4.1. Pythagoras and His Hypotenuse

Developing Ideas

1. The main event. In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.

2. Two out of three. A right triangle with legs of length 1 and 2 has hypotenuse of length $\sqrt{1^2+2^2}=\sqrt{5}$. In a right triangle with one leg of length 1 and hypotenuse of length 3, the other leg will have length x, where $\sqrt{1^2+x^2}=3$. Squaring both sides to solve for x yields $x^2 + 1 = 9$, so $x = \sqrt{8} = 2\sqrt{2}$.

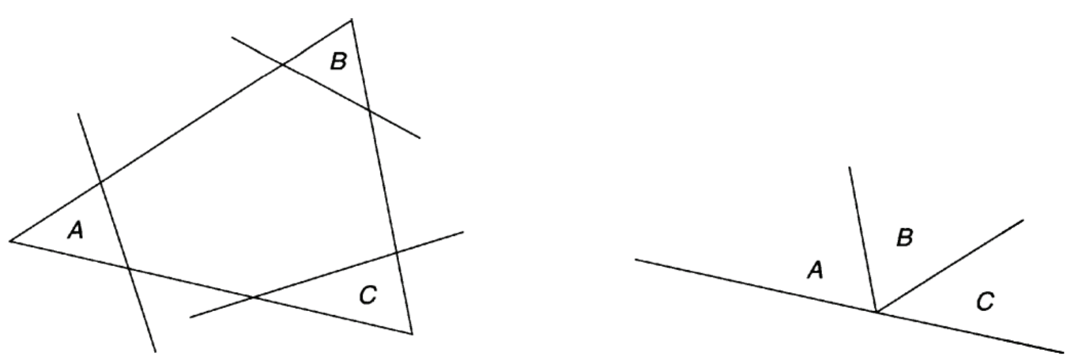
3. Hypotenuse hype. A right triangle with legs of length 1 and x has hypotenuse of length $\sqrt{x^2+1}$.

4. Assessing area. We know the base of the rectangle; we need to find the height. The diagonal divides the rectangle into two right triangles, each with one leg of length 4 and hypotenuse of length 5. The length of the other leg will be the height of the rectangle. If x denotes the length of this leg, the Pythagorean Theorem tells us that $5^2 = x^2 + 4^2$. So $x^2 = 25 - 16 = 9$, yielding $x = 3$. Thus the area of the rectangle is $4 \times 3 = 12$ square inches.

5. Squares all around. The figure shows a right triangle. Adjoined to each side of the triangle is a square with area equal to the square of that side of the triangle. The Pythagorean Theorem says that the area of the largest square (the one on the hypotenuse) is equal to the sum of the squares on the other two sides.

Solidifying Ideas

6. Operating on the triangle. See the figure. When the angles are aligned, they will form a straight line, or half of a complete rotation. Because a full rotation corresponds to 360 degrees, we have that the sum of the angles in a triangle is 180 degrees.



7. Excite your friends about right triangles. Describe the proof of the Pythagorean Theorem to someone who has never seen it before.

8. Easy as 1, 2, 3? In any right triangle, the side opposite the right angle is the largest, so in our case, the length of the hypotenuse is 3. If this is indeed a right triangle, then the

Pythagorean Theorem must hold, that is, $1^2 + 2^2 = 3^2$, but 5 isn't 9, and so we don't have a right triangle. The only set of consecutive numbers that works here is $3^2 + 4^2 = 5^2$.

9. Sky high. 'Directly above her' means that the triangle formed by the student, the kite, and the spectator is a right triangle. If the height is *H*, then the Pythagorean Theorem says that $H^2 + 90^2 = 150^2$. (If you make a 3-4-5 right triangle 30 times bigger, you get a 90-120-150 right triangle, and that makes solving for *H* easier!) The kite is flying 120 feet high.

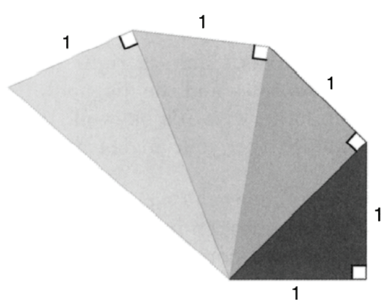
10. Sand masting. The tip of the mast, the base of the mast, and the stern of the sailboat form a right triangle. Because the backstay is the longest side of the triangle, we have $H^2 + 50^2 = 130^2$. So $H = \sqrt{(130^2 - 50^2)} = 120$. (Another useful triplet is the 5-12-13 right triangle!)

11. Getting a pole on a bus. When we measure the length of a box, we don't measure the length of the diagonal, we measure the lengths of the sides—even though the diagonal is larger than either of the sides! So Sarah somehow found a 3 foot by 4 foot box and lay Adam's fishing pole along the diagonal. Once again, the 3-4-5 triplet comes to the rescue.

12. The scarecrow. If *H* is the length of the hypotenuse, then $3^2 + 3^2 = H^2$, or $H = \sqrt{18} = 3\sqrt{2}$. In general, if the two legs have length L, then the hypotenuse will have length $L\sqrt{2}$. Note this is the 45-45-90 triangle you get when you cut a square along a diagonal.

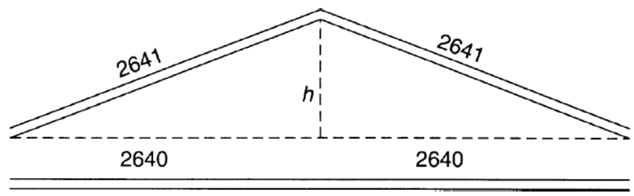
The scarecrow's assertion is invalid. He makes several errors in his attempt to state the Pythagorean Theorem. He refers to "square roots" of sides rather than "squares" of sides, and he claims that you can chose "any two sides" rather than the two shortest sides. He also refers to an "isosceles" triangle rather than a "right" triangle.

