle. In other words, explain why a triangle made of a rigid, uniformly dense material would balance at the centroid.

***2.2.8.** The three medians subdivide the triangle $\triangle ABC$ into six smaller triangles. Determine by experimentation the shape of $\triangle ABC$ for which these six subtriangles are congruent. Measure the areas of the six subtriangles and verify that the areas are always equal (regardless of the shape of the original triangle).

2.2.9. Supply a proof that the six subtriangles in the preceding exercise have equal areas.

2.3 THE ORTHOCENTER

Definition. The line determined by two vertices of a triangle is called a *sideline* for the triangle.

Definition. An *altitude* for a triangle is a line through one vertex that is perpendicular to the opposite sideline. The point at which an altitude intersects the opposite sideline of a triangle is called the *foot* of the altitude.

EXERCISES

***2.3.1.** Construct a triangle and construct the three altitudes for the triangle. Observe that no matter how the vertices of the triangle are moved around in the plane, the three altitudes continue to be concurrent.

The exercise should have confirmed the following theorem. It will be proved in Chapter 8.

Altitude Concurrence Theorem. *The three altitudes of any triangle are concurrent.*

Definition. The point of concurrency of the three altitudes is called the *orthocenter* of the triangle. It is usually denoted by H.

EXERCISES

***2.3.2.** Construct another triangle. Mark the orthocenter of your triangle and label it H. Move one vertex and watch what happens to H. Add the centroid G to your sketch and again move the vertices of the triangle. Observe that the centroid always stays inside the triangle, but that the orthocenter can be outside the triangle or even on the triangle.

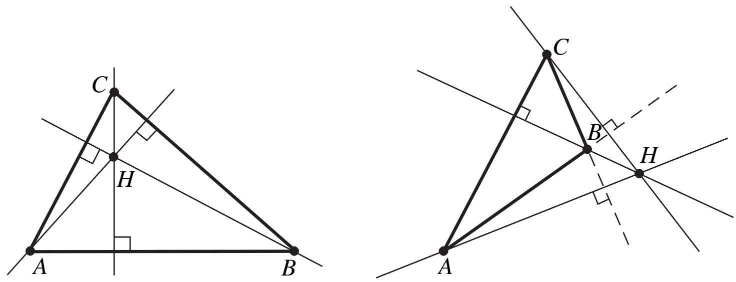

FIGURE 2.2: Two triangles with their three altitudes and orthocenter

***2.3.3.** Determine by experimentation the shape of triangles for which the orthocenter is outside the triangle. Find a shape for the triangle so that the orthocenter is equal to one of the vertices of the triangle. Observe what happens to the orthocenter when one vertex crosses over the line determined by the other two vertices. Make notes on your observations.

2.3.4. Prove that the centroid is always inside the triangle.
[Hint: The inside of the triangle is the intersection of the interiors of the three interior angles of the triangle.]

***2.3.5.** Determine by experimentation whether or not it is possible for the centroid and the orthocenter to be the same point. If it is possible, for which triangles does this happen?

2.4 THE CIRCUMCENTER

In this section you will verify that the perpendicular bisectors of the three sides of any triangle are concurrent.

Definition. The point of concurrency of the three perpendicular bisectors of the sides of a triangle is called the *circumcenter* of the triangle. The circumcenter is usually denoted by O.

The reason for the name ‘‘circumcenter’’ will become clear later when we study circles associated with triangles. The fact that the three perpendicular bisectors of any triangle are concurrent will be proved in Chapter 8.

EXERCISES

***2.4.1.** Construct a triangle $\triangle ABC$ and construct the three perpendicular bisectors for the sides of your triangle. These perpendicular bisectors should be concurrent. Mark the point of concurrency and label it O.

***2.4.2.** Move one vertex of the triangle around and observe how the circumcenter changes. Note what happens when one vertex crosses over the line determined by the other two vertices. Find example triangles which show that the circumcenter may be inside, on, or outside the triangle, depending on the shape of the triangle. Make notes on your findings.

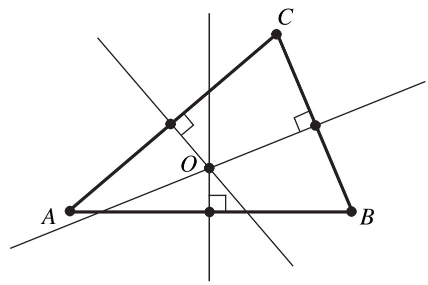

FIGURE 2.3: The three perpendicular bisectors and the circumcenter

***2.4.3.** Measure the distances OA, OB, and OC and make an observation about them.

2.4.4. Prove that the circumcenter is equidistant from the vertices of the triangle.

***2.4.5.** Determine by experimentation whether or not it is possible for the circumcenter and the centroid to be the same point. If it is possible, for which triangles does this happen?

