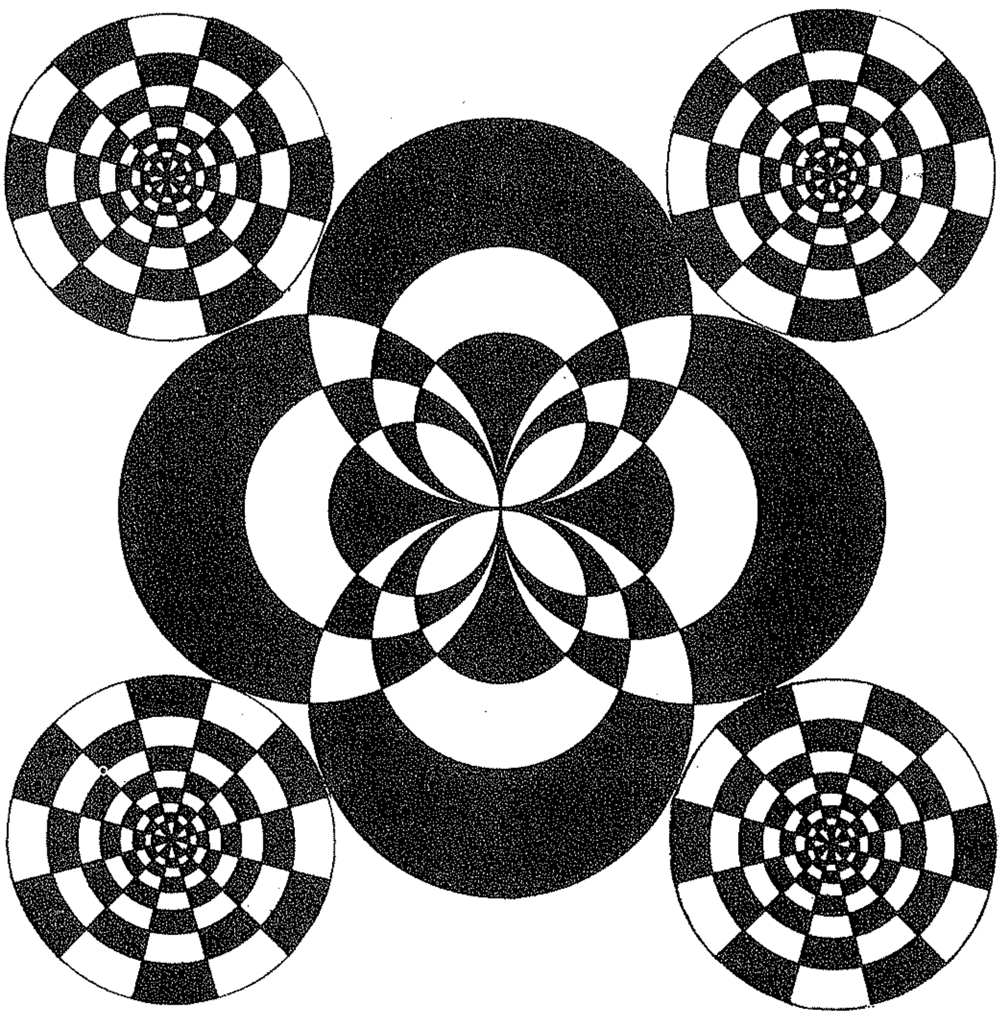
BOSTON COLLEGE MATHEMATICS INSTITUTE

Contemporary Motivated Mathematics



BOOK 3

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PREFACE

CONTEMPORARY MOTIVATED MATHEMATICS is an attempt to achieve the following objectives.

1. MATHEMATICAL SKILLS

   Students should possess some technical proficiency in the fundamental operations of arithmetic and algebra. But drill, often imposed for the sake of drill, has been one of the most unappealing experiences in mathematics. We are convinced that practice in the techniques of mathematics can be meaningful and functional if properly presented.

   Emphasis in the text is on the basic operations of the real number system. Many of the exercises in the book are open ended. They are presented in such a way that all students, at all levels of ability can do some part of the task. More capable students can solve the text problems completely and often extend the exercises into fruitful projects for participation in science fairs.

   In some problems, the computations may appear rather excessive for paper and pencil work. Students are expected to do a reasonable amount of computation under these circumstances and this is usually sufficient for the attainment of the objective of the problem. We have deliberately inserted such problems for those students who have access to desk calculators or small computers.

2. MATHEMATICAL CONTENT

   Problems in the text deal with number theory and geometry. The exercises develop skills in computation and lead to some interesting or important number patterns and mathematical conclusions. Number pattern recognition and generalizations by induction occur in many problems. Manipulative techniques are stepping stones to mathematical conclusions rather than an end in themselves.

   The element of fun and recreation in some of the problems requires no apology. Mathematics need not be dull and boring. It can be enjoyable.

3. BOOK 3 of CONTEMPORARY MOTIVATED MATHEMATICS

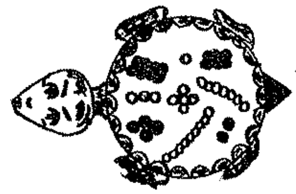
   Many of the problems in Contemporary Motivated Mathematics, Books 1 and 2 are extended or generalized in Book 3. Several new topics have been introduced. Although there is a distinct advantage for students to have completed the previous Books 1 and 2, the present Book 3 is self-contained. Each problem is explained in sufficient detail to facilitate the solution.

Many of the problems in the text come from the work of the greatest mathematicians in the world. We hope that the concepts which inspired the famous men of mathematics will turn out to be a glorious adventure in mathematical ideas for the students.

Cover design by Carl Stefani

S. J. B.
Boston College
1972

CHAPTER 1
MATHEMAGIC



1.1 THE HONORABLE TRADITION OF LO-SHU

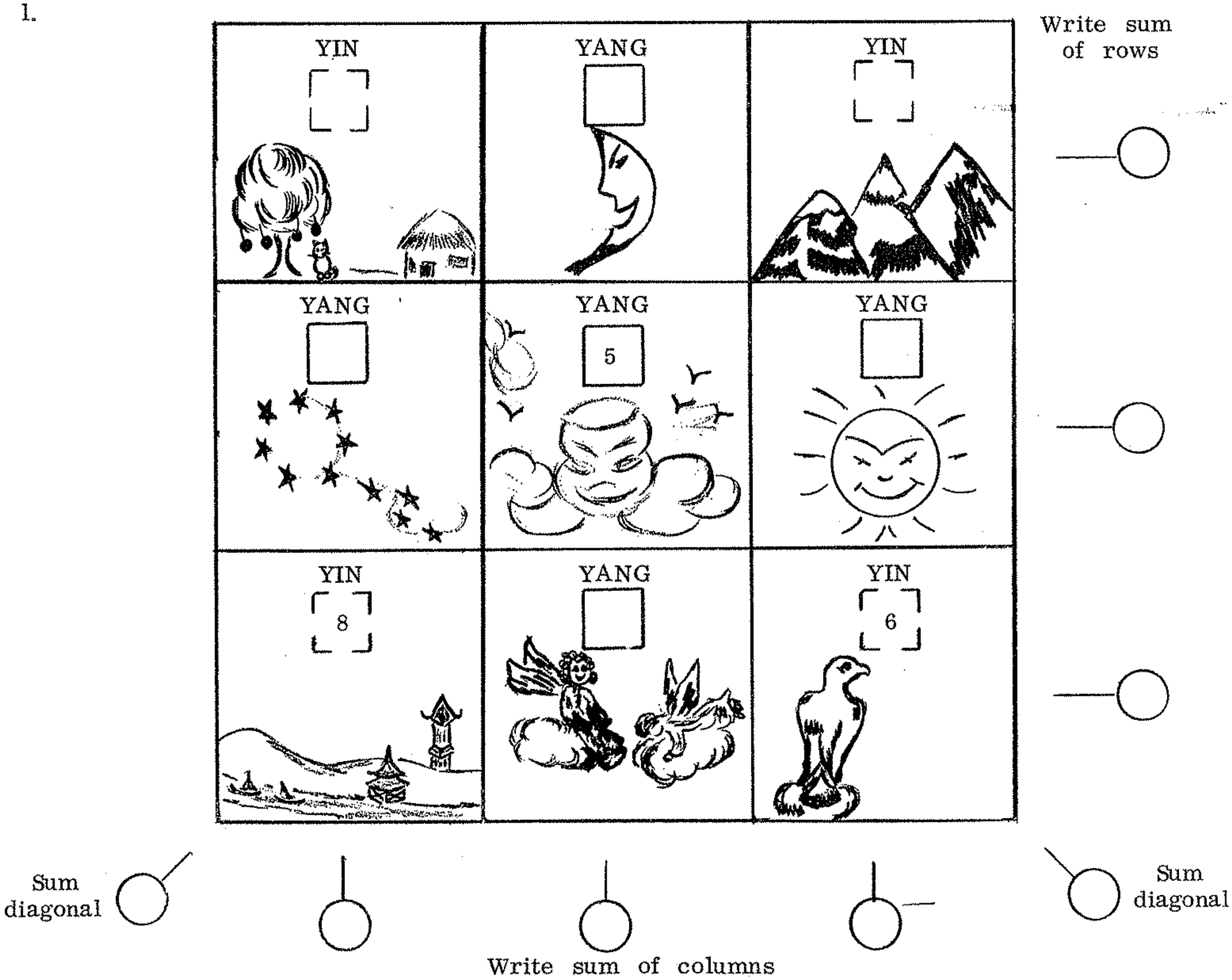
The magic Lo-Shu square was found by the Emperor Yu on the back of a tortoise. The nine numbers 1, 2, 3, . . . , 9 in the magic square array were represented by black and white knots.

> The white knots were YANG and represented the <u>odd numbers</u>.
> The YANG symbol was ——. YANG belonged to Heaven.
> The black knots were YIN and represented the <u>even numbers</u>.
> The YIN symbol was — —. YIN belonged to Earth.

YANG and YIN ruled the Heavens and the destinies of all creatures on Earth.

Fill in the cells of the Lo-Shu 3 x 3 magic square.
Magic constant of each row, column and diagonals is 15.

1.



Lo-Shu's magic square has fascinated and inspired professional and amateur mathematicians of all times and in all places. India, Arabia and Europe fell under the spell of magic patterns.

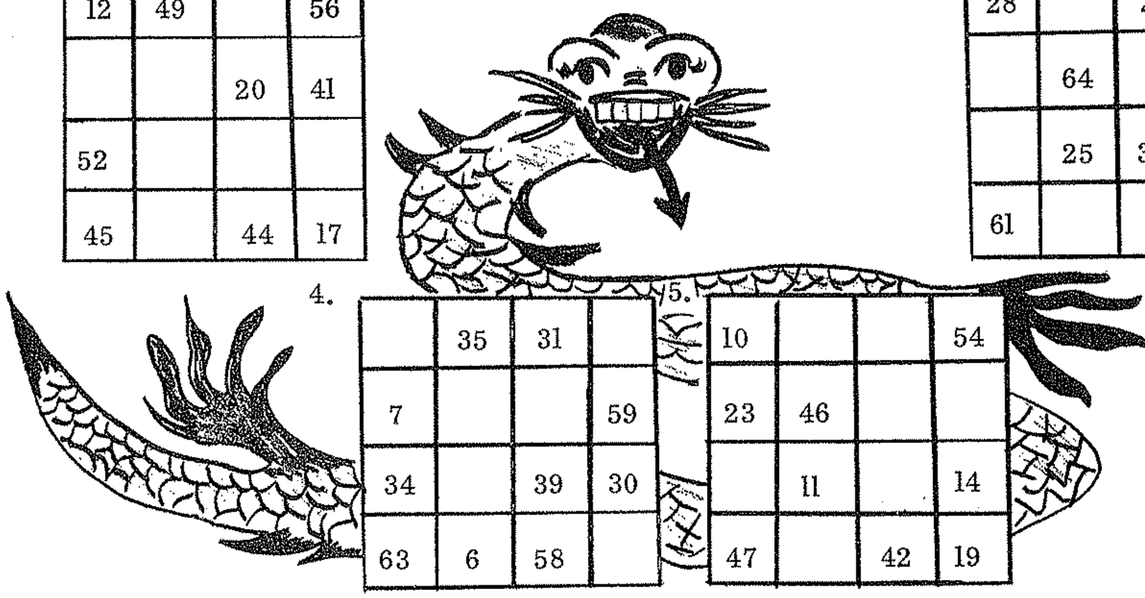
Each number 1 through 64 is used once in the 4 magic squares below. Make the magic constant of each square to be 130.

2.

12	49		56
		20	41
52			
45		44	17

3.

28		29	40
	64		
	25	37	32
61			1



4.

	35	31	
7			59
34		39	30
63	6	58	

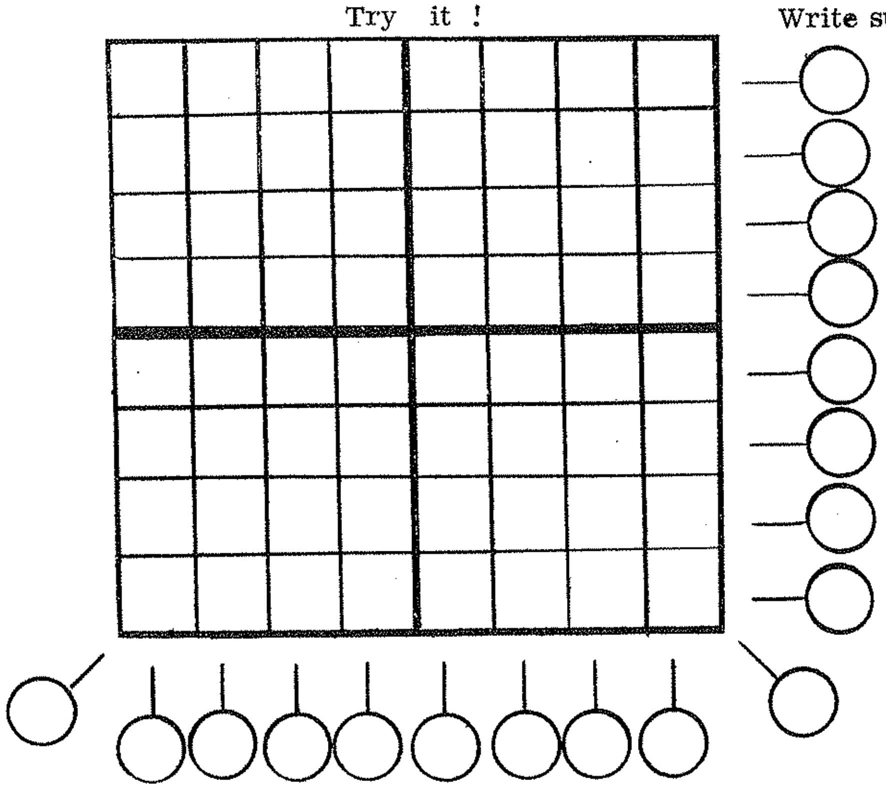
5.

10			54
23	46		
	11		14
47		42	19

6. <u>TOP POPPER</u> Mi-Mi says that the above 4 magic squares can be put together to make an 8 x 8 magic square with magic constant 260.

Try it !

Write sums



Each number 1 through 81 is used in the 9 magic squares below. Fill in the blanks. The magic constant of each square is given.

7. M constant 96

35		33
		29

8. M constant 177

		60
58		56

9. M constant 42

17		
13		11

10. M constant 69

21		25
		20

11. M constant 231

80		
75		
		74

12. M constant 204

		69
67	72	

13. M constant 15

8	1	
4		

14. M constant 123

44		
	41	43

15. M constant 150

	46	51
49		

Mi-Mi says that the above 9 magic squares can be put together to make a 9 x 9 magic square with magic constant 369. Try it! A hint is given in the square.

16.

Write sum of each row, column, diagonal.

				5				
								15
31								

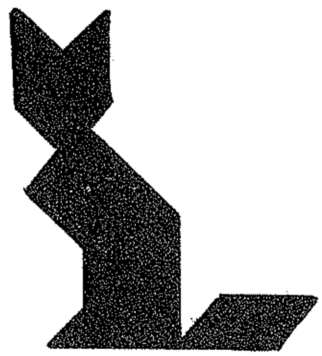
SHADES OF CONCENTRIC SQUARES

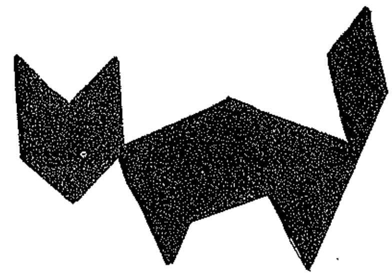
No-No , illustrious friend of Mi-Mi, had a variation on concentric squares.

Use the numbers 1 through 36.
Make the magic constant of the

4 x 4 square = 74

6 x 6 square = 111 .





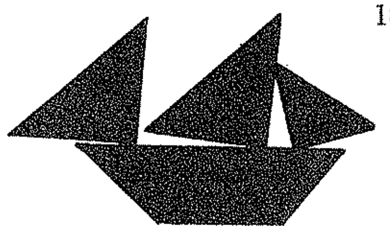
17.

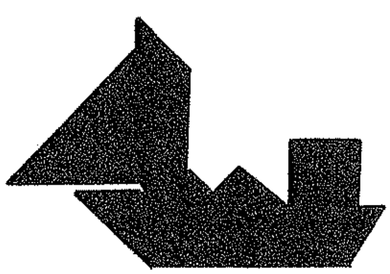
11		25			3
18		20	15	5	
22	17		19		9
			26		
1					
		10	4		29

Use the numbers 1 through 81.
Make the magic constant of the

5 x 5 square = 205

9 x 9 square = 369





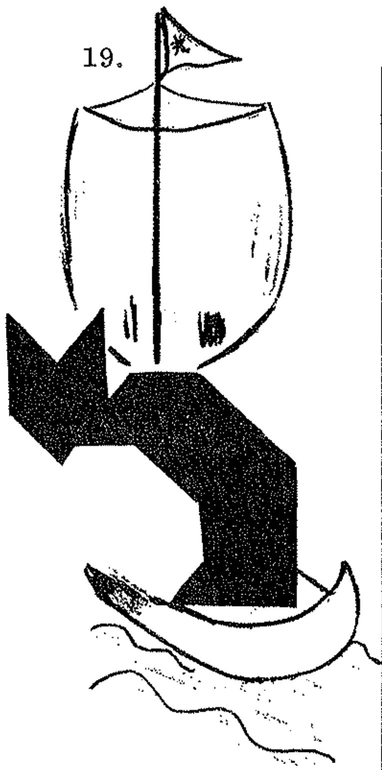
18.

71	1	51		50	2	80		79
	41			26	13		25	57
31		11				17		19
34	40		43		64		55	
48		22	54		75	7		72
33	53	15		16	44		77	5
	29	67	14		24			23
76	4	70			37	36	30	35
6			9	74	45			52

1.2 4 x 4 and 5 x 5 MAGIC SQUARE SECRET REVEALED

One day while leisurely sailing down the Hwang Ho, Cicero discovered a method for constructing 4 x 4 and 5 x 5 magic squares. The secret of Cicero has been put into modern mathematical symbols.

19.



4 x 4 SECRET MAGIC SQUARE

Write sum of rows, columns, diagonals.

$x + y + z + w$	$x - y - z + w$	$x - y - z$	$x + y + z - 2w$
$x - y + z - 2w$	$x + y - z$	$x + y - z + w$	$x - y + z + w$
$x + y - z - w$	$x - y + z - w$	$x - y + z$	$x + y - z + 2w$
$x - y - z + 2w$	$x + y + z$	$x + y + z - w$	$x - y - z - w$

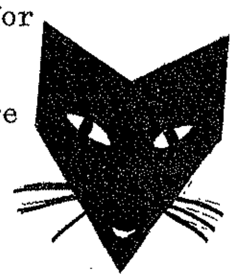
20. The general formula for the magic constant of a 4 x 4 magic square constructed from Cicero's square is ________.

Cicero constructs a 4 x 4 magic square.

Write sum of rows, columns, diagonals.

21. Substitute numbers for w, x, y, z in the secret square

Let $w = 1$
$x = 10$
$y = 5$
$z = 3$.



Magic constant ________

22. Magic constant is $4x$ if 10 is put for x. Yes ______ No ________

4 x 4 MAGIC SQUARES - NATURAL NUMBERS

Use Cicero's 4 x 4 secret square , page 5 .

23. Let w = 1 x = 12 y = 6 z = 3

Magic constant _____

24. Let w = 2 x = 16 y = 8 z = 5

Magic constant _____

25. Let w = 4 x = 34 y = 16 z = 10

Magic constant _____

4 x 4 MAGIC SQUARES - RATIONAL NUMBERS

26. Let $w = \frac{1}{4}$ $x = 2\frac{1}{8}$ $y = 1$ $z = \frac{5}{8}$

Magic constant _____

27. Let $w = \frac{1}{5}$ $x = 1\frac{7}{10}$ $y = \frac{4}{5}$ $z = \frac{1}{2}$

Magic constant _____

28. Let $w = \frac{1}{3}$ $x = 2\frac{5}{6}$ $y = 1\frac{1}{3}$ $z = \frac{5}{6}$

Magic constant _____

4 x 4 MAGIC SQUARES - DECIMALS

29. Let w = .7 x = 8.5 y = 3.1 z = 2.3

Magic constant _______

30. Let w = 1.1 x = 12.7 y = 7.3 z = 2.8

Magic constant _______

31. Let w = .3 x = 7.2 y = 2.9 z = 1.4

Magic constant _______

32. Let $w = .2$ $x = 11.3$ $y = 7.6$ $z = 1.9$

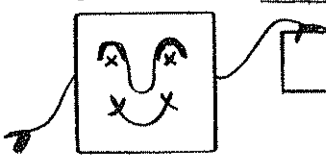
Magic constant ______

33. Let $w = 1.2$ $x = 5.6$ $y = 3.1$ $z = .9$

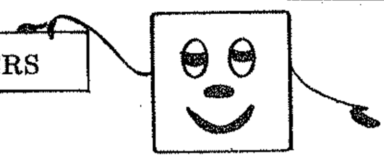
Magic constant ______

34. Let $w = 1.6$ $x = 10.2$ $y = 5.9$ $z = 2.7$

Magic constant ______



4 x 4 MAGIC SQUARES - INTEGERS



35. Let $w = 1$ $x = 5$ $y = 10$ $z = 3$

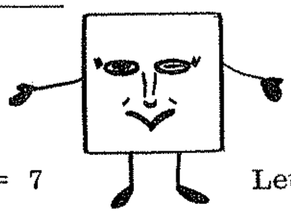
Magic constant ______

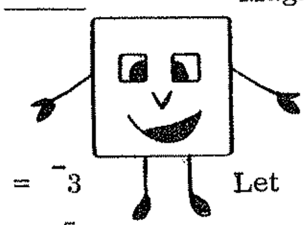
36. Let $w = {}^{-}8$ $x = {}^{-}4$ $y = {}^{-}5$ $z = {}^{-}2$

Magic constant ______

37. Let $w = {}^{-}3$ $x = 5$ $y = 11$ $z = {}^{-}1$

Magic constant ______





38. Let $w = 11$ $x = 7$ $y = 5$ $z = 8$

Magic constant ______

39. Let $w = 2$ $x = {}^{-}3$ $y = 8$ $z = 5$

Magic constant ______

40. Let $w = 7$ $x = 13$ $y = 10$ $z = 9$

Magic constant ______

CICERO'S 5 x 5 SECRET MAGIC SQUARE

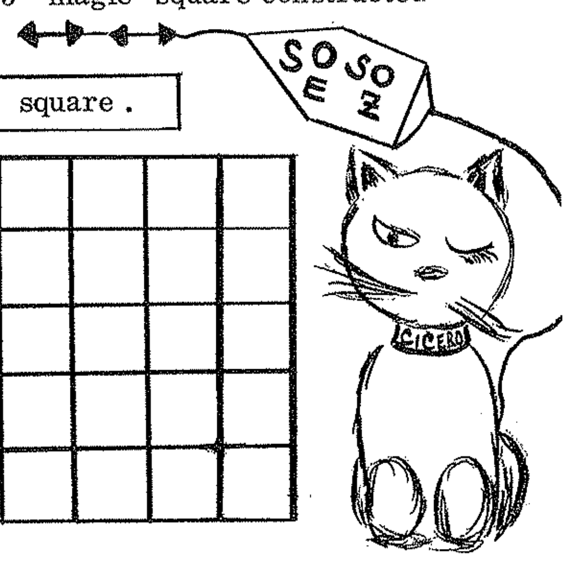
Write sum of rows, columns, diagonals .

41.

x - y - z - v - w	x + z + v	x - z	x + y + z	x + w
x + y + w	x - y	x + y + z + v	x - v	x - y - z - w
x - y + v	x + y + v	x	x - y - v	x + y - v
x + y + z + w	x + v	x - y - z - v	x + y	x - y - w
x - w	x - z - 3v	x + z	x - y - z + 2v	x + y + z + v + w

42. The general formula for the magic constant of a 5 x 5 magic square constructed from Cicero's secret square is ______________ .

Cicero constructs a 5 x 5 magic square .



43. Substitute numbers for v, w, x, y, z in Cicero's secret square .

Let v = 2 x = 13
w = 1 y = 6
z = 3

Magic constant ________

44. Magic constant is 5 x if 13 is put for x .

Yes ________ No ________

5 x 5 MAGIC SQUARES - NATURAL NUMBERS

Use Cicero's 5 x 5 secret square, page 8.

Let $v = 3$ $w = 1$ $x = 19$ $y = 9$ $z = 5$

45.

Magic constant ______

Let $v = 4$ $w = 3$ $x = 23$ $y = 9$ $z = 6$

46.

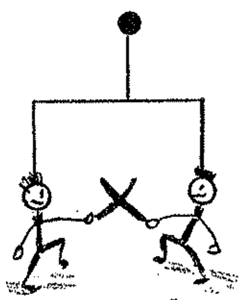
Magic constant ______

5 x 5 MAGIC SQUARES - RATIONAL NUMBERS

Let $v = \frac{3}{4}$ $w = \frac{1}{3}$ $x = 5\frac{1}{2}$ $y = 2$ $z = 1\frac{2}{3}$

47.

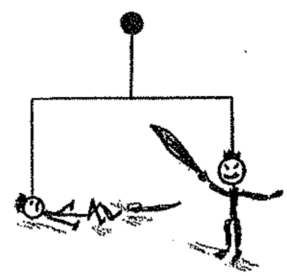
Magic constant ______



Let $v = \frac{1}{4}$ $w = \frac{1}{5}$ $x = 2$ $y = \frac{1}{2}$ $z = \frac{1}{3}$

48.

Magic constant ______



Let $v = 2\frac{1}{3}$ $w = \frac{3}{4}$ $x = 12\frac{1}{2}$ $y = 5$ $z = 4\frac{1}{5}$

49.

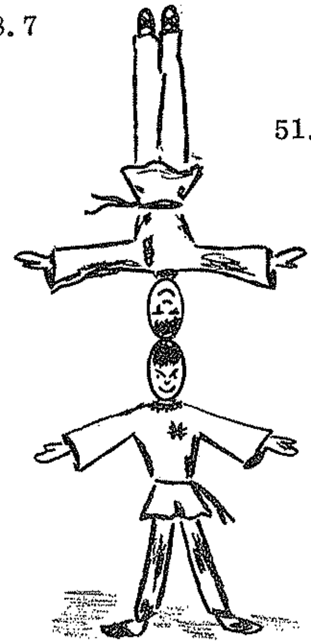
Magic constant ______

5 x 5 MAGIC SQUARES - DECIMALS

Let v = 1.3 w = .5 x = 8.7
y = 3.4 z = 1.8

50.

