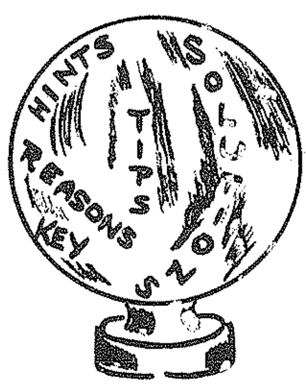
CONTEMPORARY MOTIVATED MATHEMATICS

BOOK 3

INDEX AND SELECTED ANSWERS

Solutions to many of the strictly computational problems are obvious and are not included here at present. We have given some explanations, comments and solutions for those problems which we have considered essential or unusual. If you should find that solutions to some other problems are desirable we would appreciate hearing from you.



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CONTEMPORARY MOTIVATED MATHEMATICS

Page 2 Problem 6

The squares can be arranged in any order in the 8 x 8 square to make it a magic square.

Page 3 Problem 16

71	64	69	8	1	6	53	46	51
66	68	70	3	5	7	48	50	52
67	72	65	4	9	2	49	54	47
26	19	24	44	37	42	62	55	60
21	23	25	39	41	43	57	59	61
22	27	20	40	45	38	58	63	56
35	28	33	80	73	78	17	10	15
30	32	34	75	77	79	12	14	16
31	36	29	76	81	74	13	18	11

The arrangement of the 3 x 3 squares within the 9 x 9 square can be described as follows:
Consider the magic square

8	1	6
3	5	7
4	9	2

Place the 3 x 3 square with the smallest magic constant (here 15) in the same position as 1, place the 3 x 3 square with the next smallest magic constant (here 42) in the same position as 2, and so on.

Page 5 Problem 20

The general formula for the magic constant of a 4 x 4 magic square constructed from Hu-Knos square is 4x .

Page 8 Problem 42

The general formula for the magic constant of a 5 x 5 magic square constructed from Hu-Knos secret square is 5x .

Page 19 Problem 69

Column 1	Column 2	Column 3	Column 4	Column 5	Column 6
1	2	4	8	16	32
3	3	5	9	17	33
5	6	6	10	18	34
7	7	7	11	19	35
9	10	12	12	20	36
11	11	13	13	21	37
13	14	14	14	22	38
15	15	15	15	23	39
17	18	20	24	24	40
19	19	21	25	25	41
21	22	22	26	26	42
23	23	23	27	27	43
25	26	28	28	28	44
27	27	29	29	29	45
29	30	30	30	30	46
31	31	31	31	31	47
33	34	36	40	48	48
35	35	37	41	49	49
37	38	38	42	50	50
39	39	39	43	51	51
41	42	44	44	52	52
43	43	45	45	53	53
45	46	46	46	54	54
47	47	47	47	55	55
49	50	52	56	56	56
51	51	53	57	57	57
53	54	54	58	58	58
55	55	55	59	59	59
57	58	60	60	60	60
59	59	61	61	61	61
61	62	62	62	62	62
63	63	63	63	63	63

Page 19 Problem 70

If you use the table above, the largest number that can be selected for the magic guessing is 63 .

Page 20 **Problem 72**

January 31st Age 100

```
   3 1 0 1
 x       5
 ---------
 1 5 5 0 5
 x     2 0
 ---------
 3 1 0 1 0 0
 +     1 0 0
 -----------
 3 1 0 2 0 0
```

Dini Dunit's Magic 1 does not work. You can only add a two digit age so that there will be no carrying. The reason why the scheme works is that the multiplication by 100 moves each of the 4 given digits over two places to the left leaving space for the two digit age in the tens and units position.

Page 21 **Problem 74**

January 31st Age 100

```
   3 1 0 1
 x       2
 ---------
   6 2 0 2
 +       5
 ---------
   6 2 0 7
 x     1 0
 ---------
   6 2 0 7 0
 x         5
 -----------
 3 1 0 3 5 0
 +     1 0 0
 -----------
 3 1 0 4 5 0
 -     2 5 0
 -----------
 3 1 0 2 0 0
```

Dini Dunit's Magic 2 does not work in this case. You can only add a two digit age so that there will be no carrying. The reason why the scheme works is that the steps when performed move each of the 4 digits over two places to the left, thus leaving space for the two digit age in the tens and units position.

Page 22 Problem 76

Age 100 , Number 100

$$
\begin{array}{rr}
 & 100 \\
\times & 10 \\
\hline
 & 1000 \\
+ & 10 \\
\hline
 & 1010 \\
\times & 10 \\
\hline
 & 10100 \\
+ & 100 \\
\hline
 & 10200 \\
- & 100 \\
\hline
 & 10100
\end{array}
$$

Dini Dunit's Magic 3 does not work in this case. The scheme only works for pupils numbering < 100 . As before, the procedure shifts the given digits two places to the left so that the number of pupils occupies the tens and units positions.

Page 26 Problem 83

Take	Division method	Sum - division method	Reduced sum method
799	88 9) 7 9 9 7 2 7 9 7 2 7 Replace 799 by 7 .	7 + 9 + 9 = 25 25 ÷ 9 = 2 remainder 7 Replace 799 by 7 .	7 + 9 + 9 = 25 2 + 5 = 7 Replace 799 by 7 .
957	106 9) 9 5 7 9 0 5 7 5 4 3 Replace 957 by 3 .	9 + 5 + 7 = 21 21 ÷ 9 = 2 remainder 3 Replace 957 by 3 .	9 + 5 + 7 = 21 2 + 1 = 3 Replace 957 by 3 .
985	109 9) 9 8 5 9 0 8 5 8 1 4 Replace 985 by 4 .	9 + 8 + 5 = 22 22 ÷ 9 = 2 remainder 4 Replace 985 by 4 .	9 + 8 + 5 = 22 2 + 2 = 4 Replace 985 by 4 .
1629	181 9) 1 6 2 9 9 7 2 7 2 0 9 9 0 Replace 1629 by 0 .	1 + 6 + 2 + 9 = 18 18 ÷ 9 = 2 remainder 0 Replace 1629 by 0 .	1 + 6 + 2 + 9 = 18 1 + 8 = 9 Thus, the reduced sum is 0 . Replace 1629 by 0 .

2562	284 9 ⟌ 2562 18 76 72 42 36 6 Replace 2562 by 6 .	2 + 5 + 6 + 2 = 15 15 ÷ 9 = 1 remainder 6 Replace 2562 by 6 .	2 + 5 + 6 + 2 = 15 1 + 5 = 6 Replace 2562 by 6 .

<u>Page 27</u> <u>Problem 84</u>

2 3 5 ⟶ 1 4 3 8 ⟶ 6 + 5 0 1 ⟶ 6 1 1 7 4 → ④ 13 ⟶ ④	3 2 7 ⟶ 3 8 9 0 ⟶ 8 + 5 7 4 ⟶ 7 1 7 9 1 → ⓪ 18 ⟶ ⓪	1 0 8 ⟶ 0 3 7 2 ⟶ 3 + 9 5 4 ⟶ 0 1 4 3 4 → ③ 3 ⟶ ③
3 2 7 4 ⟶ 7 8 3 0 1 ⟶ 3 + 2 7 6 5 ⟶ 2 1 4 3 3 0 → ② 12 ⟶ ③ Addition Incorrect	2 3 8 6 ⟶ 1 4 9 1 2 ⟶ 7 + 5 7 8 6 ⟶ 8 1 3 0 8 4 → ⑦ 16 ⟶ ⑦	5 3 8 7 ⟶ 5 7 0 4 9 ⟶ 2 6 1 9 7 ⟶ 5 + 8 2 0 6 ⟶ 7 2 6 8 3 9 → ① 19 ⟶ ①
1 3 5 7 ⟶ 7 6 8 4 2 ⟶ 2 9 3 2 4 ⟶ 0 + 6 2 5 3 ⟶ 7 2 2 7 7 6 → ⑥ 16 ⟶ ⑦ Addition Incorrect	2 1 3 4 ⟶ 1 5 9 0 6 ⟶ 2 6 2 8 7 ⟶ 5 + 4 9 6 4 ⟶ 5 1 9 2 9 1 → ④ 13 ⟶ ④	6 4 3 2 ⟶ 6 8 9 2 6 ⟶ 7 9 3 0 1 ⟶ 4 + 1 0 7 8 ⟶ 7 2 5 7 3 7 → ⑥ 24 ⟶ ⑥

<u>Page 28</u> <u>Problem 85</u>

268 x 52 = 13,936 ⟶ ④	7 x 7 = 49 ⟶ ④		
726 x 36 = 26,126 ⟶ ⑧	6 x 0 = 0 ⟶ ⓪	Product is incorrect.	
853 x 71 = 60,563 ⟶ ②	7 x 8 = 56 ⟶ ②		
1,276 x 47 = 59,972 ⟶ ⑤	7 x 2 = 14 ⟶ ⑤		
7,624 x 87 = 663,288 ⟶ ⑥	1 x 6 = 6 ⟶ ⑥		
3,047 x 237 = 722,139 ⟶ ⑥	5 x 3 = 15 ⟶ ⑥		
8,759 x 532 = 4,659,788 ⟶ ②	2 x 1 = 2 ⟶ ②		
9,075 x 734 = 6,661,050 ⟶ ⑥	3 x 5 = 15 ⟶ ⑥		
2,386 x 925 = 2,207,050 ⟶ ⑦	1 x 7 = 7 ⟶ ⑦		

Page 29 Problem 86

5 6 5 → 7 - 2 7 3 → -3 2 9 2 → (4) 4 → (4)	4 9 4 → 8 - 1 6 1 → -8 3 3 3 → (0) 0 → (0)	3 7 2 → 12 - 2 5 8 → - 6 1 1 4 → (6) 6 → (6)
5 1 2 → 8 - 3 5 6 → -5 1 5 6 → (3) 3 → (3)	1 2 7 4 → 5 - 9 3 2 → - 5 3 4 2 → (0) 0 → (0)	3 7 8 6 → 6 - 3 0 2 9 → - 5 7 6 7 → (2) 1 → (1) Difference is incorrect.
8 7 2 5 → 4 -5 9 6 7 → -0 2 7 5 8 → (4) 4 → (4)	7 7 7 1 → 4 → 13 -7 6 3 0 → -7 → -7 1 4 1 → (6) 6 → (6)	5 9 7 3 → 6 -2 8 4 5 → -1 3 1 2 8 → (5) 5 → (5)
	4 3 8 2 → 8 -4 2 9 5 → -2 8 7 → (6) 6 → (6)	

Page 32 Problem 91

Mathematician	Year born	Palindrome	No. of reversals
Michael Stifel	1487 A.D.	88555588	9
Franciscus Maurolycus	1494 A.D.	59895	4
Nicolo Tartaglia	1499 A.D.	56265	4
Jerome Cardan	1501 A.D.	2552	1
Robert Recorde	1510 A.D.	1661	1
Gerardus Mercator	1512 A.D.	3663	1
Ludovico Ferrari	1522 A.D.	3773	1
Raffael Bombelli	1530 A.D.	1881	1
Christopher Clavius	1537 A.D.	8888	1
Francois Vieta	1540 A.D.	1991	1
Tycho Brahe	1546 A.D.	7997	1
Simon Stevin	1548 A.D.	9999	1
John Napier	1550 A.D.	3113	2
Matteo Ricci	1552 A.D.	7117	2
Thomas Hariot	1560 A.D.	3333	2

Page 34 Problem 94

The difference of the squares of 2 consecutive natural numbers is equal to the sum of the 2 consecutive natural numbers.