13. Rooting though a spiral. The first triangle is a special case of I.7. It's a 45-45-90 triangle, and so the hypotenuse has length $\sqrt{(1^2 + 1^2)} = \sqrt{2}$. Now the hypotenuse of this triangle becomes a leg of the second triangle. The longest side of the second triangle is $\sqrt{(1^2 + (\sqrt{2})^2\ )} = \sqrt{(1 + 2)} = \sqrt{3}$. Similarly, the third triangle's hypotenuse has length $\sqrt{(1^2 + (\sqrt{3})^2)} = \sqrt{(1 + 3)} = \sqrt{4}$, etc.. The hypotenuse of the *N*th triangle has length $\sqrt{(N + 1)}$.



14. Is it right? Here we use the contrapositive of the Pythagorean Theorem: if the sum of the squares of the lengths of the shorter sides of a triangle does *not* equal the square of the length of the longest sides, then the triangle is not a right triangle. In this case, it's close! $(2.6^2 + 8.1^2) = 72.37$, while $(8.6^2) = 73.96$, so we don't have a right triangle. If the third side had length $\sqrt{72.37} = 8.507\ldots$, then we'd have a right triangle. Imagine gradually shrinking the third side from length 8.6 to 8.507. The largest angle is opposite the largest side, and when we shrink the side, the opposing angle gets smaller. At the end of our transformation the largest angle is 90 degrees, so it is intuitive that the largest angle was originally *larger* than 90 degrees. If $a^2 + b^2 < c^2$, then the largest angle is bigger than 90 degrees. If the opposite inequality holds, then all angles are less than 90 degrees.

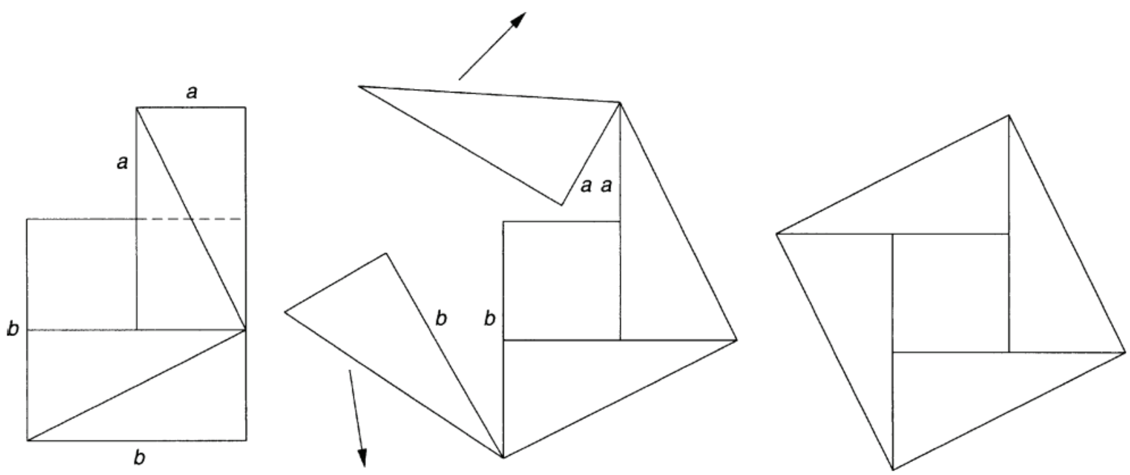
15. Train trouble. See the figure. Break the triangle into two separate right triangles, change all measurements to feet, and use the Pythagorean Theorem. We have $h^2 + 2640^2 = 2641^2$, or $h = \sqrt{(2641^2 - 2640^2)} = 72.7$ feet!



Creating New Ideas

16. Does everyone have what it takes to be a triangle? This friend of yours is lying, confused, or both. No side of a triangle can be larger than the other two sides put together. If his longest side were instead 5,641, then his triangle would have zero area, it would consist of three sides lying in a straight line.

17. Getting squared away. See the figure. Before moving the top triangle, side **a** represents the length of one side of the alleged square. After the triangle is moved, the same side represents the other side of the alleged square. This proves that the figure is indeed a square. The second square is proved in a similar manner.



18. The practical side of Pythagoras. Make a triangle out of two sides of your patio. Measure the lengths of the two sides and the diagonal. If they satisfy the Pythagorean Theorem, then your patio is true; otherwise it's skewed. If it is skewed, you can use the results of Mindscape 14 to decide whether your angle is too large or too small!

19. Pythagorean pizzas. Because the area of a circular pizza is proportional to the square of its diameter, we could rephrase the question in terms of square pizzas. This is where right triangles come in. If the area of the medium and small pizzas were equal to the area of the large pizza, then the lengths of the three diameters would form a right triangle. If the largest angle is more than 90 degrees, then you'd want the large pizza. If it's less than 90, you'd take the two smaller pizzas.

4.2. A View of an Art Gallery

Developing Ideas

1. Standing guard. A gallery with three sides will have a floor plan in the shape of a triangle requiring only one guard.

2. Art appreciation. If polygonal closed curve in the plane has v vertices, then there are $v/3$ vertices from which it is possible view every point on the interior. If $v/3$ is not an integer, then the number of vertices needed is the biggest integer less than $v/3$.

3. Upping the ante. For a gallery with 12 vertices you need at most 12/3 = 4 guards. With 13 vertices you need at most 13/3 = 4.33… guards, that is, you need at most 4 guards. With 11 vertices you need at most 11/3 = 3.66…, that is, at most 3 guards.

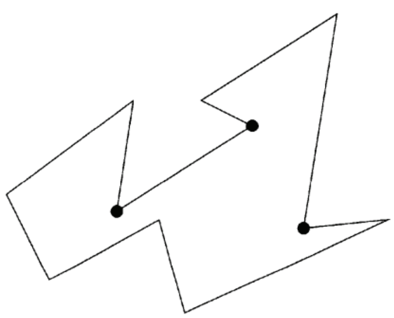
4. Create your own gallery. Answers will vary.