Magic constant ______



Let v = 1.7 w = 1.2 x = 10.4
y = 3.8 z = 2.6

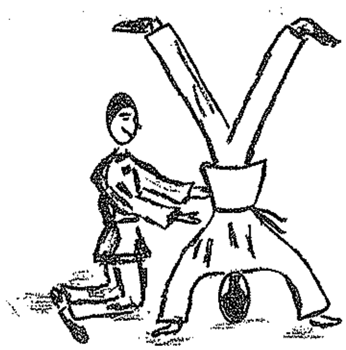
51.

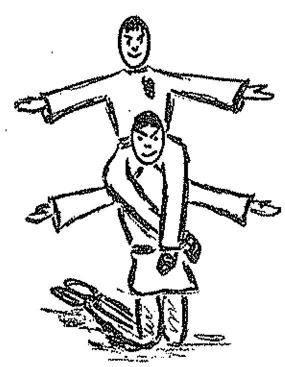
Magic constant ______

Let v = 4.1 w = 4.3 x = 16.7
y = 2.8 z = 3.9

52.

Magic constant ______





5 x 5 MAGIC SQUARES - INTEGERS

Let v = 12 w = 8
x = 1 y = 3
z = $^{-}5$

53.

Magic constant ______

Let v = 8 w = 16
x = $^{-}3$ y = 5
z = $^{-}1$

54.

Magic constant ______

55.

v = -8

w = -4

x = - 10

y = 16

z = 5

Magic constant ______

56.

v = -2

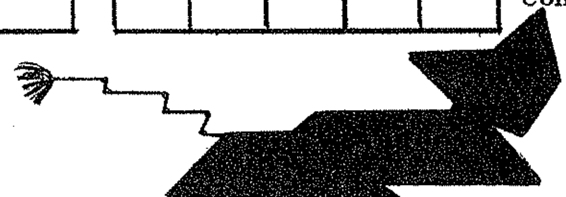
w = - 4

x = -9

y = 3

z = -8

Magic constant ______



1.3 MAGIC SQUARE SILHOUETTES

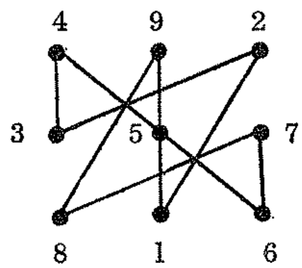
Magic square silhouettes present some interesting designs and patterns.

Study the following example.

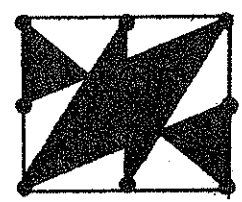
The dots are placed at the center of the cells.

4	9	2
3	5	7
8	1	6

3 x 3 magic square



Line pattern. Start at 1 and draw a line to 2 then to 3 then to 4 and so on back to 1.



Magic square silhouette
Fill in the regions enclosed by lines. Put in the frame for the silhouette.

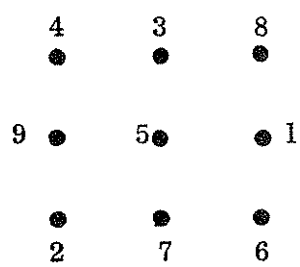
Complete the magic square silhouettes.

3 x 3 magic squares

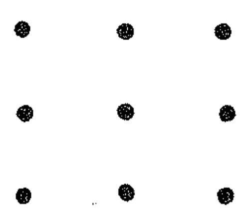
57.

4	3	8
9	5	1
2	7	6

3 x 3 magic square



Line pattern



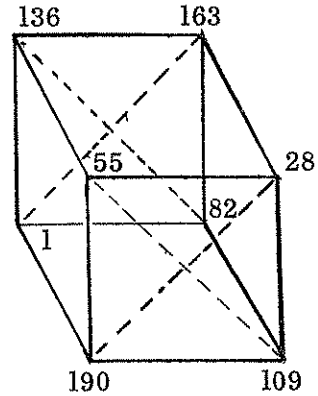
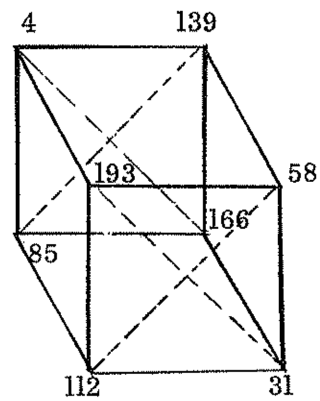
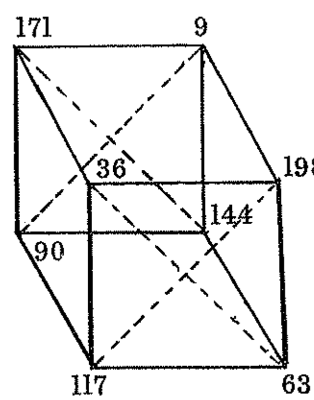
Magic square silhouette

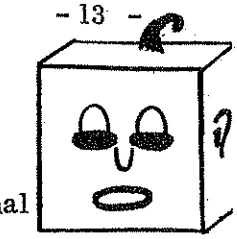
1.4 MAGIC CUBES

UNI'

Magic unit cubes are 3-din
at the 8 vertices of the c
63. planes, each with 4 number

Fill in the chart.

	Top	Bottom	B
	136 163 55 + 28	1 82 190 + 109	1 1 +
			
			



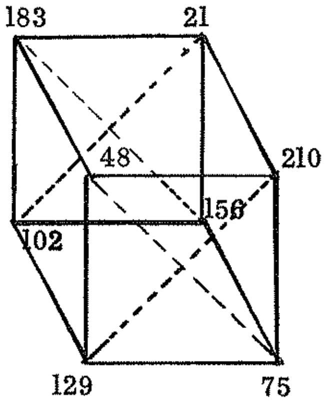
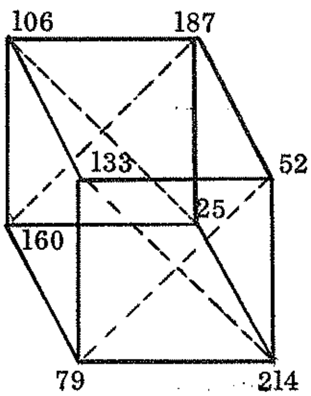
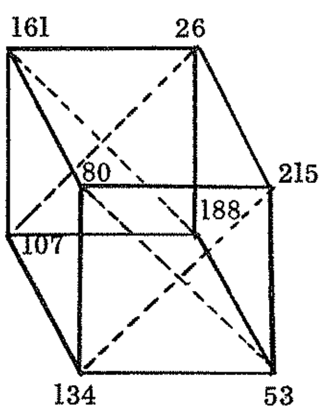
CUBES

ısional figures . Numbers are placed

e . There are 6 faces and 2 diagonal

.

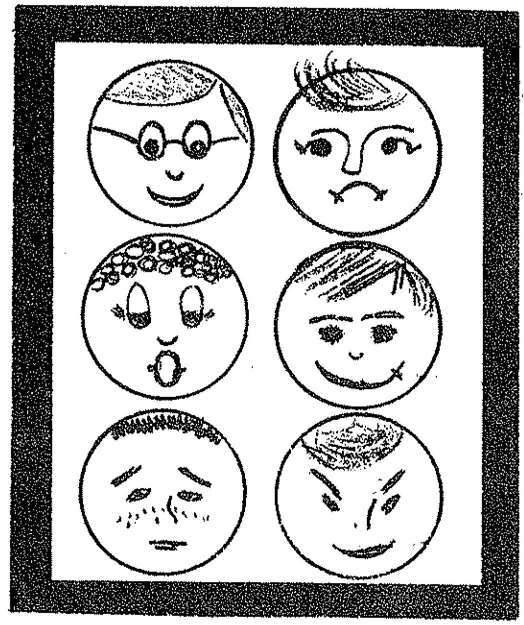
FACES				DIAGONAL PLANES	
k	Front	Left	Right	Left	Right
	55 28 190 + 109	136 1 190 + 55	163 82 28 + 109	136 55 82 + 109	163 28 1 + 190
ʌagic constant ______					
Magic constant ______					
ʌagic constant ______					

Fill in the chart	FACES						DIAGONAL PLANES	
	Top	Bottom	Back	Front	Left	Right	Left	Right
	183 21 48 + 210 ____	102 156 129 + 75 ____	183 21 102 + 156 ____	48 210 129 + 75 ____	183 48 102 + 129 ____	210 21 156 + 75 ____	183 48 156 + 75 ____	210 21 102 + 129 ____
				Magic constant ______				
								
				Magic constant ______				
								
				Magic constant ______				

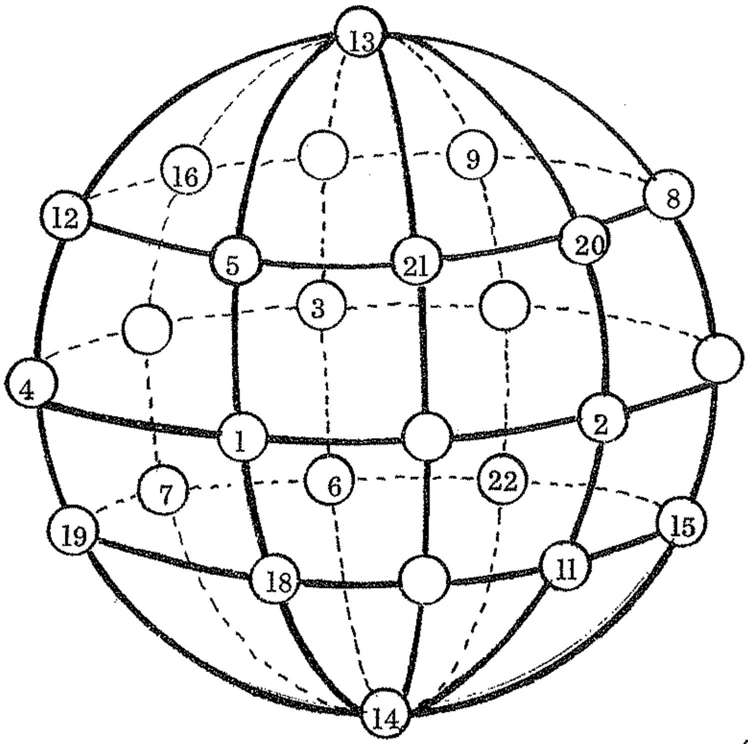
1.5 MAGIC SPHERES

Magic spheres are 3-dimensional figures. Numbers are placed at the points of intersection of circles on the sphere.

This sphere contains 7 circles, each circle with 8 numbers. The consecutive natural numbers 1 through 26 lie at the intersections of the circles. The magic constant of each circle is 108 .

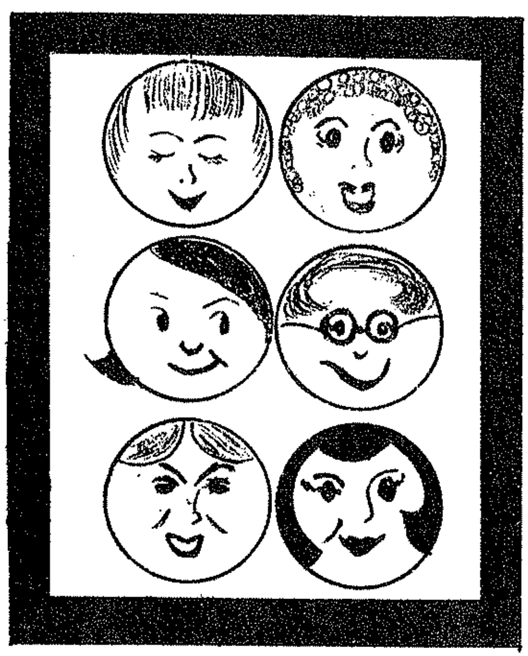


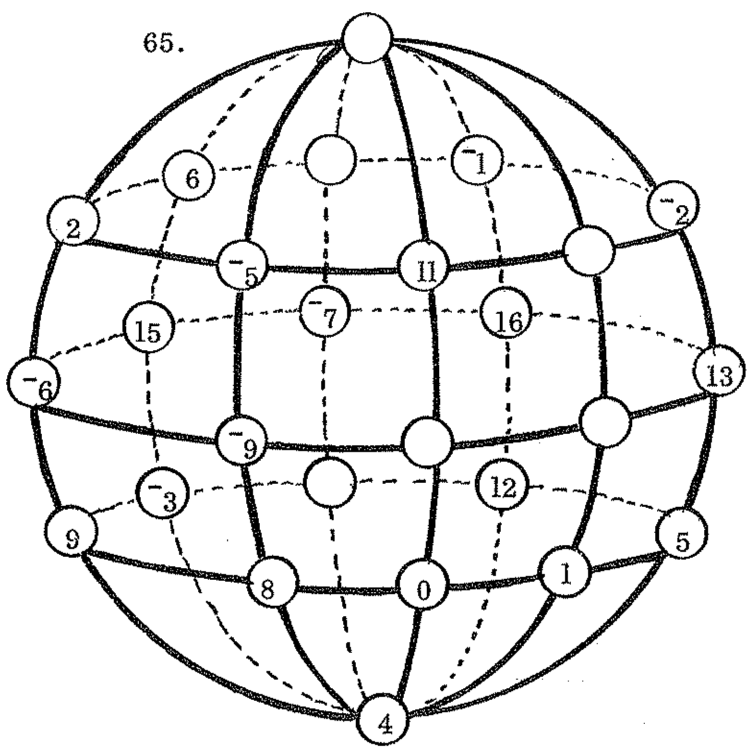
64.



The consecutive integers ¯9 through 16 lie at the intersections of the circles. The magic constant of each circle is 28 .

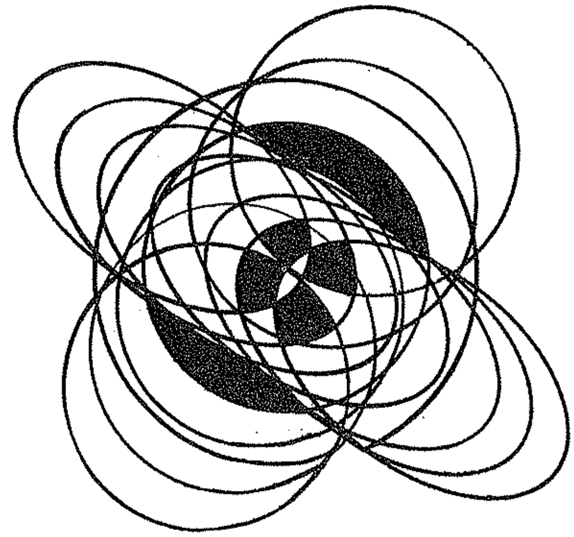
65.



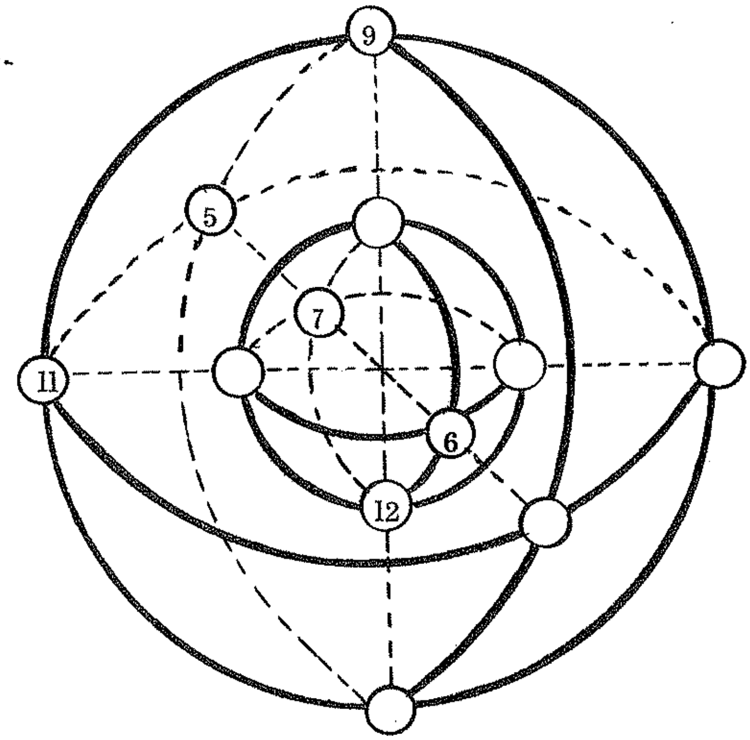


CONCENTRIC SPHERES

The consecutive natural numbers 1 through 12 are used on the concentric spheres. There are 6 great circles, each with 4 numbers. The magic constant including the 3 diameters of the larger circles is 26.

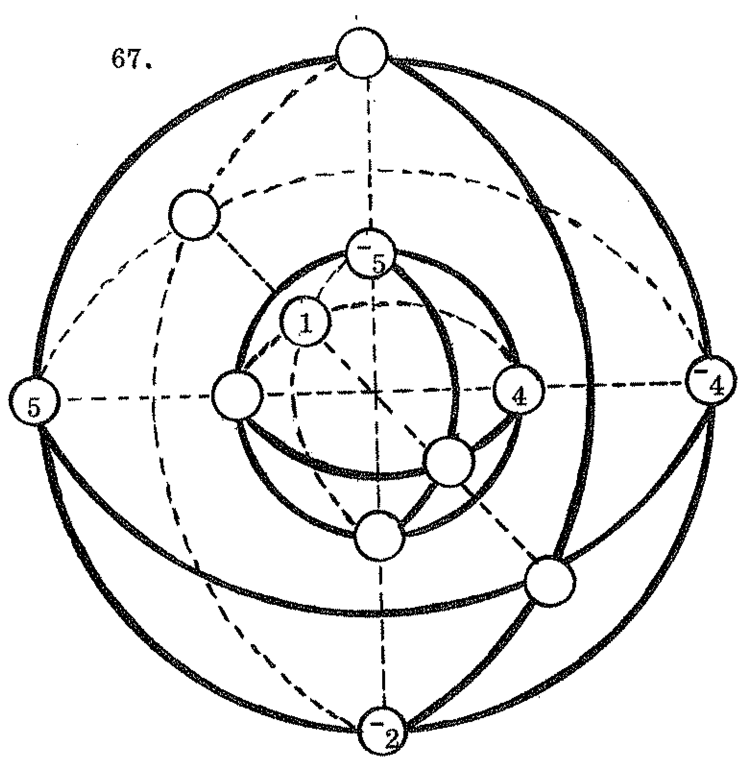


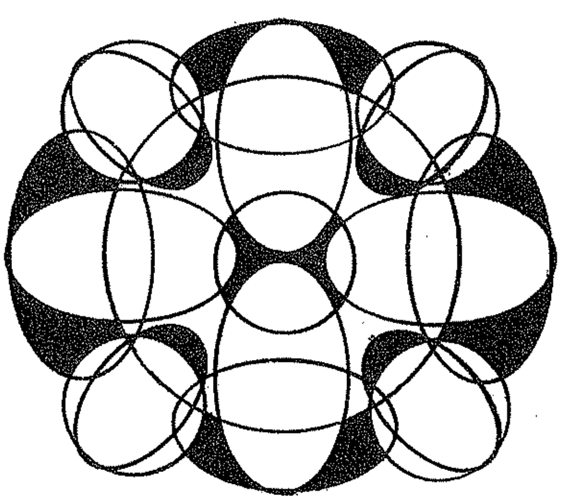
66.



The consecutive integers ¯5 through 6 are used on the concentric spheres. The magic constant including the 3 diameters of the larger circles is 2.

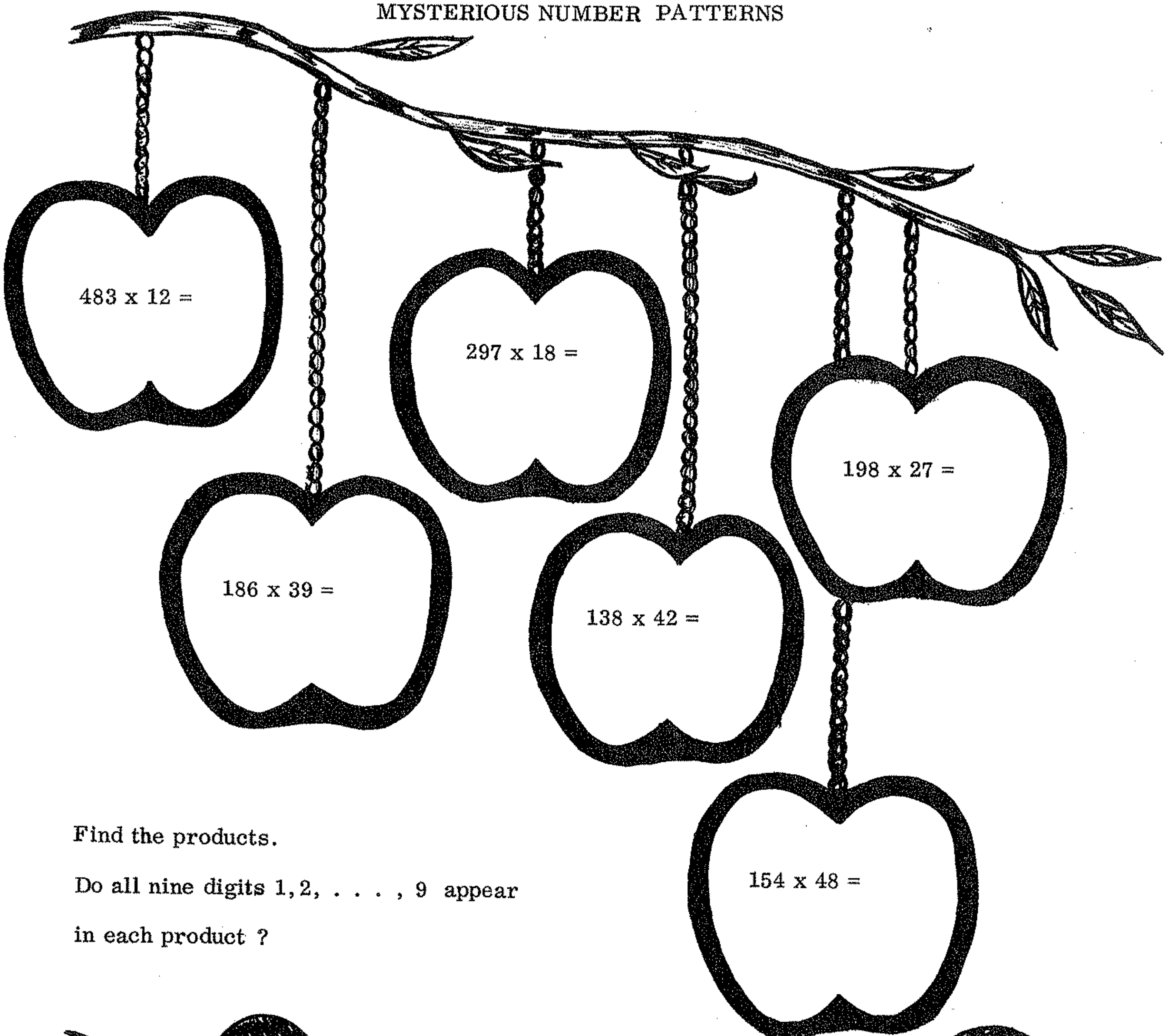
67.





CHAPTER 2

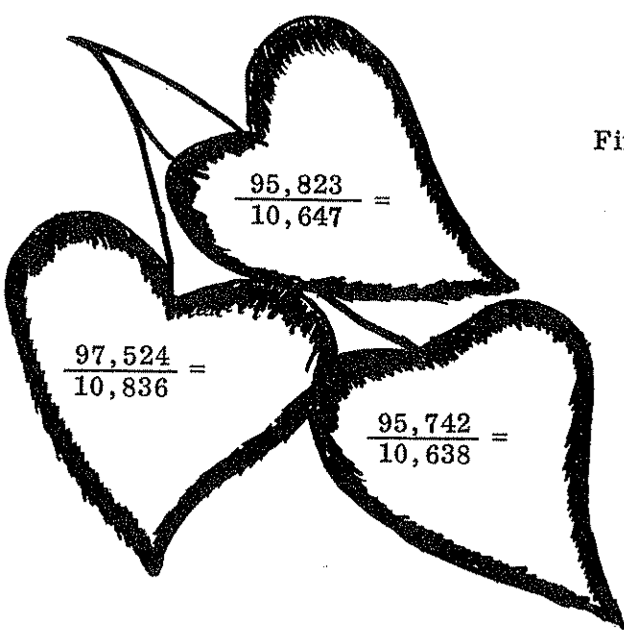
MYSTERIOUS NUMBER PATTERNS

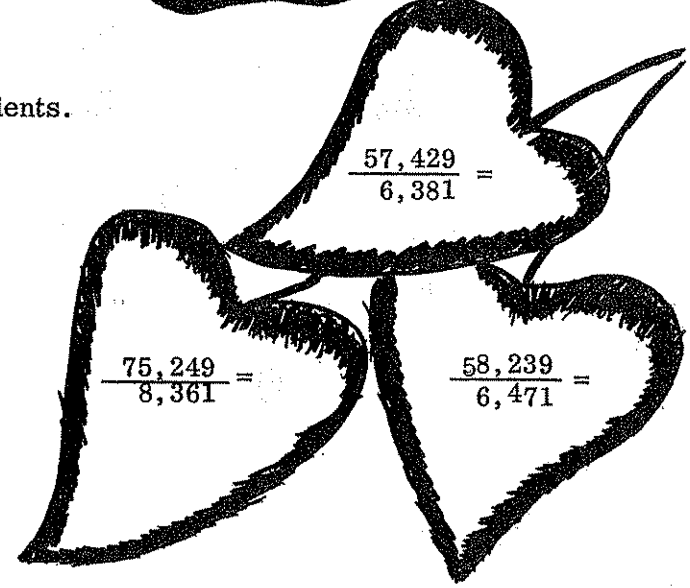


Find the products.

Do all nine digits 1, 2, . . . , 9 appear in each product ?

Find the quotients.

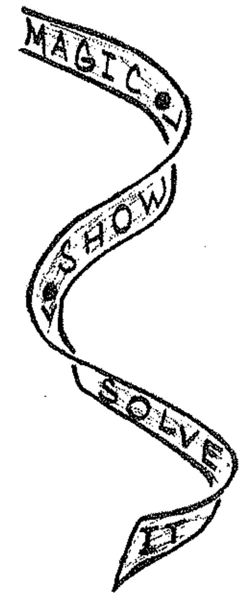




2.1 MAGIC GUESSING - NUMBERS

Here is Mr. Ree's Magic Table.

Column 1	Column 2	Column 3	Column 4	Column 5
1	2	4	8	16
3	3	5	9	17
5	6	6	10	18
7	7	7	11	19
9	10	12	12	20
11	11	13	13	21
13	14	14	14	22
15	15	15	15	23
17	18	20	24	24
19	19	21	25	25
21	22	22	26	26
23	23	23	27	27
25	26	28	28	28
27	27	29	29	29
29	30	30	30	30
31	31	31	31	31



Problem — Select a number 1 through 31. Tell Mr. Ree the columns in which your number appears. Mr. Ree will tell you the number you selected.

