If we allow complex-entried matrices and then define Σ_n^H to be the *Hermitian* matrices which are positive definite, it is clear also that our results concerning Σ_n may be modified to remain valid in Σ_n^H. In addition we should note that our notion of Π_n has no consistent analog in the complex-entried matrices. For instance, a matrix $A \in \Pi_n$ with respect to real vectors may not even have $\mathbf{x}^* A\mathbf{x}$ real when complex vectors are allowed (* means "conjugate transpose").

We have thus far formulated a theory of positive definite matrices. It is clear that we may analogously define another (disjoint) set of matrices by replacing "$>$" with "$<$" in the definition of positive definite. Such matrices are usually termed *negative definite*, and, suggestively, we might designate this set as $-\Pi_n$ since $A \in \Pi_n$ if and only if $-A \in -\Pi_n$. This, of course, is the key to the development of a theory of negative definite matrices which would proceed analogously (allowing for the peculiarities of negative numbers).

Positive (negative) *semi-definite* matrices may be defined by allowing the possibility of equality in the definition of Π_n (or $-\Pi_n$). Their theory proceeds similarly, but modified by allowance for 0 eigenvalues.

In the positive definite case, we have succeeded in establishing four characterizations through theorems: by eigenvalues ((2)); by determinants ((5)); by triangular decomposition ((6)); and by Q^TQ decomposition ((8)). This, plus the additional properties commented on, is largely sufficient to both mathematically describe and usefully apply positive definite matrices.

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LOGIC VERSUS PEDAGOGY

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Humble thyself, impotent reason.
PASCAL

1. The Current Emphasis on Logical Structure. There is no question that mathematics is distinguished from all other bodies of human knowledge in that it insists on deductive proof from explicitly stated axioms as the indispensable condition for the acceptance of its conclusions. This requirement has indeed conferred power on mathematics, for deductive proof has strengthened the

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structure. Moreover, the organization of mathematics into deductive systems has given coherence to its vast contents, and the axiomatization inherent in this type of organization has made clear precisely what is presupposed and hence where such systems are applicable. It has also suggested abstractions which embrace several structures as, for example, the theory of groups.

The deductive organization of mathematics is now a popular mode of presentation in all instruction from the fourth or fifth grade up. Deductive proof is the be-all and end-all of teaching. The enthusiasm for this mode of presentation is somewhat understandable because it was only about 75 years ago, after over 2500 years of struggle, that deductive organization and, through it, rigor were achieved. Mathematicians may be now giving vent in their textbooks to the satisfaction that Poincaré expressed in 1900 at the International Congress of Mathematicians when he gloated [11], "One may say today that absolute rigor has been attained." As an expression of enthusiasm this emphasis on deduction and rigor might indeed be excused. However, when authors are challenged as to the pedagogical wisdom of such presentations they now rejoin that this is the way to understand mathematics. In other words, the deductive approach is being defended as the pedagogical approach. Deductive organization and proof are advocated as the answer to all of the difficulties which students have had in learning mathematics and the open sesame to the subject. No longer will the memorization of techniques be necessary. Mathematics will now be accessible and understandable to almost all students.

2. The Intuitive Approach. Opposed to the deductive approach is the intuitive approach. Admittedly the nature of intuition is somewhat vague. It denotes some direct grasp of the idea, whether it be a concept or proof. There may be a special intuitive faculty distinct from the logical faculty that criticizes and reasons. Whether or not there is an intuitive faculty there are specific and explicit aids to the intuition which enable it to function. Primarily it seems to rely upon the senses, for, as Aristotle first put it, there is nothing in the intellect that was not first in the senses, except, Leibniz added, the intellect itself. Hence one of the useful devices is a picture. Consider exhibiting several triangles to inculcate the idea as opposed to the definition: the union of three non-collinear points and the line segments joining them. How much more readily is the notion of continuity grasped when presented as a curve which can be drawn with an uninterrupted motion of a pencil rather than by the $\epsilon-\delta$ definition.

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The intuition may be appealed to through physical arguments. The derivative as a velocity at an instant gives meaning to the concept, and the argument that a ball thrown up into the air must have zero velocity at its highest point suggests that the derivative must be zero at a maximum of a function.

We shall include in the intuitive approach what are often called heuristic arguments. Through experience with actual objects a child can learn that $3+4=4+3$. The generalization that $a+b=b+a$ is heuristic. Likewise the fact that if $y=x^n$, $dy/dx=nx^{n-1}$ is readily inferred from the special cases $n=2$ and $n=3$. Reasoning by analogy and even probabilistic arguments are heuristic.

Intuition is not static. Just as one's intuition about what to expect in human behavior improves with experience so does the mathematical intuition. The latter may indeed suggest, as it did to Leibniz, that the derivative of a product of two functions is the product of the derivatives. The conclusion should be tested, another heuristic measure, and of course will be found to be false. Deeper analysis will show that what holds for limits of functions does not hold for derivatives, and the intuition will be sharpened by this experience.

Clearly the intuitive approach can lead to error, but committing errors and learning to check one's results are part of the learning process. If the fear of errors is to be a deterrent, a child would never learn to walk.

It is the contention of this paper that understanding is achieved intuitively and that the logical presentation is at best a subordinate and supplementary aid to learning and at worst a decided obstacle. Intuition should fly the student to the conclusion, make a landing, and then perhaps call upon plodding logic to show the overland route to the same goals. If this contention is correct then the intuitive approach should be the primary one in introducing new subject matter *at all levels*. This recommendation may appear to be treason to mathematics, but let us withhold judgment.

3. The Historical Evidence. Though no air-tight case for the intuitive approach can be made from a study of the historical growth of mathematics a brief survey seems to offer some compelling arguments.

