5.2. The Band that Wouldn't Stop Playing:

Developing Ideas

1. One side to every story. A Möbius band is surface with one side and one edge modeled as follows: take a strip of paper, give it a half twist, then tape the two ends together.

2. Maybe Möbius. One way to determine if a loop of paper is a Möbius band is to put a pencil point down in the middle of a side and, without lifting the pencil, draw a line down the middle, all the way around until you return to where you started. If the resulting penciled circle appears on "both sides" of the band, then the band is a Möbius band because it has only one side.

3. Singin' the blues. Once you construct the band, you should see a white edge meeting a blue edge at the place where you taped the edges together. This Möbius band has one side with a blue portion and a white portion.

4. Who's blue now? This construction does not yield a Möbius band. When the edges are taped together, the white edges will meet on one side and the blue edges will meet on the other, preserving the two-sided quality of the loop.

5. Twisted sister. Your sister will get a plain loop with no twists. The second half twist in the same direction will simply undo the first one.

Solidifying Ideas

6. Record reactions. Just do it.

7. The unending proof. Read the band.

8. Two twists. There are two edges and two sides before cutting. The cutting process will cut each side in half, for a total of four sides, and create two additional edges. This is consistent with two strips of paper with two edges each. Before cutting you can see that the two edges form two linked circles. After cutting, the two strips will still be linked. In addition, the two strips also have two half twists.

9. Two twists again. This is exactly the same as Mindscape 8. You get a thin and thick copy of the original strip linked together.

10. Three twists. Because the original strip has only one side and one edge, cutting along the center line will produce two sides, and leave two edges that are as long as the single edge in the original strip. The resulting strip is a knotted circle—it is a trefoil knot.

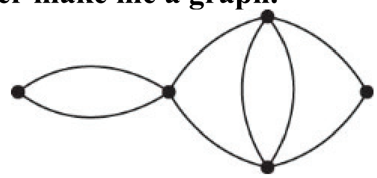
11. Möbius length. With the left and right ends identified appropriately, a rectangular strip of paper of length L represents a Möbius band whose boundary edge has length $2L$. The centerline of the Möbius band still has length L though. Cutting along this centerline produces a longer two-sided strip whose edges still have length $2L$, but the centerline of this new strip is $2L$ as well.

12. Möbius lengths. The center strip is a thinner version of the original, but the additional two-sided loop is twice as long.

Section 6.1 Circuit Training

Developing Ideas

1. **Map maker, map maker make me a graph.**



2. **Unabridged list.** AB, AB, BC, BD, CD, CD, CE, DE. (Note that each bridge can be listed with the letters in either order. For example, BC denotes the same bridge as CB.) All vertices have even degree: A and E have degree 2; B, C, and D have degree four.

3. **Will the walk work?** The requested walk is possible. Here's one way: AB, BC, CE, ED, DC, CD, DB, BA.

4. **Walk around the house.** Using Euler's Circuit Theorem, the graph does not have an Euler circuit because it has two vertices of odd degree.

5. **Where's Waldoskova?** The city of Konigsberg, Prussia, is now Kaliningrad, Russia. It lies in the territory of Oblast, which is bordered by the Baltic Sea, Poland and Lithuania, and is geographically separate from the rest of Russia.

Solidifying Ideas

6. **Walk the line.** It is possible to traverse the graph as requested. One such path is (AB)(BC)(CE)(EB)(BD)(DE)(EG)(GH)(HI)(IF)(FA)(AD)(DG)(GI)(IA)(AC)(CF)(FE)(EH)(HF)(FA).

7. **Walkabout.** It is not possible to traverse the graph as requested because vertices A and G have odd degree.

8. **Linking the loops.** The walks can be linked as follows to create a single walk that begins and ends on landmass C, traversing each edge exactly once: CAAD(DCCBBEEBBAAD)DF(FCCBBCCEEF)FC.

9. **Scenic drive.** It is not possible to add loop-d-loops to the given drive to obtain a trip that traverses each road exactly once and returns to the entrance. The given drive begins and ends at the entrance, but there is only one additional road that starts at the entrance. Any additional trip that traverses this road would not have an unused road on which to return to the entrance to complete a loop.