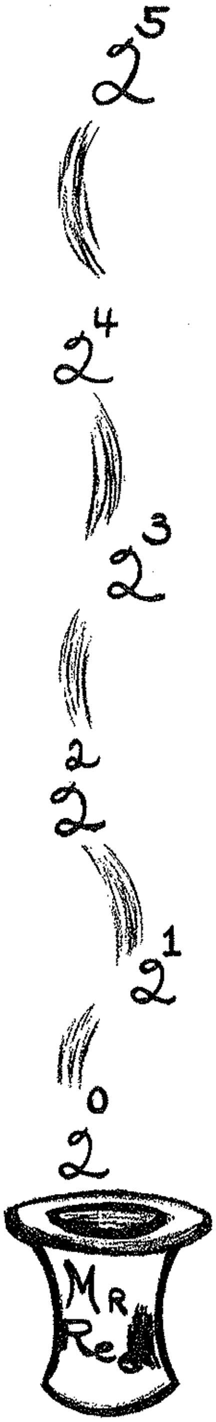
Solution — Add the numbers at the top of the columns which contain your selected number. For example, suppose you select 27.

27 is in columns	1 2 4 5
Sum of numbers at the top of these columns	1 + 2 + 8 + 16 = 27

68. Fill in the chart. Use the pattern in the first row.

Number selected	Number selected is in columns 1	2	3	4	5	Sum
9	1	+		8		9
11						
17						
21						
23						
26						
30						

69. Study Mr. Ree's Magic Table on page 18 . If you discover the secret of making up the Table, extend Mr. Ree's Table by putting in one more column . A few hints are given in the Table .



Column 1	Column 2	Column 3	Column 4	Column 5	Column 6
1	2	4	8	16	32
3	3	5	9	17	___
5	6	6	10	18	34
7	7	7	11	19	___
9	10	12	12	20	___
11	11	13	13	21	___
13	14	14	14	22	38
15	15	15	15	23	___
17	18	20	24	24	___
19	19	21	25	25	___
21	22	22	26	26	___
23	23	23	27	27	43
25	26	28	28	28	___
27	27	29	29	29	___
29	30	30	30	30	___
31	31	31	31	31	47
___	___	36	40	___	___
___	___	___	___	___	___
37	___	___	___	___	50
___	___	___	___	___	___
___	42	___	___	52	___
___	___	___	___	___	___
___	___	___	___	___	___
___	___	47	47	___	___
___	___	___	___	___	___
51	51	___	___	57	57
___	___	___	___	___	___
___	___	55	59	___	___
___	___	___	___	___	___
___	___	___	___	___	___
___	___	___	___	___	___
63	___	63	___	63	___

70. If you use the table above, what is the largest number that can be selected for the magic guessing? ______________

2.2 MAGIC GUESSING - AGES

Happiness in every day,
Especially on your Birthday!

Let January be 01, February 02, . . ., October 10, and so on.

Dini Dunit's Magic 1	Deedee Duzit Born: 26 January, Age 14	Hugh Duit Born: 15 December, Age
1. Write the number of the day and of the month as a single 4-digit number	(2 6 0 1)	(1 5 1 2)
2. Multiply by 5	1 3 0 0 5	
3. Multiply by 20	2 6 0 1 0 0	
4. Add age	2 6 0 1 1 4	

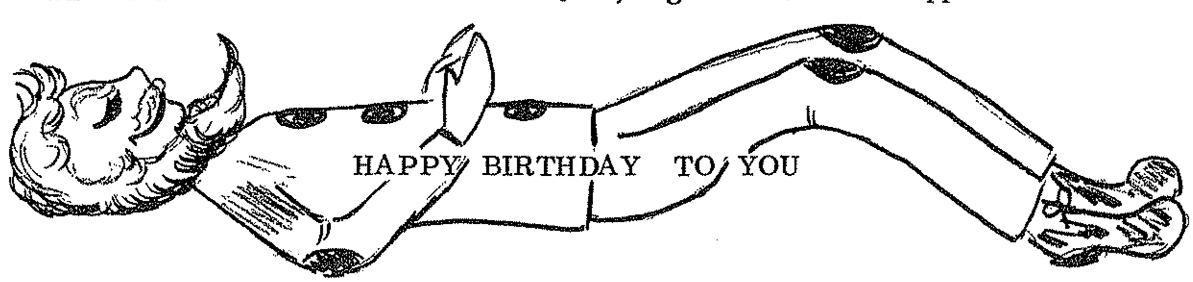
First 2 digits give day of birth. Second 2 digits give month. Third 2 digits give the age.

Try Dini Dunit's Magic 1 on the Birthdays listed below.
Follow the pattern in the first column of the chart.

71.

	10 February Age 10	20 March Age 12	1 April Age 11	31 May Age 16	18 October Age 21
1. Write the number of day and month as single 4-digit number	(1 0 0 2)				
2. Multiply by 5	5 0 1 0				
3. Multiply by 20	1 0 0 2 0 0				
4. Add age	1 0 0 2 1 0				

72. Try Dini Dunit's Magic 1 on that Catskill Dreamer, Rip Van Winkle, who was born on 31st of January, age 100. What happened?



Age in quest of the Ageless.

Dini Dunit puzzles Mr. Ree.

Let January be 01, February 02 , . . . , October 10 and so on .

Dini Dunit's Magic 2	Deedee Duzit Born: 26 November Age 14	Hugh Duit Born: 15 December Age 15
1. Write the number of the day and of the month as a single 4-digit number	(2 6 1 1)	(1 5 1 2)
2. Multiply by 2	5 2 2 2	
3. Add 5	5 2 2 7	
4. Multiply by 10	5 2 2 7 0	
5. Multiply by 5	2 6 1 3 5 0	
6. Add age	2 6 1 3 6 4	
7. Subtract 250	2 6 1 1 1 4	

First 2 digits give day of birth — Second 2 digits give month — Third 2 digits give the age

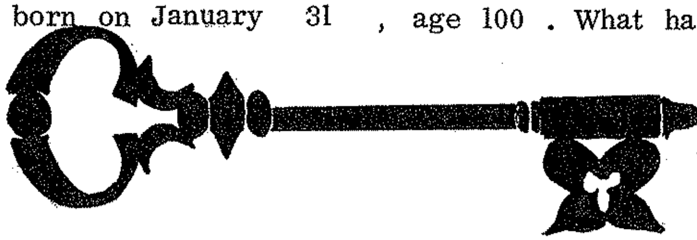
Try Dini Dunit's Magic 2 on the Birthdays listed below.

Follow the pattern in the first column of the chart.

73.

	5 May Age 5	16 July Age 13	10 August Age 17	30 September Age 20	1 April Age 1
1. Write the number of the day and of the month as a single 4-digit number	(5 0 5)				
2. Multiply by 2	1 0 1 0				
3. Add 5	1 0 1 5				
4. Multiply by 10	1 0 1 5 0				
5. Multiply by 5	5 0 7 5 0				
6. Add age	5 0 7 5 5				
7. Subtract 250	5 0 5 0 5				

74. Try Dini Dunit's Magic 2 on the notable napper, Rip Van Winkle , who was born on January 31 , age 100 . What happened ?



Another Dini Dunit age puzzle for Mr. Ree .

Dini Dunit Magic 3	Deedee Duzit : age 12 — 25 pupils in her math class	Hugh Duit : age 15 — 30 pupils in his math class
1. Take age	12	15
2. Multiply by 10	120	
3. Add 10	130	
4. Multiply by 10	1300	
5. Add number of pupils in math class	1325	
6. Subtract 100	1225	

First 2-digits represent age

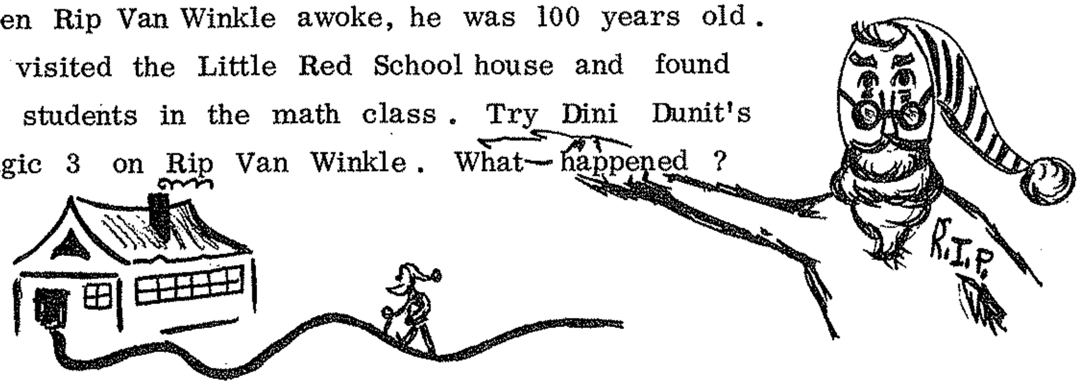
Second 2-digits represent number of pupils in math class

Try Dini Dunit's Magic 3 on the ages and the number of pupils in the math classes given below.

75.

	Age 16 Number in math class : 9	Age 35 Number in math class : 15	Age 99 Number in math class : 99
1. Take age			
2. Multiply by 10			
3. Add 10			
4. Multiply by 10			
5. Add number of pupils in math class			
6. Subtract 100			

76. When Rip Van Winkle awoke, he was 100 years old . He visited the Little Red School house and found 100 students in the math class . Try Dini Dunit's Magic 3 on Rip Van Winkle . What happened ?



2.3 MAGIC CRYSTAL BALL - SEVENS AND NINES

SECRET SEVENS

77. Find the product : 15,873 x 1 x 7 = ______________

78.

Predict the product at once	Check your prediction
15,873 x 2 x 7 = ______________	15,873 x 14
15,873 x 3 x 7 = ______________	15,873 x 21
15,873 x 4 x 7 = ______________	15,873 x 28
15,873 x 5 x 7 = ______________	15,873 x 35
15,873 x 6 x 7 = ______________	15,873 x 42
15,873 x 7 x 7 = ______________	15,873 x 49
15,873 x 8 x 7 = ______________	15,873 x 56
15,873 x 9 x 7 = ______________	15,873 x 63

79.

NOVEL NINES

Find the sums : (0 x 9) + 1 = ____   (1 x 9) + 2 = ____   (12 x 9) + 3 = ____

80.

Predict the sums at once	Check your prediction
(123 x 9) + 4 = ____	123 x 9 + 4 = ____
(1234 x 9) + 5 = ____	1234 x 9 + 5 = ____
(12345 x 9) + 6 = ____	12345 x 9 + 6 = ____
(123456 x 9) + 7 = ____	123456 x 9 + 7 = ____
(1234567 x 9) + 8 = ____	1234567 x 9 + 8 = ____
(12345678 x 9) + 9 = ____	12345678 x 9 + 9 = ____

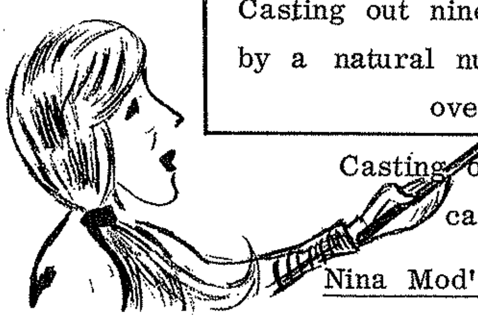
81.

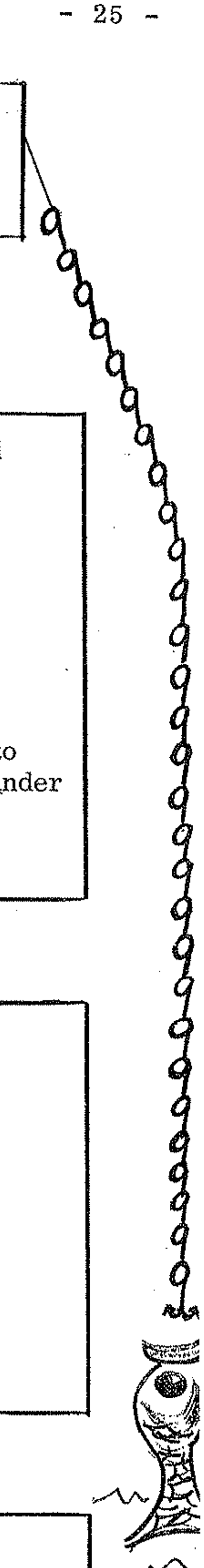
Find the differences : (21 x 9) - 8 = ____   (321 x 9) - 7 = ____

82.

Predict the differences at once	Check your prediction
(4321 x 9) - 6 = ____	4321 x 9 - 6 = ____
(54321 x 9) - 5 = ____	54321 x 9 - 5 = ____
(654321 x 9) - 4 = ____	654321 x 9 - 4 = ____
(7654321 x 9) - 3 = ____	7654321 x 9 - 3 = ____
(87654321 x 9) - 2 = ____	87654321 x 9 - 2 = ____
(987654321 x 9) - 1 = ____	987654321 x 9 - 1 = ____

2.4 CASTING OUT NINES





Casting out nines from a natural number n is to replace n by a natural number 0 through 8 which represents the excess over the number of 9 s in n .

Casting out 9 s from a natural number n is also called <u>excess of nines</u> in n .

<u>Nina Mod's division method for casting out 9 s .</u>

<u>Example 1</u> Consider the number 4 7 9 1 . Divide the number by 9 and find the remainder .

```
      5 3 2
  9 | 4 7 9 1
      4 5
        2 9
        2 7
          2 1
          1 8
           (3)
```

To cast out 9 s from 4 7 9 1 is to replace 4 7 9 1 by 3 . The remainder 3 represents the excess over the number of 9 s in 4 7 9 1 .

<u>Nina Mod's sum - division method for casting out 9 s .</u>

<u>Example 2</u> Since a natural number n is divisible by 9 if the sum of the numbers represented by the digits in n is divisible by 9 , Nina Mod used the following casting out 9 s method.

Consider the number 4 7 9 1 .

Now $4 + 7 + 9 + 1 = 21$

and $21 \div 9 = 2$ with remainder (3)

To cast out 9 s from 4 7 9 1 is to replace 4 7 9 1 by 3 , the remainder after the division .

<u>Nina Mod's reduced sum method for casting out 9 s .</u>

<u>Example 3</u> The sum of the numbers represented by the digits in the natural number n are reduced successively to a single digit number .

Consider the number 4 7 9 1 .

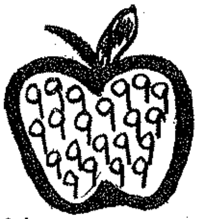
Now $4 + 7 + 9 + 1 = 21$

and for 2 1 we have $2 + 1 =$ (3)

To cast out 9 s from 4791 is to replace 4791 by 3 .

83. Casting out 9 s. Fill in the chart. Use the pattern in the first row.

Take	Division method	Sum - division method	Reduced sum method
686	7 6 9 ⟌ 6 8 6 6 3 5 6 5 4 (2) Replace 686 by 2	6 + 8 + 6 = 20 20 ÷ 9 = 2 remainder (2) Replace 686 by 2	6 + 8 + 6 = 20 and 20 is 2 + 0 = (2) Replace 686 by 2
799	9 ⟌ 7 9 9 Replace 799 by ____	Replace 799 by ____	Replace 799 by ____
957	9 ⟌ 9 5 7 Replace 957 by ____	Replace 957 by ____	Replace 957 by ____
985	9 ⟌ 9 8 5 Replace 985 by ____	Replace 985 by ____	Replace 985 by ____
1629	9 ⟌ 1 6 2 9 Replace 1629 by ____	Replace 1629 by ____	Top Popper Replace 1629 by ____
2562	9 ⟌ 2 5 6 2 Replace 2562 by ____	Replace 2562 by ____	Replace 2562 by ____



Casting out 9 s is now used to check the correctness of a computation .

ADDITION

Fill in the chart . Use the pattern in the first row .

84.

Cast out 9 s 694 → 1 456 → 6 + 345 → 3 1495 → (1) 10 → (1) Cast out 9 s. Cast out 9 s . The circled numbers are the same. Addition on left is correct.	235 438 + 501
327 890 + 574	108 372 + 954
3274 8301 + 2765 14330 Check by casting out 9 s . Addition correct _____ Incorrect _____	2386 4912 + 5786
5387 7049 6197 + 8206	1357 6842 9324 + 6253 22776 Check by casting out 9 s . Addition correct _____ Incorrect _____
2134 5906 6287 + 4964	6432 8926 9301 + 1078

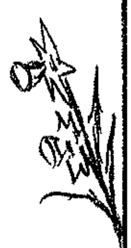
MULTIPLICATION

85. Fill in the chart. Use the pattern in the first row.

Cast out 9 s 321 → 6 x 23 → x 5 963 642 7383 → (3) 30 → (3) Cast out 9 s Cast out 9 s The circled numbers are the same. Multiplication is correct.	268 x 52
726 x 36 4356 2178 26126 Check by casting out 9 s. Product correct ____ Incorrect ____	853 x 71
1276 x 47	7624 x 87
3047 x 237	8759 x 532
9075 x 734	2386 x 925

SUBTRACTION

86. Fill in the chart. Use the pattern in the first row.

Cast out 9 s 467 → 8 - 234 → - 0 233 → (8) 8 → (8) Cast out 9 s Cast out 9 s The circled numbers are the same. The difference is correct.	Cast out 9 s 345 → 3 add 9 12 - 213 → -6 - 6 132 → (6) 6 → (6) Cast out 9 s Cast out 9 s The circled numbers are the same. The difference is correct.
565 - 273 	494 - 161
372 - 258 	512 - 356
1274 - 932 	3786 - 3029 767 Check by casting out 9 s. Difference correct _____ Incorrect ______
8725 - 5967 	7771 - 7630
5973 - 2845 	4382 - 4295

2.5 REDUCED SUM MYSTERY

> Reduce the sum of the numbers represented by the digits in a natural number n to a single digit.

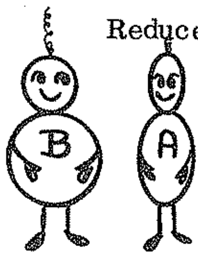
Example Take 1234. Various groupings of the numbers represented by the digits may be used in reducing the sum to a single digit.

1 2 3 4 is 1 + 2 + 3 + 4 = 10
Now 10 is 1 + 0 = 1
Reduced sum is (1).

1 234 is 1 + 234 = 235
Now 235 is 2 + 3 + 5 = 10
and 10 is 1 + 0 = 1
Reduced sum is (1).

12 34 is 12 + 34 = 46
Now 46 is 4 + 6 = 10
and 10 is 1 + 0 = 1
Reduced sum is (1).

123 4 is 123 + 4 = 127
Now 127 is 1 + 2 + 7 = 10
and 10 is 1 + 0 = 1
Reduced sum is (1).



> The reduced sum is always the same and independent of any particular grouping process of the numbers.

Find the reduced sum. Use the pattern shown in the example above. Some particular groupings are given in each problem.

87.

Take 4683 4 6 8 3 46 83 4 683 468 3	Take 5714 5 7 1 4 57 14 5 714 571 4
Take 6972 6 9 7 2 69 72 6 972 697 2	Take 8402 8 4 0 2 84 02 8 402 840 2

2.6 MORE ABADABA'S PALINDROMES

A natural number palindrome has digits which read the same when the digits are taken in reverse order.

Natural number palindromes : 0 , 5 , 121 , 1331 .

Fill in the charts . Use the pattern shown in the first row .

88.	89.	90.
111 x 11 = 1 2 2 1	222 x 11 = ________	333 x 11 = ________
1,111 x 11 = ________	2,222 x 11 = ________	3,333 x 11 = ________
11,111 x 11 = ________	22,222 x 11 = ________	33,333 x 11 = ________
111,111 x 11 = ________	222,222 x 11 = ________	333,333 x 11 = ________
1,111 x 111 = ________	2,222 x 111 = ________	3,333 x 111 = ________
11,111 x 111 = ________	22,222 x 111 = ________	33,333 x 111 = ________
111,111 x 111 =	222,222 x 111 =	333,333 x 111 =

THE SYMBOL WEAVERS

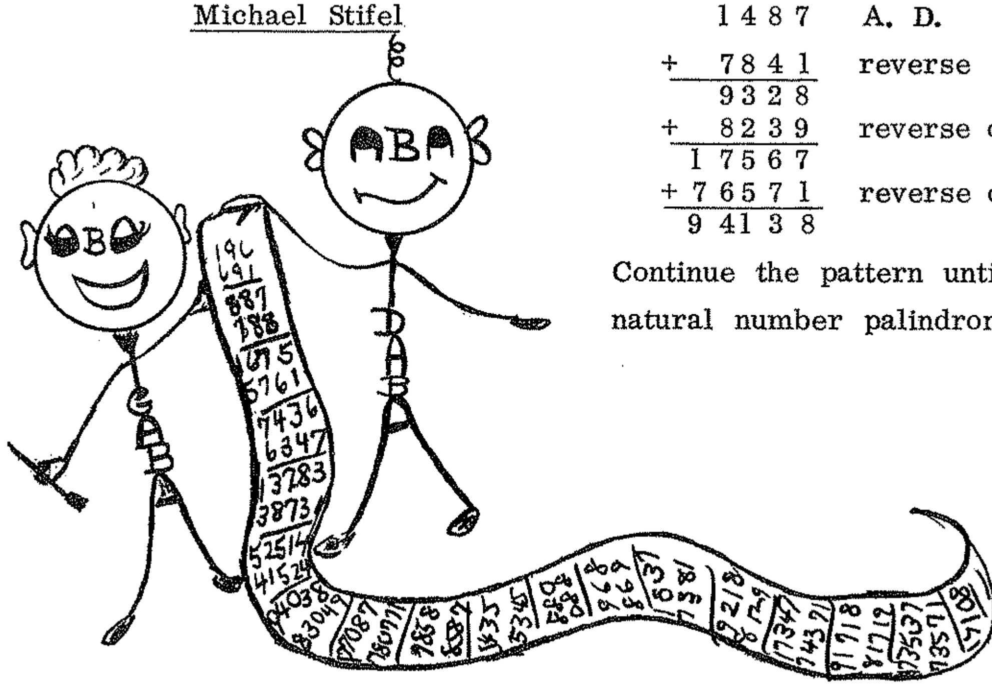
Years of Fame and Glory

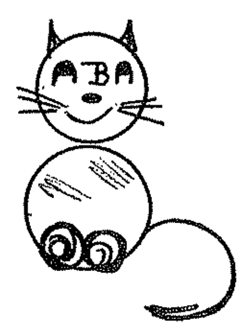
Abadaba and his sister Abagaba admired mathematicians . They wrote the year of birth for

Michael Stifel

	1 4 8 7	A. D.
+	7 8 4 1	reverse digits in 1487
	9 3 2 8	
+	8 2 3 9	reverse digits in 9328
	1 7 5 6 7	
+	7 6 5 7 1	reverse digits in 17567
	9 4 1 3 8	

Continue the pattern until you have a natural number palindrome .





Famous mathematicians are listed below and the year they were born.

Use Abadaba's addition pattern on page 31. Fill the chart.

91.

Mathematician	Year born	Palindrome	No. of reversals
Michael Stifel	1487 A. D.		9
Franciscus Maurolycus	1494 A. D.		
Nicolo Tartaglia	1499 A. D.		
Jerome Cardan	1501 A. D.		
Robert Recorde	1510 A. D.		
Gerardus Mercator	1512 A. D.		
Ludovico Ferrari	1522 A. D.		
Raffael Bombelli	1530 A. D.		
Christopher Clavius	1537 A. D.		
Francois Vieta	1540 A. D.		
Tycho Brahe	1546 A. D.		
Simon Stevin	1548 A. D.		
John Napier	1550 A. D.		
Matteo Ricci	1552 A. D.		
Thomas Hariot	1560 A. D.		
YOU 	______ A. D.		

CHAPTER 3

CURIOUS NUMBER PATTERNS

3.1 GEMS FROM THE TREASURE CHEST - ONES, TWOS, . . .

92.

987654321
x 9

987654321
x 81

987654321
x 18

987654321
x 72

$(1 + 1 + 1) \cdot 37 = \_\_\_\_\_\_$

$(2 + 2 + 2) \cdot 37 = \_\_\_\_\_\_$

$(3 + 3 + 3) \cdot 37 = \_\_\_\_\_\_$

$(4 + 4 + 4) \cdot 37 = \_\_\_\_\_\_$

$(5 + 5 + 5) \cdot 37 = \_\_\_\_\_\_$

$(6 + 6 + 6) \cdot 37 = \_\_\_\_\_\_$

$(7 + 7 + 7) \cdot 37 = \_\_\_\_\_\_$

$(8 + 8 + 8) \cdot 37 = \_\_\_\_\_\_$

$(9 + 9 + 9) \cdot 37 = \_\_\_\_\_\_$

987654321
x 27

987654321
x 63

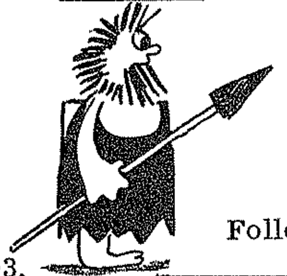
987654321
x 36

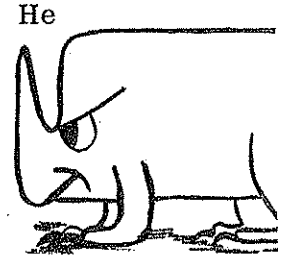
987654321
x 45

987654321
x 54

3.2 DIFFERENCE OF SQUARES OF CONSECUTIVE NATURAL NUMBERS

Willie Skware took 2 consecutive natural numbers. He squared each number. Then he took the difference of the squares. He found an interesting pattern.





Examples

(n + 1)	n	
1	0	$1^2 - 0^2 = 1 - 0 = 1 = 1 + 0$
2	1	$2^2 - 1^2 = 4 - 1 = 3 = 2 + 1$
3	2	$3^2 - 2^2 = 9 - 4 = 5 = 3 + 2$

93. Follow the pattern in the first row. Fill in the chart.

n + 1	n	$(n+1)^2 - n^2$	(n + 1) + n
4	3	$4^2 - 3^2 = 16 - 9 = 7$	4 + 3 = 7
5	4		
6	5		
7	6		
11	10		
12	11		
14	13		
16	15		
37	36		
46	45		
59	58		
83	82		
100	99		

94. Fill in the blank so that the result is a true statement.

> The difference of the squares of 2 consecutive natural numbers is equal to the ________ of the 2 consecutive natural numbers.

3.3 SQUARES OF ODD NATURAL NUMBERS

Willie Skware's search for number patterns often resulted in unexpected relations among numbers. Here is an instance.

Use the pattern in the first 3 rows. Fill in the chart.

95.

Odd numbers	Squares	Representation
1	$1^2 = 1$	$8(0) + 1 = 1$
3	$3^2 = 9$	$8(1) + 1 = 9$
5	$5^2 = 25$	$8(3) + 1 = 25$
7		
9		
11		
21		
33		
45		
57		
69		
73		

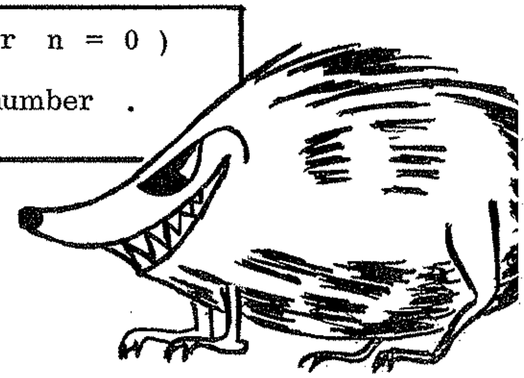
Willie discovered that:

> The square of any odd natural number can be represented in the form $8n + 1$ where n is a natural number.

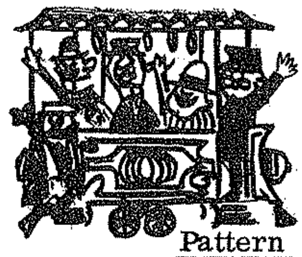
But Willie discovered something more. In $8n + 1$ the n (except for $n = 0$) was a particular kind of figurate number.