The first deductive structure was Euclid's *Elements*. Euclidean geometry, however, did not come into being in this form. It took 300 years, the period from Thales to Euclid, of exploration, fumbling, vague and even incorrect arguments before the *Elements* could be organized. Even this structure, intended to be strictly logical, rests heavily on intuitive arguments, pointless and even meaningless definitions, and inadequate proofs. That the logical structure can be devised after a subject is created and understood is not in question. What is relevant is that this deductive system came after the understanding was achieved. Moreover, it is no accident that Euclidean geometry was the first subject to receive any extensive mathematical development; the reason is that the intuition is readily applied to infer geometrical facts and the very figures suggest methods of proof.

A striking contrast is provided by the development of arithmetic and algebra.

Whole numbers and fractions and the operations with them were well accepted by the Egyptians and Babylonians, on an empirical basis, at least as far back as 2000 B.C. But irrational numbers, once their true character was recognized by the Pythagoreans, were not accepted by the classical Greeks as numbers. Why not? Because whole numbers and fractions had an obvious physical meaning whereas irrationals did not. The only intuitive meaning that one could attach to irrationals was that they represented certain geometrical lengths. What, then, did the Greeks do? They rejected irrationals as numbers and thought of them as lengths. In fact they converted all of algebra into geometry in order to work with lengths, areas and volumes that might otherwise have to be represented numerically by irrationals, and they even solved quadratic equations geometrically.

The progress that was made in the use of irrational numbers is due to the Alexandrian Greek civilization, which was a composite of the classical Greek, Egyptian and Babylonian civilizations, and to the Hindus and Arabs who were entirely empirically oriented. It was the Hindus who decided that $\sqrt{2}\sqrt{3}=\sqrt{6}$, and their argument was that these irrationals could be "reckoned with like integers," that is, like $\sqrt{4}\sqrt{9}=\sqrt{36}$. Irrational numbers were gradually accepted because of their utility and because familiarity breeds uncriticalness. The logical presentation of irrational numbers was not created until the 1870's.

Negative numbers, introduced by the practical-minded Hindus about 600 A.D., did not gain acceptance for 1000 years. The reason: they lacked intuitive support. The history of complex numbers is somewhat similar, though these did not appear until about 1540, and only about 200 years were required for these to be used somewhat freely. A remark of Gauss is very pertinent. As is well known, he was one of the men who discovered the geometrical representation of complex numbers, and about this he said in 1831 [1], "Here (in this representation) the demonstration of an intuitive meaning of $\sqrt{-1}$ is completely grounded and more is not needed in order to admit these quantities into the domain of the objects of arithmetic." Neither Descartes, Fermat, Newton, Leibniz, Euler, Lagrange, Gauss, or Cauchy could have given a definition of negative or complex numbers, or irrationals for that matter. Yet all of them managed to work with these numbers quite satisfactorily, to put it mildly, at least insofar as their times employed these numbers. In 1837 Hamilton did give the ordered couple definition of complex numbers in terms of real numbers but the logical development of the real number system itself was not constructed until the last part of the nineteenth century. The history of the entire complex number system is pertinent not only in itself but because algebra and analysis obviously utilize the number system and whatever basis there was for the latter had to serve as the basis for algebra and analysis.

The manner in which mathematics develops and is understood is beautifully exemplified by the history of the calculus. For the sake of brevity let us ignore the predecessors of Newton and Leibniz. The basic concept of the calculus is, of course, the instantaneous rate of change of a function, that is, the limit of

$\Delta y/\Delta x$ as Δx approaches 0. Where it was physically appropriate Isaac Newton thought of the limit in question as a velocity or as an acceleration, and he made great use of this fact in solving physical problems. But Newton experienced insuperable difficulties in explaining how he obtained the derivative from $\Delta y/\Delta x$. Because Δy approaches 0 when Δx does, he had to account for the fact that the quotient approached a definite number. Newton wrote three papers on the calculus and put out three editions of his famous *Mathematical Principles of Natural Philosophy*, and in each of these publications he made different explanations. In his first paper he says that his method is "shortly explained rather than accurately demonstrated." In his second paper he changed some terminology so as "to remove the harshness from the doctrine of indivisibles," but the logic is no more perspicuous. In the third paper Newton says "in mathematics minutest errors are not to be neglected." And then he gives a definition of the derivative, or fluxion as he called it, which supposedly shows that a fluxion is a precise concept. "Fluxions are, as near as we please, as the increments of fluents generated in times, as equal and as small as possible, and to speak accurately, they are in the prime ratio of nascent increments; yet they can be expressed by any lines whatever, which are proportional to them."

In the first and third editions of the *Principles* Newton says, "Ultimate ratios in which quantities vanish, are not, strictly speaking, ratios of ultimate quantities, but limits to which the ratios of these quantities decreasing without limit, approach, and which, though they can come nearer than any given difference whatever, they can neither pass over nor attain before the quantities have diminished indefinitely." He says further, "by the ultimate ratio of evanescent quantities is to be understood the ratio of the quantities, not before they vanish, nor after, but that with which they vanish." There are other statements by Newton in the published versions of his works which differ from the above. Clearly Newton struggled hard to define the derivative but scarcely succeeded in formulating a precise concept.

Leibniz worked not with the ratio $\Delta y/\Delta x$ and its limit but with differentials dx and dy which, he said, though not zero were not ordinary numbers. They were geometrically the differences in abscissa and ordinate, respectively, of two "infinitely near points." He too published many papers in which he tried to explain the meaning of the ratio dy/dx. Concerning his first paper on the calculus, published in 1684, even his friends, the Bernoulli brothers, said it was "an enigma rather than an explication."