Page 35 Problem 96

In $8n + 1$ the n (except for $n = 0$) is a triangular number.

Note 0 can be considered a triangular number since it is of the form $n(n + 1)/2$.

Page 38 Problem 102

n	no.	n	no.	n	no.	n	no.	n	no.	n	no.	n	no.	n	no.
5	61	7	127	9	217	11	331	13	469	15	631	17	817	19	1027
6	91	8	169	10	271	12	397	14	547	16	721	18	919	20	1141

Some students may discover an easier way to complete the chart without the use of $3n^2 - 3n + 1$ each time. Observe that $\left[3(n+1)^2 - 3(n+1) + 1\right] - \left[3n^2 - 3n + 1\right] = 6n$. Hence, by starting with 1 and adding successive multiples of 6 the table can be completed.

Page 39 Problem 103

$\sqrt[3]{64} = 4$ 64 - 1 63 - 7 56 -19 37 -37	$\sqrt[3]{^{-}125} = {}^{-}5$ ¯125 + 1 ¯124 + 7 ¯117 + 19 ¯98 + 37 ¯61 + 61	$\sqrt[3]{216} = 6$ 216 - 1 215 - 7 208 - 19 189 - 37 152 - 61 91 - 91	$\sqrt[3]{343} = 7$ 343 - 1 342 - 7 335 - 19 316 -37 279 - 61 218 - 91 127 -127
$\sqrt[3]{^{-}512} = {}^{-}8$ ¯512 + 1 ¯511 + 7 ¯504 + 19 ¯485 + 37 ¯448 + 61 ¯387 + 91 ¯296 + 127 ¯169 + 169	$\sqrt[3]{729} = 9$ 729 - 1 728 - 7 721 - 19 702 - 37 665 - 61 604 - 91 513 -127 386 -169 217 -217	$\sqrt[3]{1000} = 10$ 1000 - 1 999 - 7 992 - 19 973 - 37 936 - 61 875 - 91 784 - 127 657 - 169 488 - 217 271 - 271	$\sqrt[3]{^{-}1331} = {}^{-}11$ ¯1331 + 1 ¯1330 + 7 ¯1323 + 19 ¯1304 + 37 ¯1267 + 61 ¯1206 + 91 ¯1115 + 127 ¯988 + 169 ¯819 + 217 ¯602 + 271 ¯331 + 331

Page 41 Problem 104

No.	Representation	No.	Representation	No.	Representation	No.	Representation
5	$\frac{3+3}{3} + 3$	6	$(3+3)\bullet\frac{3}{3}$	7	$3+3+\frac{3}{3}$	8	$3\bullet 3 - \frac{3}{3}$
9	$\sqrt{3\bullet 3}\bullet\sqrt{3\bullet 3}$	10	$\frac{3^3+3}{3}$	11	$3! + 3! - \frac{3}{3}$	12	$(3! + 3!)\frac{3}{3}$
13	$3! + 3! + \frac{3}{3}$	14	$3!! + \frac{3+3}{3}$	15	$3!! + \frac{3\bullet 3}{3}$	16	$3!! + 3 + \frac{3}{3}$
17	$3\bullet 3! - \frac{3}{3}$	18	$3\bullet 3 + 3\bullet 3$	19	$3\bullet 3! + \frac{3}{3}$	20	$3\bullet 3! + \frac{3!}{3}$
21	$3!! + \frac{3^3}{3}$	22	$3\bullet 3! + \frac{3!!}{3}$	23	$3^3 - \frac{3!!}{3}$	24	$3\bullet 3! + 3 + 3$
25	$3^3 - \frac{3!}{3}$	26	$3^3 - \frac{3}{3}$	27	$3\bullet 3! + 3\bullet 3$		

Page 42 Problem 106

Each odd natural number can be expressed as a difference of the square of two consecutive natural numbers.

Page 43 Problem 108

On the basis of the examples given, you could say that the equality will hold if there is a single digit on either side of the decimal point and the inequality holds when there is more than one digit on a side of the decimal point.

Page 44 Problem 110

On the basis of the examples given, you could say that the equality will hold if there is a single digit on either side of the decimal point and the inequality holds when there is more than one digit on a side of the decimal point. Of course, when all digits are the same, equality holds.

Page 48 Problem 116

Rewrite the subtrahend so that it has as many digits as the minuend. The complement of this subtrahend is a number determined so that the sum of the units digits of the complement and the subtrahend is 10 and so that the sum of the digits in all other corresponding positions of the subtrahend and complement is 9 . If the units digit of the subtrahend is zero, then the units digit of the complement is zero. The sum of the numbers in the tens position must then be 10 . If the units and tens digits of the subtrahend are both zero, then the sum of the numbers in the hundreds position must be 10, etc. .

Page 49 Problem 117

$3\ 2\ 2_{five}$	$3\ 2\ 2_{five}$	$4\ 1\ 2_{five}$	$4\ 1\ 2_{five}$	$3\ 2\ 1_{five}$	$3\ 2\ 1_{five}$
$-1\ 3\ 4_{five}$	$+3\ 1\ 1_{five}$	$-2\ 4\ 3_{five}$	$+2\ 0\ 2_{five}$	$-3\ 2\ 1_{five}$	$+1\ 2\ 4_{five}$
	$\not{1}\ 1\ 3\ 3_{five}$		$\not{1}\ 1\ 1\ 4_{five}$		$\not{1}\ 0\ 0\ 0_{five}$

Page 50 Problem 118

411_{five} -43_{five}	411_{five} $+402_{five}$ ~~1~~313_{five}	324_{five} -20_{five}	324_{five} $+430_{five}$ ~~1~~304_{five}	433_{five} -4_{five}	433_{five} $+411_{five}$ ~~1~~424_{five}
3132_{five} -2433_{five}	3132_{five} $+2012_{five}$ ~~1~~0144_{five}	4213_{five} -2300_{five}	4213_{five} $+2200_{five}$ ~~1~~1413_{five}	3142_{five} -3000_{five}	3142_{five} $+2000_{five}$ ~~1~~0142_{five}
43124_{five} -21444_{five}	43124_{five} $+23001_{five}$ ~~1~~21130_{five}	31242_{five} -3143_{five}	31242_{five} $+41302_{five}$ ~~1~~23044_{five}	34124_{five} -2000_{five}	34124_{five} $+43000_{five}$ ~~1~~32124_{five}

Page 51 Problem 119

101_{two} -100_{two}	101_{two} $+011_{two}$ 1000_{two} $+1$ ~~1~~001_{two}	101_{two} -10_{two}	101_{two} $+101_{two}$ 1010_{two} $+1$ ~~1~~011_{two}	111_{two} -1_{two}	111_{two} $+110_{two}$ 1101_{two} $+1$ ~~1~~110_{two}

Page 52 Problem 120

1110_{two} -1010_{two}	1110_{two} $+0101_{two}$ 10011_{two} $+1$ ~~1~~00100_{two}	1010_{two} -110_{two}	1010_{two} $+1001_{two}$ 10011_{two} $+1$ ~~1~~0100_{two}	1001_{two} -11_{two}	1001_{two} $+1100_{two}$ 10101_{two} $+1$ ~~1~~0110_{two}
1111_{two} -100_{two}	1111_{two} $+1011_{two}$ 11010_{two} $+1$ ~~1~~1011_{two}	1100_{two} -11_{two}	1100_{two} $+1100_{two}$ 11000_{two} $+1$ ~~1~~1001_{two}	1001_{two} -1_{two}	1001_{two} $+1110_{two}$ 10111_{two} $+1$ ~~1~~1000_{two}
11011_{two} -10010_{two}	11011_{two} $+01101_{two}$ 101000_{two} $+1$ ~~1~~01001_{two}	11100_{two} -10000_{two}	11100_{two} $+01111_{two}$ 101011_{two} $+1$ ~~1~~01100_{two}	11101_{two} -11101_{two}	11101_{two} $+00010_{two}$ 11111_{two} $+1$ ~~1~~00000_{two}

Page 54 Problem 121

Roots: $x_1 = \frac{1 + \sqrt{5}}{2}$ $x_2 = \frac{1 - \sqrt{5}}{2}$

Page 54 Problem 122

Positive root: 1.61803399 Negative root: $^{-}.61803399$

Page 55 Problem 123

Length of line segment AB	Distance of Golden mean C from A	Length of CB	Summary
AB = 5 in	3.09 in	1.91 in	A C B; 3.09 in 1.91 in
AB = 10 in	6.18 in	3.82 in	A C B; 6.18 in 3.82 in
AB = 12 in	7.416 in	4.584 in	A C B; 7.416 in 4.584 in

Page 56 Problem 125

a. AC = 1.854 in

b. CB = 1,146 in

d. $\frac{AC}{CB}$ = 1.6178

e. Yes, approximately a Golden rectangle.

g. $\frac{AB}{AC}$ = 1.6181

h. Yes, approximately a Golden rectangle.

a. AC = 1.236 in

b. CB = .764 in

d. $\frac{AC}{CB}$ = 1.6178

e. Yes, approximately a Golden rectangle.

g. $\frac{AB}{AC}$ = 1.6181

h. Yes, approximately a Golden rectangle.