96. Fill in the blank so that the result is a true statement.

> In $8n + 1$ the n (except for $n = 0$) is a ______________ number.



3.4 SQUARES + 1 CURIOSITIES



In the list of curious squares,
Some work and these are very rare.

Example

	Pattern Yes	No
$(3^2 + 5^2) + 1 = (9 + 25) + 1 = 3\ 5$	✓	
$(4^2 + 3^2) + 1 = (16 + 9) + 1 = 2\ 6$		✓

For each of the following determine whether the pattern shown above exists or does not exist.

97.

Squares + 1	Pattern Yes	No
$(6^2 + 4^2) + 1 =$		
$(7^2 + 5^2) + 1 =$		
$(7^2 + 26^2) + 1 =$		
$(43^2 + 50^2) + 1 =$		
$(57^2 + 50^2) + 1 = (3249 + 2500) + 1 = 57\ 50$	✓	
$(93^2 + 26^2) + 1 =$		
$(111^2 + 206^2) + 1 =$		
$(273^2 + 446^2) + 1 =$		
$(403^2 + 491^2) + 1 =$		
$(597^2 + 491^2) + 1 =$		
$(638^2 + 576^2) + 1 =$		
$(727^2 + 446^2) + 1 =$		
$(5673^2 + 4955^2) + 1 =$		

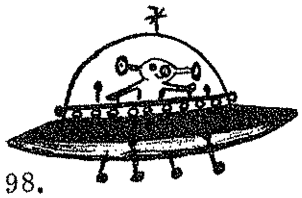
3.5 CUBE ROOTS

Ruth Qube's mini-project on cube roots.

1. Representation. The cube root of an integer z is written as $\sqrt[3]{z}$ or $z^{\frac{1}{3}}$.

2. Definition and examples. $\sqrt[3]{8} = 2$ because $2^3 = 8$

$\sqrt[3]{^-8} = {}^-2$ because $({}^-2)^3 = {}^-8$



Find the cube roots. Use the pattern in the first row.

98.

$\sqrt[3]{27}$	= 3	because	$3^3 = 27$
$\sqrt[3]{^-27}$	=	because	
$\sqrt[3]{64}$	=	because	
$\sqrt[3]{^-64}$	=	because	
$\sqrt[3]{216}$	=	because	

99.

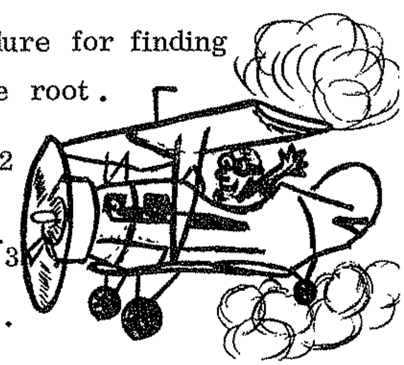
$\sqrt[3]{^-216}$	=	because	
$\sqrt[3]{343}$	=	because	
$\sqrt[3]{^-343}$	=	because	
$\sqrt[3]{512}$	=	because	
$\sqrt[3]{^-512}$	=	because	

FINDING CUBE ROOTS

Ruth Qube had two methods for getting cube roots. One procedure for finding cube roots was to make a chart based on the definition of cube root.

Since $2^3 = 2 \times 2 \times 2 = 8$ then $\sqrt[3]{8} = 2$

$({}^-3)^3 = {}^-3 \times {}^-3 \times {}^-3 = {}^-27$ then $\sqrt[3]{^-27} = {}^-3$



Fill in the chart. Use the pattern in the first row.

100.

Since 4^3 = 64		then	$\sqrt[3]{64} = 4$
$({}^-7)^3$ =	_____	then	__________
8^3 =	_____	then	__________
$({}^-9)^3$ =	_____	then	__________
10^3 =	_____	then	__________

101.

Since $({}^-11)^3$ =	_____	then	__________
12^3 =	_____	then	__________
$({}^-13)^3$ =	_____	then	__________
14^3 =	_____	then	__________
$({}^-15)^3$ =	_____	then	__________

Ruth Qube remembered Ruth Skware's method for finding square roots.

Take $\sqrt{4}$. Now

$$\begin{array}{r} 4 \\ -\ 1 \\ \hline 3 \\ -\ 3 \\ \hline 0 \end{array}$$

Subtract consecutive odd positive integers from 4.

Here there are 2 subtractions. Thus,

$\sqrt{4} = 2$.

Take $\sqrt{9}$. Now

$$\begin{array}{r} 9 \\ -\ 1 \\ \hline 8 \\ -\ 3 \\ \hline 5 \\ -\ 5 \\ \hline 0 \end{array}$$

Subtract consecutive odd positive integers from 9.

Here there are 3 subtractions. Thus,

$\sqrt{9} = 3$.

Ruth Qube wondered: What integers would one use to get cube roots ?

After working on the problem, Ruth Qube found that the integers are given by the formula $3n^2 - 3n + 1$ where n takes on the values 1, 2, 3, ..

102. Fill in the blanks. Use the formula and the given values of n .

n	number	n	number	n	number	n	number
1	1	6		11		16	
2	7	7		12		17	
3	19	8		13		18	
4	37	9		14		19	
5		10		15		20	

Take $\sqrt[3]{8}$. Now

$$\begin{array}{r} 8 \\ -\ 1 \\ \hline 7 \\ -\ 7 \\ \hline 0 \end{array}$$

To find the cube root of a positive integer, subtract consecutive numbers in the chart. Here 2 subtractions.

Thus, $\sqrt[3]{8} = 2$.

Take $\sqrt[3]{^{-}27}$. Now

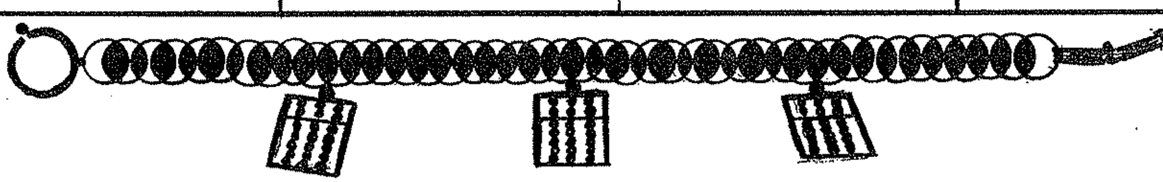
$$\begin{array}{r} ^{-}27 \\ +\ \ 1 \\ \hline ^{-}26 \\ +\ \ 7 \\ \hline ^{-}19 \\ +\ 19 \\ \hline 0 \end{array}$$

To find the cube root of a negative integer, add consecutive numbers in the chart. Here 3 additions.

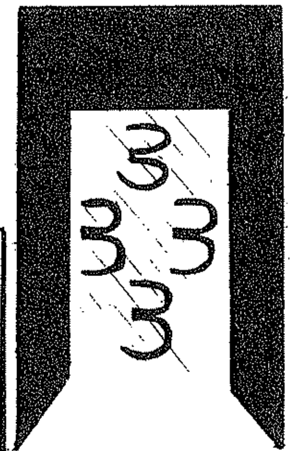
Thus, $\sqrt[3]{^{-}27} = {}^{-}3$

103. Use the chart and Ruth Qube's method. Find the following cube roots.

$\sqrt[3]{64}$ 64 $-\ 1$ $\sqrt[3]{64}$ = ______	$\sqrt[3]{^-125}$ $^-125$ $+\ 1$ $\sqrt[3]{^-125}$ = ______	$\sqrt[3]{216}$ 216 $-\ 1$ $\sqrt[3]{216}$ = ______	$\sqrt[3]{343}$ 343 $-\ 1$ $\sqrt[3]{343}$ = ______
$\sqrt[3]{^-512}$ $^-512$ $\sqrt[3]{^-512}$ = ______	$\sqrt[3]{729}$ 729 $\sqrt[3]{729}$ = ______	$\sqrt[3]{1000}$ 1000 $\sqrt[3]{1000}$ = ______	$\sqrt[3]{^-1331}$ $^-1331$ $\sqrt[3]{^-1331}$ = ______



3.6 REPRESENTATION OF NATURAL NUMBERS - FOUR THREES



FOUR THREES

A fascinating challenge !

Natural number	Representation using four 3 s
0	$0 = (3 - 3) + (3 - 3)$
1	$1 = \frac{3 + 3}{3 + 3}$
2	$2 = \frac{(3)\,3!}{3 \cdot 3}$
3	$3 = \frac{3 - 3}{3} + 3$

Many natural numbers can be represented using four 3 s under the following conditions .

1. Use all four 3 s and only four 3 s in each representation of a natural number .
2. You may use any combination of the following operations
   - a. $+$, $\times$, $-$, $\div$
   - b. $\sqrt{}$ also $\sqrt[3]{}$
   - c. exponents involving 3 s , for example, 3^3 , 33^3
   - d. factorials involving 3 s and bifactorials involving 3 s ,

     Note factorials $3! = 1 \times 2 \times 3$

     $n! = 1 \times 2 \times 3 \times \ldots \times n$

     bifactorials $3!! = 1!\,2!\,3!$

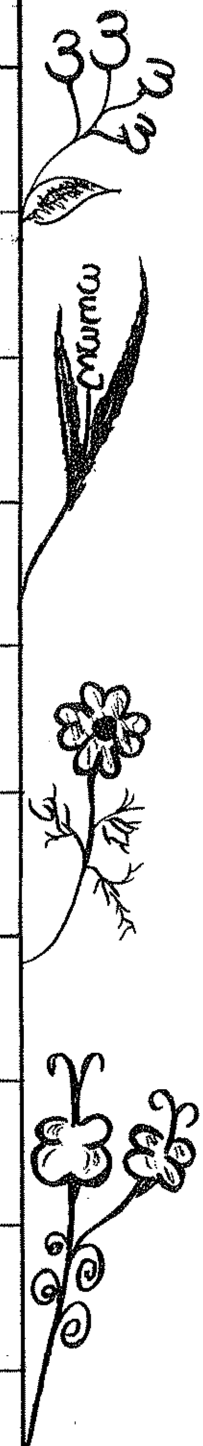
     $n!! = 1!\,2!\,3! \ldots n!$

     Also $0! = 1$
3. You may use
   - a. decimals, that is, .3 , .33 , 3.3 and so on ,
   - b. juxtapositions of the 3 s , that is , 33 , 333 , 3333 ,
   - c. symbols of grouping , that is , ( ) , { } .

For any given natural number more than one representation is often possible using four 3 s .

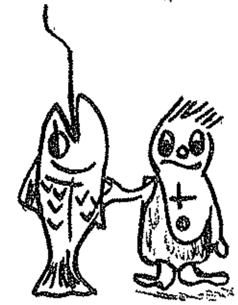
104. Represent each of the natural numbers below using four 3 s .

Number	Representation	Number	Representation
4	$\sqrt{3 \cdot 3} + \frac{3}{3}$	16	
5		17	
6		18	
7		19	
8		20	
9		21	
10		22	
11		23	
12		24	
13		25	
14		26	
15		27	



3.7 REPRESENTATION OF ODD NATURAL NUMBERS - DIFFERENCE OF SQUARES

An immediate consequence of Willie Skware's discovery on page 34 is the following .



$1 = 1^2 - 0^2$ $\qquad$ $5 = 3^2 - 2^2$

$3 = 2^2 - 1^2$ $\qquad$ $7 = 4^2 - 3^2$

105. Follow the pattern in the first row . Fill in the chart .

$9 = 5^2 - 4^2 = 25 - 16 = 9$
11 =
13 =
21 =
23 =
25 =
31 =
33 =
43 =
45 =
55 =
57 =
59 =

106. Fill in the blanks so that the result is a true statement .

Each odd natural number can be expressed as a ____________ of the _________ of two ____________ natural numbers .

3.8 FRACTURED DECIMALS - SUMS AND DIFFERENCES

The following decimal patterns are a challenging curiosity.

SUMS

Example 1   4.2 + 2.4 = 6.6

1.1 (4 + 2) = 1.1 (6) = 6.6   Equality here.

Example 2   1.12 + 21.2 = 22.22

1.1 (1 + 21) = 1.1 (22) = 24.2   Inequality here.

107. Follow the pattern in the first row. Fill in the chart.

	Equality	Inequality
8.3 + 3.8 = 12.1 1.1 (8 + 3) = 1.1 (11) = 12.1	✓	
2.7 + 7.2 = 1.1 (2 + 7) =		
12.3 + 3.21 = 1.1 (12 + 3) =		
13.26 + 62.31 = 1.1 (13 + 62) =		
5.5 + 5.5 = 1.1 (5 + 5) =		
9.7 + 7.9 = 1.1 (9 + 7) =		
1.25 + 52.1 = 1.1 (1 + 52) =		
6.1 + 1.6 = 1.1 (6 + 1) =		
9.8 + 8.9 = 1.1 (9 + 8) =		

108. TOP POPPER   Explain why the equality or inequality holds.

DIFFERENCES

Example 1

4.2 - 2.4 = 1.8

.9 (4 - 2) = .9 (2) = 1.8

Equality here

Example 2

12.5 - 5.21 = 7.29

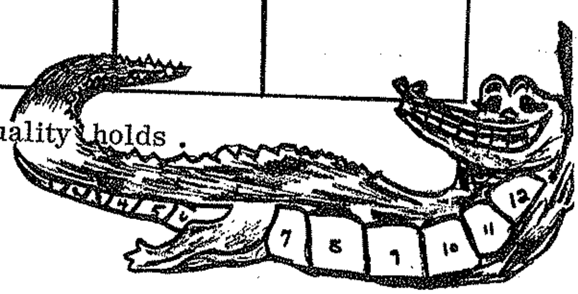
.9 (12 - 5) = .9 (7) = 6.3

Inequality here

109. Follow the pattern in the first row. Fill in the chart.

	Equality	Inequality
5.8 - 8.5 = $^{-}2.7$ .9 (5 - 8) = .9 ($^{-}3$) = $^{-}2.7$	✓	
8.3 - 3.8 = .9 (8 - 3) =		
9.6 - 6.9 = .9 (9 - 6) =		
4.8 - 8.4 = .9 (4 - 8) =		
17.5 - 5.71 = .9 (17 - 5) =		
1.23 - 32.1 = .9 (1 - 32) =		
7.4 - 4.7 = .9 (7 - 4) =		
11.1 - 1.11 = .9 (11 - 1) =		
22.22 - 22.22 = .9 (22 - 22) =		

110. **TOP POPPER** Explain why the equality or inequality holds.

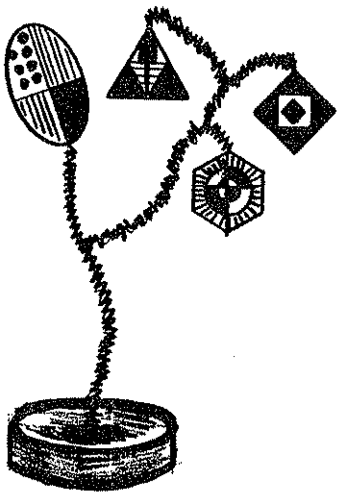


3.9 ADDITION SYMMETRIES

SIM TREE'S SYMMETRIC PATTERNS

The rows of numbers are symmetric with respect to the vertical line .

111. Which sum is the greater ?



1	1
2 1	1 2
3 2 1	1 2 3
4 3 2 1	1 2 3 4
5 4 3 2 1	1 2 3 4 5
6 5 4 3 2 1	1 2 3 4 5 6
7 6 5 4 3 2 1	1 2 3 4 5 6 7
8 7 6 5 4 3 2 1	1 2 3 4 5 6 7 8
+ 9 8 7 6 5 4 3 2 1	+ 1 2 3 4 5 6 7 8 9

Here are some other Sim Tree's symmetry patterns. The numbers in the rows are symmetric with respect to the vertical line . Find the sums .

112.

562 34 + 1	265 43 + 1	721 36 + 2	127 63 + 2	843 25 + 1	348 52 + 1
836 21 + 2	638 12 + 2	936 42 + 1	639 24 + 1	516 61 + 1	615 16 + 1
5134 821 43 + 1	4315 128 34 + 1	6253 712 21 + 2	3526 217 12 + 2	4651 103 13 + 2	1564 301 31 + 2
46012 2301 421 53 + 1	21064 1032 124 35 + 1	65411 3043 212 11 + 2	11456 3403 212 11 + 2	30103 2311 402 71 + 2	30103 1132 204 17 + 2

3.10 SUM PATTERNS IN + AND x TABLES

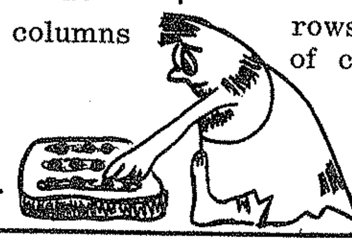
The abbreviated addition and multiplication tables below are formed in the usual way.

+	5	6	7	Sum of rows
5	10	11	12	33
6	11	12	13	36
7	12	13	14	39
Sum of columns	33	36	39	108

108 Total of rows = total of columns .

x	5	6	7	Sum of rows
5	25	30	35	90
6	30	36	42	108
7	35	42	49	126
Sum of columns	90	108	126	324

324 Total of rows = tot of column



Fill in the addition and multiplication tables .
Find the sums of the rows, columns and the totals.

113.

+	8	9	11	Sum of rows
4				
7				
12				
Sum of columns				Total

x	8	9	11	Sum of rows
4				
7				
12				
Sum of columns				Total

+	1	3	8	7	Sum of rows
4					
11					
9					
5					
Sum of columns					Total

x	1	3	8	7	Sum of rows
4					
11					
9					
5					
Sum of columns					Total

3.11 SUBTRACTION BY COMPLEMENTS - BASE TEN, FIVE, TWO.

TEN: Subtraction can be reduced to addition by the method of complements.

Example 1

```
  3 2 4   —— Recopy ——————————▶   3 2 4
- 1 0 8   Write complement      + 8 9 2
                                 1̸ 2 1 6

                                   2 1 6
```

Recopy the minuend.
Write the complement of the subtrahend.

Put down the number which added to 8 is 10 (here 2).

Put down the number which added to 0 is 9 (here 9).

Put down the number which added to 1 is 9 (here 8).

Perform the addition and discard the first digit. Answer is: 2 1 6.

Check

```
  3 2 4
- 1 0 8
  2 1 6
```

Example 2

```
  3 2 4     3 2 4  —— Recopy ————▶   3 2 4
-   6 9   - 0 6 9   Complement     + 9 3 1
                                   1̸ 2 5 5

                                     2 5 5
```

Recopy the minuend.
Subtrahend must always have as many digits as the minuend. Add 0 s.

Put down the number which added to 9 is 10 (here 1).

Put down the number which added to 6 is 9 (here 3).

Put down the number which added to 0 is 9 (here 9).

Perform the addition. Discard the first digit. Answer: 2 5 5

Check

```
  3 2 4
-   6 9
  2 5 5
```

Examples in the brief. Please study.


```
  3 2 4   —— Recopy ——▶   3 2 4
- 1 5 0   Complement    + 8 5 0
                        1̸ 1 7 4

                          1 7 4
```

Check

```
  3 2 4
- 1 5 0
  1 7 4
```

```
  3 2 4 —— Recopy ——▶   3 2 4
- 1 0 0   Complement  + 9 0 0
                      1̸ 2 2 4

                        2 2 4
```

Check

```
  3 2 4
- 1 0 0
  2 2 4
```

114. Find the following differences by the method of complements.

```
  5 6 2          5 6 2
- 3 7 9        + ______

Check   5 6 2
      - 3 7 9
```

```
  7 8 1          7 8 1
- 6 9 5        + ______

Check   7 8 1
      - 6 9 5
```

```
  8 8 8          8 8 8
- 8 8 8        + ______

Check   8 8 8
      - 8 8 8
```

115. Find the following differences by the method of complements.

4372 − 2315 4372 + ____ Check 4372 − 2315	6067 − 379 6067 + ____ Check 6067 − 379	8432 − 760 8432 + ____ Check 8432 − 760
9372 − 1900 9372 + ____ Check 9372 − 1900	8984 − 78 8984 + ____ Check 8984 − 78	7689 − 6000 7689 + ____ Check 7689 − 6000
37893 − 32779 37893 + ____ Check 37893 − 32779	54671 − 4671 54671 + ____ Check 54671 − 4671	63105 − 7210 63105 + ____ Check 63105 − 7210

116. <u>TOP POPPER</u> Restate the procedure for finding complements in a simple way.

FIVE Subtraction by the method of complements in base five follows the pattern for base ten with the changes noted in the examples.

<u>Example 1</u>

$$\begin{array}{r} 3\,4\,3_{five} \\ -\;3\,2\,1_{five} \\ \hline \end{array} \quad \text{Recopy} \longrightarrow \quad \text{Write complement} \quad \begin{array}{r} 3\,4\,3_{five} \\ +\;1\,2\,4_{five} \\ \hline \not{1}\,0\,2\,2_{five} \\ \\ 2\,2_{five} \end{array}$$

Recopy the minuend.
Write the complement of the subtrahend.

Put down the number which added to 1 is 5 (here 4).

Put down the number which added to 2 is 4 (here 2).

Put down the number which added to 3 is 4 (here 1).

Perform the addition in base five, discard the first digit. Answer is: 22_{five}.

<u>Check</u>

$$\begin{array}{r} 3\,4\,3_{five} \\ -\;3\,2\,1_{five} \\ \hline 2\,2_{five} \end{array}$$

<u>Example 2</u>

$$\begin{array}{r} 4\,2\,1_{five} \\ -\;\;3\,2_{five} \\ \hline \end{array} \quad \begin{array}{r} 4\,2\,1_{five} \\ -\;0\,3\,2_{five} \\ \hline \end{array} \quad \text{Recopy} \longrightarrow \quad \text{Comp.} \quad \begin{array}{r} 4\,2\,1_{five} \\ +\;4\,1\,3_{five} \\ \hline \not{1}\,3\,3\,4_{five} \\ 3\,3\,4_{five} \end{array}$$

Recopy the minuend.
Subtrahend must always have as many digits as the minuend. Add 0 s.

Put down the number which added to 2 is 5 (here 3).

Put down the number which added to 3 is 4 (here 1).

Put down the number which added to 0 is 4 (here 4).

Perform the addition in base five, discard the first digit. Answer is: $3\,3\,4_{five}$.

<u>Check</u>

$$\begin{array}{r} 4\,2\,1_{five} \\ -\;\;3\,2_{five} \\ \hline 3\,3\,4_{five} \end{array}$$

<u>Examples in the brief. Please study.</u>

$$\begin{array}{r} 4\,2\,3_{five} \\ -\;2\,3\,0_{five} \\ \hline \end{array} \quad \text{Recopy} \longrightarrow \quad \text{Comp.} \quad \begin{array}{r} 4\,2\,3_{five} \\ +\;2\,2\,0_{five} \\ \hline \not{1}\,1\,4\,3_{five} \\ 1\,4\,3_{five} \end{array}$$

<u>Check</u>

$$\begin{array}{r} 4\,2\,3_{five} \\ -\;2\,3\,0_{five} \\ \hline 1\,4\,3_{five} \end{array}$$

$$\begin{array}{r} 3\,4\,1_{five} \\ -\;2\,0\,0_{five} \\ \hline \end{array} \quad \text{Recopy} \longrightarrow \quad \text{Comp.} \quad \begin{array}{r} 3\,4\,1_{five} \\ +\;3\,0\,0_{five} \\ \hline \not{1}\,1\,4\,1_{five} \\ 1\,4\,1_{five} \end{array}$$

<u>Check</u>

$$\begin{array}{r} 3\,4\,1_{five} \\ -\;2\,0\,0_{five} \\ \hline 1\,0\,1_{five} \end{array}$$

117. Find the following differences by the method of complements.

$$\begin{array}{r} 3\,2\,2_{five} \\ -\;1\,3\,4_{five} \\ \hline \end{array} \quad \begin{array}{r} 3\,2\,2_{five} \\ +\;\underline{\qquad} \end{array}$$

<u>Check</u>

$$\begin{array}{r} 3\,2\,2_{five} \\ -\;1\,3\,4_{five} \\ \hline \end{array}$$

$$\begin{array}{r} 4\,1\,2_{five} \\ -\;2\,4\,3_{five} \\ \hline \end{array} \quad \begin{array}{r} 4\,1\,2_{five} \\ +\;\underline{\qquad} \end{array}$$

<u>Check</u>

$$\begin{array}{r} 4\,1\,2_{five} \\ -\;2\,4\,3_{five} \\ \hline \end{array}$$

$$\begin{array}{r} 3\,2\,1_{five} \\ -\;3\,2\,1_{five} \\ \hline \end{array} \quad \begin{array}{r} 3\,2\,1_{five} \\ +\;\underline{\qquad} \end{array}$$

<u>Check</u>

$$\begin{array}{r} 3\,2\,1_{five} \\ -\;3\,2\,1_{five} \\ \hline \end{array}$$

118.  Find the following differences by the method of complements.

$411_{five} - 43_{five}$ $411_{five} + \underline{\quad\quad}$ Check $411_{five} - 43_{five}$	$324_{five} - 20_{five}$ $324_{five} + \underline{\quad\quad}$ Check $324_{five} - 20_{five}$	$433_{five} - 4_{five}$ $433_{five} + \underline{\quad\quad}$ Check $433_{five} - 4_{five}$
$3132_{five} - 2433_{five}$ $3132_{five} + \underline{\quad\quad}$ Check $3132_{five} - 2433_{five}$	$4213_{five} - 2300_{five}$ $4213_{five} + \underline{\quad\quad}$ Check $4213_{five} - 2300_{five}$	$3142_{five} - 3000_{five}$ $3142_{five} + \underline{\quad\quad}$ Check $3142_{five} - 3000_{five}$
$43124_{five} - 21444_{five}$ $43124_{five} + \underline{\quad\quad}$ Check $43124_{five} - 21444_{five}$	$31242_{five} - 3143_{five}$ $31242_{five} + \underline{\quad\quad}$ Check $31242_{five} - 3143_{five}$	$34124_{five} - 2000_{five}$ $34124_{five} + \underline{\quad\quad}$ Check $34124_{five} - 2000_{five}$

TWO Subtraction by the method of complements in base two is very simple.