Other papers and efforts to clarify his ideas did not accomplish any more. In a letter to Wallis Leibniz says: "It is useful to consider quantities infinitely small such that when their ratio is sought, they may not be considered zero but which are rejected as often as they occur with quantities incomparably greater. Thus if we have $x+dx$, dx is rejected. But it is different if we seek the difference between $x+dx$ and x. Similarly we cannot have xdx and $dxdx$ standing together. Hence, if we are to differentiate xy we write $(x+dx)(y+dy)-xy=xdy+ydx+dxdy$. But here $dxdy$ is to be rejected as incomparably less than $xdy+ydx$.

Thus in any particular case, the error is less than any finite quantity."

In the absence of satisfactory definitions he resorted to analogies to explain his differentials. At one time he referred to dy and dx as momentary increments or as vanishing or incipient magnitudes. These are Newtonian phrases. By way of additional explanation he said that as a point adds nothing to a line so differentials of higher order, e.g., $dxdx$, add nothing to dx. Alternatively dx is to x as a point to the Earth or as the radius of the Earth to that of the heavens. There are many other statements by Leibniz which are equally obscure.

There were many attacks on Leibniz's and Newton's work. Newton did not respond but Leibniz did. He objected to "overprecise critics" and argued that we should not be led by excessive scrupulousness to reject the fruits of invention. The phrases infinitely large and infinitely small signify no more than quantities which one can take as great or as small as one wishes. And then he adds that one can use these ultimate quantities, the actual infinite and the infinitely small, as a tool much as the algebraists use the imaginary with great profit. He also said that if one prefers to reject infinitely small quantities, it was possible instead to assume them to be as small as one judges necessary in order that they should be incomparable and that the error produced should be of no consequence or less than any given magnitude.

Of course the successors of Newton and Leibniz were aware of the lack of rigor in the calculus. Euler in the first classic text on the calculus, his *Introductio in Analysin Infinitorum* (1748), and again in his *Institutiones Calculi Differentialis* (1755) and *Institutiones Calculi Integralis* (1768–70), Lagrange in his *Théorie des fonctions analytiques* (1797) and d'Alembert in his article *Limite* in the *Encyclopédie* all struggled manfully but futilely to clarify the basic concepts of the calculus. Even Cauchy, the founder of rigor, gave definitions in his *Cours d'Analyse Algébrique* (1821) that would be considered loose and intuitive today. For example he says, "When the successive values attributed to a variable approach indefinitely a fixed value so as to end by differing from it by as little as one wishes, this last is called the limit of all the others." He defines continuity essentially by the requirement that the numerical value of the difference $f(x_0+\alpha)-f(x_0)$ decrease indefinitely with that of α. Because he was far more intuitive than rigorous Cauchy failed to distinguish between continuity and differentiability and even after his attention was called to this fact he persisted for twenty years in using differentiability where he had assumed only continuity. Clearly Cauchy's own rigor was beyond his comprehension. Cauchy also failed to recognize the necessity for the uniform convergence of series in order to integrate series term-by-term and to assert that the sum of a convergent series of continuous functions is continuous. Nor did he question that a function of two independent variables has a limit in both variables if it has a limit in each variable separately.

It is interesting that with respect to continuity and differentiability most texts of the nineteenth century including those written by the best mathematicians either followed Cauchy or "proved" that continuity implied differentiability. This is why the mathematical world was shocked when Weierstrass in 1872 produced the example of a function that is continuous for all real values of x but has no derivative at any value of x. Luckily this example came late in the development of the calculus for, as Emile Picard said in 1905 [8], "If Newton and Leibniz had known that continuous functions need not necessarily have a derivative, the differential calculus would never have been created."

In view of the vague, unclear, and even incorrect foundations of the calculus one might expect that the subject would collapse. But before Weierstrassian rigor became known through his lectures at Berlin in the 1870's not only had the calculus been extended and applied but the subjects of ordinary and partial differential equations, the calculus of variations, differential geometry, and the theory of functions of a complex variable had been erected on the calculus. How did the mathematicians achieve these tremendous victories? Clearly they thought intuitively.

We could examine the developments in projective geometry, non-Euclidean geometry and other areas but the story would be about the same. One can safely say that no proof given up to at least 1850 in any area of mathematics, except in the theory of numbers, and even there the logical foundation was missing, would be regarded as satisfactory by the standards of 1900, to say nothing about today's standards. Yet the mathematics created by the men was surely understood by them. The history teaches us then that the intuition of great men is far more successful than their logic.

One could of course argue that the growth of mathematics may indeed have proceeded as described but now that we have the proper logical structures for the number system, algebra, analysis and the various branches of geometry we need not ask students to repeat the fumblings of the masters. We can give them the correct approaches and they will understand them. This argument can be countered with the fact that many mathematicians did try to build logical foundations for the various subjects—witness Euler, Lagrange and Cauchy in the calculus—and their failure to do so ought to be some evidence that the logical approaches are not easy to grasp. Of course our students are superior to the best mathematicians of the past.

There is not much doubt that the difficulties the great mathematicians encountered are precisely the stumbling blocks that students experience and that no attempt to smother these difficulties with logical verbiage will succeed. If it took mathematicians 1000 years from the time that first class mathematics appeared to arrive at the concept of negative numbers, and it did, and if it took another 1000 years for mathematicians to accept negative numbers, as it did, we may be sure that students will have difficulties with negative numbers. Moreover, the students will have to master these difficulties in about the same way that the mathematicians did, by gradually accustoming themselves to the new concepts, by working with them and by taking advantage of all the intuitive support that the teacher can muster.