Page 57 Problem 127

Ratio of successive Fibonacci numbers	Number of places ratio matches Golden ratio 1.618 033 989 ...	Ratio of successive Fibonacci numbers	Number of places ratio matches Golden ratio 1.618 033 989 ..
$\frac{F_9}{F_8}$ = 1.619047619	3	$\frac{F_{18}}{F_{17}}$ = 1.618033813	7
$\frac{F_{12}}{F_{11}}$ = 1.617977528	3	$\frac{F_{20}}{F_{19}}$ = 1.618033963	8
$\frac{F_{15}}{F_{14}}$ = 1.618037135	6	$\frac{F_{25}}{F_{24}}$ = 1.618033988	9

Page 58 Problem 128

a	b	c	d	x = ad	y = 2bc	z = cd - ab	$z^2 = x^2 + y^2$
2	3	5	8	16	30	34	1,156 = 900 + 256
3	5	8	13	39	80	89	7,921 = 6,400 + 1,521
5	8	13	21	105	208	233	54,289 = 43,264 + 11,025
8	13	21	34	272	546	610	372,100 = 298,116 + 73,984
13	21	34	55	715	1,428	1,597	2,550,409 = 2,039,184 + 511,225
21	34	55	89	1,869	3,740	4,181	17,480,761 = 13,987,600 + 3,493,161

Page 59 Problem 129

Sides	Name											Formula
4	Square	1	4	9	16	25	36	49	64	81	100	n^2
5	Pentagonal	1	5	12	22	35	51	70	92	117	145	$\frac{n}{2}(3n-1)$
6	Hexagonal	1	6	15	28	45	66	91	120	153	190	$n(2n-1)$
7	Heptagonal	1	7	18	34	55	81	112	148	189	235	$\frac{n}{2}(5n-3)$
8	Octagonal	1	8	21	40	65	96	133	176	225	280	$n(3n-2)$
9	Nonagonal	1	9	24	46	75	111	154	204	261	325	$\frac{n}{2}(7n-5)$
10	Decagonal	1	10	27	52	85	126	175	232	297	370	$n(4n-3)$
11	Undecagonal	1	11	30	58	95	141	196	260	333	415	$\frac{n}{2}(9n-7)$
12	Dodecagonal	1	12	33	64	105	156	217	288	369	460	$n(5n-4)$
13	Tridecagonal	1	13	36	70	115	171	238	316	405	505	$\frac{n}{2}(11n-9)$
14	Tetradecagonal	1	14	39	76	125	186	259	344	441	550	$n(6n-5)$
15	Pentadecagonal	1	15	42	82	135	201	280	372	477	595	$\frac{n}{2}(13n-11)$
16	Hexadecagonal	1	16	45	88	145	216	301	400	513	640	$n(7n-6)$
17	Heptadecagonal	1	17	48	94	155	231	322	428	549	685	$\frac{n}{2}(15n-13)$
18	Octadecagonal	1	18	51	100	165	246	343	456	585	730	$n(8n-7)$
19	Nondecagonal	1	19	54	106	175	261	364	484	621	775	$\frac{n}{2}(17n-15)$
20	Icosagonal	1	20	57	112	185	276	385	512	657	820	$n(9n-8)$

Page 60 Problem 130

Yes, every prime number except 2, can be expressed in the form $2n + 1$, where n is some natural number.

Page 60 Problem 131

No, every number of the form $2n + 1$ where $n = 1, 2, 3, \ldots$ is not a prime number. For example, $2 \cdot 4 + 1 = 8 + 1 = 9$ is not a prime.

Page 60 Problem 132

b. Except for the prime number 3, columns 1 and 2 will contain the other prime numbers.

c. Every prime number except 3 can be expressed in the form $3n + 1$ or $3n + 2$, n = some natural number.

d. No, every number of the forms given in (c) is not a prime number. For example, $3 \cdot 3 + 1 = 9 + 1 = 10$ is not a prime, $3 \cdot 2 + 2 = 6 + 2 = 8$ is not a prime.

Page 60 Problem 133

b. Except for the prime number 2, columns 1 and 3 will contain the other prime numbers.

c. Every prime number except 2 can be expressed in the form $4n + 1$ or $4n + 3$, n = some natural number.

d. No, every number of the forms given in (c) is not a prime number. For example, $4 \cdot 2 + 1 = 8 + 1 = 9$ is not a prime, $4 \cdot 3 + 3 = 12 + 3 = 15$ is not a prime.

Page 61 Problem 134

b. Except for the prime number 5, columns 1, 2, 3, 4 will contain the other prime numbers.

c. Every prime number except 5 can be expressed in the forms: $5n + 1$, $5n + 2$, $5n + 3$, $5n + 4$, n = some natural number.

d. No, every number that is of the forms given in (c) is not a prime number. For example, $5 \cdot 3 + 1 = 15 + 1 = 16$ is not a prime, $5 \cdot 2 + 2 = 10 + 2 = 12$ is not a prime number, $5 \cdot 1 + 3 = 5 + 3 = 8$ is not a prime, $5 \cdot 1 + 4 = 5 + 4 = 9$ is not a prime.

Page 61 Problem 135

Except for the prime numbers 2 and 3, every prime number can be expressed in the form $6n + 1$ or $6n + 5$, where n = some natural number.

Page 61 Problem 135

No, every number of the forms $6n + 1$ or $6n + 5$ is not a prime number. For example, $6 \cdot 4 + 1 = 24 + 1 = 25$ is not a prime, $6 \cdot 5 + 5 = 30 + 5 = 35$ is not a prime.

Page 61 Problem 137

b. Except for the prime number 7, columns 1, 2, 3, 4, 5, 6 will contain the other prime numbers.

c. Every prime number except 7 can be expressed in the forms $7n + 1$, $7n + 2$, $7n + 3$, $7n + 4$, $7n + 5$, $7n + 6$, n = some natural number.

d. No, every number that is of the forms given in (c) is not a prime number. For example,
$7 \cdot 0 + 1 = 0 + 1 = 1$ is not a prime, $7 \cdot 1 + 2 = 7 + 2 = 9$ is not a prime,
$7 \cdot 1 + 3 = 7 + 3 = 10$ is not a prime, $7 \cdot 0 + 4 = 0 + 4 = 4$ is not a prime,
$7 \cdot 1 + 5 = 7 + 5 = 12$ is not a prime, $7 \cdot 0 + 6 = 0 + 6 = 6$ is not a prime.

Page 63 Comment on Text

The powers of five can be found in each row of the two tree to the right of the decimal point. Note that

$$\frac{1}{2^n} \cdot 10^n = 5^n .$$

Page 63 Problem 138

The secret is $1 + 2 + \ldots + 2^{n-1} = 2^n - 1$. The formula can be verified by means of the geometric series.

Page 64 Problem 140

a. Simon Degree's offer is better since
$1 + 2 + 4 + \ldots + 2^{30} = 2^{31} - 1 =$ 2,147,483,647 cents = 21,474,836.47 dollars .

b. D. Lemma would get \$ 20,474,836.47 more from Simon Degree.

Page 66 Problem 144

p	$2^p - 1$	Prime Yes	Prime No
3	$2^3 - 1 = 8 - 1 = 7$	✓	
5	$2^5 - 1 = 32 - 1 = 31$	✓	
7	$2^7 - 1 = 128 - 1 = 127$	✓	
11	$2^{11} - 1 = 2,048 - 1 = 2,047$		✓
13	$2^{13} - 1 = 8,192 - 1 = 8,191$	✓	

Page 67 Problem 145

n	Euclid Perfect number = $2^{n-1}(2^n - 1)$
3	$2^2(2^3 - 1) = 4 \cdot 7 = 28$
5	$2^4(2^5 - 1) = 16 \cdot 31 = 496$
7	$2^6(2^7 - 1) = 64 \cdot 127 = 8,128$
13	$2^{12}(2^{13} - 1) = 4,096 \cdot 8,191 = 33,550,336$
17	$2^{16}(2^{17} - 1) = 65,536 \cdot 131,071 = 8,589,869,056$
19	$2^{18}(2^{19} - 1) = 262,144 \cdot 524,287 = 137,438,691,328$
31	$2^{30}(2^{31} - 1) = 1,073,744,824 \cdot 2,147,483,647 = 2,305,843,008,139,952,128$

Page 67 Problem 146

Perfect number from chart above	Sum of cubes of consecutive odd positive integers
496	$1^3 + 3^3 + 5^3 + 7^3 = 1 + 27 + 125 + 343$
8,128	$1^3 + 3^3 + 5^3 + 7^3 + 11^3 + 13^3 + 15^3 = 1 + 27 + 125 + 343 + 729 + 1,331 + 2,197 + 3,375$

Page 68 Problem 147

n	$2^{2^n} + 1$	Prime Yes	Prime No
1	$2^{2^1} + 1 = 2^2 + 1 = 5$	✓	
2	$2^{2^2} + 1 = 2^4 + 1 = 16 + 1 = 17$	✓	
3	$2^{2^3} + 1 = 2^8 + 1 = 256 + 1 = 257$	✓	
4	$2^{2^4} + 1 = 2^{16} + 1 = 65,536 + 1 = 65,537$	✓	

Page 68 Problem 148

n	p	$n^{p-1} - 1$	$\frac{n^{p-1} - 1}{p}$
2	5	$2^{5-1} - 1 = 2^4 - 1 = 16 - 1 = 15$	$\frac{15}{5} = 3$
2	7	$2^{7-1} - 1 = 2^6 - 1 = 64 - 1 = 63$	$\frac{63}{7} = 9$
2	11	$2^{11-1} - 1 = 2^{10} - 1 = 1,024 - 1 = 1,023$	$\frac{1,023}{11} = 93$
2	13	$2^{13-1} - 1 = 2^{12} - 1 = 4,096 - 1 = 4,095$	$\frac{4,095}{13} = 315$
2	17	$2^{17-1} - 1 = 2^{16} - 1 = 65,536 - 1 = 65,535$	$\frac{65,535}{17} = 3,855$

Page 69 Problem 153

You can not distribute the set $\{1,2,3,4,5,6,7,8,9\}$ into 2 groups according to the given rules since

Group 1	Group 2
1	3
2	5
4	6
8	7

is the only way of arranging 1 to 8 in two groups. Note that it is not possible to add 9 to either group.