5. Keep it safe. Only one camera is needed. Place it at one of the four inside vertices.)

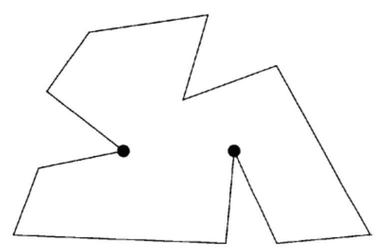
Solidifying Ideas

6. Klee and friends. Answers will vary.

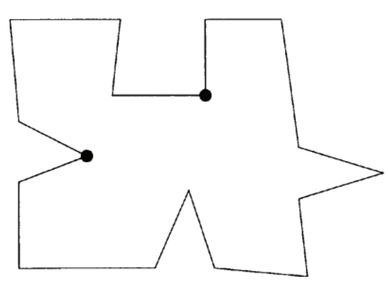
7. Putting guards in their place. See the figures. There are several ways to do this.



a. 3 guards

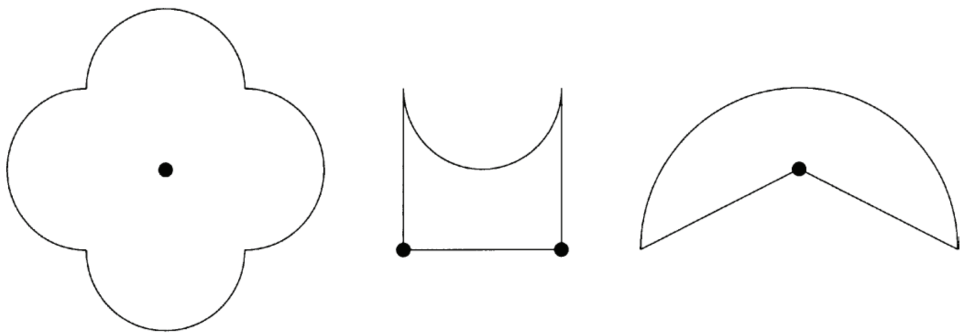


b. 2 guards



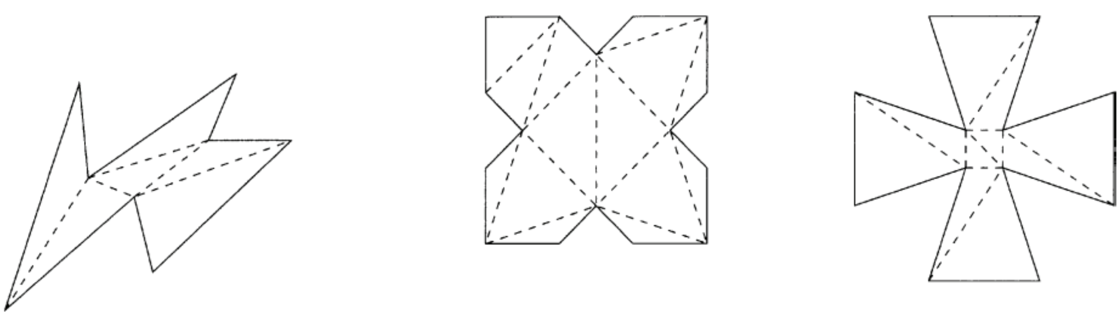
c. 2 guards

8. Guarding the Guggenheim. See the figures.

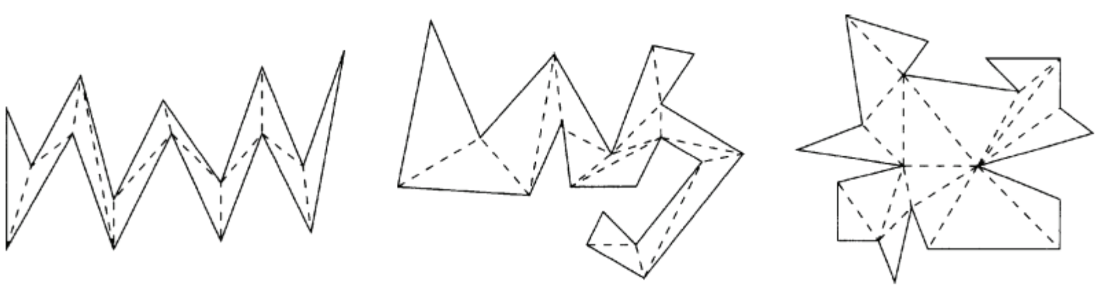


a. 1 guard **b.** 2 guards **c.** 1 guard

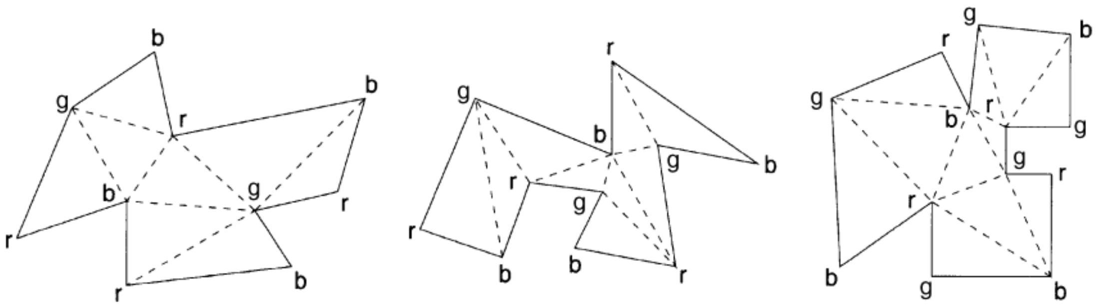
9. Triangulating the Louvre. See the figures. There are numerous correct triangulations.