These conclusions have been reached by many great mathematicians who have concerned themselves with pedagogy. Poincaré said [12], "The zoologists maintain that in a brief period the development of the embryo of an animal recapitulates the history of its ancestors of all geological epochs. It appears that it is the same in the development of the mind. The task of the educator is to make the mind of the child go through what his fathers have experienced, to pass rapidly through certain stages but not to omit any. For this purpose, the history of the science ought to be our guide." Even Hilbert, the founder of modern axiomatics, granted the high pedagogic and heuristic value of the genetic method [2].

But we shall not insist on the evidence of history. There are other weighty arguments.

4. The Distortions of the Deductive Approach to Mathematics. Far from being the pedagogically sound representation of mathematics, the deductive approach introduces distorted views of the subject. First of all, mathematics is primarily a creative activity, and this calls for imagination, geometric intuition, experimentation, judicious guessing, trial and error, the use of analogies of the vaguest sort, blundering and fumbling. Even when a mathematician is convinced that a result must be correct he must still create to find the proof. As Gauss put it, "I have got my result but I do not know yet how to get it." Every mathematician knows that the hard work, tribulations and real thinking are required by, and the sense of achievement derives from, the creative effort. Writing up the final deductive formulation is a boring task.

Creativity presupposes flexibility in solving problems and any ideas from any domain of mathematics should be entertained whether or not they fall within the confines of a particular axiomatic structure. The latter, in fact, acts as a straitjacket on the mind.

What does logic contribute to the creation of concepts? Suppose one wishes to define the curvature of a surface. This definition is not arrived at by a deduction from axioms. It requires some deep insight to appreciate that this concept can be effectively represented by the product of the maximum and minimum curvatures of the curves through a point on the surface. As a matter of fact this definition was created by Euler who paid no attention to axiomatics.

Some of the greatest ideas in mathematics are not at all a matter of logic. Perhaps the best example is the realization that non-Euclidean geometry is applicable to physical space. The logical side, namely, pursuing the consequences of assuming a non-Euclidean parallel axiom, was a relatively simple task and was performed by Saccheri, Lambert, Legendre, Schweikart, Taurinus and many others. But it was Gauss who first recognized that these new geometries are as applicable as Euclidean geometry. The consequences for mathematics were as revolutionary as the very creation of mathematics itself.

It is true that some mathematicians, for example, Weierstrass in part of his work, Peano and Frege, produced rigorous theorems or axiomatic deductive structures. But in this work they were only reformulating what was already known and their goal was to rigorize what was well understood. New ideas were never obtained in this manner. Logic discovers nothing, neither the statement of a theorem nor its proof, even in the construction of axiomatic formulations of known results. Thus the concentration on the deductive approach omits the real activity. The logical formulation does dress up this activity but conceals the flesh and blood. It is like the clothes which make the woman but are not the woman. It is the last act in the development of a branch of mathematics and, as one wise professor put it, when this is performed the subject is ready for burial. Logic may be a standard and an obligation of mathematics but it is not the essence.

The student should be creating mathematics. Of course he will be re-creating it and with the aid of a teacher. This recreativity on the part of the student is more popularly termed discovery today. Every teacher professes to espouse discovery. The student can be gotten to do this if he is allowed to think intuitively but he cannot be expected to discover within the framework of a logical development that is almost always a highly sophisticated and artificial reconstruction of the original creative work.

The logical version is a distortion of mathematics for another reason. The concepts, theorems and proofs emerged from the real world. It is the uses to which the mathematics is put that tell us what is correct. Thus we add fractions by finding a common denominator and not by adding numerators and adding denominators though we do multiply fractions by multiplying numerators and multiplying denominators. Likewise, the uses to which matrices are put determine that multiplication is to be noncommutative though we can devise purely mathematical multiplications of matrices that are commutative. After we have determined what properties mathematical concepts and operations must possess on the basis of the uses of these concepts and operations we *then* invent a logical structure, however artificial it must be, which yields these properties. Hence, the logic does *not* dictate the content of mathematics. The uses determine the logical structure. The logical organization is an afterthought. As Jacques Hadamard remarked, logic merely sanctions the conquests of the intuition. Or, as Weyl put, "logic is the hygiene which the mathematician practices to keep his ideas healthy and strong."

In fact, if a student is really bright and he is told to cite the commutative law to justify, say $3 \cdot 4 = 4 \cdot 3$, he may very well ask, Why is the commutative law correct? The true answer is, of course, that we accept the commutative law because our experience with groups of objects tells us that $3 \cdot 4 = 4 \cdot 3$. In other words the commutative law is correct because $4 \cdot 3 = 3 \cdot 4$ and not the other way around. The normal student will parrot the words commutative law, and he will, as Pascal put it in his *Provincial Letters*, "fix this term in his memory because it means nothing to his intelligence."

The deductive development of a branch of mathematics is often so artificial that it is meaningless. No example is more pertinent than the deductive development of the real number system. There were good reasons to axiomatize the number system, but the introduction of fractions and negative numbers as couples with special definitions of the operations with these couples and the introduction of the irrationals by Cantor sequences or Dedekind cuts, clever as they may be, are so artificial, trumped-up and foreign to the intuitive meaning and uses of these numbers as to preclude understanding.

In such developments -2 is often introduced as the number which when added to 2 gives 0 or, as the modern mathematics texts put it, -2 is the unique additive inverse to 2. Such a definition induces no more understanding of -2 than the statement, anti-matter is that which added to matter produces a vacuum, gives any understanding of anti-matter. One doesn't learn even about dogs from a definition of dogs.