8.2. Predicting the Future in an Uncertain World

Developing Ideas

1. Black or white? The probability he wears the black pair on a Wednesday is 1 minus the probability he wears the white pair on a Wednesday, or 2/5.

2. Eleven cents. The outcomes are HH, HT, TH, and TT, where the first letter denotes the dime and the second denotes the penny. The event E is the set of outcomes HH, HT, and TH. The probability that E occurs is 3/4.

3. Yumm. The probability of getting green is (number of green candies)/(total number of candies) = 2/14 or 1/7. The probability of getting blue is 5/14.

4. Rubber duckies. There are eleven numbers (60, 61, …, 70) out of 75 that result in a stuffed duck; so the probability of winning a stuffed duck is 11/75 = 14666…. There are 59 numbers (1, 2, …59) that yield a consolation prize, so the probability of neither stuffed duck nor banana is 59/75 = 78666….

5. Legally large. If an experiment is repeated a large number of times, then the relative frequency of a particular outcome will tend to be close to the probability of that particular outcome.

Solidifying Ideas

6. Lincoln takes a hit. As of 2009, there have been 44 presidents, one of whom was Lincoln. If you assume each president is an equally likely target, the probability your dart hits Lincoln is 1/44.

7. Giving orders. Completely fulfilling all promises (0.00001), Picking the Queen of Hearts (1/52), Seeing a full moon (1/30), Selecting an ace (1/13), Rolling a 6 (1/6), Picking a black card (1/2), Flying safely (0.9999999). Parentheses indicate non-rigorous guesses at the probabilities.

8. Two Heads are Better. About 1/3 of the time, two heads should appear.

9. Tacky probabilities. In our experiments, about half of the tacks landed on their heads. It seemed that the shorter the fall, the greater the chance they would land on their heads.

10. BURGER AND STARBIRD. 17 letters, 4 Rs, 2 Bs, 9 in the first half (A-M) and 5 vowels. There is 4/17 chance of getting an R, 2/17 chance of getting a B, 9/17 chance of pulling a letter from the first half of the alphabet and 5/17 chance of pulling a vowel.

11. Monty Hall. The experiments should reflect the analysis of the text. Sticking wins 1/3 of the time, and switching wins 2/3 of the time. The probabilities of sticking and winning should add to 1.

12. 7 or 11. Answer: 2/9. There are 6 ways to roll a seven, (1-6, 2-5, 3-4, 4-3, 5-2, 6-1) and 2 ways to roll an eleven (5-6,6-5). Because there are 36 equally likely ordered rolls, the chances of rolling 7 or 11 is 8/36 or 2/9. If these numbers come up on your first roll in the casino game craps, you automatically win.

13. D and D.

H	H	H	H	H	H	T	T	T	T	T	T
1	2	3	4	5	6	1	2	3	4	5	6

Of the 12 equally likely outcomes, there is only one way to get a head and a 4 (H, 4) and so the probability is 1/12. Once you know that a 2 is showing, there are only two possible (equally likely) outcomes (H, 2) and (T, 2). So the chances that the dime is showing heads are 1/2.

14. The top 10. There is one 3, four prime numbers (2, 3, 5, 7), five even numbers, and zero numbers evenly divisible by 13. So, the corresponding probabilities are 1/10, 4/10 = 2/5, 5/10 = 1/2, and 0/10 = 0, respectively.

15. One five and dime.

Penny	H	H	H	H	T	T	T	T
Nickel	H	H	T	T	H	H	T	T
Dime	H	T	H	T	H	T	H	T

Three presidents: 1/8. Exactly two presidents: 3/8. If you only know that a head is showing, then you can rule out the (T, T, T) outcome, so that there are really only 7 equally likely outcomes. Now the probability of three presidents is 1/7. Knowing that Lincoln is showing eliminates four of the equally likely outcomes. Given this, the probability of seeing three heads is reduced to 1/4.

16. Five flip. Given this new information (assuming this person is trustworthy), the probability of five heads is 1/1. Without the information, the probability of such a feat is rare indeed, 1/32.

17. Flipped out. If the coin were normal, then it would land on heads roughly half the time. In fact, the standard deviation in this case is less than 1600 flips, so there is definitely something going on with this coin. Take it to a physicist, take to a séance, but don't spend it!

