Review of[2]
Set Theory: A First Course
by Daniel W. Cunningham
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Review by

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When I first "learned" a thing or two about "set theory" in junior high school, it all seemed obvious, trivial, and of no apparent use whatsoever. And at that level, I think, I was right. One can barely scratch the surface with *naïve* naïve set theory, and I now regard the attempt to introduce the concept (devoid of any application) into the curriculum at such an early grade as misguided, at best. Set theory in and of itself can only be interesting to one in possession of enough mathematical background and sophistication to appreciate its more subtle aspects, e.g., the nature of infinite sets; in short, what might be expected of a math or CS major in their second year (or later) of their undergraduate eduction.

On the other hand, mathematics is built so essentially on the bedrock of set theory, and we use it in almost everything we do without thinking deeply about it, that it's easy to take for granted. While naïve set theory is probably all that most math or CS students (and even researchers) will need through their careers, there is much to be said in favor of introducing set theory in some depth as a more routine component of undergraduate curricula. This is not because working mathematicians routinely draw on techniques such as transfinite induction or ordinal arithmetic; but rather, because the subject provides an ideal medium for thinking abstractly, and cultivating skills in the development of rigorous mathematical proofs. As Cunningham points out in his preface, a background in set theory gives the mathematics student a distinct advantage in subsequent courses in analysis or algebra. It is also an intrinsically beautiful theory, which starts with virtually nothing and ascends to some of the deepest ideas and most universal themes in mathematics.

Cunningham's book develops axiomatic set theory truly *ab initio*, assuming little on the part of the reader beyond mathematical maturity. To get the best sense of how this is done, here is a run-down of the contents of the book, chapter by chapter:

1. *Introduction*: We begin with naïve set theory. There is also an introduction to propositional and predicate calculus, using them to establish a formal language for set theory (in which everything, including any element of a set, is a set). The chapter concludes with the statement of the Zermelo-Fraenkel axioms. All is very well-motivated historically and logically. Throughout, great care is taken to give examples. The ZF axioms are each briefly discussed in turn, and it is explained how they will be used, e.g., the regularity axiom is not used explicitly until chapter 8.

2. *Basic Set-Building Axioms and Operations*. This considers the first 6 axioms, further explaining each and then using them to build up useful facts and constructions. For example, the Subset Axiom leads to a handy theorem for proving when a class is a set (my impression is that this Theorem (2.1.3) is applied more than any other single theorem in the book). The Union Axiom can be used to construct

both the union and the intersection of the elements of a given set. These tools and the axioms are further applied to obtain basic results such as the De Morgan and Distributive laws.

3. *Relations and Functions* begins with Kuratowski's construction of ordered pairs entirely within the axioms of set theory. It moves on to the definition of relation and various associated notions such as the inverse, image, and restriction; and properties such as single-rootedness; and equivalence relations and their relationship to partitions. This is followed by the basic definitions of functions, including properties such as one-to-one functions and bijections. We then consider indexed functions and the axiom of choice, including a brief discussion of the distinction between ZF and ZFC. The chapter concludes with routine fundamentals on order relations (i.e., partial and total orders), congruence, and preorder relations.

4. *The Natural Numbers*. Up to this point the book covers subjects that are pretty standard fare in a discrete math course, but here we are entering somewhat deeper waters. We begin with the construction of the natural number (as sets, of course) and define the set of natural numbers ω and inductive sets, out of which the principle of mathematical induction emerges naturally. This is used to prove the Recursion Theorem, which leads to the Peano axioms and the theory of arithmetic and order on ω. The power of the techniques is illustrated by numerous applications of the same basic tools, in this case inductive arguments.

5. *On the Size of Sets* begins with careful definition of the size of a finite set in terms of a one-to-one function from the set to a natural number (each natural number being represented as a set here and throughout the book). Various properties of these functions and finite sets are explored, including the Pigeonhole Principle. Along the way, one obtains as a corollary that ω is infinite. The main agenda of the chapter is Cantor's theory generalizing the notion of the size of a set to infinite sets. Thus we study countable sets, and diverse ways to build them (e.g., from the countable union of countable sets, as finite sequences of countable sets, etc.). The proof of Cantor's theorem stating the uncountability of all functions from ω to $\{0, 1\}$ is very clearly stated and explained.