Poincaré makes this point too [9]. "In becoming rigorous mathematical science assumes a character so artificial as to strike every one. It forgets its historical origins; we see how the questions can be answered, but we no longer see how and why they were put."

Poincaré also notes [12] that in building up the number system from the integers there are many different constructions one can make. Why do we take one rather than another? "The choice is guided by the recollection of the intuitive notion in which this construction took place; without this recollection, the choice appears unjustified. But to understand a theory it is not sufficient to show that the path that one follows does not present obstacles; it is necessary to take account of the reasons that one chooses that path. Can one ever understand a theory if one builds it up right from the start in the definitive form that rigorous logic imposes, without some indications of the attempts which led to it? No; one does not really understand it; one cannot even retain it or one retains it only by learning it by heart."

Many teachers might retort that the student has already learned the intuitive facts about the number system and is now ready for the appreciation of the deductive version, which exemplifies mathematics. If the student really understands the number system intuitively the logical development will not only not enhance his understanding but will destroy it. As an example of mathematical structure no poorer choice could be made because the construction is so contrived. The development is so full of details and so stilted that it not only stultifies the mind but obscures the real ideas. Yet just this topic has now become the chief one in high school and college mathematics courses.

Actually this deductive approach is even misleading. In extending the number system from the natural numbers to the various other types we insist that the commutative and associative properties of the operations be retained. Why do we insist on these properties? We teachers know that the uses of the numbers call for these properties but the student gets the impression that these are necessary properties of all mathematical quantities. Why then do we not extend the order properties to complex numbers and the commutative property to matrices? The logical approach gives the student an entirely false impression of how mathematics develops.

The insistence on a deductive approach deceives the student in another way. He is led to believe that mathematics is created by geniuses who start with axioms and reason directly from the axioms to the theorems. The student feels humbled and baffled, but the obliging teacher is fully prepared to demonstrate genius in action. Perhaps most of us do not need to be told how mathematics is created but it may help to listen to the words of Felix Klein [3]. "You can often hear from non-mathematicians, especially from philosophers, that mathematics consists exclusively in drawing conclusions from clearly stated premises; and that in this process, it makes no difference what these premises signify, whether they are true or false, provided only that they do not contradict one another. But a person who has done productive mathematical work will talk quite differently. In fact those people are thinking only of the crystallized form into which finished mathematical theories are finally cast. The investigator himself, however, in mathematics as in every other science, does not work in this rigorous deductive fashion. On the contrary, he makes essential use of his imagination and proceeds inductively aided by heuristic expedients. One can give numerous examples of mathematicians who have discovered theorems of the greatest importance which they were unable to prove. Should one then refuse to recognize this as a great accomplishment and in deference to the above definition insist that this is not mathematics? After all it is an arbitrary thing how the word is to be used, but no judgment of value can deny that the inductive work of the person who first announces the theorem is at least as valuable as the deductive work of the one who first proves it. For both are equally necessary, and the discovery is the presupposition of the later conclusion."

The deductive approach produces practical complications. If a student has to show, for example, that $4ab(ab+3ac)=4a^2b^2+12a^2bc$ and if he has to justify each step, he will have to think carefully and give reasons for so many steps that he will take minutes to do what he should do almost automatically on the basis of experience with numbers. It is far preferable that the student should become so familiar with the basic properties such as distributivity, commutativity and associativity that he does not realize he is using them. Likewise many students of calculus have learned (by heart) the proof that a continuous function on a closed interval has a maximum and a minimum but cannot find the maxima and the minima of simple functions.

We should be grateful that students accept unquestioningly facts that seem entirely reasonable to them whether on the basis of experience with numbers or intuitive arguments. In fact we should do all we can to make the elementary operations so habitual that students do not have to think about them any more than one thinks when he ties his shoelaces. If students do not see readily that $3\cdot x=x\cdot 3$, it is not because they lack familiarity with the commutative principle but rather because they fail to understand that x is just a number. (Of course, one should say, x is a placeholder for a number.) When the time to teach a noncommutative operation arrives, then commutativity can be stressed.

The need to make some of the work automatic was stressed by a man who certainly understood the role of axiomatics. Alfred North Whitehead says

[15], "It is a profoundly erroneous truism, repeated by all copybooks, and by eminent people when they are making speeches, that we should cultivate the habit of thinking of what we are doing. The precise opposite is the case. Civilization advances by extending the number of important operations which we can perform without thinking about them. Operations of thought are like cavalry charges in a battle—they are strictly limited in number, they require fresh horses and must only be made at decisive moments."

Modern texts are not content just to present mathematics deductively. They incorporate the rigor that meets the professional's standards. Thus the deductive approach to geometry now requires that the axioms overlooked by Euclid, the order axioms for example, be included to justify the steps. The consequence is that a host of trivial theorems must be proved before one reaches the significant ones. Thus the student must prove that there is a unique midpoint for each line segment and that there is an inside and an outside of a triangle. Even worse is the fact that many of the theorems are more obvious than the axioms used to establish them. Hence, the less obvious is used to prove the more obvious. But as far as the student is concerned the whole point of proof is just the reverse. Students will question what is being accomplished and perhaps even wonder whether we teachers are sane. That for two thousand years Euclidean geometry, as formulated by the presumably careless or naive Euclid, was regarded by the best mathematicians as the paradigm of rigor, bears no weight with the advocates of precise axiomatics. Today's students, we are apparently supposed to believe, are sharper and will not be satisfied with proofs that fail to mention details whose absence no one noticed for so many centuries.