18. Spinning wheel. The chance of a 13 is 1/38. There are 18 red spaces, 18 black spaces, and 2 green spaces. So, the chance of a ball landing on a red space 18/38 = 9/19.

19. December 9. Let's call the two people Tip and Kirk. The chance that Tip was born on December 9 is 1/365 (forget leap year). The chance that Kirk was born on this day is also 1/365. The chance that both Tip AND Kirk were born on this day is (1/365)(1/365). So the answer is $(1/365)^2$.

20. High roller. Answer: 5/6. It is easier to find the probability that the sum *doesn't* exceed 4. The only such sums are (1-1), (1-2), (2-1), (1-3), (3-1), (2-2). So, the chance of not exceeding 4 is 6/36 = 1/6. As soon as we talk about the 36 equally likely outcomes, we are treating 2-1 and 1-2 differently and 3-1 and 1-3 differently, which is why it is necessary to count them separately in the analysis.

21. Double dice. Of the 36 equally likely ordered rolls, there are six ways to get doubles. So, the chance is 6/36 = 1/6. Alternatively, we could think of this as a birthday problem in which there were only six calendar days to the year. It doesn't matter what the first die rolls. The chance that the second die matches it is 1/6.

22. Silly puzzle. We hope you didn't spend too long on this one. She knew that there were twins in the class.

23. Just do it. Do birthday problem survey.

24. No matches. The chance that the second person doesn't match the first is 365/366. Two birthdays are used up, so the chance that the third doesn't match either of the first two is 364/366. The probability that the second doesn't match *and* the third doesn't match is (365/366)x(364/366). Continuing in this way, the probability that the first 40 people don't match is $(365/366)(364/366)\ldots(327/366) = (365!/326!)/366^{39} = 0.1087681902\ldots$

25. Spinner winner. Four of the 10 equally wide places are numbered 6 or higher. Therefore, the probability of winning is 4/10 or .2.

Creating New Ideas

26. Flip side. Probabilities are easiest to calculate when we start with a list of equally likely outcomes. The simplest way to do this is to view the coins as *different* and then order the outcomes. With this in mind, there are four equally likely outcomes that contain at least two heads. (T,H,H), (H,T,H), (H,H,T), and (H,H,H). So, the chance of three heads is 1/4.

27. Other flip side. Number the coins 1,2, and 3, and consider the eight equally likely ordered outcomes. The person has only eliminated one of the eight outcomes, which leaves a probability of 1/7 that three heads came up.

28. Blackjack. The remaining deck has 50 cards, 12 of which are face cards. So, the probability of getting a face card is 12/50 or 6/25.

29. Be rational. No, the probability is 1/3. Following the Expanded Hint we note that when the original numbers are chosen, the resulting fraction will fall into one of four equally likely categories: even/even, even/odd, odd/even, or odd/odd. For the reduced fraction to contain only odd numbers, both the numerator and the denominator must have originally had the same number of 2's in their prime factorizations. Half the numbers from 1 to 1,000,000 are even and half are odd. Thus half the numbers have no 2's in their prime factorization. Of the half that *are* divisible by two, half of them are divisible by four and half are not. Together this means that 1/4 of all the numbers have exactly one 2 in their prime factorization. Classify each number by the number of 2's in its prime factorization:

Num of 2's in factorization	0	1	2	3	…	N	…
Probability	1/2	1/4	1/8	1/16		$1/2^{N+1}$	…

Now to answer the Mindscape, we need only answer the question: "What is the probability that the chosen numerator and denominator both had zero 2's in their factorization OR both had one 2 in their factorization OR both had two 2's in their factorization… etc. ?" This translates into $P = (1/2)^2 + (1/4)^2 + (1/8)^2 + \ldots + (1/2^n)^2 + \ldots = 1/4 + (1/4)^2 + .. + (1/4)^n + \ldots = 1/3$.

30. Well red. This is non-intuitive. The probability that the other side is blue is 1/3. Remember that our notions of probability stem from manipulating sets of *equally likely* outcomes. Red-red, red-blue, blue-red, and blue-blue are not all equally likely outcomes!. Assume that all the cards and colors are different; for example, (pink, red), (blue, navy), (ruby-red, sky-blue). You are shown a side of a card with one of the red hues. It might be the