Page 70 Problem 163

$\{1,2,3,\ldots,15\}$

Group 1	Group 2	Group 3
1	3	9
2	5	10
4	6	11
8	7	12
	14	13
	15	

Problem 164

$\{1,2,3,\ldots,16\}$

Group 1	Group 2	Group 3
1	3	7
2	5	9
4	6	11
8	10	13
16	12	15
	14	

Problem 165

$\{1,2,3,\ldots,17\}$

Group 1	Group 2	Group 3
1	3	9
2	5	10
4	6	12
8	7	13
11		14
16		15
		17

Page 71 Problem 166

$\{1,2,3,\ldots,18\}$

Group 1	Group 2	Group 3
1	3	9
2	5	10
4	6	12
8	7	13
11		14
16		15
		17
		18

Problem 167

$\{1,2,3,\ldots,19\}$

Group 1	Group 2	Group 3
1	3	9
2	5	10
4	6	12
8	7	13
11	19	14
16		15
		17
		18

Problem 168

$\{1,2,3,\ldots,20\}$

Group 1	Group 2	Group 3
1	3	9
2	5	10
4	6	12
8	7	13
11	19	14
16		15
		17
		18
		20

Page 71 Problem 169

$\{1,2,3,\ldots,21\}$

Group 1	Group 2	Group 3
1	3	9
2	5	10
4	6	12
8	7	13
11	19	14
16	21	15
		17
		18
		20

Problem 170

$\{1,2,3,\ldots,22\}$

Group 1	Group 2	Group 3
1	3	9
2	5	10
4	6	12
8	7	13
11	19	14
16	21	15
22		17
		18
		20

Problem 171

$\{1,2,3,\ldots,23\}$

Group 1	Group 2	Group 3
1	3	9
2	5	10
4	6	12
8	7	13
11	19	14
16	21	15
22	23	17
		18
		20

Page 71 Problem 172

It appears to be impossible to distribute the set $\{1,2,3,\ldots,24\}$ into 3 groups according to the given rules.

Page 72 Problem 173

There will be 2 numbers in the chain for a prime number.

Page 72 Problem 174

Number	Chain length	Chain
10	4	10,8,7,1
12	7	12,16,15,9,4,3,1
14	5	14,10,8,7,1
15	5	15,9,4,3,1
16	6	16,15,9,4,3,1
18	4	18,21,11,1
20	7	20,22,14,10,8,7,1

Page 72 Problem 175

Number	Chain length	Chain
38	7	38,22,14,10,8,7,1
45	7	45,33,15,9,4,3,1
46	8	46,26,16,15,9,4,3,1
52	9	52,46,26,16,15,9,4,3,1
62	9	62,34,20,22,14,10,8,7,1
80	7	80,106,56,64,63,41,1
84	6	84,140,196,203,37,1
86	9	86,46,26,16,15,9,4,3,1
90	10	90,144,259,45,33,15,9,4,3,1

Page 72 Problem 176

Number	Chain length	Chain
30	15	30,42,54,66,78,90,144,259,45,33,15,9,4,3,1
42	14	42,54,66,78,90,144,259,45,33,15,9,4,3,1
54	13	54,66,78,90,144,259,45,33,15,9,4,3,1
60	11	60,108,172,136,134,70,74,40,50,43,1
66	12	66,78,90,144,259,45,33,15,9,4,3,1
78	11	78,90,144,259,45,33,15,9,4,3,1
95	infinite	95,25,6,6,6,. . .
102	18	102,114,126,186,198,270,450,759,393,135,105,87,33,15,9,4,3,1
114	17	114,126,186,198,270,450,759,393,135,105,87,33,15,9,4,3,1

Page 72 Problem 177

The chain length for 284 is infinite. 284,220,284,220,...

Page 72 Problem 178

The chain length for 138 is apparently infinite.

Page 72 Problem 179

The chain length for 276 is apparently infinite.

With the use of a computer a chain length of 100 for 138 and 276 has been tabulated. In each case, the numbers in the chains are increasing in size with the length, hence indicating an infinite length.

Page 74 **Problem 182**

Prime	4n + 1	Sum of squares
17	4•4 + 1	$4^2 + 1^2 = 17$
29	4•7 + 1	$5^2 + 2^2 = 29$
37	4•9 + 1	$6^2 + 1^2 = 37$
41	4•10 + 1	$5^2 + 4^2 = 41$
53	4•13 + 1	$7^2 + 2^2 = 53$
61	4•15 + 1	$6^2 + 5^2 = 61$
73	4•18 + 1	$8^2 + 3^2 = 73$
89	4•22 + 1	$8^2 + 5^2 = 89$
97	4•24 + 1	$9^2 + 4^2 = 97$
101	4•25 + 1	$10^2 + 1^2 = 101$
109	4•27 + 1	$10^2 + 3^2 = 109$
113	4•28 + 1	$8^2 + 7^2 = 113$
137	4•34 + 1	$11^2 + 4^2 = 137$

Problem 183

Many answers are possible.

Prime	4n + 1	Sum of squares
149	4•37 + 1	$10^2 + 7^2 = 149$
157	4•39 + 1	$11^2 + 6^2 = 157$
173	4•43 + 1	$13^2 + 2^2 = 173$
181	4•45 + 1	$10^2 + 9^2 = 181$
193	4•48 + 1	$12^2 + 7^2 = 193$
197	4•49 + 1	$14^2 + 1^2 = 197$
229	4•57 + 1	$15^2 + 2^2 = 229$
233	4•58 + 1	$13^2 + 8^2 = 233$
241	4•60 + 1	$15^2 + 4^2 = 241$
257	4•64 + 1	$16^2 + 1^2 = 257$
269	4•67 + 1	$13^2 + 10^2 = 269$
277	4•69 + 1	$14^2 + 9^2 = 277$
281	4•70 + 1	$16^2 + 5^2 = 281$
293	4•73 + 1	$17^2 + 2^2 = 293$

Page 75 **Problem 184**

Consecutive primes		Sum	Square
P_3	5	= 10	
	+		← $4^2 = 16$
P_4	7	= 17	
	+		← $5^2 = 25$
P_5	11	= 28	
	+		← $6^2 = 36$
P_6	13	= 41	
	+		← $7^2 = 49$
P_7	17	= 58	
	+		← $8^2 = 64$
P_8	19	= 77	
	+		← $9^2 = 81$
P_9	23	= 100	
	+		← $10^2 = 100$ or $11^2 = 121$
P_{10}	29	= 129	
	+		← $12^2 = 144$
P_{11}	31	= 160	
	+		← $13^2 = 169$ or $14^2 = 196$
P_{12}	37	= 197	
	+		← $15^2 = 225$
P_{13}	41	= 238	

Problem 185

Consecutive primes		Sum	Square
P_{13}	41	= 238	
	+		← $16^2 = 256$
P_{14}	43	= 281	
	+		← $17^2 = 289$ or $18^2 = 324$
P_{15}	47	= 328	
	+		← $19^2 = 361$
P_{16}	53	= 381	
	+		← $20^2 = 400$
P_{17}	59	= 440	
	+		← $21^2 = 441$ or $22^2 = 484$
P_{18}	61	= 501	
	+		← $23^2 = 529$
P_{19}	67	= 568	
	+		← $24^2 = 576$ or $25^2 = 625$
P_{20}	71	= 639	
	+		← $26^2 = 676$
P_{21}	73	= 712	
	+		← $27^2 = 729$ or $28^2 = 784$

P_{22}	79	=	791	
	+		←	$29^2 = 841$
P_{23}	83	=	874	$30^2 = 900$
	+		← or	$31^2 = 961$
P_{24}	89	=	963	
	+		←	$32^2 = 1,024$
P_{25}	97	=	1,060	

Page 76 Problem 186

	Primes - Yes	No
$4^2 + 3^2 = 25$ $4^2 - 3^2 = 7$		✓
$5^2 + 4^2 = 41$ $5^2 - 4^2 = 9$		✓
$6^2 + 5^2 = 61$ $6^2 - 5^2 = 11$	✓	
$8^2 + 7^2 = 113$ $8^2 - 7^2 = 15$		✓
$10^2 + 9^2 = 181$ $10^2 - 9^2 = 19$	✓	
$13^2 + 12^2 = 313$ $13^2 - 12^2 = 25$		✓
$15^2 + 14^2 = 421$ $15^2 - 14^2 = 29$	✓	
$19^2 + 18^2 = 685$ $19^2 - 18^2 = 37$		✓
$30^2 + 29^2 = 1,741$ $30^2 - 29^2 = 59$	✓	

	Primes - Yes	No
$31^2 + 30^2 = 1,861$ $31^2 - 30^2 = 61$	✓	
$36^2 + 35^2 = 2,521$ $36^2 - 35^2 = 71$	✓	
$8^2 + 5^2 = 89$ $8^2 - 5^2 = 39$		✓
$14^2 + 10^2 = 296$ $14^2 - 10^2 = 96$		✓
$15^2 + 4^2 = 241$ $15^2 - 4^2 = 209$		✓
$16^2 + 1^2 = 257$ $16^2 - 1^2 = 255$		✓
$20^2 + 10^2 = 500$ $20^2 - 10^2 = 300$		✓
$25^2 + 0^2 = 625$ $25^2 - 0^2 = 625$		✓

Page 76 Problem 187

True. As shown above, for some consecutive natural numbers, the sum and difference of the squares are prime numbers.

Page 76 Problem 188

False. The difference $a^2 - b^2$ of any nonconsecutive natural numbers a, b is factorable and hence nonprime.