Example 1	Example 2
$\begin{array}{r} 111_{two} \\ -\ 101_{two} \\ \hline \end{array}$ — Recopy → $\begin{array}{r} 111_{two} \\ +\ 010_{two} \\ \hline 1001_{two} \\ +\ \ \ 1_{two} \\ \hline \not{1}010_{two} \\ 10_{two} \end{array}$ (Write complement)	$\begin{array}{r} 101_{two} \\ -\ \ 11_{two} \\ \hline \end{array}$ $\begin{array}{r} 101_{two} \\ -\ 011_{two} \\ \hline \end{array}$ — Recopy → $\begin{array}{r} 101_{two} \\ +\ 100_{two} \\ \hline 1001_{two} \\ +\ \ \ 1_{two} \\ \hline \not{1}010_{two} \\ 10_{two} \end{array}$
Recopy the minuend. Write the complement of the subtrahend. To get the complement of the subtrahend, replace each 1 by a 0 and each 0 by a 1. Perform the addition, then add 1. Discard the first digit. Answer is 10_{two}.	Recopy the minuend. Subtrahend must always have as many digits as the minuend. Add 0 s. To get the complement of the subtrahend, replace each 1 by a 0 and each 0 by a 1. Perform the addition, then add 1. Discard the first digit. Answer is: 10_{two}
Check $\begin{array}{r} 111_{two} \\ -\ 101_{two} \\ \hline 10_{two} \end{array}$	Check $\begin{array}{r} 101_{two} \\ -\ \ 11_{two} \\ \hline 10_{two} \end{array}$

119. Find the following differences by the method of complements.

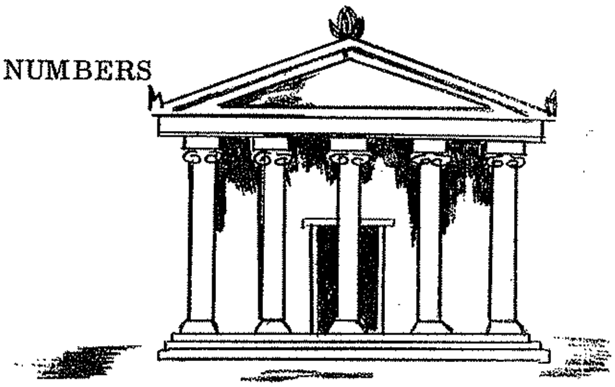
$\begin{array}{r} 101_{two} \\ -\ 100_{two} \\ \hline \end{array}$ $\begin{array}{r} 101_{two} \\ +\ \qquad \\ \hline \end{array}$	$\begin{array}{r} 101_{two} \\ -\ \ 10_{two} \\ \hline \end{array}$ $\begin{array}{r} 101_{two} \\ +\ \qquad \\ \hline \end{array}$	$\begin{array}{r} 111_{two} \\ -\ \ \ 1_{two} \\ \hline \end{array}$ $\begin{array}{r} 111_{two} \\ +\ \qquad \\ \hline \end{array}$
Check $\begin{array}{r} 101_{two} \\ -\ 100_{two} \\ \hline \end{array}$	Check $\begin{array}{r} 101_{two} \\ -\ \ 10_{two} \\ \hline \end{array}$	Check $\begin{array}{r} 111_{two} \\ -\ \ \ 1_{two} \\ \hline \end{array}$

120. Find the following differences by the method of complements.

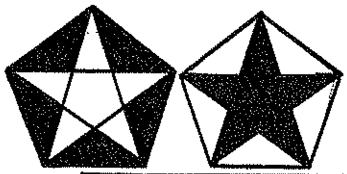
1110_{two} $-\ 1010_{two}$ 1110_{two} $+$ ______ **Check** 1110_{two} $-\ 1010_{two}$	1010_{two} $-\ 110_{two}$ 1010_{two} $+$ ______ **Check** 1010_{two} $-\ 110_{two}$	1001_{two} $-\ 11_{two}$ 1001_{two} $+$ ______ **Check** 1001_{two} $-\ 11_{two}$
1111_{two} $-\ 100_{two}$ 1111_{two} $+$ ______ **Check** 1111_{two} $-\ 100_{two}$	1100_{two} $-\ 11_{two}$ 1100_{two} $+$ ______ **Check** 1100_{two} $-\ 11_{two}$	1001_{two} $-\ 1_{two}$ 1001_{two} $+$ ______ **Check** 1001_{two} $-\ 1_{two}$
11011_{two} $-\ 10010_{two}$ 11011_{two} $+$ ______ **Check** 11011_{two} $-\ 10010_{two}$	11100_{two} $-\ 10000_{two}$ 11100_{two} $+$ ______ **Check** 11100_{two} $-\ 10000_{two}$	11101_{two} $-\ 11101_{two}$ 11101_{two} $+$ ______ **Check** 11101_{two} $-\ 11101_{two}$

CHAPTER 4

WONDER - FULL WORLD OF NUMBERS



4.1 FIBONACCI AND THE GOLDEN SECTION



> " Geometry has two great treasures : one is the Theorem of Pythagoras, the other is the division of a line into extreme and mean ratio. The first we may compare to a measure of gold, the second we may name a precious jewel . "
>
> J. Kepler (1571 - 1630)

The Pythagoreans first tangled with the problem in constructing the pentagram or star pentagon, the symbol of recognition for the Brotherhood .

Euclid gives a solution of the problem in his Elements , Bk. 4, Prop. 11 , Bk. 6, Prop. 30 .

THE PROBLEM

Given : Line segment AB

A ——————————— B

Find : Point C on AB

A ——————————— C ——— B

such that

$$AB : AC = AC : CB$$

or

$$\frac{AB}{AC} = \frac{AC}{CB}$$

The above division of a finite line segment is the "division of a line segment in mean and extreme ratio " . This long description was later replaced by 'the section' , then by the 'Golden section ' .

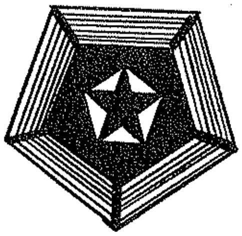
Fra Luca Pacioli (1445-1514) called AB : AC = AC : CB the "Divine Proportion" .

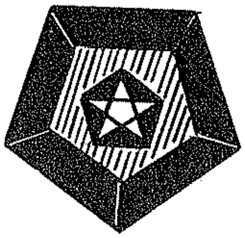
The point C is called the 'Golden Mean' , and x where

$$x = \frac{AB}{AC} = \frac{AC}{CB} \qquad \ldots (1)$$

is the 'Golden Ratio' .

Now to find the value of the Golden Ratio.





From equation (1) page 53	$AC = \dfrac{AB}{x}$	. . . (2)
and	$CB = \dfrac{AC}{x} = \dfrac{AB}{x^2}$	. . . (3)
Now	$AB = AC + CB$	. . . (4)
Substitute (2), (3) in (4)	$AB = \dfrac{AB}{x} + \dfrac{AB}{x^2}$	. . . (5)
Divide by AB	$1 = \dfrac{1}{x} + \dfrac{1}{x^2}$	. . . (6)
and clearing	$x^2 - x - 1 = 0$	. . . (7)

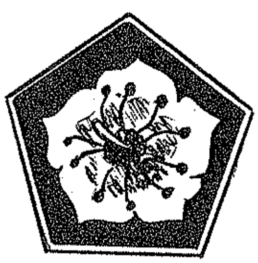
121. Find the two roots of the quadratic equation : $x^2 - x - 1 = 0$ using the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Roots: $x_1 =$ ____________ $x_2 =$ ____________

122. Compute the roots to 8 decimal places using $\sqrt{5} = 2.23606798 \ldots$.

Positive root : ____________ Negative root: ____________



In the computations below, we will use the approximate values

Golden ratio $x \cong 1.6180$

$$\frac{1}{x} \cong \frac{1}{1.6180} \cong 0.6180 \qquad \frac{1}{x^2} \cong \frac{1}{(1.6180)^2} \cong 0.3820$$

Example Given: line segment AB

Length of AB is 4 in .

(a) How far from point A is the Golden mean C ?

(b) What is the length of line segment CB ?

Solution From equation (2) at the top of the page, we have

$$AC = \frac{AB}{x} = \frac{1}{x} \bullet AB \qquad \text{where} \quad AB = 4 \text{ in}, \quad \frac{1}{x} = 0.6180$$

Thus, $AC = 0.6180\,(4 \text{ in}) = 2.4720$ in .

(a) Point C, the Golden mean , is 2.4720 in from point A .

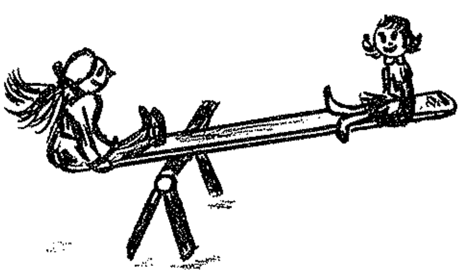
(b) Length of $CB = AB - AC = 4 \text{ in} - 2.4720 \text{ in} = 1.5280$ in .

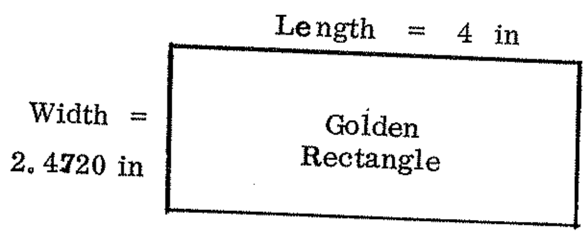
123. Fill in the chart on the pattern in the first row.

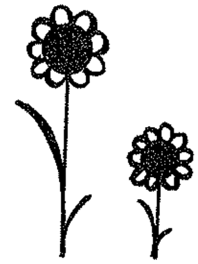
Length of line segment AB	Distance of Golden mean C from A	Length of CB	Summary
A ———— C ———— B AB = 4 in	2.4720 in	1.5280 in	A —— 2.4720 in —— C —— 1.5280 in —— B
A ———— C ———— B AB = 5 in			
A ———— C ———— B AB = 10 in			
A ———— C ———— B AB = 12 in			

GOLDEN RECTANGLE

Use the data in the first problem in the chart to construct the following rectangle.







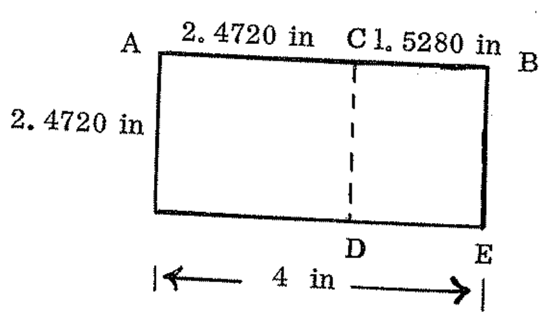
$$\frac{\text{Length}}{\text{Width}} = \frac{4 \text{ in}}{2.4720 \text{ in}} \cong 1.6180$$

Golden Rectangle: A Golden Rectangle is a rectangle in which the ratio of the Length to the Width is the Golden ratio, that is,

$$\frac{\text{Length}}{\text{Width}} = x = 1.618\,033\,989\ldots$$

Consider the Golden rectangle above. If the square of side 2.4720 in is removed from the rectangle the result is the rectangle CDEB.

124.

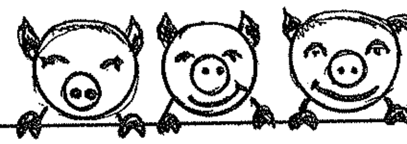


Compute to 4 decimal places

$$\frac{\text{Length}}{\text{Width}} = \frac{BE}{DE} = \frac{2.4720 \text{ in}}{1.5280 \text{ in}} = \_\_\_\_\_\_$$

Is the rectangle CDEB a Golden rectangle?

Yes ________ No ________

125.  Fill in the chart. 

Given: line segment AB = 3 in

a. Find the length of AC, where the point C is the Golden mean.

AC = ____________

b. Find the length of CB.

CB = ____________

c. Draw accurately a rectangle with

AC = Length CB = Width

d. Compute to 4 decimal places: $\frac{AC}{CB}$ = ________

e. Do you have a Golden rectangle ? Yes___ No___

f. Draw accurately a rectangle with

AB = Length AC = Width

g. Compute to 4 decimal places: $\frac{AB}{AC}$ = ________

h. Do you have a Golden rectangle? Yes____ No___

Given: line segment AB = 2 in

a. Find the length of AC, where the point C is the Golden mean.

AC = ____________

b. Find the length of CB.

CB = ____________

c. Draw accurately a rectangle with

AC = Length CB = Width

d. Compute to 4 decimal places : $\frac{AC}{CB}$ = ________

e. Do you have a Golden rectangle?

Yes ______ No ______

f. Draw accurately a rectangle with

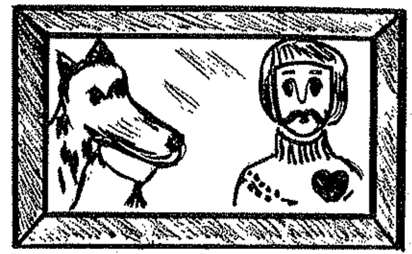
AB = Length AC = Width

g. Compute to 4 decimal places: $\frac{AB}{AC}$ = ________

h. Do you have a Golden rectangle ?

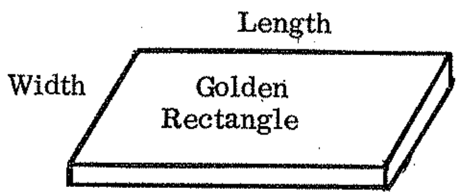
Yes ______ No ________

For the Greeks, the Golden rectangle was one of the most attractive and most pleasing of all rectangles. Temples and state buildings were often built with the Golden rectangle as a base.



Medieval painters often painted their pictures in a Golden rectangle where the important part of the picture was seldom centrally located- it was placed at the Golden mean .

Because of the pleasant appearance of the Golden rectangle, some manufacturers package their goods in rectangular boxes where the face rectangle is a Golden rectangle .



$$\frac{\text{Length}}{\text{Width}} = x = 1.618\ 033\ 989\ldots$$

126. Measure the face rectangle of 4 different rectangular boxes .

Box 1	Box 2	Box 3	Box 4
L = ______	L = ______	L = ______	L = ______
W = ______	W = ______	W = ______	W = ______
$\frac{L}{W}$ = ______	$\frac{L}{W}$ = ______	$\frac{L}{W}$ = ______	$\frac{L}{W}$ = ______

Fibonacci numbers : 1, 1, 2, 3, 5, 8, 13, 21, . . .
F_1, F_2, F_3 F_4, F_5, F_6, F_7, F_8, . . .

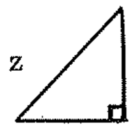
127. Fill in the chart on the pattern of the first row .

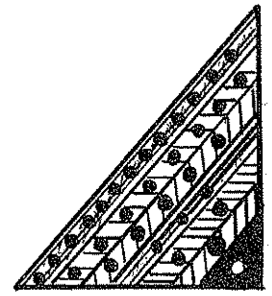
Ratio of successive Fibonacci numbers	Number of places ratio matches Golden ratio 1.618 033 989 . . .	Ratio of successive Fibonacci numbers	Number of places ratio matches Golden ratio 1.618 033 989 . . .
$\frac{F_2}{F_1} = \frac{1}{1} = 1$	1	$\frac{F_{15}}{F_{14}} = \frac{610}{377} =$ ______	
$\frac{F_6}{F_5} = \frac{8}{5} = 1.6$	2	$\frac{F_{18}}{F_{17}} = \frac{2584}{1597} =$ ______	
$\frac{F_9}{F_8} = \frac{34}{21} =$ ______		$\frac{F_{20}}{F_{19}} = \frac{6765}{4181} =$ ______	
$\frac{F_{12}}{F_{11}} = \frac{144}{89} =$ ______		$\frac{F_{25}}{F_{24}} = \frac{75025}{46368} =$ ______	

4.2 FIBONACCI PYTHAGOREAN TRIANGLES

A Pythagorean triangle is a right triangle in which x, y, z are positive integers and

$$z^2 = x^2 + y^2$$

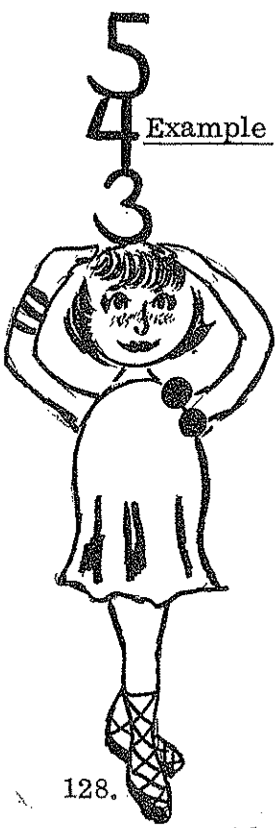




Gigi Luigi discovered a method for getting Pythagorean triangles from the

FIBONACCI NUMBERS

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,...



128.

Example Gigi took 4 successive Fibonacci numbers and lettered them

a	b	c	d
1	1	2	3

Gigi set	$x = ad$	$y = 2bc$	$z = cd - ab$
so that	$x = 1(3) = 3$	$y = 2(1)(2)$	$z = 2(3) - 1(1) = 5$
and thus	$x = 3$	$y = 4$	$z = 5$

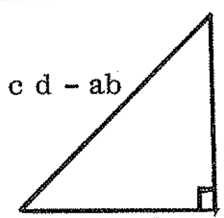
Now x, y, z are the sides of a Pythagorean triangle

since $z^2 = x^2 + y^2$

becomes $5^2 = 3^2 + 4^2$

or $25 = 9 + 16$

a valid relation.



Successive Fibonacci numbers

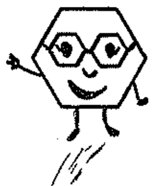
Fill in the chart. Follow the pattern in the first row.

a	b	c	d	$x = ad$	$y = 2bc$	$z = cd - ab$	$z^2 = x^2 + y^2$
1	2	3	5	5	12	13	169 = 25 + 144 = 169
2	3	5	8				
3	5	8	13				
5	8	13	21				
8	13	21	34				
13	21	34	55				
21	34	55	89				

4.3 FIGURATE NUMBERS - RELATIONS AND SUMMARY

Tuto Figaro found he could get all the figurate numbers listed in the chart below from the first row of triangular numbers using the following simple rule .

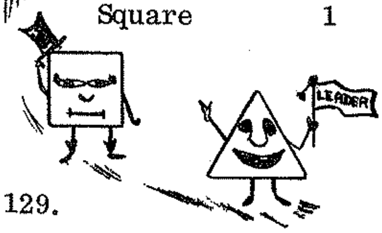
> Each figurate number is equal to the sum of the figurate number immediately above it and the triangular number at the top of the preceding column .



Example 1

To get 2nd, 3rd, square numbers

Triangular	1 ← + 3	← + 6	
Square	1	[4]	[9]
		3 + 1	6 + 3



Example 2

To get 2nd, 3rd, pentagonal numbers

Triangular	1	3	6
Square	1	4	9
Pentagonal	1	[5]	[12]
		4 + 1	9 + 3

129. Fill in the chart . Some figurate numbers are given .

Sides	Name											Formula
3	Triangular	1	3	6	10	15	21	28	36	45	55	$\frac{n}{2}(n+1)$
4	Square	1	4	9				49				n^2
5	Pentagonal	1	5	12							145	$\frac{n}{2}(3n-1)$
6	Hexagonal	1					66					$n(2n-1)$
7	Heptagonal	1		18								$\frac{n}{2}(5n-3)$
8	Octagonal	1			40			133				$n(3n-2)$
9	Nonagonal	1										$\frac{n}{2}(7n-5)$
10	Decagonal	1			52							$n(4n-3)$
11	Undecagonal	1							260			$\frac{n}{2}(9n-7)$
12	Dodecagonal	1		33								$n(5n-4)$
13	Tridecagonal	1										$\frac{n}{2}(11n-9)$
14	Tetradecagonal	1		39						441		$n(6n-5)$
15	Pentadecagonal	1										$\frac{n}{2}(13n-11)$
16	Hexadecagonal	1			88						640	$n(7n-6)$
17	Heptadecagonal	1					231					$\frac{n}{2}(15n-13)$
18	Octadecagonal	1										$n(8n-7)$
19	Nonadecagonal	1							484			$\frac{n}{2}(17n-15)$
20	Icosagonal	1			112							$n(9n-8)$

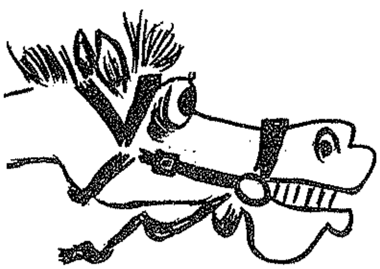
4.4 PRIME NUMBERS - FORMULAS

A challenge of the centuries

> Is there a formula involving n, where n is a natural number, which yields prime numbers only ?

As of the present, no one has found such a formula. Bea Prime was interested in the challenge and did some experimenting .

Experiment Bea Prime wrote several consecutive natural numbers in 2 columns and then circled the prime numbers .



Column 1	Column 2
1	(2)
(3)	4
(5)	6
(7)	8
9	10
(11)	12
.	.
2n + 1	

Bea saw that all the prime numbers except 2 would be in the 1st column. Each number in column 1 was of the form 2n + 1 , where n = 0, 1, 2, 3, ...

130. Every prime number except 2, can be expressed in the form 2n + 1 , where n is some natural number.
Yes ________ No ________

131. Is every number of the form 2n + 1 where n = 1, 2, 3 . . . a prime number?
Yes ________ No ________

Continue Bea's experiment.

132.

Column 1	Column 2	Column 3
1	2	3
4	5	6
7	8	9
10	11	12
.	.	.

a. Circle the prime numbers .

b. Except for the prime number _____ in what columns will the other prime numbers appear ? Ans. ________________

c. Every prime number except _______ can be expressed in the form ________ or ____________, n = ____________

d. Is every number of the form you wrote above a prime number ?
Yes ________ No ________

133.

Column 1	Column 2	Column 3	Column 4
1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16
.	.	.	.

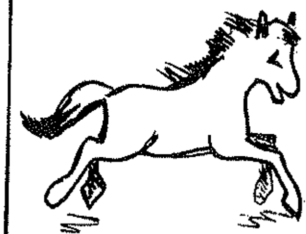
a. Circle the prime numbers .

b. Except for the prime number _____ in what columns will the other prime numbers appear ? Ans . ________________

c. Every prime number except _______ can be expressed in the form ________ or ____________, n = ____________

d. Is every number of the form you wrote above a prime number ?
Yes ________ No ________

134.



Columns

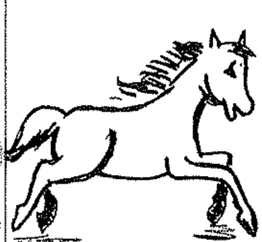
1	2	3	4	5
1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
.	.	.	.	.

a. Circle the prime numbers .

b. Except for the prime number ______ in what columns will the other prime numbers appear ? Ans. ______________________

c. Every prime number except ________ can be expressed in the forms : __________ , __________ , __________ , __________ n = __________

d. Is every number of the form you wrote above a prime number ? Yes ________ No ________



135. Write consecutive natural numbers in 6 columns on the pattern above .

Except for the prime numbers _____ and ____ every prime number can be expressed in the form ____________ or ____________ where n = ____________.

136.

Is every number of the form you wrote at the left a prime number ? Yes ________ No __________

137.

Columns

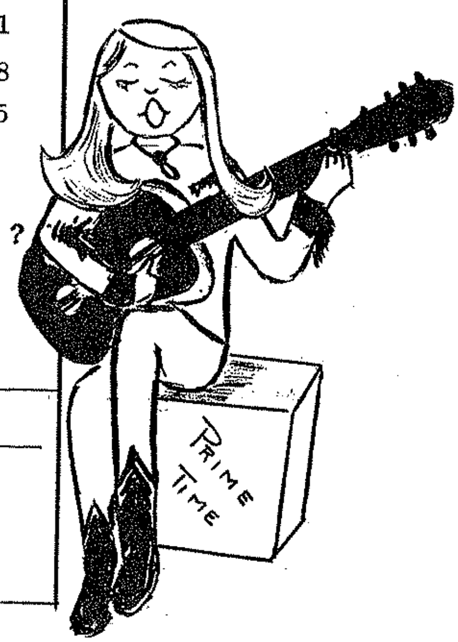
1	2	3	4	5	6	7
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31	32	33	34	35
.	.	.	.	.	.	.

a. Circle the prime numbers

b. Except for the prime number ______ in what columns will the other prime numbers appear ? Ans. ______________________

c. Every prime number except _______ can be expressed in the form __________, __________ __________ , __________, __________ , __________ n = __________ .

d. Is every number of the form you wrote above a prime number ? Yes ________ No ________



4.5 TWO TREE - POWERS AND RECIPROCALS

The Two Tree will prove very useful for many computations. Note: $2^{-n} = \frac{1}{2^n}$

n	2^n	2^{-n}
0	1	1.0
1	2	0.5
2	4	0.25
3	8	0.125
4	16	0.062 5
5	32	0.031 25
6	64	0.015 625
7	128	0.007 812 5
8	256	0.003 906 25
9	512	0.001 953 125
10	1 024	0.000 976 562 5
11	2 048	0.000 488 281 25
12	4 096	0.000 244 140 625
13	8 192	0.000 122 070 312 5
14	16 384	0.000 061 035 156 25
15	32 768	0.000 030 517 578 125
16	65 536	0.000 015 258 789 062 5
17	131 072	0.000 007 629 394 531 25
18	262 144	0.000 003 814 697 265 625
19	524 288	0.000 001 907 348 632 812 5
20	1 048 576	0.000 000 953 674 316 406 25
21	2 097 152	0.000 000 476 837 158 203 125
22	4 194 304	0.000 000 238 418 579 101 562 5
23	8 388 608	0.000 000 119 209 289 550 781 25
24	16 777 216	0.000 000 059 604 644 775 390 625
25	33 554 432	0.000 000 029 802 322 387 695 312 5
26	67 108 864	0.000 000 014 901 161 193 847 656 25
27	134 217 728	0.000 000 007 450 580 596 923 828 125
28	268 435 456	0.000 000 003 725 290 298 461 914 062 5
29	536 870 912	0.000 000 001 862 645 149 230 957 031 25
30	1 073 741 824	0.000 000 000 931 322 574 615 478 515 625
31	2 147 483 648	0.000 000 000 465 661 287 307 739 257 812 5
32	4 294 967 296	0.000 000 000 232 830 643 653 869 628 906 25
33	8 589 934 592	0.000 000 000 116 415 321 826 934 814 453 125
34	17 179 869 184	0.000 000 000 058 207 660 913 467 407 226 562 5
35	34 359 738 368	0.000 000 000 029 103 830 456 733 703 613 281 25
36	68 719 476 736	0.000 000 000 014 551 915 228 366 851 806 640 625
37	137 438 953 472	0.000 000 000 007 275 957 614 183 425 903 320 312 5
38	274 877 906 944	0.000 000 000 003 637 978 807 091 712 951 660 156 25
39	549 755 813 888	0.000 000 000 001 818 989 403 545 856 475 830 078 125

The Two Tree (page 62) also contains powers of 5 in the column under 2^{-n}. Find the secret and do the following problem.

5^3 = ________ 5^7 = ________ 5^{12} = ________

5^5 = ________ 5^9 = ________ 5^{20} = ________

138. There is an easy method for finding the following sums from the Two Tree.

Series	Sum of the series
1 + 2 + 4 =	
1 + 2 + 4 + 8 =	
1 + 2 + 4 + . . . + 16 =	
1 + 2 + 4 + . . . + 32 =	
1 + 2 + 4 + . . . + 128 =	
1 + 2 + 4 + . . . + 512 =	
What is the secret ?	

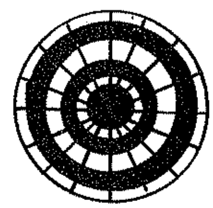
The sum of series such as those above can also be found from the

FORMULA

$$\text{Sum} = \frac{r^n - 1}{r - 1}$$

where n is the number of terms in the series

r = 2 for the series .



Example Take 1 + 2 + 4 Three terms, thus n = 3 and from above r = 2 .

Substitute in the formula :

$$\text{Sum} = \frac{r^n - 1}{r - 1} = \frac{2^3 - 1}{2 - 1} = \frac{8 - 1}{2 - 1} = \frac{7}{1} = 7$$

Find the sum for each of the following series.

139. Use the formula . r = 2 for each series .

Series	n	Sum of series
1 + 2 + 4 + . . . + 32		
1 + 2 + 4 + . . . + 128		
1 + 2 + 4 + . . . + 512		
1 + 2 + 4 + . . . + 2048		
1 + 2 + 4 + . . . + 8192		

140. D. Lemma's Decision

Luce Cass and Simon Degree want to hire D. Lemma as a consultant in mathematics for the whole month of May.