2.5 THE EULER LINE

In this section we will investigate the relationship between the three triangle centers that was mentioned at the beginning of the chapter. The theorem was discovered by the German mathematician Leonhard Euler (1707–1783).

EXERCISES

***2.5.1.** Construct a triangle $\triangle ABC$ and construct all three centers G, H and O. Hide any lines that were used in the construction so that only the triangle and the three centers are visible. Put a line through two of the centers and observe that the third also lies on that line. Verify that G, H, and O continue to be collinear even when the shape of the triangle is changed.

***2.5.2.** Measure the distances HG and GO. Calculate HG/GO. Leave the calculation visible on the screen as you change the shape of your triangle. Observe what happens to HG/GO as the triangle changes.

In the exercises above you should have discovered the following theorem.

Euler Line Theorem. *The orthocenter H, the circumcenter O, and the centroid G of any triangle are collinear. Furthermore, G is between H and O (unless the triangle is equilateral, in which case the three points coincide) and* $HG = 2GO$.

Definition. The line through H, O, and G is called the *Euler line* of the triangle.

The proof of the Euler Line Theorem is outlined in the following exercises. Since the existence of the three triangle centers depends on the concurrency theorems stated earlier in the chapter, those results are implicitly assumed in the proof.

EXERCISES

2.5.3. Prove that a triangle is equilateral if and only if its centroid and circumcenter are the same point. In case the triangle is equilateral, the centroid, the orthocenter, and the circumcenter all coincide.

2.5.4. Fill in the details in the following proof of the Euler Line Theorem. Let $\triangle ABC$ be a triangle with centroid G, orthocenter H, and circumcenter O. By the previous exercise, it may be assumed that $G \neq O$ (explain why). Choose a point H' on $\overrightarrow{OG}$ such that G is between O and H' and $GH' = 2OG$. The proof can be completed by showing that $H' = H$ (explain). It suffices to show that H' is on the altitude through C (explain why this is sufficient). Let F be the midpoint of $\overline{AB}$. Use Exercise 2.2.6 and the SAS Similarity Criterion to prove that $\triangle GOF \sim \triangle GH'C$. Conclude that $\overleftrightarrow{CH'} \parallel \overleftrightarrow{OF}$ and thus $\overleftrightarrow{CH'} \perp \overleftrightarrow{AB}$.

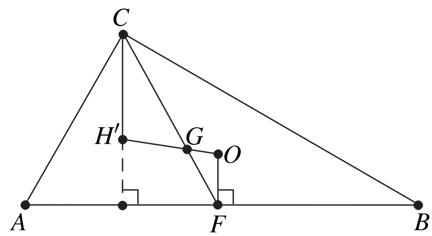

FIGURE 2.4: Proof of Euler Line Theorem

***2.5.5.** Figure 2.4 shows a diagram of the proof of the Euler Line Theorem in case the original triangle is acute. Use GSP to experiment with triangles of other shapes to determine what the diagram looks like in case $\triangle ABC$ has some other shape.

2.5.6. Prove that a triangle is isosceles if and only if two medians are congruent (i.e., have the same length).

2.5.7. Prove that a triangle is isosceles if and only if two altitudes are congruent.

***2.5.8.** Construct a triangle $\triangle ABC$ and the bisectors of angles at A and B. Mark the points at which the angle bisectors intersect the opposite sides of the triangle and label the points D and E, respectively. The two segments $\overline{AD}$ and $\overline{BE}$ are called *internal angle bisectors*. Measure the lengths of the two internal angle bisectors and then measure the lengths of the sides $\overline{BC}$ and $\overline{AC}$. Use your measurements to verify the following theorem: *A triangle is isosceles if and only if two internal angle bisectors are congruent.*

The theorem in the last exercise is known as the *Steiner-Lehmus Theorem* because the question was proposed by C. L. Lehmus in 1840 and the theorem was proved by the Swiss mathematician Jacob Steiner in 1842. A proof of the theorem and discussion of the interesting history of the proof may be found in §1.5 of [3].