Poincaré struck at this very folly [12]. "When a student commences seriously to study mathematics, he believes he knows what a fraction is, what continuity is, and what the area of a curved surface is; he considers as evident, for example, that a continuous function cannot change its sign without vanishing. If, without any preparation, you say to him: No, that is not at all evident; I must demonstrate it to you; and if the demonstration rests on premises which do not appear to him more evident than the conclusion, what would this unfortunate student think? He will think that the science of mathematics is only an arbitrary accumulation of useless subtleties; either he will be disgusted with it or he will amuse himself with it as a game and arrive at a state of mind analogous to that of the Greek sophists."

The rigorous approach requires such a multitude of minor theorems that the larger features of the subject fail to stand out. As Poincaré put it [10], "In the edifices built up by our masters, of what use is it to admire the work of the mason if we cannot comprehend the plan of the architect? Now pure logic cannot give us the appreciation of the total effect; this we must ask of the intuition."

In many areas the present emphasis on the logical approach is sheer hypocrisy. What mathematician uses the logical development of the complex number system to justify his operations with real or complex numbers? Yet this is what is taught to students as the way to learn the "truth" about numbers. How many mathematicians have ever satisfied themselves that $\sqrt{2}^{\sqrt{3}}$ is defined in the theory of irrational numbers or even that $\sqrt{2}\sqrt{3}=\sqrt{3}\sqrt{2}$? How many have ever worked through a rigorous development of Euclidean geometry (as opposed to the pseudo-rigorous developments found in modern texts)? Felix Klein did not hestitate to admit [5], "To follow a geometrical argument purely logically without having the figure on which the argument bears constantly before me is for me impossible."

As a matter of fact the attempt to be completely deductive ensnares the teacher in a trap. It is often necessary to include a proof which even the rigor-oriented teachers concede to be too difficult for the student, such as the proof of the formula for the area of a circle in plane geometry. Many texts evade the issue by adopting an axiom. As a consequence numerous elementary geometry texts contain as many as 70 or 80 axioms. Surely if one can adopt axioms at will there is no need to prove anything. The only lesson the student will learn from such presentations is that if he is stuck he can adopt an axiom. The mathematics teacher can no more afford to be profligate with axioms than to be parsimonious. Likewise in the presentation of the real number system the high school texts proceed axiomatically from the natural numbers. But when they get to the irrational numbers, whose logical development the authors recognize to be too difficult for the student, they resort to the number line and speak of points which have no numbers assigned to them. These are designated by the irrational numbers. If the logical presentation of the rational numbers had any value it is dissipated by this meaningless introduction of the irrationals.

One of the gravest defects in the teaching of mathematics is the lack of motivation. Mathematics proper, as Weyl described it, has the inhuman quality of starlight, brilliant and sharp, but cold. Consequently very few students are attracted to the subject. In fact most of those taking high school and college mathematics do so because it is required or because they are prospective scientists or engineers. These students would prefer to learn more about the fruits than the roots of mathematics. Proper pedagogy requires that these students be shown why they should be studying particular topics and subjects. To assure them that the material will prove useful at some later time is hardly an incentive to take it seriously. The mathematician or the rare student who finds intellectual challenge or aesthetic satisfactions in the subject may be intrigued to learn that there are only five regular polyhedra. But very few students are excited by this fact. As far as they are concerned the world would be just as well off if there were an infinite number of them. As a matter of fact there is an infinite number of regular polygons and no one seems depressed by this fact.

Although the subject of motivation is a vast one its relevance here is simply that it is relatively easy to give a genuine or significant motivation to a mathematical topic when this is introduced intuitively or heuristically because historically there were significant motivations, whereas it is very difficult to do so in a logical presentation because the latter is many stages removed from reality and, as we have already pointed out, is often artificial. How does one motivate the concept of a fraction when it is to be introduced as an ordered couple of natural numbers? How can a student see the point of the $\epsilon-\delta$ definition of continuity if this is his introduction to the concept? In fact the logical approach often destroys the motivation. One may motivate the integral as the method of finding the area under a curve. But if one defines the area as an integral one begs the whole question of doing something significant with the integral.

Many texts and teachers claim that they do provide motivation even for a logical approach. Thus they "motivate" the introduction of negative, irrational and complex numbers by stating that we wish to solve equations such as $x^2+2=0$. But for students who have no reason to solve even $x-2=0$, the challenge of solving $x^2+2=0$ certainly isn't exciting. Moreover, the bright student can come back at the teacher and ask, "Why can't we solve $5/x=0$ by introducing ∞ as a number?" If we can invent definitions to operate with $\sqrt{-2}$, we can invent definitions to operate with ∞.

The presentation of theorems without the motivation robs the student of insight. Even on the somewhat advanced level where we deal with students who have some leaning toward mathematics, to present theorems without the motivation, whether it does or does not kill off the interest in mathematics, certainly leaves them with no more than a meaningless collection of theorems and proofs and without the power to think for themselves. Thus in linear algebra texts the subject of eigenvalues of matrices is always treated. I have not found one text which indicates why one wants to learn anything about the eigenvalues. The simultaneous reduction of two quadratic forms to sums of squares is another seemingly meaningless topic. At least the origin in mechanical problems might be suggested. Equivalent, congruent, and similar matrices are treated for no apparent reason. (See the article by R. J. Jarvis: A Case for Applications of Linear Algebra and Group Theory, this MONTHLY, 73, 1966, 654–656.)