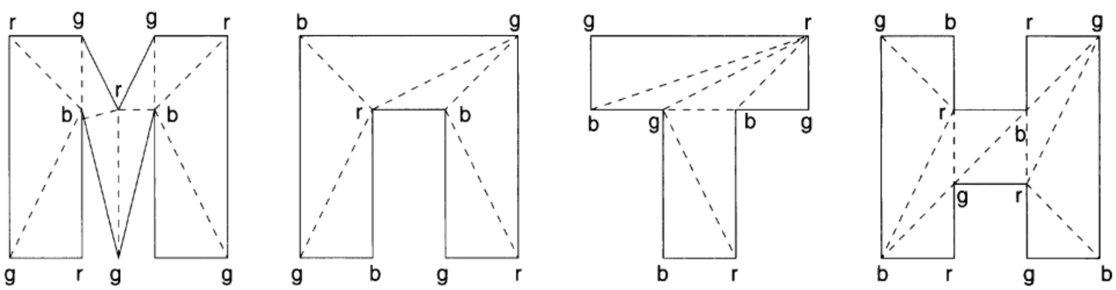
10. Triangulating the Clark. The triangulations in the figures are examples of correct triangulations. There are a variety of correct answers.



11. Tricolor me. You are free to pick the colors of two adjacent vertices, but after that, the colors of the rest of the vertices are determined. See the figure for colorings. All colorings will look like these with the exception of a permutation of the three colors (for example, your blue might be my red).



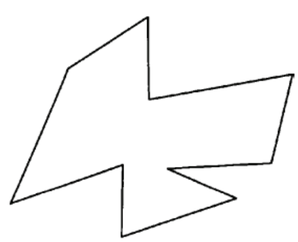
12. Tricolor hue. Pick a side and arbitrarily label the two vertices with different colors. The other vertices are now determined. Locate the triangle containing the chosen side, and color the remaining vertex. You've colored two new sides. Find any triangles containing these new sides and color the remaining (uncolored) vertex. Repeat this process until you're done. See the figures.



13. One-third. (The relevance? Suppose we triangulate and color a 6-sided museum. If R, Y, and B represent the number of red, yellow, and blue vertices, then we have R + Y + B = 6. In the text we proved that one of these numbers must be smaller than or equal to v/3 = 6/3 = 2.) There are three ways to express 6 as a sum of three numbers: 1 + 1 + 4, 1 + 2 + 3, 2 + 2 + 2. In each case, we've got **at least** one number less than or equal to 2.

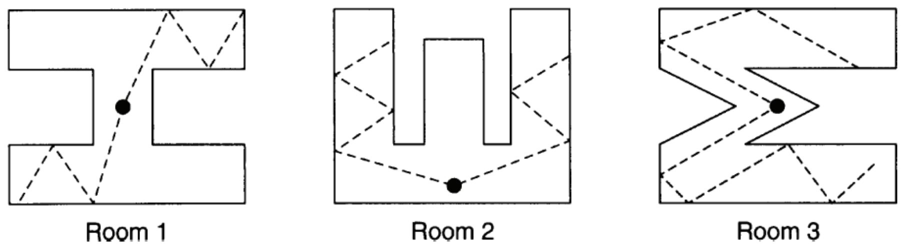
14. Easy watch. Draw a regular hexagon, and put the guard in the center.

15. Two watches. See the zig-zag museum below.

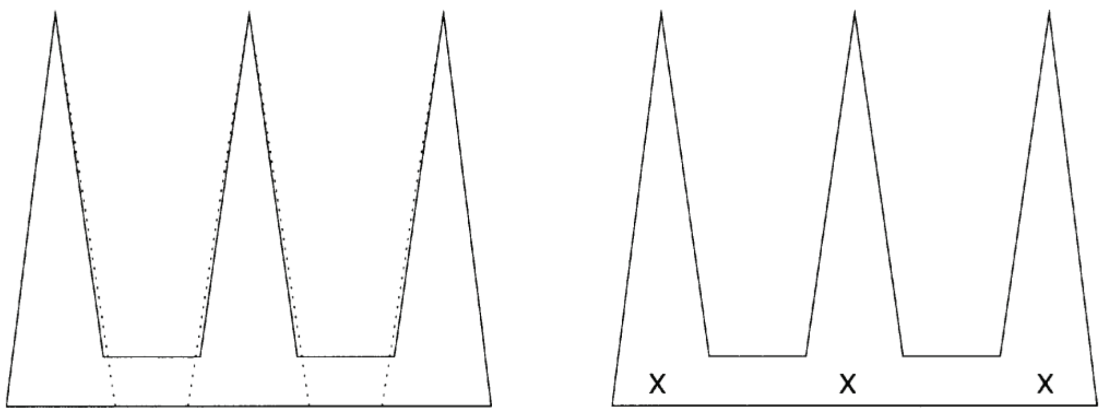


Creating New Ideas

16. Mirror mirror on the wall. Any point will work for any of these figures.

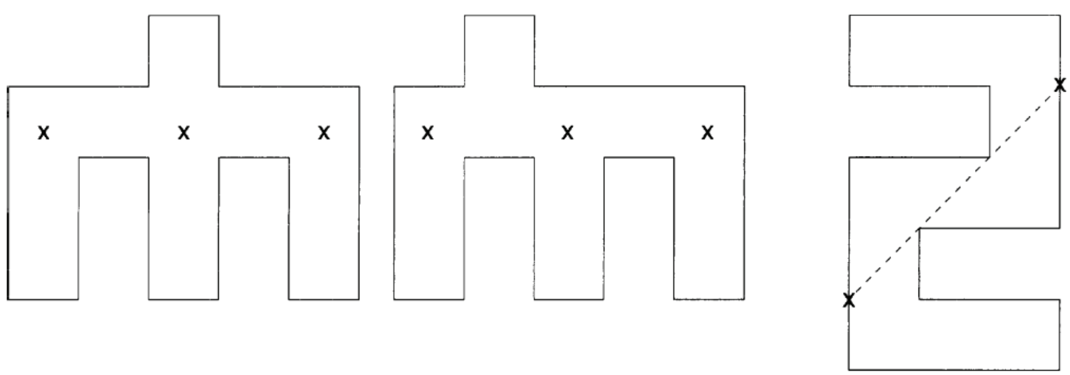


17. Nine needs three. See the comb-shaped museum in the figure. It is necessary to have a guard standing inside each of the extended triangular regions. Because these extended triangles don't intersect, there is no way to make do with fewer guards. Try generalizing this shape to draw a 12-sided room needing at least 4 guards.