6. *Transfinite Recursion.* While the material gets significantly more difficult at this point, it also gets a lot more interesting. This is an exceptionally cogent presentation of transfinite recursion. The existence of functions defined by transfinite recursion is proved for ordinary functions and then generalized to "class functions" (a class that is functional in the sense that the defining relation is single-valued). The subtleties in going from the set form to the class form are very clearly explained. This is the first instance in which the Replacement Axiom is applied. The Transfinite Recursion Theorem is used to prove that every set is a subset of its transitive closure.

7. *The Axiom of Choice (Revisited).* The main theme of this chapter is Zorn's Lemma, its relationship to AC, and some of its consequences. The text proves that AC implies Zorn's Lemma; the opposite implication (from which equivalence follows) is relegated to an exercise. Applications include the comparability theorem (the cardinality of any two sets are comparable) and the order extension theorem (any partial order on a set can be extended to a total order). Filters, ultrafilters, and ideals are introduced and their interrelationships are explored in light of Zorn's Lemma. E.g., filters and ideals can be easily constructed one from the other, and the ideals corresponding to ultrafilters are prime. The chapter closes with a proof of the Well-Ordering Theorem from Zorn's Lemma.

8. *Ordinals.* The ordinal numbers are defined (effortlessly, given the background provided in earlier chapters) in terms of well-ordered transitive sets. Properties of ordinals are derived, and the ordinal

type belonging to any well-ordered structure is characterized. The ideas of supremum/infimum, and successor and limit ordinals are introduced, and the Burali-Forti theorem (that the class of ordinals is not a set) is proved. Transfinite induction is formulated as ordinal induction, resulting in the ordinal recursion theorem. The latter is applied to normal class functions, resulting in Veblen's Fixed Point Theorem. Transfinite recursion is then used to define arithmetic operations on the ordinals (by analogy with recursion on ω, used to define arithmetic on ω back in Chapter 4). Finally, the cumulative hierarchy is defined, its basic properties are proved, and it is identified (in part thanks to the applications of the regularity axiom promised in Chapter 1) as the universe of all sets.

9. *Cardinals.* Again, the cardinals are effortlessly defined given the previous chapters. What follows is, perhaps, the most technically challenging chapter. After some preliminaries, we proceed quickly to Hartogs' Theorem (the class of cardinals is proper), and, via ordinal class recursion, to the $\aleph$ numbers. The ideas of cofinality and regular and singular cardinals are introduced (in a sense *re*-introduced in the case of cofinality, as the notion of "cofinal" first appears in Chapter 7), and applied to the $\aleph$ numbers and the cardinality of (cartesian) products of ordinals. The next section constructs the arithmetic operations for cardinals. Exponentiation facilitates precise statements of the cardinality of the reals, the continuum hypothesis, and the generalized continuum hypothesis. König's Theorem (i.e., the theorem of that name which states that any infinite cardinal κ is contained in the cofinality of 2^{κ}) is proved, appealing to AC. The final section of the book deals with unbounded, closed and club sets, and proves some results on club filters and stationary sets.

As far as I can see, Cunningham neglects no opportunity to make the subject as accessible as possible. The mathematical development is rigorous, as it should be, but not excessively so. Although he starts from zero, that is not to say the book is easy, but any difficulty that arises is in the nature of the subject, and is no fault of the author's. Throughout the book, he offers many appropriate examples (or non-examples), and provides numerous and diverse exercises, which often prove results that are later used in the body of the text, drawing the reader into the subject. The reader participation increases in later chapters, in which a very significant fraction of the proofs is done in exercises. This is eminently suitable for a course at the undergraduate level, although once one hits Chapter 6 and beyond, decidedly at the junior or senior level. It would also be great for self-study.

In short, this is an excellent book! You will know a good deal about basic set theory after working through it.