Luce Cass offers to pay D. Lemma \$1,000,000 on the 31st of May.

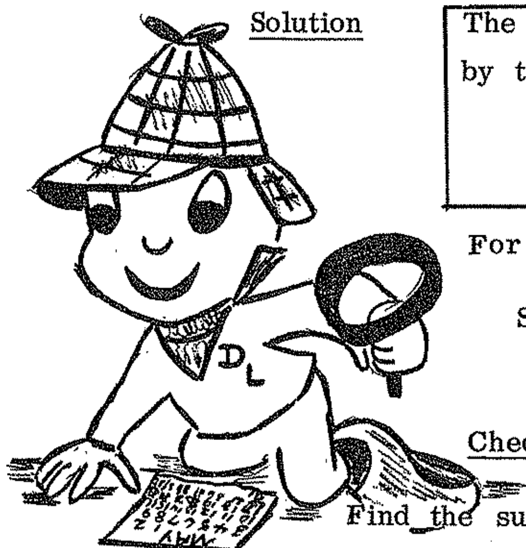
Simon Degree offers to pay D. Lemma

1 cent on May 1st.
2 cents on May 2nd
4 cents on May 3rd
8 cents on May 4th
16 cents on May 5th
and so on through May 31 st.

a. From whose offer would D. Lemma get more money ? ____________

b. How much more ? ______________________________

Example Find the sum of the series $1 + \frac{1}{2} + \frac{1}{4}$.



Solution

The sum of the series of the type shown above is given by the formula

$$\text{Sum} = 2 - 2^{-n+1}$$

where n is the number of terms in th series.

For the series shown above n = 3 and we have

$\text{Sum} = 2 - 2^{-n+1} = 2 - 2^{-3+1} = 2 - 2^{-2}$, use the T Tree

$= 2 - 0.25 = \underline{1.75}$

Check $1 + \frac{1}{2} + \frac{1}{4} = 1 + 0.5 + 0.25 = \underline{1.75}$

141. Find the sum of each of the following series. Use the Two Tree and follow the pattern shown in the first 3 rows.

Series	n	$2 - 2^{-n+1}$
1	1	$2 - 2^{-1+1} = 2 - 2^0 = 2 - 1 = 1$
$1 + \frac{1}{2}$	2	$2 - 2^{-2+1} = 2 - 2^{-1} = 2 - 0.5 = 1.5$
$1 + \frac{1}{2} + \frac{1}{4}$	3	$2 - 2^{-3+1} = 2 - 2^{-2} = 2 - 0.25 = 1.75$
$1 + \frac{1}{2} + \ldots + \frac{1}{8}$		
$1 + \frac{1}{2} + \ldots + \frac{1}{32}$		
$1 + \frac{1}{2} + \ldots + \frac{1}{64}$		
$1 + \frac{1}{2} + \ldots + \frac{1}{512}$		

Example Find the sum of the series $1 + \frac{1}{2} + \frac{1}{4}$

Solution

The sum of the series of the type shown above is given by the formula

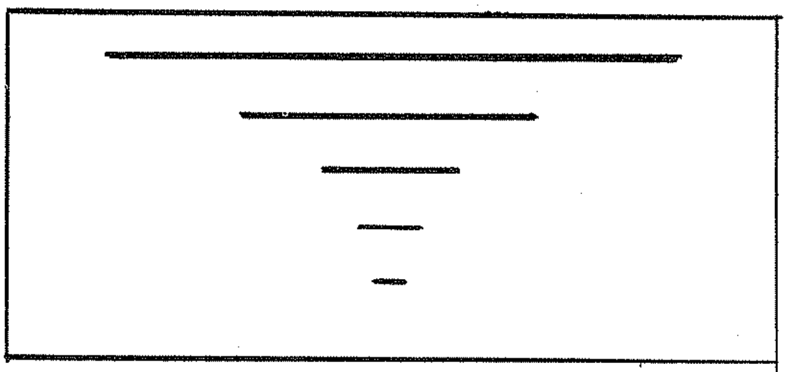
$$\text{Sum} = \frac{r^n - 1}{r - 1}$$

where n is the number of terms in the series, $r = \frac{1}{2}$ for the series.

142. Find the sum of each of the following series. Use the Two Tree (page 62). Follow the pattern in the first row

Series	n	$\frac{r^n - 1}{r - 1}$ where $r = 1/2$
$1 + \frac{1}{2} + \frac{1}{4}$	3	$\frac{(1/2)^3 - 1}{(1/2) - 1} = \frac{(1/8) - 1}{-(1/2)} = \frac{0.125 - 1}{-0.5} = 1.75$
$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$		
$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$		

Cy Delic is making a Pop Poster from narrow strips of paper with lengths as shown below.

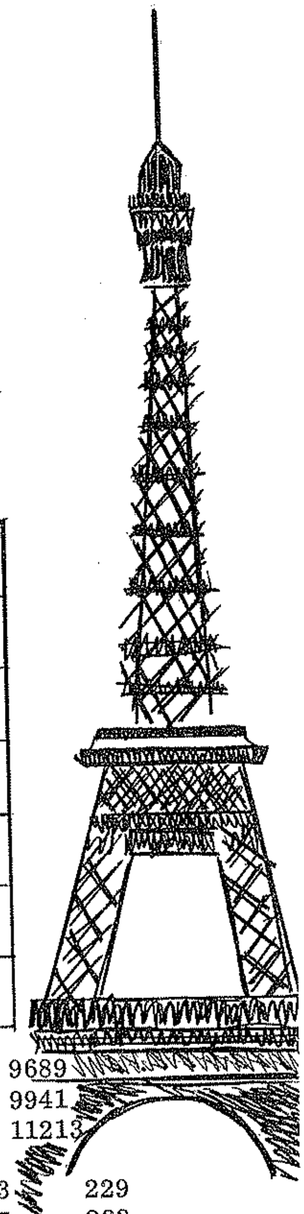


Lengths
1 foot
1/2 foot
1/4 foot
1/8 foot
1/16 foot
and so on

143. Fill in the chart below. Hint: see pp. 64-65.

Number of strips of paper		Total length of paper needed
5	$(1 + \frac{1}{2} + \ldots + \frac{1}{16})$	
7	$(1 + \frac{1}{2} + \ldots + \frac{1}{64})$	
10	$(1 + \frac{1}{2} + \ldots + \frac{1}{512})$	
15	$(1 + \frac{1}{2} + \ldots + \frac{1}{16384})$	
18	$(1 + \frac{1}{2} + \ldots + \frac{1}{131\,072})$	
20	$(1 + \frac{1}{2} + \ldots + \frac{1}{524\,288})$	

4.6 MERSENNE, FERMAT AND PERFECT NUMBERS



MERSENNE NUMBERS

In 1644, Marin Mersenne (1588 - 1648) in his book Cogitata gave a formula which seemed to yield prime numbers.

> **Mersenne numbers:** A Mersenne number is a number of the form
>
> $2^p - 1$ where p is a prime 2, 3, 5, . . .

Fill in the chart. Find the Mersenne numbers for the values of p given. Use the Two Tree on page 62.

144.

p	$2^p - 1$	Prime Yes	Prime No
2	$2^2 - 1 = 4 - 1 = 3$	✓	
3			
5			
7			
11			
13			

Interest element

$2^p - 1$ prime for p :

17	61	127	1279	3217	9689
19	89	521	2203	4253	9941
31	107	607	2281	4423	11213

$2^p - 1$ composite for p :

23	53	83	131	163	193	229
29	59	97	137	167	197	233
37	67	101	139	173	199	239
41	71	103	149	179	211	241
43	73	109	151	181	223	
47	79	113	157	191	227	

$2^n - 1$ is always composite when n is composite.

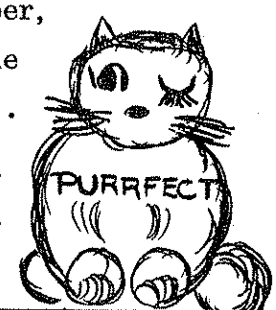
PERFECT NUMBERS

Approximately 1900 years before Mersenne, Euclid (c. 300 B.C.) derived a formula for even perfect numbers in the form

> **Perfect number** $= 2^{n-1}(2^n - 1)$
>
> where n must be a number which makes $(2^n - 1)$ a prime number.

A perfect number is a number that is equal to the sum of its proper divisors. 6, 28, are perfect numbers.

Notice that in Euclid's formula, $(2^n - 1)$, for n = prime number, is a Mersenne number. Since some Mersenne numbers are prime they can be used in Euclid's formula for even perfect numbers.



Compute each of the following. Use the Two Tree on page 62. Each product below is one of the first 8 even perfect numbers.

145.

n	Euclid Perfect number $= 2^{n-1}(2^n - 1)$
2	$2^1(2^2 - 1) = 2(4-1) = 2(3) = 6$
3	
5	
7	
13	
17	
19	
31	

146. Except for 6, each of the above perfect numbers can be written as the sum of the cubes of consecutive odd positive integers. Fill in the chart.

Perfect number from chart above	Sum of cubes of consecutive odd positive integers
28	$1^3 + 3^3 = 1 + 27 = 28$
5th perfect number	$1^3 + 3^3 + \ldots + 127^3$
6th perfect number	$1^3 + 3^3 + \ldots + 511^3$
7th perfect number	$1^3 + 3^3 + \ldots + 1023^3$
8th perfect number	$1^3 + 3^3 + \ldots + 65,536^3$

FERMAT NUMBERS

In 1640, Pierre de Fermat (1601 - 1665) thought he had discovered a formula that yielded prime numbers.

> Fermat numbers : A Fermat number is a number of the form $2^{2^n} + 1$ where $n = 0, 1, 2, \ldots$

Definition $\quad 2^{2^n} = 2^{\overbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdots 2}^{n \text{ of these}}} \qquad 2^{2^0} = 2^1 = 2$

147. Fill in the chart. Find the Fermat numbers. Use the Two Tree page 62.

n	$2^{2^n} + 1$	Prime Yes	Prime No
0	$2^{2^0} + 1 = 2^1 + 1 = 3$	✓	
1			
2			
3			
4			
5	$2^{2^5} + 1 = 2^{32} + 1 = 4\ 294\ 967\ 296 + 1 = 4\ 294\ 967\ 297 = 641(6,700,417)$		✓

Interest element $\quad 2^{2^n} + 1$ composite for n : 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 18, 23, 36, 38, 73.

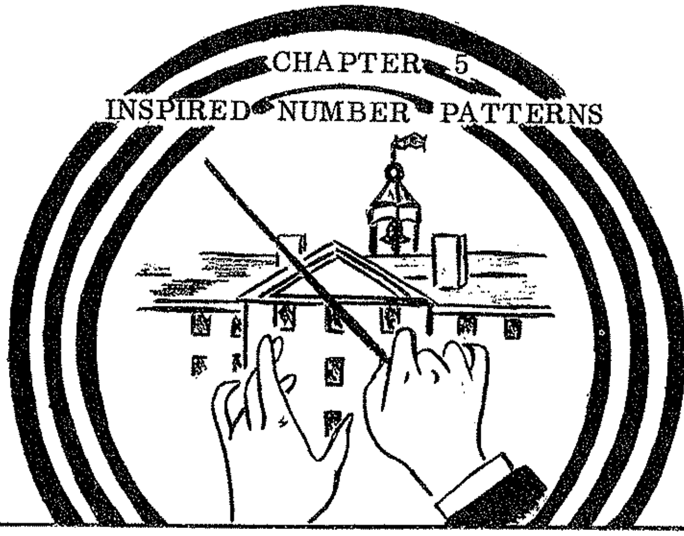
Fermat numbers are used in the formula for regular polygons which can be constructed with straightedge and compass.

Fermat's Theorem

> If n is a positive integer not divisible by the prime p, then $n^{p-1} - 1$ is exactly divisible by the prime p.

148. Fill in the chart. Use the pattern in the first row.

n	p	$n^{p-1} - 1$	$\frac{n^{p-1} - 1}{p}$
2	3	$2^{3-1} - 1 = 2^2 - 1 = 4 - 1 = 3$	$\frac{3}{3} = 1$
2	5		
2	7		
2	11		
2	13		
2	17		



CHAPTER 5
INSPIRED NUMBER PATTERNS

5.1 CONSECUTIVE NATURAL NUMBERS - 2 AND 3 GROUPS

Versa Tilly found an interesting and challenging problem.

> Distribute the set of consecutive natural numbers { 1, 2, 3, . . . , n } into 2 groups .
> No natural number may be placed in a group if it is the sum of any 2 other numbers in that group.

TWO GROUPS

Example The set of consecutive natural numbers is shown above each chart. More than one solution is often possible .

{ 1, 2 }

Group 1	Group 2
1	2

{ 1, 2, 3 }

Group 1	Group 2
1 2	3

{ 1, 2, 3, 4 }

Group 1	Group 2
1	2 3 4

Distribute the given sets of consecutive natural numbers into 2 groups. Follow the rules.

149. {1, 2, 3, . . . 5 }

Group 1	Group 2

150. { 1, 2, 3, . . . 6 }

Group 1	Group 2

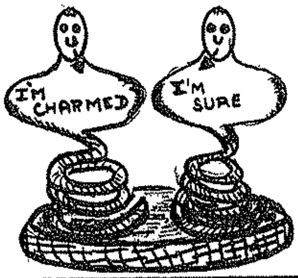
151. { 1, 2, 3, . . . 7 }

Group 1	Group 2

152. { 1, 2, 3, . . . 8 }

Group 1	Group 2

153. TOP POPPER Can you distribute the set { 1, 2, 3, 4, 5, 6, 7, 8, 9 } into 2 groups according to the above rules ? Try it !



Distribute the set of consecutive natural numbers {1, 2, 3, . . . , n } into 3 groups. No natural number may be placed in a group if it is the sum of any 2 other numbers in that group.

THREE GROUPS

Example The set of consecutive natural numbers is shown above each chart. More than one solution is often possible.

{ 1, 2, 3 }

Group 1	Group 2	Group 3
1	2	3

{ 1, 2, 3, 4 }

Group 1	Group 2	Group 3
1	2	3
	4	

{ 1, 2, 3, . . . , 5 }

Group 1	Group 2	Group 3
1	2	3
4	5	

Distribute the given sets of consecutive natural numbers into 3 groups. Follow the rul

154. {1, 2, 3, . . . , 6 }

Group 1	Group 2	Group 3

155. { 1, 2, 3, . . . , 7 }

Group 1	Group 2	Group 3

156. { 1, 2, 3, . . . , 8 }

Group 1	Group 2	Group 3

157. { 1, 2, 3, . . . , 9 }

Group 1	Group 2	Group 3

158. { 1, 2, 3, . . . , 10 }

Group 1	Group 2	Group 3

159. { 1, 2, 3, . . . , 11 }

Group 1	Group 2	Group 3

160. { 1, 2, 3, . . . , 12 }

Group 1	Group 2	Group 3

161. { 1, 2, 3, . . . , 13 }

Group 1	Group 2	Group 3

162. { 1, 2, 3, . . . , 14 }

Group 1	Group 2	Group 3

163. {1, 2, 3, . . . , 15 }

Group 1	Group 2	Group 3

164. { 1, 2, 3, . . . , 16 }

Group 1	Group 2	Group 3

165. { 1, 2, 3, . . . , 17 }

Group 1	Group 2	Group

166. { 1, 2, 3, . . . , 18 }

Group 1	Group 2	Group 3

167. { 1, 2, 3, . . . , 19 }

Group 1	Group 2	Group 3

168. { 1, 2, 3, . . . , 20 }

Group 1	Group 2	Group 3

169. { 1, 2, 3, . . . , 21 }

Group 1	Group 2	Group 3

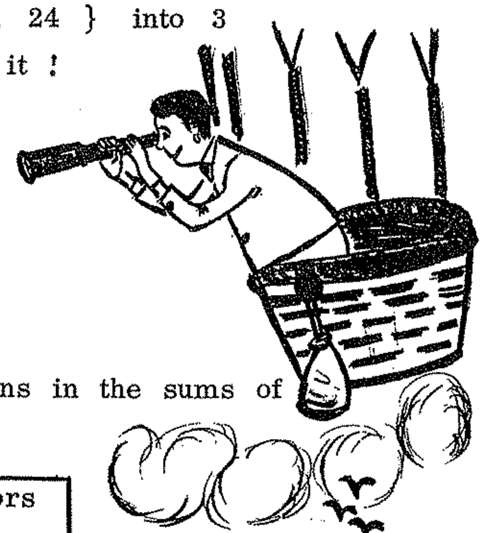
170. { 1, 2, 3, . . . , 22 }

Group 1	Group 2	Group 3

171. { 1, 2, 3, . . . , 23 }

Group 1	Group 2	Group 3

172. TOP POPPER Can you distribute the set { 1, 2, 3, . . . , 24 } into 3 groups according to the above rules ? Try it !



5.2 SUMS OF PROPER DIVISORS OF NUMBERS

A proper divisor of a natural number n is any exact divisor of n except n itself.

Proper divisors of 6 are : 1, 2, 3 .

Seymour Matt found some surprising and remarkable patterns in the sums of the proper divisors of natural numbers .

Seymour's experiment

Find the sum of the proper divisors of a natural number .
Repeat the procedure for the sum .
Continue the process as shown in the example .
Discover what happens !

Example PD in the chart stands for 'Proper Divisors'

n	PD	Sum PD	PD	Sum PD	
3	1 →	1			2 numbers in the chain for 3
4	1, 2 →	3	1 →	1	3 numbers in the chain for 4
5	1 →	1			2 numbers in the chain for 5
6	1, 2, 3 →	6	1, 2, 3 →	6 , pattern repeats	An infinity of 6 s in the chain for 6 .

173. How many numbers will there be in the chain for a prime number ? Ans. ________

Fill in the chart . Follow the procedure shown in the example above .

174.

Number	Chain length
8	3
9	4
10	
12	
14	
15	
16	
18	
20	

175.

Number	Chain length
38	
45	
46	
52	
62	
80	
84	
86	
90	

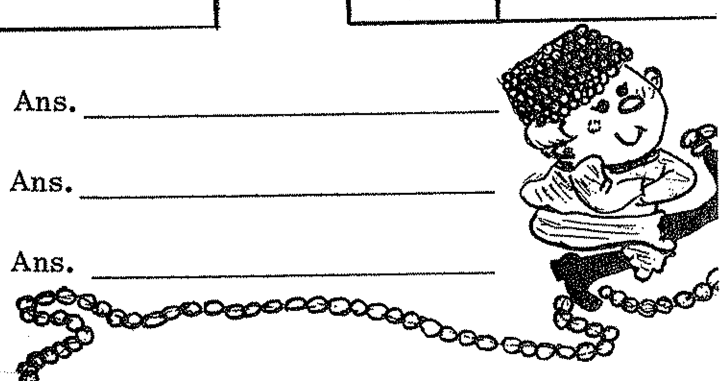
176.

Number	Chain length
30	
42	
54	
60	
66	
78	
95	
102	
114	

177. What is the chain length for 284 ? Ans. ________

178. What is the chain length for 138 ? Ans. ________

179. What is the chain length for 276 ? Ans. ________



5.3 PRIMES LESS THAN OR EQUAL TO A NATURAL NUMBER n

Zoe Prime and Bea Prime knew that the prime natural numbers were not distributed regularly among the natural numbers. One day they asked the interesting question :

> How many prime natural numbers are there less than or equal to a given natural number n ?

180. Fill in the chart. Follow the pattern in the first few rows.

n	Prime numbers ≤ n	Total number
0	none	0
1	none	0
2	2	1
3	2, 3	2
4	2, 3	2
5		
6		
7		
8		
9		
10		
11		
12		
13		
14		
15		
16		
17		
18		
19		

181.

n	Prime numbers ≤ n	Total number
20		
21		
22		
23		
24		
25		
26		
27		
28		
29		
30		

Interest element

> The prime natural numbers less than or equal to n are not distributed regularly. There is no formula that gives the total number of prime natural numbers less than or equal to n .

5.4 PRIMES 4 n + 1 AND SQUARES. SQUARES BETWEEN SUMS OF PRIMES

From the problems on page 60, Izzo Prime found that many prime numbers can be written in the form $4n + 1$, where n is a natural number.

Example Take 5, then $5 = 4 \cdot 1 + 1$

13, then $13 = 4 \cdot 3 + 1$.

After some experimentation, Izzo Prime made the following discovery:

> A prime natural number which can be expressed in the form $4n + 1$, n a natural number, can always be written as the sum of 2 perfect squares of natural numbers.

Fill in the chart. Follow the pattern in the first row. Find the squares.

182.

Prime	4 n + 1	Sum of squares
5	$4 \cdot 1 + 1$	$2^2 + 1^2 = 5$
13	$4 \cdot 3 + 1$	$3^2 + 2^2 = 13$
17		
29		
37		
41		
53		
61		
73		
89		
97		
101		
109		
113		
137		

183.

Find primes of the form 4 n + 1		
Prime	4 n + 1	Sum of squares

An item for a Math Treasure Chest.

Let

$P_1 = 2$	$= 2$	where P_1 is the sum of the 1st prime
$P_2 = 2 + 3$	$= 5$	P_2 is the sum of the first 2 primes
$P_3 = 2 + 3 + 5$	$= 10$	P_3 is the sum of the first 3 primes

and so on.

From above

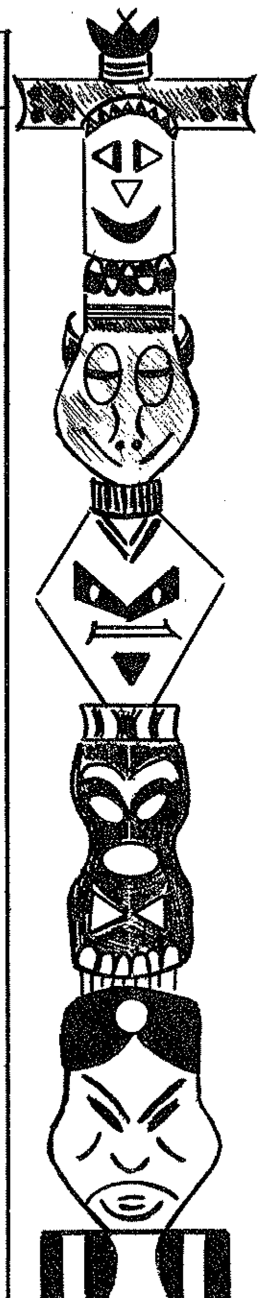
$P_1 = 2$, $P_2 = 5$ ▶ and between P_1 and P_2 there is a perfect square $2^2 = 4$

$P_2 = 5$, $P_3 = 10$ ▶ and between P_2 and P_3 there is a perfect square $3^2 = 9$

Fill in the charts. Study and follow the pattern.

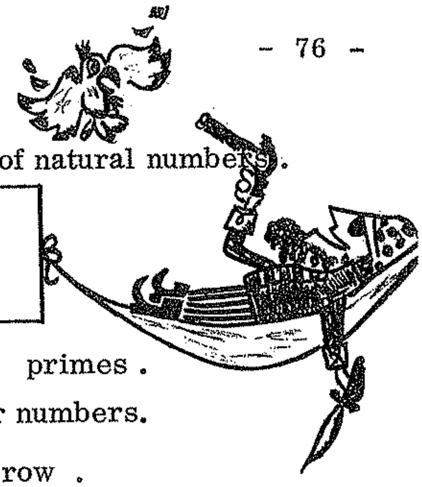
184.

Consecutive primes		Sum	Square
P_1	2	= 2	
	+		← $2^2 = 4$
P_2	3	= 5	
	+		← $3^2 = 9$
P_3	5	= 10	
	+		
P_4	7	=	
	+		
P_5	11	=	
	+		
P_6	13	=	
	+		
P_7	17	=	
	+		
P_8	19	=	
	+		
P_9	23	=	
	+		
P_{10}	29	=	
	+		
P_{11}	31	=	
	+		
P_{12}	37	=	
	+		
P_{13}	41	=	



185.

Consecutive primes		Sum	Square
P_{13}	41	=	
	+		
P_{14}	43	=	
	+		
P_{15}	47	=	
	+		
P_{16}	53	=	
	+		
P_{17}	59	=	
	+		
P_{18}	61	=	
	+		
P_{19}	67	=	
	+		
P_{20}	71	=	
	+		
P_{21}	73	=	
	+		
P_{22}	79	=	
	+		
P_{23}	83	=	
	+		
P_{24}	89	=	
	+		
P_{25}	97	=	



5.5 SUMS AND DIFFERENCES OF SQUARES : PRIMES

Willie Skware found the sum and difference of two squares of natural numbers.

Sum of 2 squares :	$2^2 + 1^2 = 4 + 1 = 5$
Their difference :	$2^2 - 1^2 = 4 - 1 = 3$

Zoe Prime noticed that the sum and difference were primes. Willie Skware and Zoe Prime experimented with other numbers.

186. Fill in the charts. Follow the pattern in the first row.

Squares	Primes Yes	Primes No	Squares	Primes Yes	Primes No
$3^2 + 2^2 = 9 + 4 = 13$ $3^2 - 2^2 = 9 - 4 = 5$	✓		$30^2 + 29^2 =$ $30^2 - 29^2 =$		
$4^2 + 3^2 =$ $4^2 - 3^2 =$			$31^2 + 30^2 =$ $31^2 - 30^2 =$		
$5^2 + 4^2 =$ $5^2 - 4^2 =$			$36^2 + 35^2 =$ $36^2 - 35^2 =$		
$6^2 + 5^2 =$ $6^2 - 5^2 =$			$8^2 + 5^2 =$ $8^2 - 5^2 =$		
$8^2 + 7^2 =$ $8^2 - 7^2 =$			$14^2 + 10^2 =$ $14^2 - 10^2 =$		
$10^2 + 9^2 =$ $10^2 - 9^2 =$			$15^2 + 4^2 =$ $15^2 - 4^2 =$		
$13^2 + 12^2 =$ $13^2 - 12^2 =$			$16^2 + 1^2 =$ $16^2 - 1^2 =$		
$15^2 + 14^2 =$ $15^2 - 14^2 =$			$20^2 + 10^2 =$ $20^2 - 10^2 =$		
$19^2 + 18^2 =$ $19^2 - 18^2 =$			$25^2 + 0^2 =$ $25^2 - 0^2 =$		

187. For some consecutive natural numbers, the sum and difference of the squares are prime numbers. True________ False ________

TOP POPPER

188. For some nonconsecutive natural numbers, the sum and difference of the squares are prime numbers. True________ False ________

5.6 DIVISIBILITY HINTS: 2 THROUGH 11

The following divisibility hints often prove useful for finding the divisors of a composite natural number .