18. One-third again. Suppose the natural number $n = a + b + c$. If $a < n/3$, $b < n/3$, and $c < n/3$, then $a + b + c < n$, which contradicts the original condition.

19. Square museum. See the figures for possible guard placements. As always, there are several ways to do this. The museums that look like 3's need at least 3 guards, the museum that resembles the number 2 needs only 2 guards!



20. Worst squares. We'll modify the comb-shaped museum used for Mindscape 17. See the figure below for the 20-sided museum. Note that this is easily modified to generate the 12- and 16-sided museums.

4.3. The Sexiest Rectangle:

Developing Ideas

1. Defining gold. A rectangle is Golden if the ratio of the longer side to the short side is exactly $\varphi = \frac{1+\sqrt{5}}{2}$, the Golden Ratio.

2. Approximating gold. The Golden Ratio is approximately 1.618, so the number 1.67 is closer than any of the other numbers.

3. Approximating again. In decimal form, the four ratios are 5/3 = 1.66…, 11/8.5 = 1.29…, 14/11 = 1.2727…, and 1.5454…. Thus the first object, the 3 × 5 card, has proportions closest to those of a Golden Rectangle.

4. Same solution. There are several ways to see that these equations have the same solutions. Inverting both sides of one equation yields the other. Also, cross-multiplying both equations yields the same quadratic: $\varphi^2 - \varphi = 1$.

5. X marks the unknown.

(a) Given $\frac{2x}{1} = \frac{1}{x-1}$, multiply both sides by x-1 to get $2x^2$-2x = 1 or $2x^2$-2x-1 = 0. The quadratic formula gives solutions $x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-1)}}{2(2)} = \frac{2 \pm \sqrt{12}}{4} = \frac{2 \pm 2\sqrt{2}}{4} = \frac{1 \pm \sqrt{2}}{2}$.

(b) Given, $\frac{x}{3} = \frac{2}{x-4}$ cross multiply to get x^2-4x=6 or x^2-4x-6=0. The quadratic formula gives solutions $x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-6)}}{2(1)} = \frac{4 \pm \sqrt{40}}{2} = \frac{4 \pm 2\sqrt{10}}{2} = 2 \pm \sqrt{10}$.

(c) Given, $\frac{3x}{2} = \frac{1}{x+1}$ cross multiply to get $3x^2 + 3x = 2$ or $3x^2 + 3x - 2 = 0$. The quadratic formula gives solutions $x = \frac{3 \pm \sqrt{3^2 - 4(3)(-2)}}{2(3)} = \frac{3 \pm 2\sqrt{33}}{6}$.

Solidifying Ideas

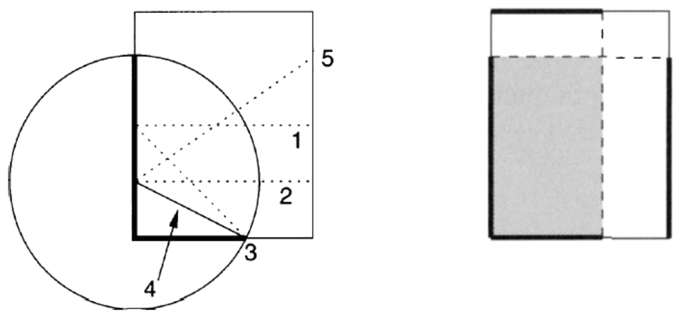
6. In search of gold. Find three examples of Golden Rectangles.

7. Golden art. Several Golden Rectangles can be created using lines in the painting.

8. A cold tall one? Yes. Stand a Golden Rectangle on its shorter side, and the base is shorter than the height. Stand it on its longer side and the height is shorter. To be Golden, the ratio of the longest side to the shortest side has to be the Golden ratio.

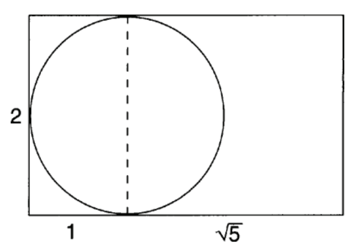
9. Fold the gold. The dimensions of the new rectangle are exactly half of the original. The ratio between the longer and shorter sides remains unchanged and so still represents a Golden Rectangle.

10. Sheets of gold. If you don't mind a severely folded piece of paper, you can create your Golden Rectangle with only one rectangular sheet of paper without recourse to a straight-edge. For a pristine Golden Rectangle, it's necessary to cannibalize the second sheet (see the figure). Using two 8.5 × 11 inch papers, make four folds on one paper to get the correct lengths; mark the lengths on the second sheet, and use the straight edge and scissors to cut out the rectangle.