Page 78 **Problem 189**

Number	Divisible by
484	2, 4, 11
1331	11
270	2, 3, 5, 6, 9, 10
343	7
518	2, 7
612	2, 3, 4, 6, 9
924	2, 3, 4, 6, 7, 11
250	2, 5, 10
665	5, 7
880	2, 4, 5, 8, 10, 11
195	3, 5
247	none
990	2, 3, 5, 6, 9, 10, 11
792	2, 3, 4, 6, 9, 11

Problem 190

Number	Divisible by
746	2
445	5
623	7
3003	3, 7, 11
760	2, 4, 5, 8, 10
770	2, 5, 7, 10, 11
597	3
2310	2, 3, 5, 6, 7, 10, 11
273	3, 7
540	2, 3, 4, 5, 6, 9, 10
810	2, 3, 5, 6, 9, 10
2520	2, 3, 4, 5, 6, 7, 8, 9, 10
168	2, 3, 4, 6, 7, 8
126	2, 3, 6, 7, 9
27720	2, 3, 4, 5, 6, 7, 8, 9, 10, 11

Page 82 **Problem 195**

$2^3 \cdot 3^1 = 24$

$2^0 = 1$	$2^1 \cdot 3^1$
2^1	2^3
3^1	$2^2 \cdot 3^1$
2^2	$2^3 \cdot 3^1$

$2^3 \cdot 3^2 = 72$

$2^0 = 1$	2^3
2^1	$2^2 \cdot 3^1$
3^1	$2^1 \cdot 3^2$
2^2	$2^3 \cdot 3^1$
3^2	$2^2 \cdot 3^2$
$2^1 \cdot 3^1$	$2^3 \cdot 3^2$

$2^3 \cdot 3^3 = 216$

$2^0 = 1$	$2^1 \cdot 3^2$
2^1	3^3
3^1	$2^3 \cdot 3^1$
2^2	$2^2 \cdot 3^2$
3^2	$2^1 \cdot 3^3$
$2^1 \cdot 3^1$	$2^2 \cdot 3^3$
2^3	$2^3 \cdot 3^2$
$2^2 \cdot 3^1$	$2^3 \cdot 3^3$

$2^1 \cdot 3^1 \cdot 5^1 = 30$

$2^0 = 1$
2^1
3^1
5^1
$2^1 \cdot 3^1$
$2^1 \cdot 5^1$
$3^1 \cdot 5^1$
$2^1 \cdot 3^1 \cdot 5^1$

$2^2 \cdot 3^1 \cdot 5^1 = 60$

$2^0 = 1$	$2^1 \cdot 5^1$
2^1	$3^1 \cdot 5^1$
3^1	$2^1 \cdot 3^1 \cdot 5^1$
5^1	$2^2 \cdot 3^1$
2^2	$2^2 \cdot 5^1$
$2^1 \cdot 3^1$	$2^2 \cdot 3^1 \cdot 5^1$

$2^2 \cdot 3^2 \cdot 5^1 = 180$

$2^0 = 1$	$2^1 \cdot 3^1$	$3^2 \cdot 5^1$
2^1	$2^1 \cdot 5^1$	$2^1 \cdot 3^1 \cdot 5^1$
3^1	$3^1 \cdot 5^1$	$2^2 \cdot 3^2$
5^1	$2^2 \cdot 3^1$	$2^2 \cdot 3^1 \cdot 5^1$
2^2	$2^2 \cdot 5^1$	$2^1 \cdot 3^2 \cdot 5^1$
3^2	$2^1 \cdot 3^2$	$2^2 \cdot 3^2 \cdot 5^1$

Page 82 Problem 195

$2^2 \cdot 3^2 \cdot 5^2 = 900$

$2^0 = 1$	$3 \cdot 5$	$2^2 \cdot 5^2$
2	$2^2 \cdot 3$	$3^2 \cdot 5^2$
3	$2^2 \cdot 5$	$2^2 \cdot 3 \cdot 5$
5	$2 \cdot 3^2$	$2 \cdot 3^2 \cdot 5$
2^2	$3^2 \cdot 5$	$2 \cdot 3 \cdot 5^2$
3^2	$2 \cdot 5^2$	$2^2 \cdot 3^2 \cdot 5$
5^2	$3 \cdot 5^2$	$2^2 \cdot 3 \cdot 5^2$
$2 \cdot 3$	$2 \cdot 3 \cdot 5$	$2 \cdot 3^2 \cdot 5^2$
$2 \cdot 5$	$2^2 \cdot 3^2$	$2^2 \cdot 3^2 \cdot 5^2$

$2^3 \cdot 3 \cdot 5 = 120$

$2^0 = 1$	2^3
2	$2^2 \cdot 3$
3	$2^2 \cdot 5$
5	$2 \cdot 3 \cdot 5$
2^2	$2^3 \cdot 3$
$2 \cdot 3$	$2^3 \cdot 5$
$2 \cdot 5$	$2^2 \cdot 3 \cdot 5$
$3 \cdot 5$	$2^3 \cdot 3 \cdot 5$

$2^3 \cdot 3^2 \cdot 5 = 360$

$2^0 = 1$	$3 \cdot 5$	$2^3 \cdot 5$
2	2^3	$2^2 \cdot 3^2$
3	$2^2 \cdot 3$	$2^2 \cdot 3 \cdot 5$
5	$2^2 \cdot 5$	$2 \cdot 3^2 \cdot 5$
2^2	$2 \cdot 3^2$	$2^3 \cdot 3^2$
3^2	$3^2 \cdot 5$	$2^3 \cdot 3 \cdot 5$
$2 \cdot 3$	$2 \cdot 3 \cdot 5$	$2^2 \cdot 3^2 \cdot 5$
$2 \cdot 5$	$2^3 \cdot 3$	$2^3 \cdot 3^2 \cdot 5$

$2^3 \cdot 3^2 \cdot 5^2 = 1800$

$2^0 = 1$	$3 \cdot 5$	$2^3 \cdot 3$	$2^3 \cdot 5^2$
2	2^3	$2^3 \cdot 5$	$2^3 \cdot 3 \cdot 5$
3	$2^2 \cdot 3$	$2^2 \cdot 3^2$	$2^2 \cdot 3^2 \cdot 5$
5	$2^2 \cdot 5$	$2^2 \cdot 3 \cdot 5$	$2^2 \cdot 3 \cdot 5^2$
2^2	$2 \cdot 3^2$	$3^2 \cdot 5^2$	$2 \cdot 3^2 \cdot 5^2$
3^2	$3^2 \cdot 5$	$2 \cdot 3^2 \cdot 5$	$2^3 \cdot 3^2 \cdot 5$
5^2	$2 \cdot 5^2$	$2^2 \cdot 5^2$	$2^3 \cdot 3 \cdot 5^2$
$2 \cdot 3$	$3 \cdot 5^2$	$2 \cdot 3 \cdot 5^2$	$2^2 \cdot 3^2 \cdot 5^2$
$2 \cdot 5$	$2 \cdot 3 \cdot 5$	$2^3 \cdot 3^2$	$2^3 \cdot 3^2 \cdot 5^2$

$2^3 \cdot 3^3 \cdot 5 = 1080$

$2^0 = 1$	$3 \cdot 5$	$2^3 \cdot 3$	$2^3 \cdot 3 \cdot 5$
2	2^3	$2^3 \cdot 5$	$2^2 \cdot 3^3$
3	3^3	$2^2 \cdot 3^2$	$2^2 \cdot 3^2 \cdot 5$
5	$2^2 \cdot 3$	$2^2 \cdot 3 \cdot 5$	$2 \cdot 3^3 \cdot 5$
2^2	$2^2 \cdot 5$	$2 \cdot 3^3$	$2^3 \cdot 3^3$
3^2	$2 \cdot 3^2$	$3^3 \cdot 5$	$2^3 \cdot 3^2 \cdot 5$
$2 \cdot 3$	$3^2 \cdot 5$	$2 \cdot 3^2 \cdot 5$	$2^2 \cdot 3^3 \cdot 5$
$2 \cdot 5$	$2 \cdot 3 \cdot 5$	$2^3 \cdot 3^2$	$2^3 \cdot 3^3 \cdot 5$

$2^3 \cdot 3^3 \cdot 5^2 = 5400$

$2^0 = 1$	$2^2 \cdot 3$	$2^2 \cdot 3 \cdot 5$	$2 \cdot 3^2 \cdot 5^2$
2	$2^2 \cdot 5$	$2 \cdot 3^2 \cdot 5$	$2 \cdot 3^3 \cdot 5$
3	$2 \cdot 3^2$	$2 \cdot 3 \cdot 5^2$	$2^3 \cdot 3^3$
5	$3^2 \cdot 5$	$2^2 \cdot 5^2$	$2^3 \cdot 3^2 \cdot 5$
2^2	$2 \cdot 5^2$	$3^2 \cdot 5^2$	$2^3 \cdot 3 \cdot 5^2$
3^2	$3 \cdot 5^2$	$2^3 \cdot 3^2$	$2^2 \cdot 3^3 \cdot 5$
5^2	$2 \cdot 3 \cdot 5$	$2^3 \cdot 5^2$	$2 \cdot 3^3 \cdot 5^2$
$2 \cdot 3$	$2^3 \cdot 3$	$2^2 \cdot 3^3$	$2^2 \cdot 3^2 \cdot 5^2$
$2 \cdot 5$	$2^3 \cdot 5$	$3^3 \cdot 5^2$	$2^3 \cdot 3^3 \cdot 5$
$3 \cdot 5$	$2 \cdot 3^3$	$2^3 \cdot 3 \cdot 5$	$2^3 \cdot 3^2 \cdot 5^2$
2^3	$3^3 \cdot 5$	$2^2 \cdot 3^2 \cdot 5$	$2^2 \cdot 3^3 \cdot 5^2$
3^3	$2^2 \cdot 3^2$	$2^2 \cdot 3 \cdot 5^2$	$2^3 \cdot 3^3 \cdot 5^2$

$2^3 \cdot 3^3 \cdot 5^3 = 27,000$

$2^0 = 1$	$3^2 \cdot 5$	$2^3 \cdot 3^2$	$2^3 \cdot 3 \cdot 5^2$
2	$2 \cdot 5^2$	$2^2 \cdot 3^3$	$2^2 \cdot 3^3 \cdot 5$
3	$3 \cdot 5^2$	$2^3 \cdot 5^2$	$2 \cdot 3^3 \cdot 5^2$
5	$2 \cdot 3 \cdot 5$	$3^3 \cdot 5^2$	$2^2 \cdot 3 \cdot 5^3$
2^2	$2^3 \cdot 3$	$2^2 \cdot 5^3$	$2 \cdot 3^2 \cdot 5^3$
3^2	$2^3 \cdot 5$	$3^2 \cdot 5^3$	$2^2 \cdot 3^2 \cdot 5^2$
5^2	$2 \cdot 3^3$	$2^3 \cdot 3 \cdot 5$	$2^3 \cdot 3^3 \cdot 5$
$2 \cdot 3$	$3^3 \cdot 5$	$2 \cdot 3^3 \cdot 5$	$2^3 \cdot 3 \cdot 5^3$
$2 \cdot 5$	$2 \cdot 5^3$	$2 \cdot 3 \cdot 5^3$	$2 \cdot 3^3 \cdot 5^3$
$3 \cdot 5$	$3 \cdot 5^3$	$2^2 \cdot 3^2 \cdot 5$	$2^3 \cdot 3^2 \cdot 5^2$
2^3	$2^2 \cdot 3^2$	$2^2 \cdot 3 \cdot 5^2$	$2^2 \cdot 3^3 \cdot 5^2$
3^3	$2^2 \cdot 5^2$	$2 \cdot 3^2 \cdot 5^2$	$2^2 \cdot 3^2 \cdot 5^3$
5^3	$3^2 \cdot 5^2$	$2^3 \cdot 3^3$	$2^3 \cdot 3^3 \cdot 5^2$
$2^2 \cdot 3$	$2^2 \cdot 3 \cdot 5$	$2^3 \cdot 5^3$	$2^3 \cdot 3^2 \cdot 5^3$
$2^2 \cdot 5$	$2 \cdot 3^2 \cdot 5$	$3^3 \cdot 5^3$	$2^2 \cdot 3^3 \cdot 5^3$
$2 \cdot 3^2$	$2 \cdot 3 \cdot 5^2$	$2^3 \cdot 3^2 \cdot 5$	$2^3 \cdot 3^3 \cdot 5^3$