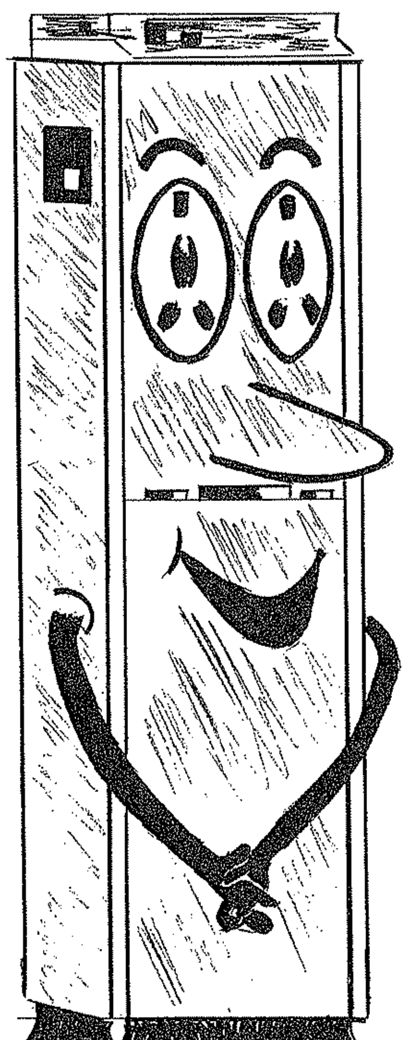
2	A natural number N is divisible exactly by 2 if it ends in 0, 2, 4, 6, 8 .	Example Let N = 124 . Here N ends in 4. N is divisible exactly by 2. 124 ÷ 2 = 62 .
3	A natural number N is divisible exactly by 3 if the sum of the numbers represented by the digits in N is divisible exactly by 3.	Example Let N = 123 . Here the sum of the digits 1 + 2 + 3 = 6 and 6 is divisible exactly by 3. 123 ÷ 3 = 41 .
4	A natural number N is divisible exactly by 4 if the number represented by the last 2 digits of N is divisible exactly by 4 .	Example Let N = 132 . Here 32 is divisible exactly by 4 . 132 ÷ 4 = 33 .
5	A natural number N is divisible exactly by 5 if it ends in 0 or 5 .	Example Let N = 135 . Here 135 ends in a 5 . 135 is divisible exactly by 5 . 135 ÷ 5 = 27 .
6	An even number N which is divisible exactly by 3 is divisible exactly by 6 . Note: No odd number is divisible by 6 .	Example Let N = 138 . Here 138 is even. The sum 1 + 3 + 8 = 12 is divisible exactly by 3 and so is 138 . Thus, 138 is divisible exactly by 6. 138 ÷ 6 = 23 .
7	SEE SUBTLE SEVENS ON PAGE 79.	
8	A natural number N is divisible exactly by 8 if the number represented by the last 3 digits of N is divisible exactly by 8 .	Example Let N = 1136 . Here 136 is divisible exactly by 8 : 136 ÷ 8 = 17. 1136 ÷ 8 = 142 .
9	A natural number N is divisible exactly by 9 if the sum of the numbers represented by the digits in N is divisible exactly by 9 .	Example Let N = 918 . Here the sum of the digits 9 + 1 + 8 = 18 is divisible exactly by 9 . 918 ÷ 9 = 102 .

10	A natural number N is divisible exactly by 10 if it ends in 0 .	Example Let N = 230 . Here 230 ends in 0. $230 \div 10 = 23$.
11	A natural number N is divisible exactly by 11 if the difference of the sums of the digits in the even positions and odd positions is divisible exactly by 11 . If $N = d_5 d_4 d_3 d_2 d_1$ where the d's are the digits in N then d_2 , d_4 are in even positions, d_1, d_3 , d_5 are in odd positions .	Example Let N = 9394 . Sum of digits in odd positions $3 + 4 = 7$, sum in even positions $9 + 9 = 18$, the difference $18 - 7 = 11$. $9394 \div 11 = 854$.

Fill in the charts. Use the hints on pages 77, 78 . State which numbers 2 through 11 divide each given number exactly . See first row .

189.

Number	Divisible by
330	2, 3, 5, 6, 10, 11
484	
1331	
270	
343	
518	
612	
924	
250	
665	
880	
195	
247	
990	
792	



190.

Number	Divisible by
746	
445	
623	
3003	
760	
770	
597	
2310	
273	
540	
810	
2520	
168	
126	
27720	

SUBTLE SEVENS

7

Example 1

Take any one digit number	4
Double the number represented by the digit : 2 x 4 = 8	8
Place 8 in front of the 4	84
Result, 84 , is divisible exactly by 7.	84 ÷ 7 = 12 .

191. Use the procedure in the example . Follow the pattern in the first 2 frames.

Take 1	Take 2	Take 3	Take 5	Take 6
Double 2	Double 4	Double	Double	Double
Place in front 21	Place in front 42	Place in front	Place in front	Place in front
Result 21 ÷ 7 = 3	Result 42 ÷ 7 = 6	Result	Result	Result

Take 7	Take 8	Take 9
Double	Double	Double
Place in front	Place in front	Place in front
Result	Result	Result

Example 2

Take any two digit number	14
Triple the number: 3 x 14 = 42	42
Place 42 in front of 14	4214
Result: 4214 is divisible exactly by 7 .	4214 ÷ 7 = 602

192. Use the procedure in the example . Follow the pattern in the first frame .

Take 20	Take 22	Take 41
Triple 60	Triple	Triple
Place in front 6020	Place in front	Place in front
Result 6020 ÷ 7 = 860	Result	Result

Take 55	Take 63	Take 76	Take 87	Take 98
Triple	Triple	Triple	Triple	Triple
Place in front	Place in front	Place in front	Place in front	Place in front
Result	Result	Result	Result	Result

Example 1

Take any three digit number	131
Place 131 in front of 131	131131
Result : 131131 is divisible exactly by 7.	131131 ÷ 7 = 18733

193. Use the procedure in the example. Follow the pattern in the first frame.

Take 123	Take 224	Take 321
Place 123123	Place	Place
Result	Result	Result
123123 ÷ 7 = 17589		

The general procedure and results of example 1 on page 79 are now used to test a natural number N for divisibility by 7 .

Example 2 Take N = 2625

```
  2 6 2|5     Mark off last digit 5
-   1 0|      Double 5, subtract
  2 5|2       Repeat. Mark off last
-   4|          digit, double, subtract
  2 1         21 is divisible exactly
                by 7 and so is
                2625 .
```

Note: In row 2, we are actually subtracting 105 , and in row 4 we are subtracting 42 .

Thus, 2625 ÷ 7 = 375

Example 3 Take N = 1573

```
  1 5 7|3     Mark off last digit 3
-     6|      Double 3, subtract
  1 5|1       Repeat . Mark off last
-   2|          digit, double, subtract.
  1 3         13 is not divisible exactly
                by 7 and neither is 1573.
```

Note: In row 2, we are actually subtracting 63 , and in row 4 we are subtracting 21 .

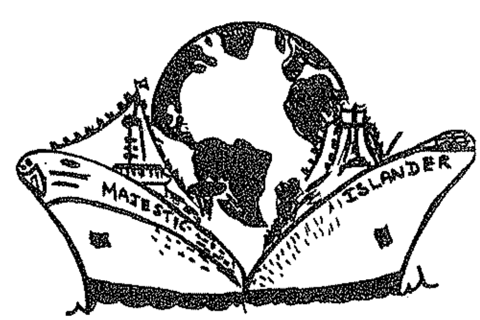
Thus, 1573 ÷ 7 = $224\frac{5}{7}$

Abbreviated form

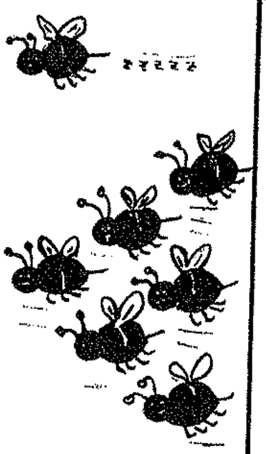
Let N = 947

```
  9 4|7
- 1 4|
    8|0
  - 0|
    8       7 does not divide
              8 exactly .
```

Thus, 947 ÷ 7 = $135\frac{2}{7}$



194. Use examples 2, 3 on page 80. Follow the abbreviated form.



Let N = 399

3 9 9

399 ÷ 7 =

Let N = 3241

3 2 4 1

3241 ÷ 7 =

Let N = 25,252

2 5 2 5 2

25,252 ÷ 7 =

Let N = 342,342

3 4 2 3 4 2

342,342 ÷ 7 =

Let N = 425,361

4 2 5 3 6 1

425,361 ÷ 7 =

Let N = 684,320

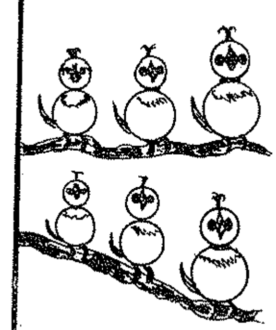
6 8 4 3 2 0

684,320 ÷ 7 =

Let N = 876,543

8 7 6 5 4 3

876,543 ÷ 7 =



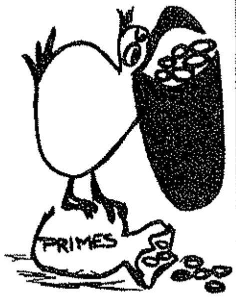
5.7 EXACT DIVISORS FROM PRIME FACTORIZATION

All the exact divisors of a natural number N can be found from the
195. prime factorization of N. The pattern is shown in the first 4 columns.

<u>Hint :</u> Examine the pattern in the sum of the exponents of the exact divisors given in the first 4 columns. Fill in the chart.

N 512	6	12	36	24	72	216	30	60	180
2^9	$2^1 \cdot 3^1$	$2^2 \cdot 3^1$	$2^2 \cdot 3^2$	$2^3 \cdot 3^1$	$2^3 \cdot 3^2$	$2^3 \cdot 3^3$	$2^1 \cdot 3^1 \cdot 5^1$	$2^2 \cdot 3^1 \cdot 5^1$	$2^2 \cdot 3^2 \cdot 5^1$
$2^0 = 1$	$2^0 = 1$	$2^0 = 1$	$2^0 = 1$	$2^0 = 1$	$2^0 = 1$	$2^0 = 1$	$2^0 = 1$	$2^0 = 1$	$2^0 = 1$
2^1	2^1	2^1	2^1						
2^2	3^1	3^1	3^1						
2^3	$2^1 \cdot 3^1$	2^2	2^2						
.		$2^1 \cdot 3^1$	3^2						
.		$2^2 \cdot 3^1$	$2^1 \cdot 3^1$						
.			$2^2 \cdot 3^1$						
.			$2^1 \cdot 3^2$						
2^9			$2^2 \cdot 3^2$						

$2^2 \cdot 3^2 \cdot 5^2$ (900)	
$2^3 \cdot 3^1 \cdot 5^1$ (120)	
$2^3 \cdot 3^2 \cdot 5^1$ (360)	
$2^3 \cdot 3^2 \cdot 5^2$ (1800)	
$2^3 \cdot 3^3 \cdot 5^1$ (1080)	
$2^3 \cdot 3^3 \cdot 5^2$ (5400)	
$2^3 \cdot 3^3 \cdot 5^3$ (27,000)	
$2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1$ (210)	



If the prime factorization of a natural number N is

$N = a^r b^s c^t d^u$, where a, b, c, d are primes

r, s, t, u are natural numbers

then the total number of exact divisors of N is given by the product: $(r+1)(s+1)(t+1)(u+1)$.

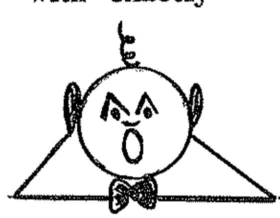
Example If $N = 2^3 \cdot 5^2 \cdot 7^1$, then the total number of exact divisors of N is: $(3+1)(2+1)(1+1) = 24$.

196. Fill in the chart.

N	Prime factorization	Number of exact divisors	Set of exact divisors
36	$36 = 2^2 \cdot 3^2$	$(3)(3) = 9$	1, 2, 3, 4, 9, 6, 12, 18, 36
54			
80			
84			
120			
294			
336			
504			
1200			

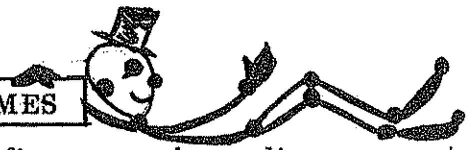
TOP POPPER Find a natural number with exactly

197. a. 11 exact natural number divisors ________
b. 15 exact natural number divisors ________
c. 18 exact natural number divisors ________
d. 25 exact natural number divisors ________



198. Every square of a prime number has exactly ______ exact prime number divisors.

199. Every cube of a prime number has exactly ______ exact prime number divisors.



5.8 FACTOR LATTICES - 4 PRIMES

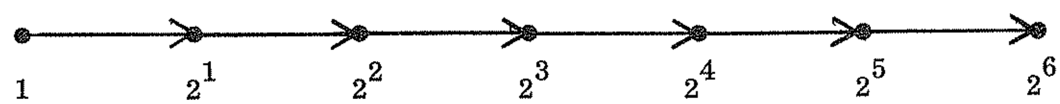
Mazie Dots used geometric figures such as line segments, squares, cubes and so on, to represent the exact natural number divisors of a natural number N .

Mazie Dots called such figures Factor Lattices .

ONE PRIME

Particular instance of the general factor lattice for $N = a^r$, where a is a prime and r is a natural number .

Factor lattice for $N = 64 = 2^6$. $D_{64} = \{ 1 , 2^1, 2^2, 2^3 , 2^4 , 2^5 , 2^6 \}$



Each number is a divisor of the succeeding number.
The numbers are written in exponential form. See page 82 .

200. List 8 composite numbers whose factor lattice has the same pattern as that shown above. Ans. $N = 3^3 = 27$ ________ ________ ________

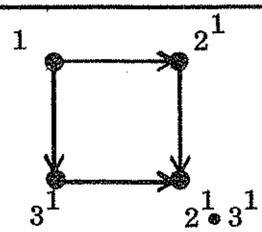
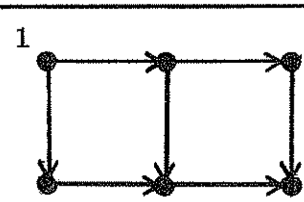
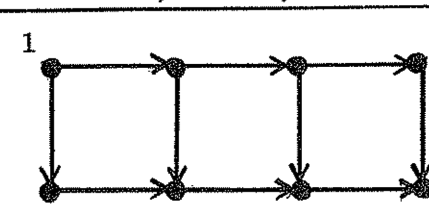
________ ________ ________ ________

TWO PRIMES

Particular instance of the general factor lattice for $N = a^r b^s$, where a, b are primes and r, s are natural numbers .

Follow the pattern in the first frame . Fill in the numbers in the factor lattices. The topmost horizontal line has increasing powers of 2 . The leftmost vertical line has increasing powers of 3 . All other points of the factor lattice are associated with the least common multiple of the numbers directed to the point . Write the numbers in exponential form. See page 82.

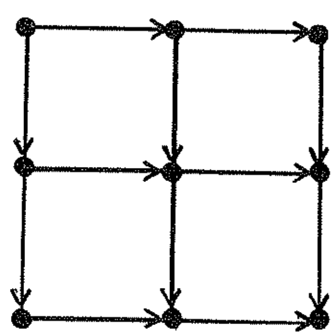
201.

$N = 6 = 2^1 \bullet 3^1$ $D_6 = \{ 1 , 2^1 , 3^1, 2^1 \bullet 3^1 \}$	$N = 12 = 2^2 \bullet 3^1$ $D_{12} = \{ 1, 2^1 , 3^1 , 2^2, 2^1 \bullet 3^1 , 2^2 \bullet 3^1 \}$	$N = 24 = 2^3 \bullet 3^1$ $D_{24} = \{ 1 , 2^1 , 3^1 , 2^2 , 2^1 \bullet 3^1 , 2^3 , 2^2 \bullet 3^1 , 2^3 \bullet 3^1 \}$
		
List 8 composite numbers with the same lattice pattern .	List 8 composite numbers with the same lattice pattern .	List 8 composite numbers with the same lattice pattern .

202.

$N = 36 = 2^2 \cdot 3^2$

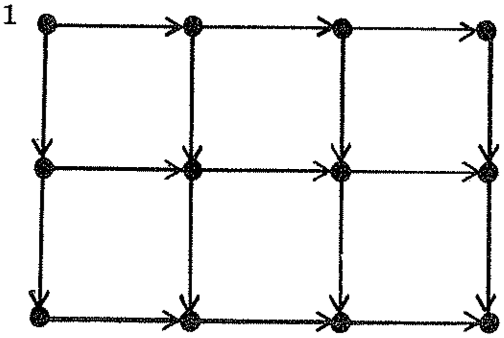
$D_{36} = \{ 1,\ 2^1,\ 3^1,\ 2^2,\ 3^2,\ 2^1 \cdot 3^1,\ 2^2 \cdot 3^1,\ 2^1 \cdot 3^2,\ 2^2 \cdot 3^2 \}$



List 8 composite numbers with the same lattice pattern.

$N = 72 = 2^3 \cdot 3^2$

$D_{72} = \{ 1,\ 2^1,\ 3^1,\ 2^2,\ 3^2,\ 2^1 \cdot 3^1,\ 2^3,\ 2^2 \cdot 3^1,\ 2^1 \cdot 3^2,\ 2^2 \cdot 3^2,\ 2^3 \cdot 3^1,\ 2^3 \cdot 3^2 \}$

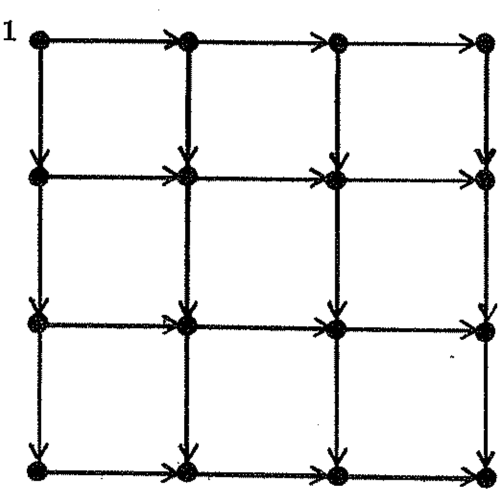


List 8 composite numbers with the same lattice pattern.

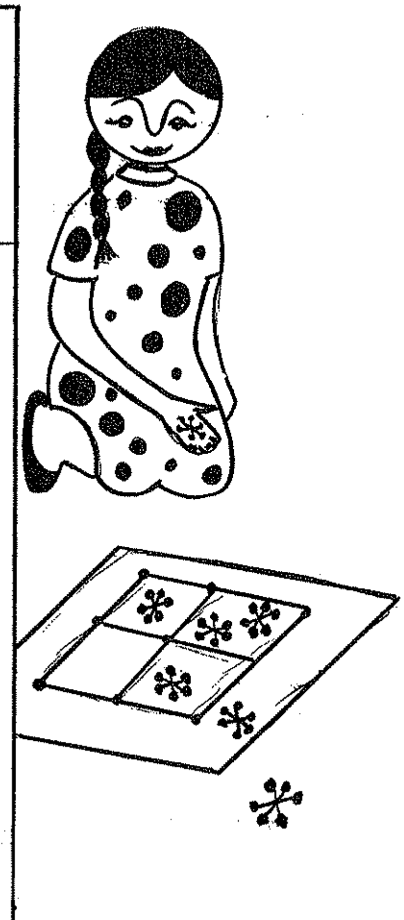
203. Find the divisors. Use exponential form.

$N = 1000 = 2^3 \cdot 5^3$

$D_{1000} = \{ \qquad \}$



List 8 composite numbers with the same lattice pattern.



THREE PRIMES

Particular instance of the general factor lattice for $N = a^r b^s c^t$ where a, b, c are primes and r, s, t are natural numbers.

Follow the pattern in the first frame. Fill in the numbers in the factor lattices

Each of the 3 primes a, b, c is at the end of a line segment from 1.

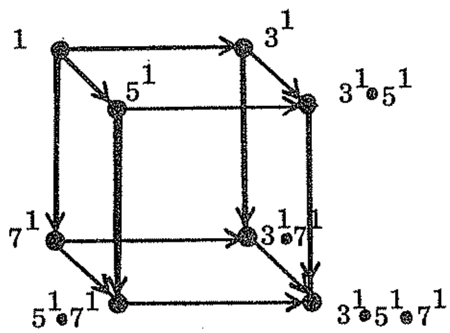
Powers of a, b, c lie along the extensions of the line segments containing a, b, c. Other points of the factor lattice are associated with the least common multiple of the numbers directed to the point.

Write the numbers in exponential form. See page 82.

204.

$N = 105 = 3^1 \bullet 5^1 \bullet 7^1$

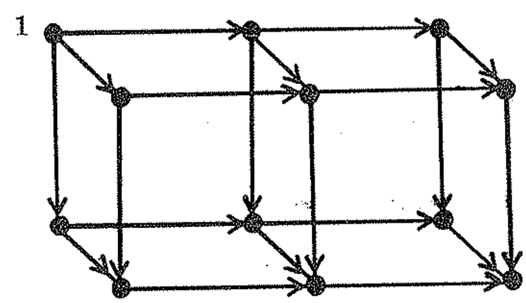
$D_{105} = \{ 1,\ 3^1,\ 5^1,\ 7^1,\ 3^1 \bullet 5^1,\ 3^1 \bullet 5^1,\ 5^1 \bullet 7^1,\ 3^1 \bullet 5^1 \bullet 7^1 \}$



List 8 composite numbers with the same lattice pattern.

$N = 315 = 3^2 \bullet 5^1 \bullet 7^1$

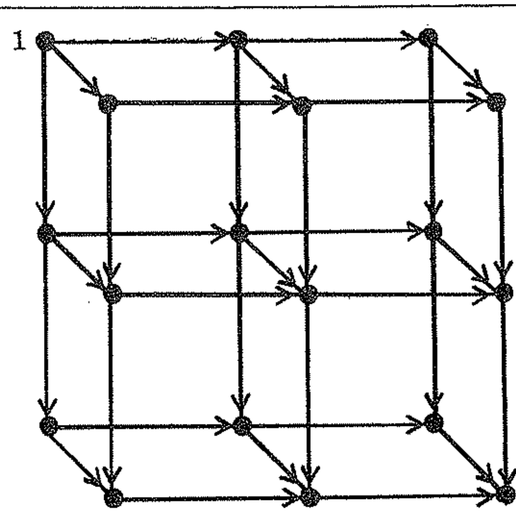
$D_{315} = \{ \qquad \}$



List 8 composite numbers with the same lattice pattern.

$N = 1575 = 3^2 \bullet 5^2 \bullet 7^1$

$D_{1575} = \{ \qquad \}$



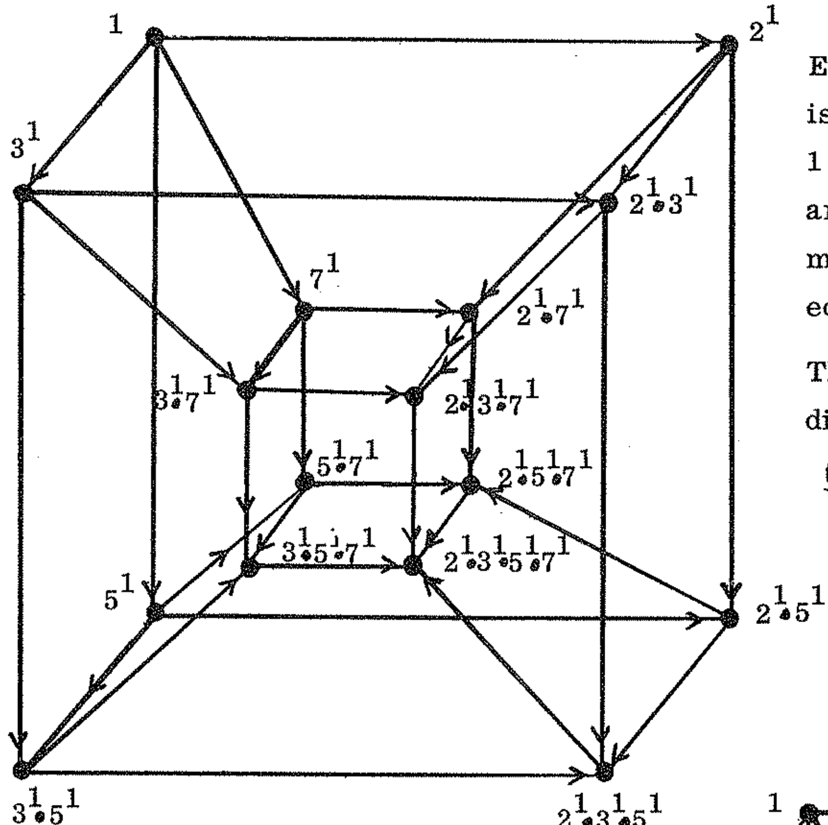
List 8 composite numbers with the same lattice pattern.

FOUR PRIMES

Particular instance of the general factor lattice for $N = a^r b^s c^t d^u$, where a, b, c, d are primes and r, s, t, u are natural numbers .

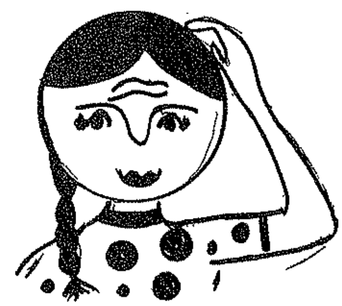
Example $N = 210 = 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1$. Number of divisors $= (1 + 1)(1 + 1)(1 + 1)(1 + 1) = 16$

$D_{210} = \{ 1,\ 2^1,\ 3^1,\ 5^1,\ 7^1,\ 2^1 \cdot 3^1,\ 2^1 \cdot 5^1,\ 2^1 \cdot 7^1,\ 3^1 \cdot 5^1,\ 3^1 \cdot 7^1,\ 5^1 \cdot 7^1,$
$2^1 \cdot 3^1 \cdot 5^1,\ 2^1 \cdot 3^1 \cdot 7^1,\ 2^1 \cdot 5^1 \cdot 7^1,\ 3^1 \cdot 5^1 \cdot 7^1,\ 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1 \}$



Each of the 4 primes a, b, c, d is at the end of a line segment from 1 . Other points of the factor lattice are associated with the least common multiple of the numbers directed to the point .

The figure at the left is a four dimensional figure called a <u>tesseract or hypercube</u> .



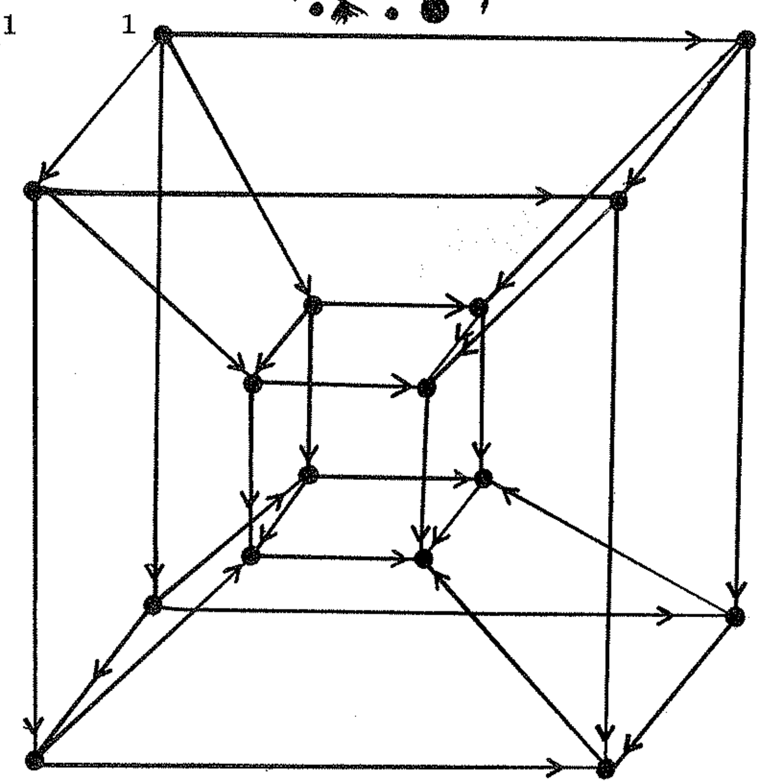
205. Follow the example above . Fill in the numbers for the factor lattice . Write the numbers in exponential form.