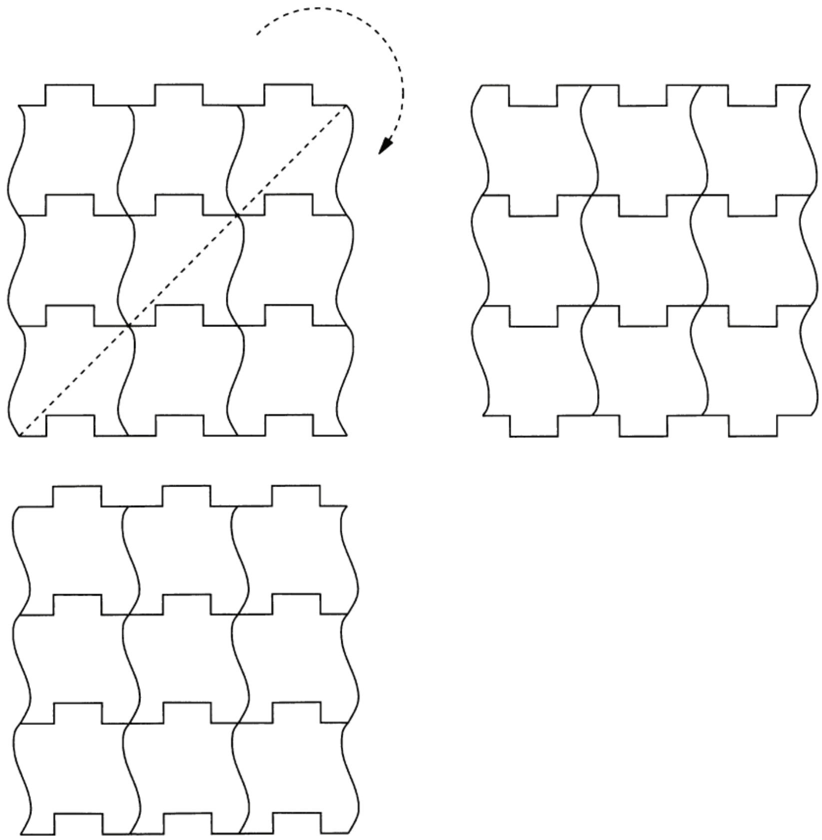
1. Fold in half lengthwise.
2. Fold again in half (let 1/2 of the length represent 1 unit of length).
3. Create the diagonal fold (3) by moving the lower left corner to the first fold (marking the lower thickened edge of length 2 units).
4. Mark the solid line to be used in the next fold ($\sqrt{5}$ units).
5. Make the fourth fold (5) by bringing the solid line in alignment with the left edge of the paper. (This marks the end of the long thickened line).
6. Transfer the lengths of the thickened edges to the other paper, use a straight-edge (or fold) to complete the Golden Rectangle.

11. Circular logic? See the figure. If the original rectangle had lengths 2 and $(1 + \sqrt{5})$, then the new rectangles would have dimensions 2×1 and $2 \times \sqrt{5}$, neither of which is a Golden Rectangle.



12. Growing gold. Note that after we add the square, we have a rectangle with the following property: If we remove the largest square, we are left with a Golden Rectangle. Though not proved in the text, the Golden Rectangle is the *only* rectangle that has this feature, and so the original rectangle is indeed a Golden Rectangle. If the original sides had length 1 and φ, then the new rectangle has sides of length $1 + \varphi$ and φ. Verify by calculator or by algebra that $(1 + \varphi)/\varphi = \varphi$ to prove that the new rectangle is also golden.

13. Counterfeit gold? In Mindscape 12, we start with a Golden Rectangle and find that repeatedly adding squares doesn't change the Golden nature of the rectangles. If we start with **any** rectangle, the process of adding larger and larger squares makes the resulting equivalent to flipping about a horizontal line. Reversing the steps is equivalent to flipping about a vertical line.



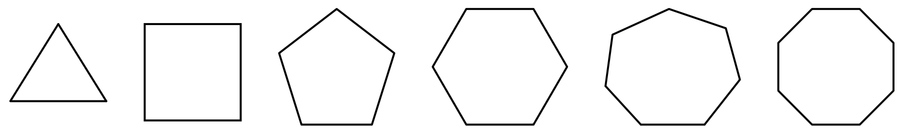
Further Challenges

21. Penrose tiles. Here is the start of a Penrose tiling.

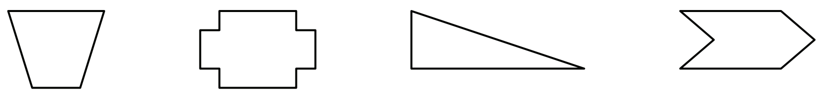
4.5. The Platonic Solids Turn Amorous

Developing Ideas

1. It's nice to be regular. A polygon is regular if all its sides are the same length and all angles have equal measure. Here are regular polygons with 3, 4, 5, 6, 7, and 8 sides. (Note that the regular 7-gon is particularly hard to draw.)



Here are some polygons that aren't regular:

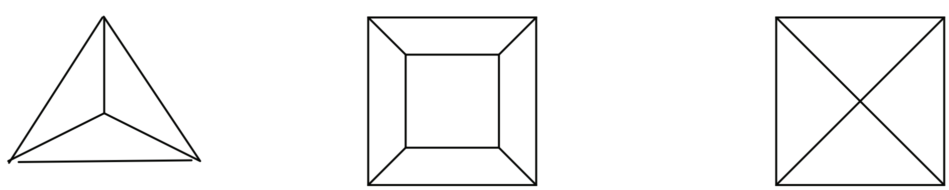


2. Keeping it Platonic. A solid is regular if all its faces are identical regular polygons meeting at the same number of edges at each vertex.

3. Count 'em up. A cube has six faces, twelve edges, and eight vertices. A tetrahedron has four faces, six edges, and four vertices.

4. Defending duality. The cube and the octahedron are duals for the following reason. If you start with a cube and put a vertex in the middle of each face, then join by an edge any two vertices whose faces are adjacent, you obtain an octahedron. Similarly, start with an octahedron, put a vertex on each face, then join by an edge any two vertices who faces are adjacent. The result is a cube.

5. Eye of the beholder. The solid on the left is a tetrahedron viewed by looking down on a vertex. The middle solid is a cube viewed by looking straight down at a face (the perspective here is skewed, because the face closer to you should actually look larger than the face farther away.) The third solid is an octahedron viewed by looking directly at a vertex.