5. The Role of Logic in Pedagogy. In view of the many pedagogical shortcomings in the logical approach to mathematics it is not surprising that many perceptive mathematicians (there are nonperceptive ones) have spoken out against the logical approach. Descartes deprecated logic in rather severe language. "I found that, as for Logic, its syllogisms and the majority of its other precepts are useful rather in the communication of what we already know or . . . in speaking without judgment about things of which one is ignorant." Roger Bacon said, "Argument concludes a question but it does not make us feel certain, or acquiesce in the contemplation of a truth, except the truth also be found to be so by experience." Pascal pointed out that "Reason is the slow and tortuous method by which those who do not understand the truth discover it."

Is there then no role for logic or proof? Should it be rejected all together? Not at all. The first approach to any subject should indeed be intuitive. As

Poincaré put it [9], "I have already had occasion to insist on the place intuition should hold in the teaching of the mathematical sciences. Without it young minds could not make a beginning in the understanding of mathematics; they could not learn to love it and would see in it only a vain logomachy; above all without intuition they would never become capable of applying mathematics. . . .

"We need a faculty which makes us see the end from afar and intuition is this faculty."

Proof should enter but only gradually. Moreover, the level of rigor must be suited to the level of the student's mathematical development. The proof need only convince the student. The capacity to appreciate rigor is a function of the mathematical age of the student and not of the age of mathematics. This appreciation is acquired gradually, and the student must have the same freedom to make intuitive leaps that the mathematicians had. Rigor will not refine an intuition that has not been allowed to function freely. Proofs of whatever nature should be invoked only where the students think they are required. A proof is meaningful when it answers doubts. Felix Klein has stressed this point [6]: "It is my opinion that in teaching it is not only admissible, but absolutely necessary, to be less abstract at the start, to have constant regard for applications, and to refer to the refinements only gradually as the student becomes able to understand them. This is, of course, nothing but a universal pedagogical principle to be observed in all mathematical instruction." As Professor Max M. Schiffer of Stanford University has stated it, "Never put logical carts before heuristic horses."

The level of rigor can, of course, be advanced as the student progresses. Poincaré makes this point too [12]. "On the other hand, when he is more advanced, when he becomes familiar with mathematical reasoning and his mind will be matured by this very experience, the doubts will be born of themselves and then your demonstration will be well received. It will awaken new doubts and the questions will arise successively to the child as they arose successively to our fathers to the point where only perfect rigor can satisfy him. It is not sufficient to doubt everything; it is necessary to know why one doubts."

6. Why is the Deductive Approach Favored? Despite the pedagogical defects of the deductive approach, the criticisms of big mathematicians, and the claims of many mathematicians that they do teach discovery, the prevailing practice, if I may judge from the textbooks and hundreds of talks with professors, is to present mathematics rigorously and to emphasize the axiomatic method. Indeed this is the essence of the so-called reform known as "modern mathematics" or the "new mathematics." Why do teachers use this approach?

There is no doubt that some teachers actually believe that the axiomatic deductive presentation is the essence of mathematics. Whether they acquired this limited view through the instruction they themselves received or have been induced to adopt it because the textbooks now favor it, they are at least sincere if not effective pedagogues. One has the sneaking suspicion that a few teachers enjoy presenting the familiar number system in the recondite axiomatic form because they understand the simple mathematics it represents and yet can appear to be presenting profound mathematics. Certainly much of the rigor in modern texts comes from limited men who seek to conceal shallowness by giving a facade of profundity to the obvious and from pedants who mask their pedantry under the guise of rigor.

Many young teachers believe that now that we have the correct, polished version of mathematics it is sufficient to give the axiomatic or rigorous approach and that students will absorb it. These very same teachers would have been swamped by such a presentation but having learned the correct version they can no longer recall and appreciate the difficulties they encountered in learning the rigorous versions.

Some teachers, knowing the rigorous proofs, feel uneasy about presenting a convincing argument which they, at least, know is incomplete. But it is not the teacher who is to be satisfied; it is the student. Good pedagogy demands such compromises.

Other teachers want to give students the whole truth at once so that they should not have to unlearn what they once learned. But one cannot teach even English or History by starting at the top. The A that a high school student might earn for an English composition would most likely be rated C at the college level.

For whatever reason teachers insist on presenting to young people a modern rigorous proof they are deceiving themselves. There is no ultimate rigorous proof. This fact derives from the very way in which mathematics develops. Felix Klein has described it [4]: "In fact, mathematics has grown like a tree, which does not start at its tiniest rootlets and grow merely upward, but rather sends its roots deeper and deeper at the same time and rate that its branches and leaves are spreading upward. . . . *We see, then, that as regards the fundamental investigations in mathematics, there is no final ending, and therefore on the other hand, no first beginning, which could offer an absolute basis for instruction.*" Poincaré expressed a similar view. There are no solved problems; there are only problems that are more or less solved. Mathematics is as correct as human beings are and humans are fallible.

At no time in the history of mathematics have we been less certain of what rigor is. Hence no proof is really complete, and the teacher must compromise in any case. It would be interesting to know how many teachers are aware that set theory, which they now regard as the indispensable beginning to any rigorous approach to mathematics, has been the source of our deepest and thus far insuperable logical difficulties [16]. Those who are not aware of the foundational problems might at least note the words of Hermann Weyl [14]: "The question of the ultimate foundations and the ultimate meaning of mathematics remains open; we do not know in what direction it will find its final solution nor even whether a final objective answer can be expected at all. 'Mathematizing' may well be a creative activity of man, like language or music, of primary originality, whose historical decisions defy complete objective rationalization."

Many teachers favor a logical presentation, particularly, an abstract one, such as group theory, because it is supposed to be efficient. They are under the impression that if a student is taught abstract groups he will in one swoop learn the properties of the rational, real and complex numbers, matrices, congruences, transformations and other topics. But of course a student who learns group theory could not on this basis add fractions. Nor does offering an example or two of a group save the day. The concrete cases must be thoroughly understood *before* one introduces an abstract development which unifies several concrete ones. To introduce as examples concrete material which is as yet unfamiliar to the student is of no help at all in making the abstract notion clearer. In every case learning proceeds from the concrete to the abstract and not vice versa.