Page 82 **Problem 195**

$2 \cdot 3 \cdot 5 \cdot 7 = 210$

$2^0 = 1$	$3 \cdot 5$
2	$3 \cdot 7$
3	$5 \cdot 7$
5	$2 \cdot 3 \cdot 5$
7	$2 \cdot 3 \cdot 7$
$2 \cdot 3$	$2 \cdot 5 \cdot 7$
$2 \cdot 5$	$3 \cdot 5 \cdot 7$
$2 \cdot 7$	$2 \cdot 3 \cdot 5 \cdot 7$

Page 83 **Problem 196**

N	Prime factorization	Number of exact divisors	Set of exact divisors
54	$2 \cdot 3^3$	8	1, 2, 3, 6, 9, 18, 27, 54
80	$2^4 \cdot 5$	10	1, 2, 4, 5, 8, 10, 16, 20, 40, 80
84	$2^2 \cdot 3 \cdot 7$	12	1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, 84
120	$2^3 \cdot 3 \cdot 5$	16	1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120
294	$2 \cdot 3 \cdot 7^2$	12	1, 2, 3, 6, 7, 14, 21, 42, 49, 98, 147, 294
336	$2^4 \cdot 3 \cdot 7$	20	1, 2, 3, 4, 6, 7, 8, 12, 14, 16, 21, 24, 28, 42, 48, 56, 84, 112, 168, 336
504	$2^3 \cdot 3^2 \cdot 7$	24	1, 2, 3, 4, 6, 7, 8, 9, 12, 14, 18, 21, 24, 28, 36, 42, 56, 63, 72, 84, 126, 168, 252, 504
1200	$2^4 \cdot 3 \cdot 5^2$	30	1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, 25, 30, 40, 48, 50, 60, 75, 80, 100, 120, 150, 200, 240, 300, 400, 600, 1200

Page 83 **Problem 197**

Many answers are possible. For example,

a) $1024 = 2^{10}$ since $10 + 1 = 11$ 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024

b) $144 = 2^4 \cdot 3^2$ since $(4 + 1)(2 + 1) = 15$ 1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 36, 48, 72, 144

c) $180 = 2^2 \cdot 3^2 \cdot 5$ since $(2 + 1)(2 + 1)(1 + 1) = 18$ 1, 2, 3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 30, 36, 45, 60, 90, 180

d) $1296 = 2^4 \cdot 3^4$ since $(4 + 1)(4 + 1) = 25$ 1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, 36, 48, 54, 72, 81, 108, 144, 162, 216, 324, 432, 648, 1296

Page 83 **Problem 198**

Every square of a prime number has exactly 3 exact prime number divisors.

Page 83 **Problem 199**

Every cube of a prime number has exactly 4 exact prime number divisors.

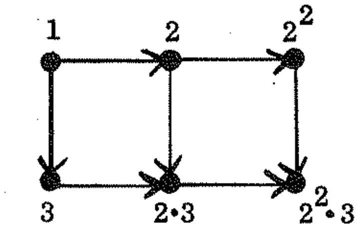
Page 84 **Problem 201**

$N = 6 = 2 \cdot 3$

8 composite numbers with the same lattice pattern as 6 are:

$10 = 2 \cdot 5$	$22 = 2 \cdot 11$
$14 = 2 \cdot 7$	$33 = 3 \cdot 11$
$15 = 3 \cdot 5$	$35 = 5 \cdot 7$
$21 = 3 \cdot 7$	$55 = 5 \cdot 11$

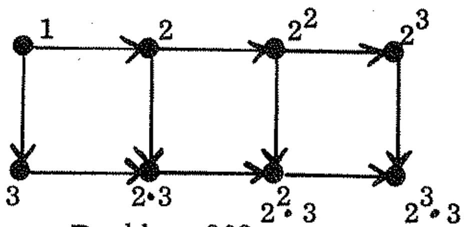
$N = 12 = 2^2 \cdot 3$



8 composite numbers with the same lattice pattern as <u>12</u> are:

$18 = 2 \cdot 3^2$ $50 = 2 \cdot 5^2$
$20 = 2^2 \cdot 5$ $63 = 3^2 \cdot 7$
$28 = 2^2 \cdot 7$ $75 = 3 \cdot 5^2$
$45 = 3^2 \cdot 5$ $175 = 5^2 \cdot 7$

$N = 24 = 2^3 \cdot 3$

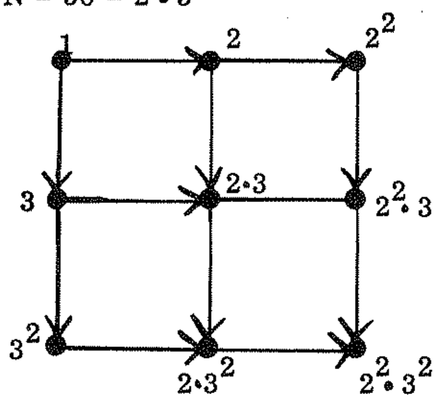


8 composite numbers with the same lattice pattern as <u>24</u> are:

$40 = 2^3 \cdot 5$ $189 = 3^3 \cdot 7$
$54 = 2 \cdot 3^3$ $250 = 2 \cdot 5^3$
$56 = 2^3 \cdot 7$ $375 = 3 \cdot 5^3$
$135 = 3^3 \cdot 5$ $686 = 2 \cdot 7^3$

<u>Page 85</u> <u>Problem 202</u>

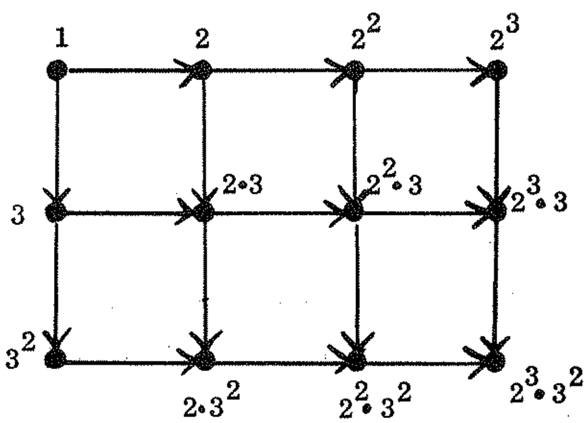
$N = 36 = 2^2 \cdot 3^2$



8 composite numbers with the same lattice pattern as <u>36</u> are:

$100 = 2^2 \cdot 5^2$ $484 = 2^2 \cdot 11^2$
$196 = 2^2 \cdot 7^2$ $1,225 = 5^2 \cdot 7^2$
$225 = 3^2 \cdot 5^2$ $1,089 = 3^2 \cdot 11^2$
$441 = 3^2 \cdot 7^2$ $3,025 = 5^2 \cdot 11^2$

$N = 72 = 2^3 \cdot 3^2$



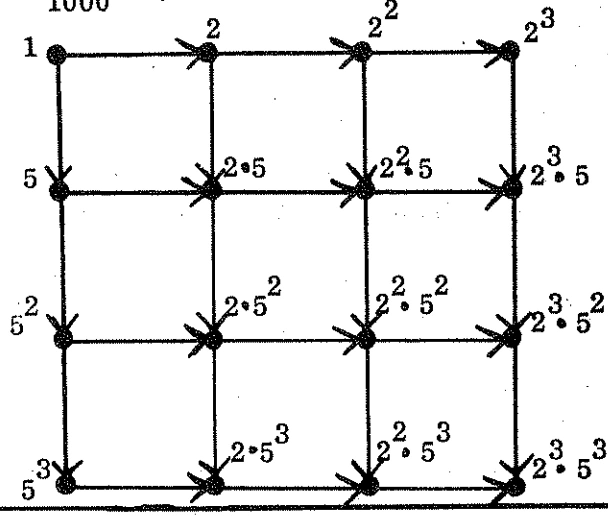
8 composite numbers with the same lattice pattern as <u>72</u> are:

$108 = 2^2 \cdot 3^3$ $675 = 3^3 \cdot 5^2$
$200 = 2^3 \cdot 5^2$ $1,125 = 3^2 \cdot 5^3$
$392 = 2^3 \cdot 7^2$ $1,323 = 3^3 \cdot 7^2$
$500 = 2^2 \cdot 5^3$ $6,125 = 5^3 \cdot 7^2$

Page 85 **Problem 203**

$N = 1000 = 2^3 \cdot 5^3$

$D_{1000} = \{1, 2, 5, 2^2, 5^2, 2\cdot5, 2^3, 5^3, 2^2\cdot5, 2\cdot5^2, 2^3\cdot5, 2\cdot5^3, 2^2\cdot5^2, 2^2\cdot5^3, 2^3\cdot5^2, 2^3\cdot5^3\}$

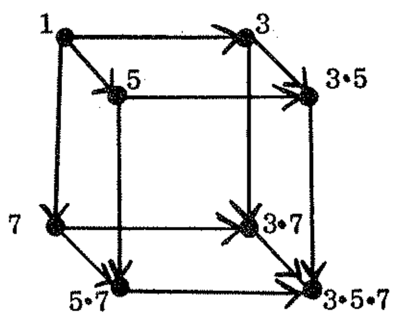


8 composite numbers with the same lattice pattern as <u>1000</u> are:

$216 = 2^3\cdot3^3$	$10,648 = 2^3\cdot11^3$
$2,744 = 2^3\cdot7^3$	$35,937 = 3^3\cdot11^3$
$3,375 = 3^3\cdot5^3$	$42,875 = 5^3\cdot7^3$
$9,261 = 3^3\cdot7^3$	$166,375 = 5^3\cdot11^3$

Page 86 **Problem 204**

$N = 3\cdot5\cdot7$

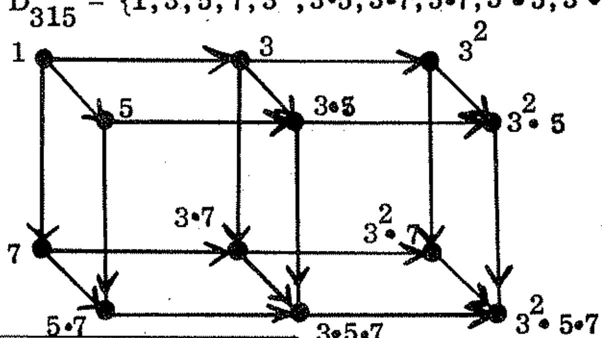


8 composite numbers with the same lattice pattern as <u>105</u> are:

$30 = 2\cdot3\cdot5$	$154 = 2\cdot7\cdot11$
$42 = 2\cdot3\cdot7$	$165 = 3\cdot5\cdot11$
$66 = 2\cdot3\cdot11$	$231 = 3\cdot7\cdot11$
$110 = 2\cdot5\cdot11$	$385 = 5\cdot7\cdot11$

$N = 315 = 3^2\cdot5\cdot7$

$D_{315} = \{1, 3, 5, 7, 3^2, 3\cdot5, 3\cdot7, 5\cdot7, 3^2\cdot5, 3^2\cdot7, 3\cdot5\cdot7, 3^2\cdot5\cdot7\}$

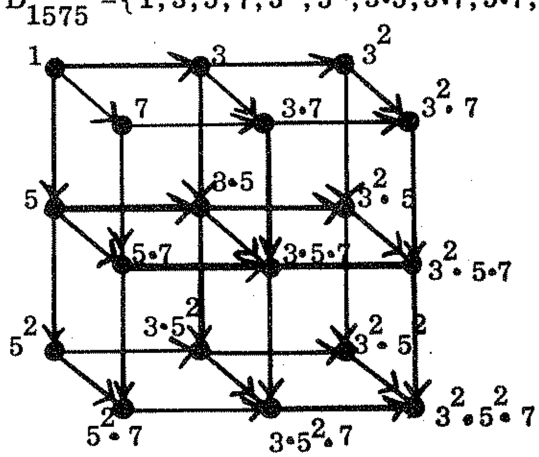


8 composite numbers with the same lattice pattern as <u>315</u> are:

$60 = 2^2\cdot3\cdot5$	$140 = 2^2\cdot5\cdot7$
$84 = 2^2\cdot3\cdot7$	$150 = 2\cdot3\cdot5^2$
$90 = 2\cdot3^2\cdot5$	$294 = 2\cdot3\cdot7^2$
$126 = 2\cdot3^2\cdot7$	$350 = 2\cdot5^2\cdot7$

$N = 1575 = 3^2\cdot5^2\cdot7$

$D_{1575} = \{1, 3, 5, 7, 3^2, 5^2, 3\cdot5, 3\cdot7, 5\cdot7, 3^2\cdot5, 3^2\cdot7, 3\cdot5^2, 5^2\cdot7, 3\cdot5\cdot7, 3^2\cdot5^2, 3^2\cdot5\cdot7, 3\cdot5^2\cdot7, 3^2\cdot5^2\cdot7\}$



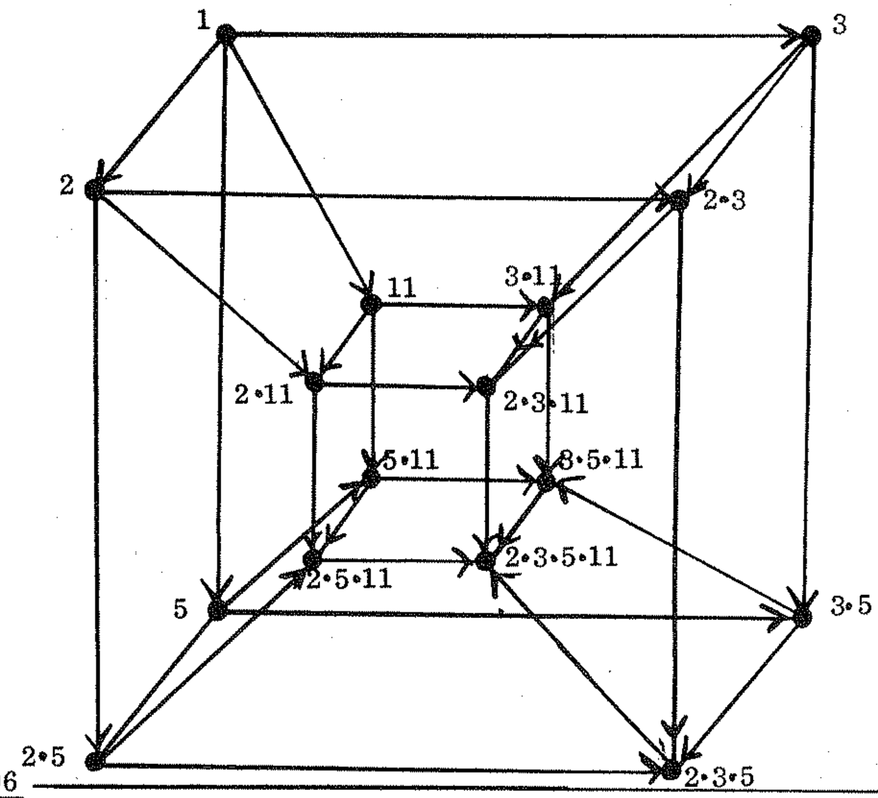
8 composite numbers with the same lattice pattern as <u>1575</u> are:

$180 = 2^2\cdot3^2\cdot5$	$588 = 2^2\cdot3\cdot7^2$
$252 = 2^2\cdot3^2\cdot7$	$700 = 2^2\cdot5^2\cdot7$
$300 = 2^2\cdot3\cdot5^2$	$882 = 2\cdot3^2\cdot7^2$
$450 = 2\cdot3^2\cdot5^2$	$980 = 2^2\cdot5\cdot7^2$

Page 87 Problem 205

$N = 330 = 2\cdot3\cdot5\cdot11$

$D_{330} = \{1, 2, 3, 5, 11, 2\cdot3, 2\cdot5, 2\cdot11, 3\cdot5, 3\cdot11, 5\cdot11, 2\cdot3\cdot5, 2\cdot3\cdot11, 2\cdot5\cdot11, 3\cdot5\cdot11, 2\cdot3\cdot5\cdot11\}$



5 composite numbers with the same lattice pattern as <u>330</u> are:

390 = 2•3•5•13

462 = 2•3•7•11

546 = 2•3•7•13

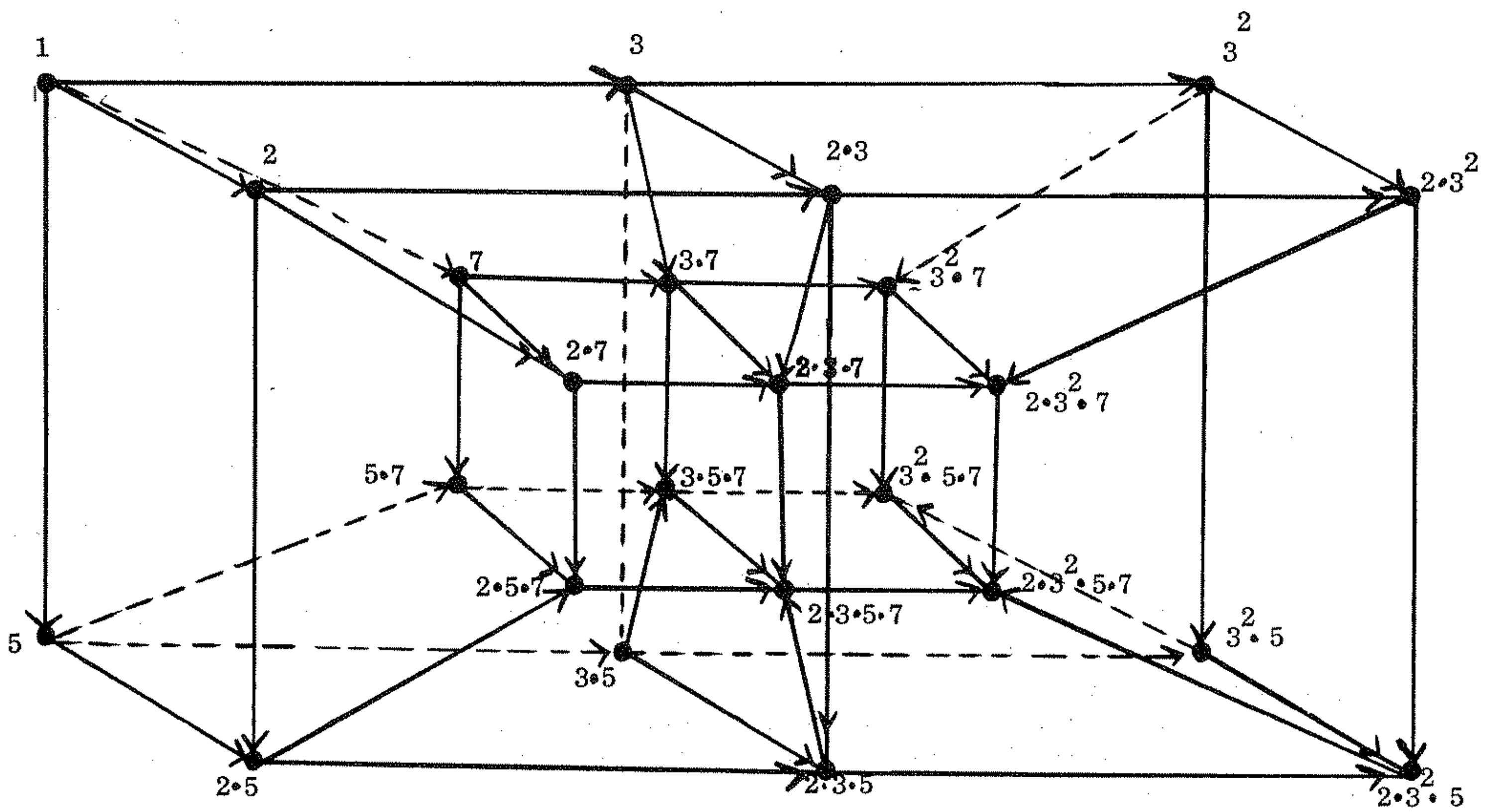
770 = 2•5•7•11

1,155 = 3•5•7•11

Page 88 Problem 206

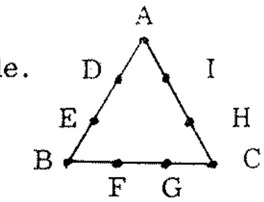
$N = 630 = 2\cdot3^2\cdot5\cdot7$

$D_{630} = \{1, 2, 3, 5, 7, 3^2, 2\cdot3, 2\cdot5, 2\cdot7, 3\cdot5, 3\cdot7, 5\cdot7, 2\cdot3^2, 3^2\cdot5, 3^2\cdot7, 2\cdot3\cdot5, 2\cdot3\cdot7, 2\cdot5\cdot7, 3\cdot5\cdot7, 2\cdot3^2\cdot5,$
$2\cdot3^2\cdot7, 3^2\cdot5\cdot7, 2\cdot3\cdot5\cdot7, 2\cdot3^2\cdot5\cdot7\}$