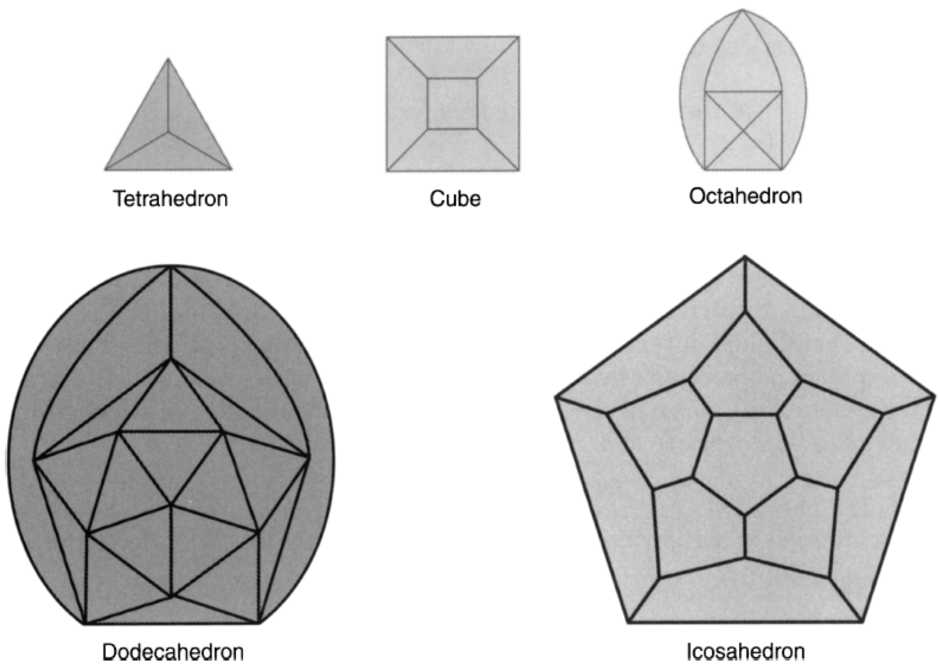


Solidifying Ideas

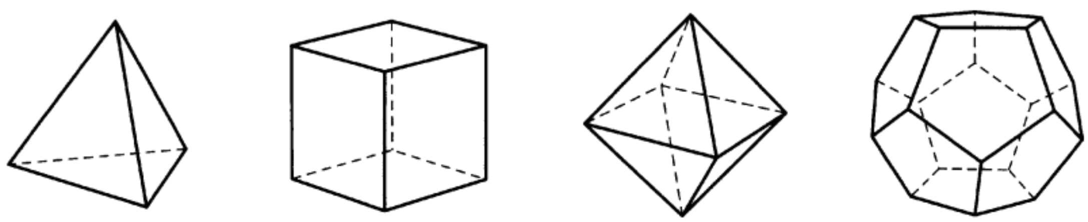
6. Build them. Make a complete set of the Platonic solids.

7. Unfold them. Pages 274 – 275 of the text show one way to unfold each of the regular solids. Many other unfoldings are possible.

8. Edgy drawing.



9. Drawing solids. See the figures.



10. Life drawing. Draw the regular solids.

11. Count. 2 is the magic number. Tetrahedron, 4 – 6 + 4 = 2. Cube, 8 – 12 + 6 = 2. Octagon, 6 – 12 + 8 = 2. Dodecahedron, 20 – 30 + 12 = 2. Icosahedron, 12 – 30 + 20 = 2.

12. Soccer counts. There are 12 pentagons and 20 hexagons on the soccer ball, for a total of 32 faces. Each pentagon has 5 vertices and each vertex lies on exactly one pentagon, for a total of $12 \times 5 = 60$ vertices. Each vertex has exactly 3 edges coming out of it, for a total of $60 \times 3 = 180$ edges, except that each edge is incident with two vertices. So this calculation counted each edge twice. Thus there are 90 edges. So the number of vertices minus the number of edges plus the number of faces is 60 – 90 + 32 = 2.

Here's another way to answer the question. View the soccer ball as an icosahedron with its vertices sliced off. The 12 vertices of the icosahedron become the 12 pentagons on the soccer ball, and the 20 triangular faces become hexagons, for a total of 32 faces. The vertices come in groups of five, each corresponding to a vertex of the original icosahedron. So there are $12 \times 5 = 60$ vertices. There are $12 \times 5 = 60$ edges between the hexagons and the pentagons, and 20 × 3/2 edges between hexagons (20 hexagons, 3 such edges per hexagon, and we overcount by a factor of 2), for a total of 90 edges. Once again, 60 – 90 + 32 = 2.

13. Golden rectangles. Assume that the icosahedron has unit length. Because the base of the first rectangle is an edge, it also has unit length. Focus on the vertices of one of the vertical edges of the rectangle. There is exactly one pentagon formed by the edges of the icosahedron containing these two vertices, and this imaginary vertical edge cuts the pentagon in two. The length of the vertical side is 2 x sin(54) = ($\sqrt{5}$ + 1)/2, the Golden Ratio.

14. A solid slice. Cubes, tetrahedrons, and dodecahedrons yield triangles when their vertices are sliced. Octagons yield squares, and icosahedrons yield pentagons. The number of sides of the boundary correspond to the number of faces that meet at a vertex.

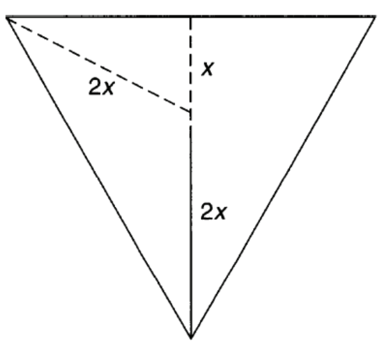
15. Siding on the cube.
Faces: 6 faces of the cube × 4 glued faces = 24 total faces.
Vertices: 8 original vertices of the cube + 1 new vertex for each of the 6 faces of the cube = 14 total vertices.
Edges: 12 original edges of the cube + 4 new edges for each of the 6 faces = 36 total edges.
$V - E + F = 14 - 36 + 24 = 2$.