However, the major reason for the popularity of the axiomatic rigorous approach is that it is easier to teach. The entire body of material is laid out in a clear, clean-cut sequence and all the teacher has to do is repeat it. He has but to offer a canned body of material. I have heard teachers complain that many students, particularly engineers, wish to be told how to perform the processes they are asked to learn and then want to hand back the processes. But the teachers who teach the logical presentation because it avoids such difficulties as teaching discovery, leading students to participate in a constructive process, explaining the reasons for proceeding one way rather than another, and finding convincing arguments, are more reprehensible than the students who wish to avoid thinking and prefer just to repeat mechanically learned processes. Postulating properties has the advantage, as Bertrand Russell put it, of theft over honest toil. Pedagogically it is worse because the theft produces no gain in understanding. The logical approach to teaching is reminiscent of a reply that Samuel Johnson gave to a man who asked Johnson for further explanation of some argument he had given. Johnson barked, "I have found you an argument but I am not obliged to find you an understanding."

Many of us know the story of the professor who was presenting a logical proof to his class, got stuck in the course of the proof, went over to the corner of the blackboard where he drew some pictures, erased the pictures, and then continued the proof. Whether the import of this story for pedagogy has been noted is doubtful.

Many mathematicians prefer to present rigorous axiomatic approaches which, for example, use a minimal set of axioms, because they favor their own professional interest at the expense of the student. Even if such systems can be made understandable to young people the time required to teach them could be spent on more significant material. In this matter as well as in presenting sophisticated rigorous proofs they are using the classroom to challenge themselves. These professors are serving themselves rather than the students not only in the form in which they present the various subjects but also in the premature teaching of abstractions such as abstract algebraic concepts, linear vector spaces, finite geometries, set theory, symbolic logic and functional analysis, because these subjects lend themselves to axiomatic treatments. Is it any wonder that students become alienated and question the relevance of what they are being taught?

There are many indications that professors who present rigorous material are really uncertain as to the wisdom of doing so. A number of calculus books begin with rigorous definitions and theorems, for example, those concerning limits and continuity, and then never refer to this material. Thereafter they use the cookbook presentation. The most charitable view of such books is that the authors wish to ease their own consciences or to give the students some idea of what rigor means. Perhaps an unfairly severe view is that these books offer only a pretense of rigor in order to appeal to both markets, the one that demands rigor and the one that is satisfied to teach mechanical procedures.

Other texts adopt another "compromise." In the body of the text the presentation is mechanical with perhaps an occasional condescension to an intuitive explanation. The real "explanation" is given in rigorous proofs but these are put in appendices and presented so compactly that they are certain to be totally ununderstandable to the student. However, the authors have salved their consciences. Such books are no different from the old mechanical presentations. They do contribute to understanding in one respect, namely, they show that competent mathematicians are inept in pedagogy.

Perhaps, after all, there is some merit to the logical approach to mathematics. It has been said of rigor that "The virtue of a logical proof is not that it compels belief but that it suggests doubts and the proof tells us where to concentrate our doubts." Or as Bertrand Russell put it [**13**], "It is one of the chief merits of proofs that they instill a certain scepticism as to the result proved." Lebesgue pointed out another value of rigorous proof [**7**]. "Logic makes us reject certain arguments but it cannot make us believe any argument." One must respect but suspect mathematical proofs. Since one of the main objectives of mathematics education is to instill scepticism in the student, he is deriving at least one benefit from the current logical extravaganzas.

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MATHEMATICAL NOTES

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A COROLLARY TO THE GELFAND-MAZUR THEOREM

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We wish to establish the following result as a corollary to the Gelfand-Mazur Theorem:

Let $A \neq 0$ be a complex commutative normed algebra such that $\|xy\| = \|x\| \, \|y\|$ for all $x, y \in A$. Then A is isomorphic and isometric to the complex field.

Note that there is no assumption that A contains an identity. In fact the major important step in the construction of the proof is to demonstrate that under the given norm condition on A, the algebra has an identity. (If A were assumed to have an identity, then the above corollary would merely be a special case of a well-known result which appears in the literature. See for example [**3**, Cor. 1.7.3, p. 39]. Also under the assumption of an identity in A, Lorch has a proof of the above corollary [**1**, Th. 5-2, p. 129].)

We shall use an algebraic approach to allow us to demonstrate the existence of the identity in A. To obtain the result, the Gelfand-Mazur Theorem will be applied. The conclusion of the corollary follows directly.

Proof. By the norm condition, A has no zero divisors. Let $S = A - \{0\}$. Let $T = \{(a, s) \mid a \in A, s \in S\}$. As in Herstein [**2**, pp. 101–103] define an equivalence relation $\sim$ on T as follows: $(a, s) \sim (a_1, s_1)$ if and only if $s_1 a = s a_1$. Denote the equivalence class of (a, s) by a/s. We define $S^{-1}A = \{a/s \mid (a, s) \in T\}$. Then $S^{-1}A$ is the field of quotients of A under operations $+, \cdot$, now to be defined. Let a/s, $a_1/s_1 \in S^{-1}A$.

$$a/s + a_1/s_1 \equiv (s_1 a + s a_1)/s s_1 \tag{1}$$

$$(a/s) \cdot (a_1/s_1) \equiv a a_1/s s_1 = a_1 a/s_1 s = (a_1/s_1) \cdot (a/s). \tag{2}$$