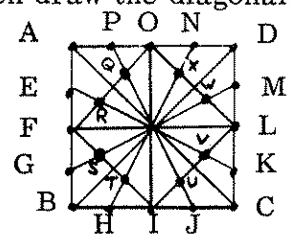
Page 89. Problem 207.

Triangle: 1. Construct an equilateral triangle.
2. Trisect each side.



3. Join D and I, E and F, G and H to form three small equilateral triangles at each vertex.
4. Join D and F, F and H, H and D to form an equilateral triangle in the middle of the original triangle. You should now have four equilateral triangles.
5. Trisect the sides of each and then join as above in steps 3 and 4. Continue this procedure until your figure matches the one given and color where needed.

Square: 1. Construct a square.
2. Divide each side into four equal parts and then draw the diagonals AC and BD. Connect O and I, F and L.

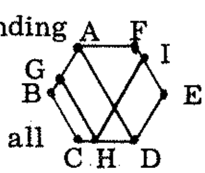


3. Connect O and F, F and I, I and L, L and O.
4. Connect E and K, G and M, H and N, J and P.
5. Connect Q and X and extend until it intersects AC and BD.
Connect R and S and extend until it intersects AC and BD.
Connect T and U and extend until it intersects AC and BD.
Connect V and W and extend until it intersects AC and BD.
6. Connect Q and T, R and W, S and V, U and X. You should now have two smaller squares within square ABCD. Their sides are already bisected by OI and FL.
7. Connect the appropriate points to finish the figure. Color it to match the given figure. You will have to erase some extra lines as a final step.

Pentagon: 1. Construct a regular pentagon.
2. Draw all diagonals. This will produce a smaller pentagon in the middle of the original pentagon.
3. Draw all diagonals of this small pentagon extending them until they intersect the sides of the given pentagon. This will result in five small pentagons at each of the vertices of the given pentagon.
4. Draw all diagonals of each of the small pentagons. This will produce a smaller pentagon in the middle of each of the small pentagons.
5. Draw all diagonals of each of the smallest pentagons and then color appropriately.

Hexagon: 1. Construct a regular hexagon.
2. Divide each side into 9 equal parts.
3. Draw lines parallel to a side of the hexagon connecting corresponding points.

Ex: Connect G and H, A and D, H and I, etc.



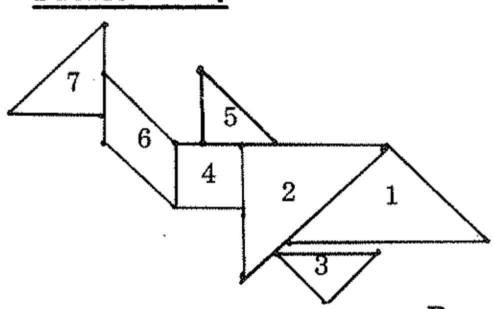
4. Color in the figure appropriately following the picture. You have all the necessary points. You will have to erase some extra lines.

Circle: 1. Draw a circle.
2. Construct 18 equally spaced points on the circle.
3. Draw all possible chords using these 18 points.

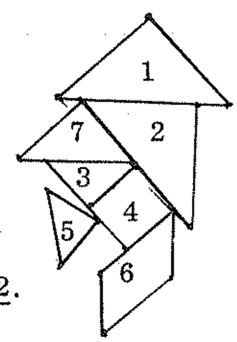
Page 90. Problem 208

Pieces 1, 2	Pieces 3, 5
The inch-measure of $\overline{AH}$ is $\sqrt{2}$. The inch-measure of $\overline{BH}$ is $\sqrt{2}$. The inch-measure of $\overline{AB}$ is 2.	The inch measure of $\overline{DF}$ is 1. The inch-measure of $\overline{FG}$ is $\sqrt{2}/2$. The inch-measure of $\overline{DG}$ is $\sqrt{2}/2$.
Piece 7. The inch-measure of $\overline{FC}$ is 1. The inch-measure of $\overline{CE}$ is 1. The inch-measure of $\overline{FE}$ is $\sqrt{2}$.	**Piece 4** The inch-measure of $\overline{GF}$ is $\sqrt{2}/2$. The inch-measure of $\overline{FJ}$ is $\sqrt{2}/2$. The inch-measure of $\overline{JH}$ is $\sqrt{2}/2$. The inch measure of $\overline{HG}$ is $\sqrt{2}/2$.
Piece 6 The inch measure of $\overline{JE}$ is $\sqrt{2}/2$. The inch-measure of $\overline{IB}$ is $\sqrt{2}/2$. The inch-measure of $\overline{JI}$ is 1. The inch-measure of $\overline{EB}$ is 1.	

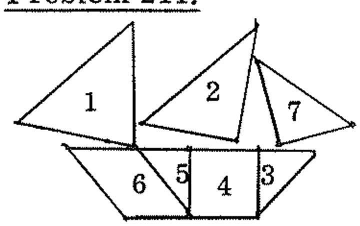
Page 91. Problem 209.



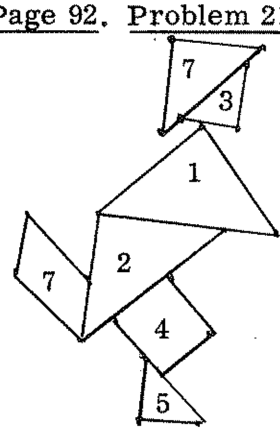
Page 91. Problem 210.



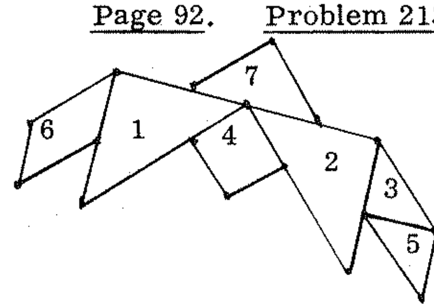
Page 91. Problem 211.



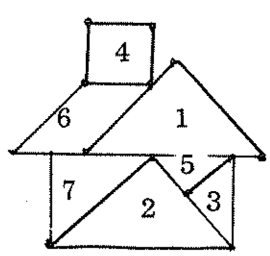
Page 92. Problem 212.



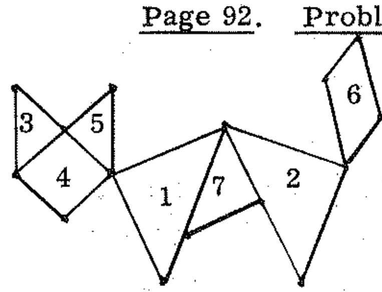
Page 92. Problem 213.



Page 92. Problem 214.



Page 92. Problem 215.



Page 93 Problem 216

Number of unit cubes forming the large cube	Total number of faces	Total number of faces painted	Total number of unpainted faces
$3^3 = 27$	6 x 27 = 162	6 x 9 = 54	162 - 54 = 108
$4^3 = 64$	6 x 64 = 384	6 x 16 = 96	384 - 96 = 288
$5^3 = 125$	6 x 125 = 750	6 x 25 = 150	750 - 150 = 600
$6^3 = 216$	6 x 216 = 1,296	6 x 36 = 216	1,296 - 216 = 1,080
$7^3 = 343$	6 x 343 = 2,058	6 x 49 = 294	2,058 - 294 = 1,764
$8^3 = 512$	6 x 512 = 3,072	6 x 64 = 384	3,072 - 384 = 2,688
$9^3 = 729$	6 x 729 = 4,374	6 x 81 = 486	4,374 - 486 = 3,888
n^3 where $n = 2,3,4,\ldots$	$6 \times n^3$	$6 \times n^2$	$6n^3 - 6n^2 = 6n^2(n - 1)$

Page 95 Problem 217

Points distant from O by	Total number of points
3 units	12
4 units	16
5 units	20
6 units	24
n units where n = 1,2,3,...	4n

Page 96 Problem 218

2. (5 + 4) - (3 - 2) x 1 = 8
3. [(5 + 8) x 17] ÷ (13 x 17) = 1
4. (4 x 4) + (3 ÷ 3) + 2 = 19
5. (1 x 3) + 5 + (7 x 9) = 71
6. [(24 ÷ 8) x (35 ÷ 7)] - 1 = 14
7. [(7 + 9) ÷ (12 ÷ 3)] x 5 = 20
8. [{ (12 + 2) + 10 } ÷ 3] + 7 = 15
9. [(28 + 21) + 14] ÷ (6 + 1) = 9
10. { [(9 + 13) ÷ 11] x 14 } - 3 = 25
11. { [(10 + 8) - 6] ÷ 4 } - 2 = 1
12. [(3 x 6) - 12] + (9 x 2) = 24
13. { [(4 + 8) + 4] ÷ 8 } x 4 = 8
14. [(1 ÷ 10) x 50] - (40 ÷ 10) = 1
15. [(13 - 3) + (15 - 5)] x 1 = 20
16. { (3 + 25) ÷ 10 } x (12 - 2) = 28
17. { (7 - 1) x (11 - 1) } - 13 = 47
18. { 9 + [(11 - 3) - 5] } + 6 = 18