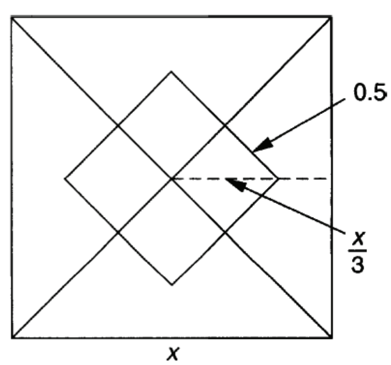
Creating New Ideas

16. Cube slices. Slicing off a vertex generates a triangle because the cutting plane intersects three sides of the cube. If we continue making parallel cuts that triangle will get larger until additional sides intersect the cutting plane. At this point, the slices will look like triangles with one or more vertices cut off. If your cuts generate growing equilateral triangles, then when you are halfway through the cube, the slice will be a hexagon. Depending on where you cut, you can get a wide variety of 3-, 4-, 5-, and 6-sided shapes.

17. Dual quads. The edges of the octagon connect the centers of the square faces. Each edge is the hypotenuse of a right triangle with legs of length 1/2. Therefore, by the Pythagorean Theorem, the edges of the octagon have length $\sqrt{2}/2$.

18. Super dual. Octahedron edges have length $3\sqrt{2}/2$. Draw a vertical line from the vertex of an equilateral triangle to the base of the opposite side. The center of the triangle cuts this vertical line so that the longer piece is twice the length of the shorter (calculate lengths explicitly). This 2:1 proportion is unchanged when we view the triangle from other perspectives. View the surrounding octahedron from a vertex (see the figure). Let x represent the length of the new edges, and use the 45-45-90 triangle to solve for x.





19. Self-duals. The edge lengths are a third of the original tetrahedron. As in the previous problem, it helps to view the tetrahedron from a vertex, and to note that the center of an equilateral triangle is a third as high as the vertex.

20. Not quite regular. Two types of answers. Technically, you could subdivide the faces of an icosahedron into smaller triangles so that most vertices (the new ones) are surrounded by six triangles, while some vertices (the old ones) are surrounded by five. Many dome structures are made this way. This is slightly unsatisfying because if six equilateral triangles meet at a vertex, you've really just got one hexagonal face! As an alternative, arrange things so that more than 6 triangles surround a vertex. This can be done by progressively replacing equilateral triangles with the top part of a tetrahedron. (One triangle gets replaced with three.) The more triangles added, the more the surface undulates. You lose convexity, but you get real, honest to goodness sides.

Further Challenges

21. Truncated Solids.

Solid (pretruncated)	Number of Vertices in Truncated Solid	Number of Edges in Truncated Solid	Number of Faces in Truncated Solid
Tetrahedron	12	18	8
Cube	14	36	24
Octahedron	14	36	24
Dodecahedron	32	90	60
Icosahedron	32	90	60

New F = Old F + Old V
New E = Old E + Old $V \times$ number of faces that meet at a vertex
New V = Old $V \times$ number of faces that meet at a vertex
Each old vertex is cashed in for a new face and as many new vertices as there are faces that meet at a vertex.

22. Stellated Solids.

Solid (prestellated)	Number of Vertices in Stellated Solid	Number of Edges in Stellated Solid	Number of Faces in Stellated Solid
Tetrahedron	8	18	12
Cube	24	36	14
Octahedron	24	36	14
Dodecahedron	60	90	32
Icosahedron	60	90	32

New V = Old V + Old F
New E = Old E + Old $F \times$ number of sides per face
New F = Old $F \times$ number of sides per face
Each old face is cashed in for a new vertex and as many new triangular faces as there were vertices per old face. Note that these solids are the duals of the solids in the previous problem.

For the Algebra Lover

27. **Soccer solid.** Let h equal the number of hexagons on a soccer ball. Then the number of pentagons is $h - 8$. Thus the total number of sides is $32 = h + h - 8$. So $40 = 2h$ and thus $h = 20$. So a soccer ball has 20 hexagonal sides and 12 pentagonal sides.

28. **How do you pronounce that?** Let s equal the number of square faces on a Great R solid. Then the number of octagonal faces is $s/2$ and the number of hexagonal faces is $2s/3$. Thus the total number of faces is $s + s/2 + 2s/3 = 26$. Multiply both sides by 6 to obtain $6s + 3s + 4s = 156$. So we have $13s = 156$, which gives us $s = 12$. So the Great R has 12 square faces, 6 octagonal faces, and 8 hexagonal faces.

29. **The Great R returns.** To solve the two equations $E + V = 120$ and $5E - 2V = 264$ for E and V, multiply the first equation by 2 and add the second to obtain $7E = 240 + 264$. Thus $7E = 504$ and $E = 72$. Therefore $V = 120 - 72 = 48$.

30. **Rhombi-what?** Let t equal the number of triangular faces on a Small R solid. Then the number of square faces is $(3/2)t$ and the number of pentagonal faces is $(3/5)t$. Thus the total number of faces is $t + 3t/2 + 3t/5 = 62$. Multiply both sides by 10 to obtain $10t + 15t + 6t = 620$. So we have $31t = 620$, which means $t = 20$. So the Small R has 20 triangular faces, 30 square faces, and 12 pentagonal faces.

31. **Small again.** To solve the equations $E + 2V = 240$ and $3E - V = 300$ for E and V, multiply the second equation by 2 and add it to the first equation to obtain $7E = 240 +600$. Thus $7E = 840$ and $E = 120$. Therefore $V = 3E - 300 = 60$.