$N = 330 =$

Number of divisors:

$D_{330} = \{$

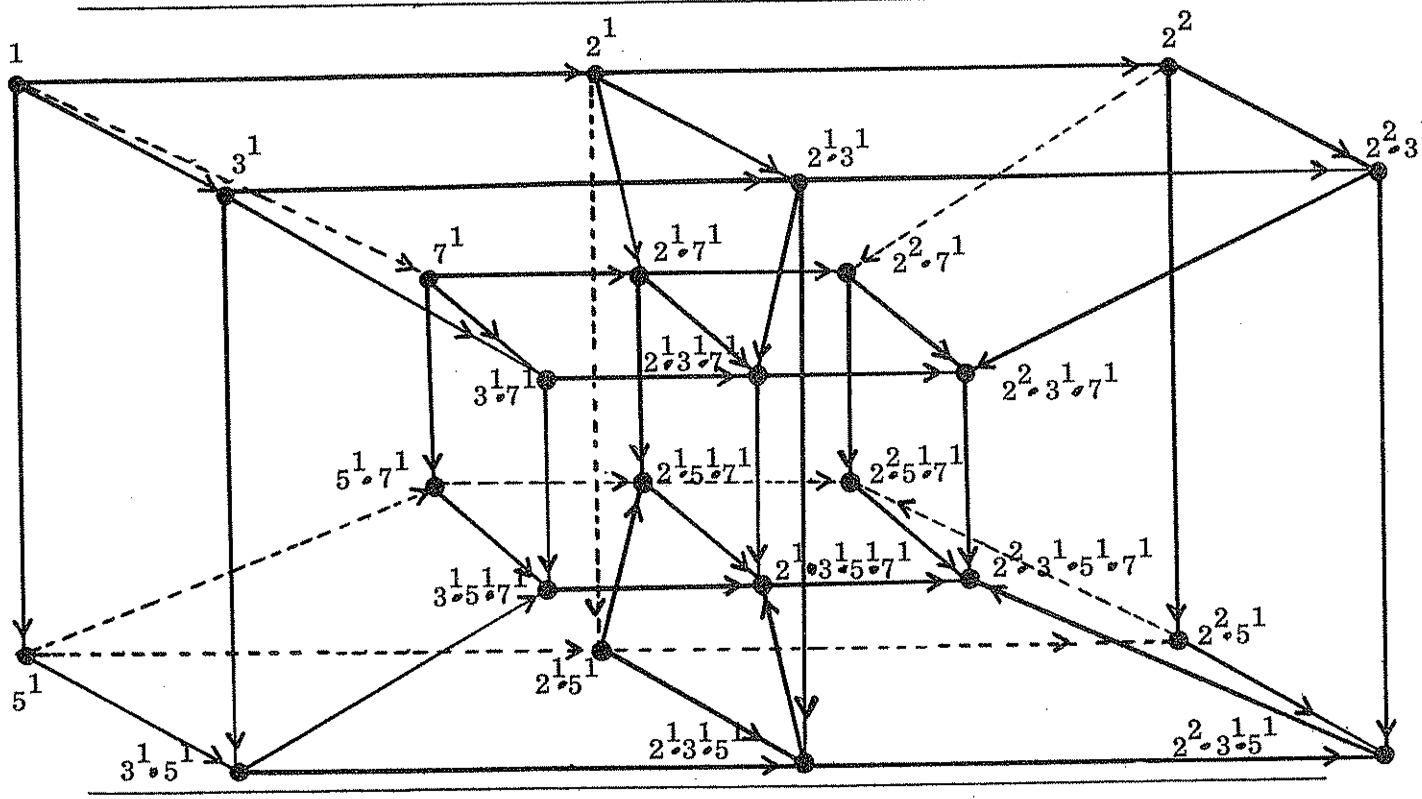
$\}$



List 5 composite numbers with the same lattice pattern .

Example $N = 420 = 2^2 \cdot 3^1 \cdot 5^1 \cdot 7^1$. Number of divisors: $(2+1)(1+1)(1+1)(1+1) = 24$

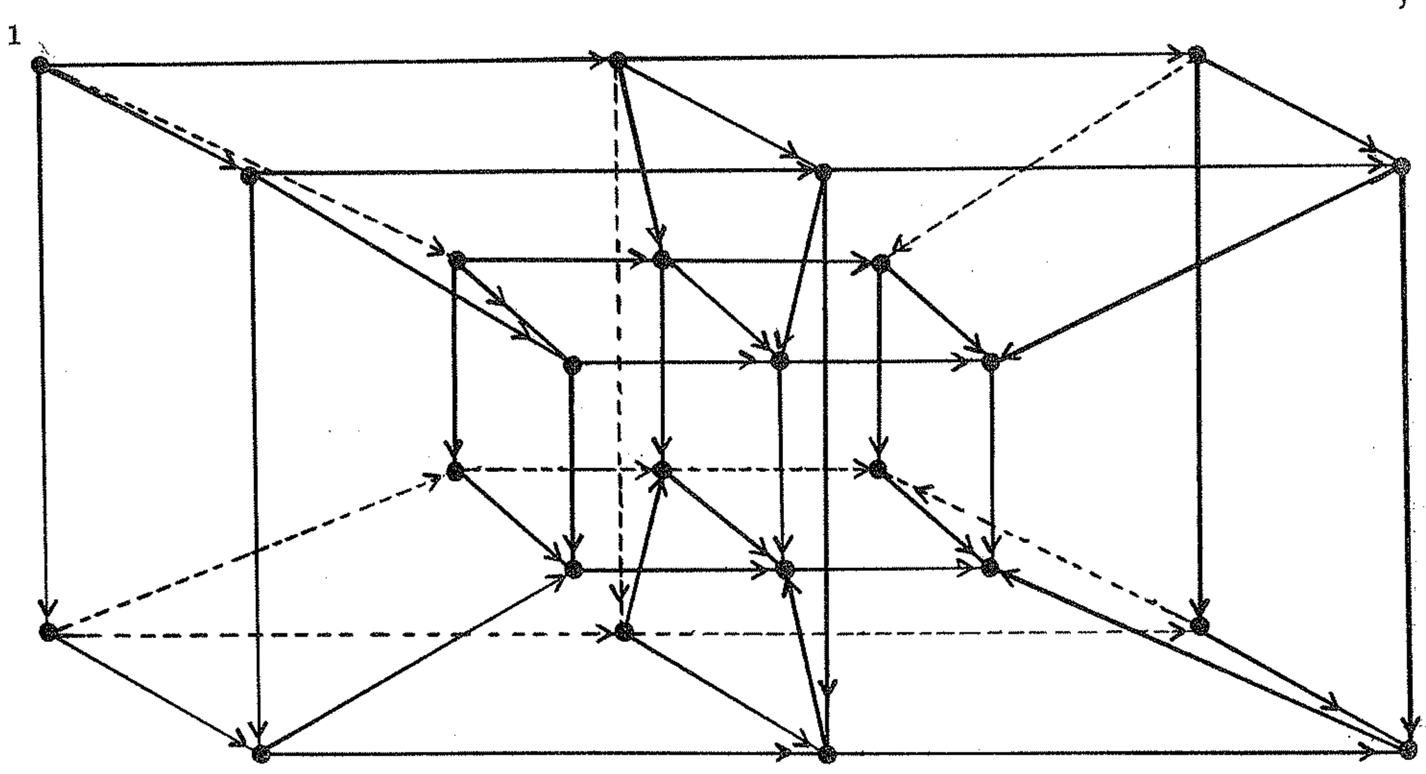
$D_{420} = \{ 1,\ 2^1,\ 3^1,\ 5^1,\ 7^1,\quad 2^2,\ 2^1 \cdot 3^1,\ 2^1 \cdot 5^1,\ 2^1 \cdot 7^1,\ 3^1 \cdot 5^1,\ 3^1 \cdot 7^1,\ 5^1 \cdot 7^1,$
$2^2 \cdot 3^1,\ 2^2 \cdot 5^1,\ 2^2 \cdot 7^1,\ 2^1 \cdot 3^1 \cdot 5^1,\ 2^1 \cdot 3^1 \cdot 7^1,\ 2^1 \cdot 5^1 \cdot 7^1,\ 3^1 \cdot 5^1 \cdot 7^1,$
$2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1,\ 2^2 \cdot 3^1 \cdot 5^1,\ 2^2 \cdot 3^1 \cdot 7^1,\ 2^2 \cdot 5^1 \cdot 7^1,\ 2^2 \cdot 3^1 \cdot 5^1 \cdot 7^1 \}$



206. Fill in the numbers for the factor lattice. $N = 630 =$

$D_{630} = \{$

$\}$

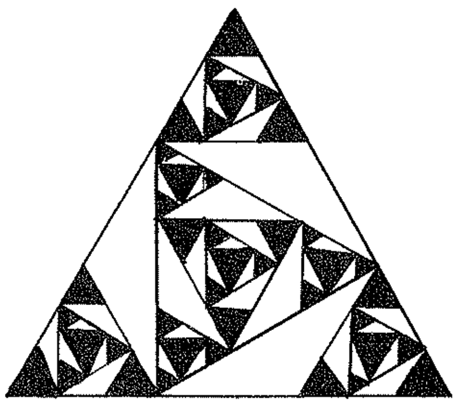


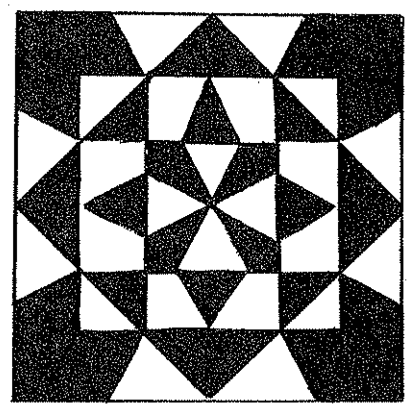
CHAPTER 6

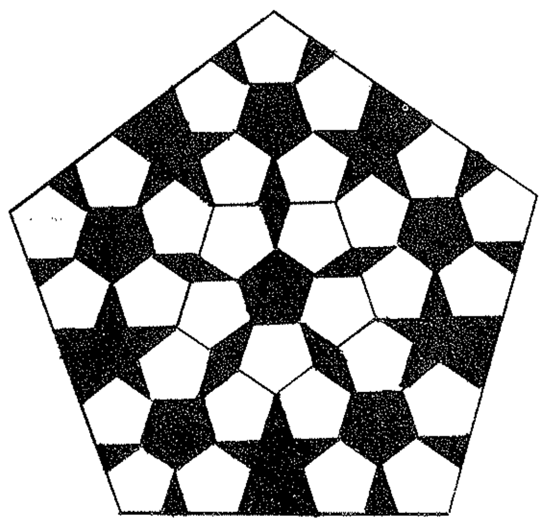
ROYAL AND OTHER ROADS

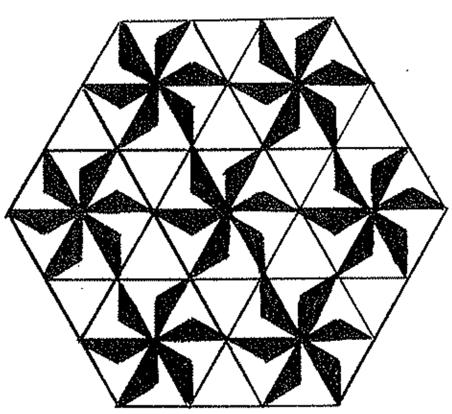
6.1 EUCLID AND POP DESIGNS

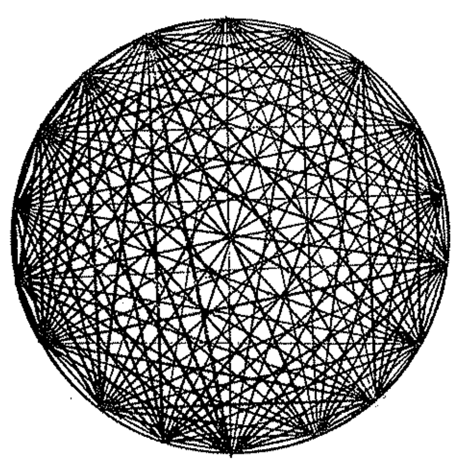
207. Construct each of the following figures using a straightedge and compass. Enlarge the figures if you wish. Copy the block-ins as shown or select your own.











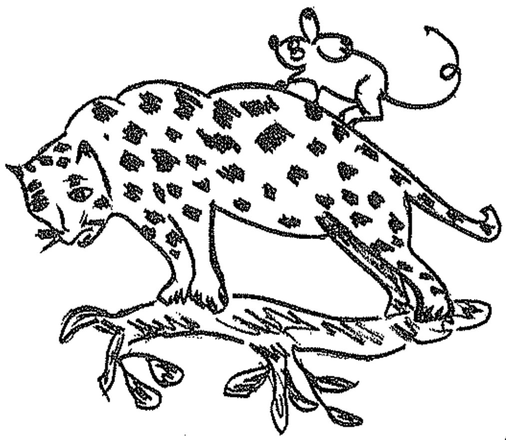
6.2 CHINESE TANGRAMS

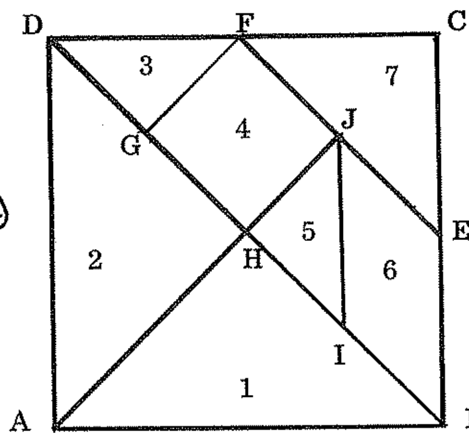
The Chinese Tangram is a delightful and entertaining puzzle. It resembles to some extent our modern jigsaw puzzles. But whereas a jigsaw puzzle is completed in only one way, the charm of the Tangram lies in the extraordinary variety of ways in which the pieces of the puzzle can be put together.

The Tangram became popular during the 19th century.

7 PIECE TANGRAM

Take a square piece of stiff cardboard 2 inches by 2 inches. Mark the square as shown below. Carefully cut out the pieces.

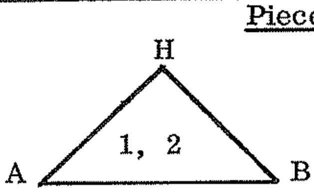




$\overline{BD}$	diagonal of square ABCD	1, 2	large congruent isosceles right triangles
E, F	midpoints of $\overline{BC}$ and $\overline{CD}$	3, 5	small congruent isosceles right triangles
H	midpoint of $\overline{BD}$	4	a square
G	midpoint of $\overline{DH}$	6	a parallelogram
I	midpoint of $\overline{BH}$	7	a medium isosceles right triangle
J	midpoint of $\overline{EF}$		

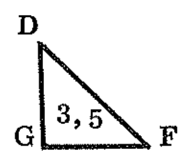
208. Compute the inch-measures of the sides of each of the 7 pieces of the Tangram.

Pieces 1, 2



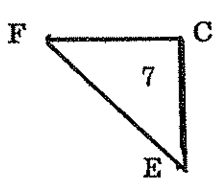
Sides	Inches
$\overline{AH}$	______
$\overline{BH}$	______
$\overline{AB}$	______

Pieces 3, 5



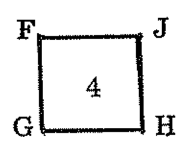
Sides	Inches
$\overline{DF}$	______
$\overline{FG}$	______
$\overline{DG}$	______

Piece 7



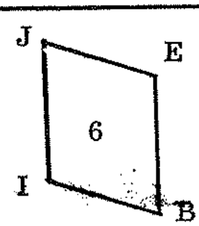
Sides	Inches
$\overline{FC}$	______
$\overline{CE}$	______
$\overline{FE}$	______

Piece 4



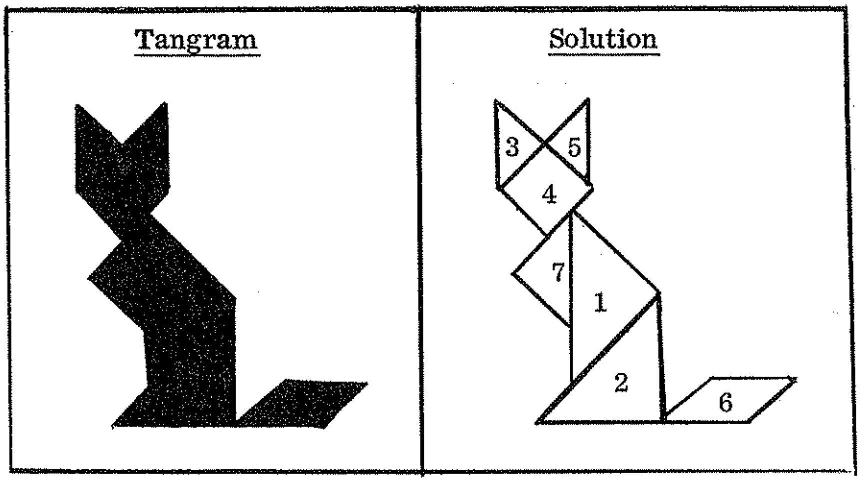
Sides	Inches
$\overline{GF}$	______
$\overline{FJ}$	______
$\overline{JH}$	______
$\overline{HG}$	______

Piece 6

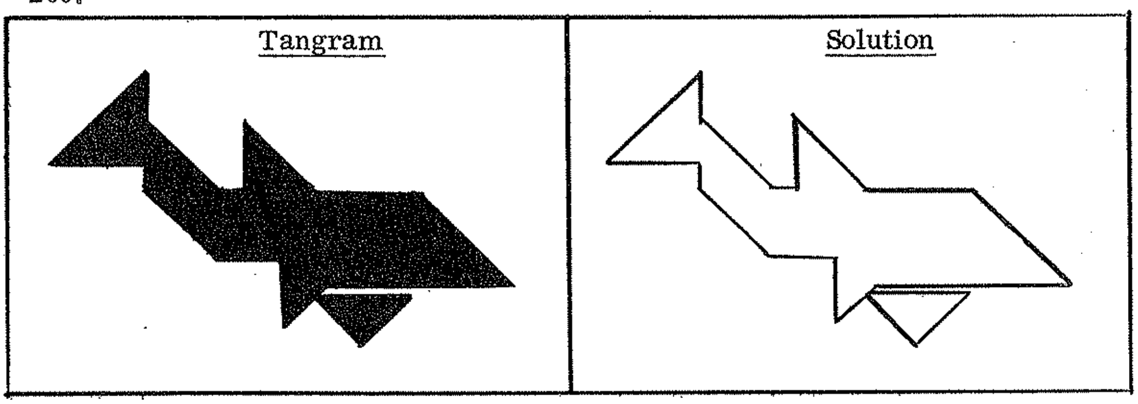


Sides	Inches
$\overline{JE}$	______
$\overline{IB}$	______
$\overline{JI}$	______
$\overline{EB}$	______

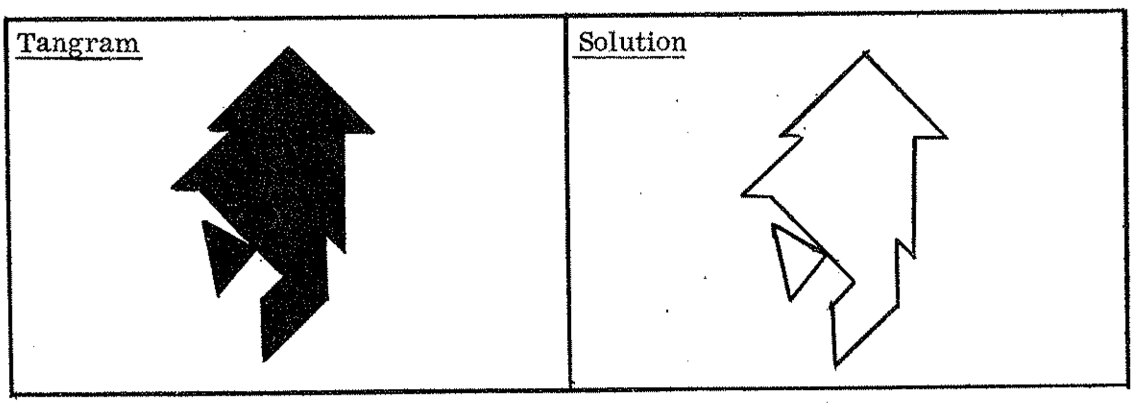
Example Put together the 7 pieces to make the Tangram in the first frame.



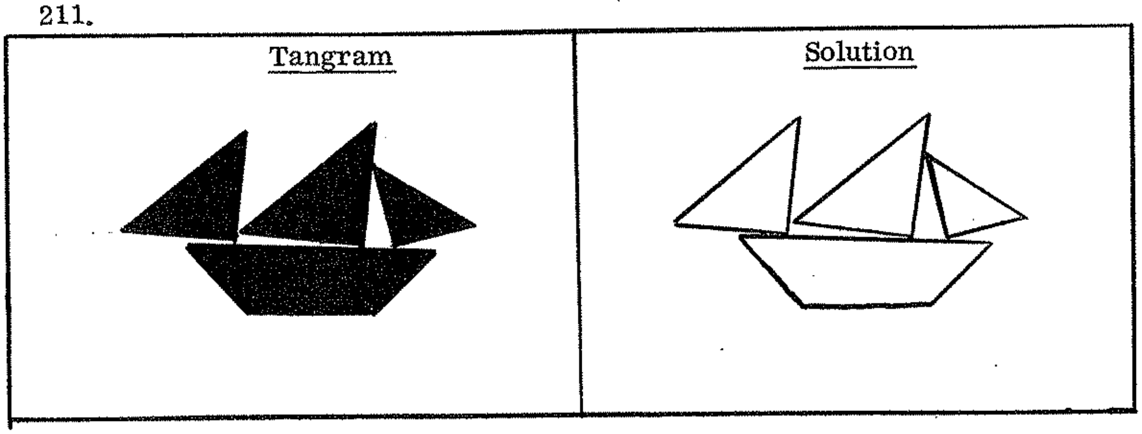
209.



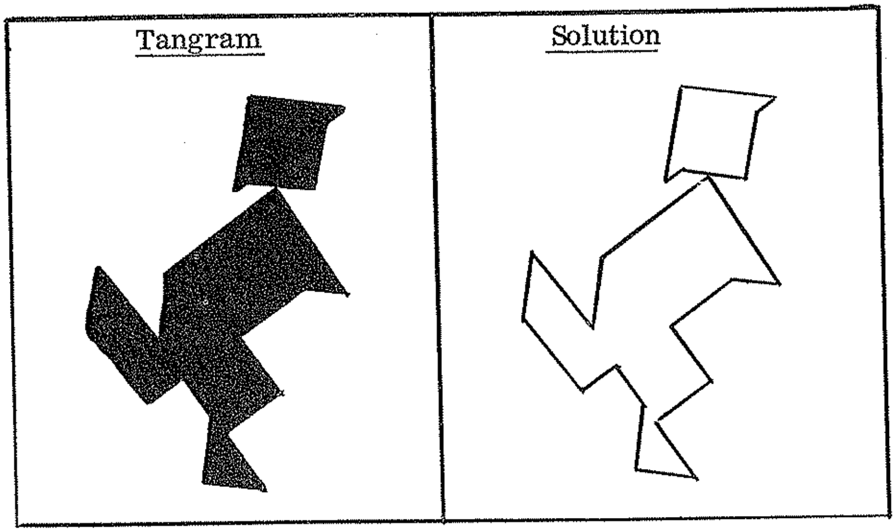
210.

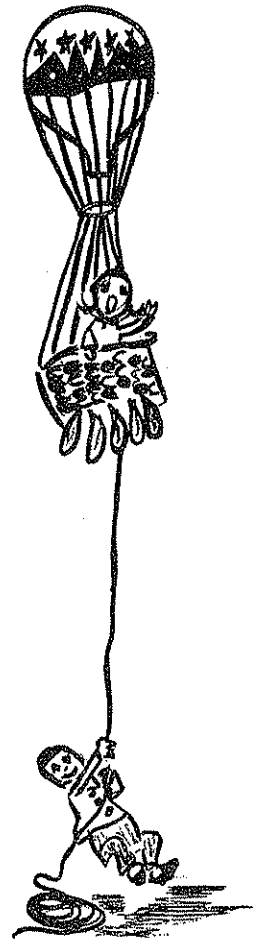


211.

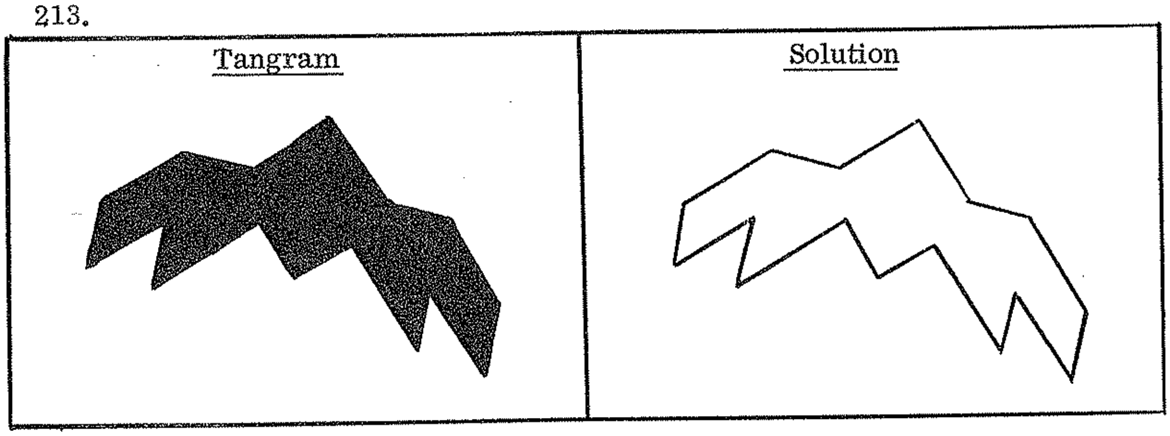


212.

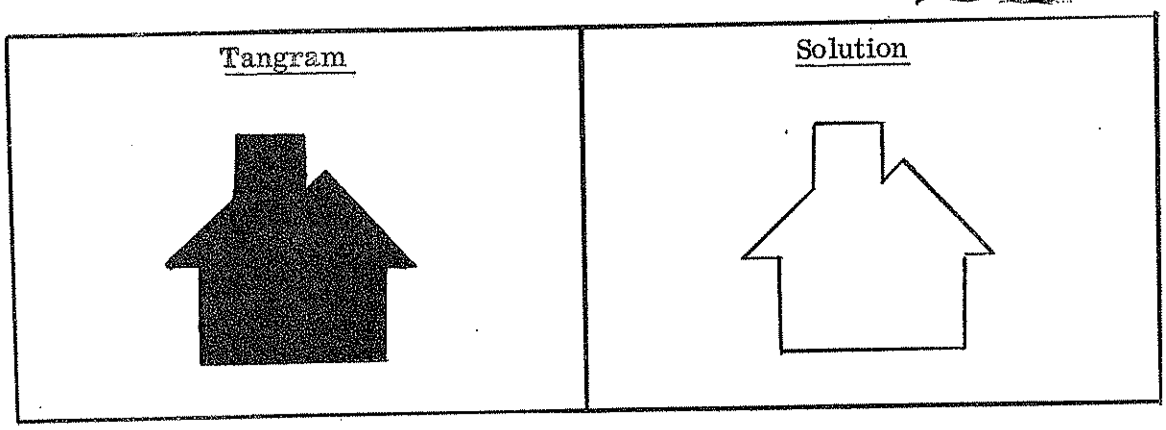




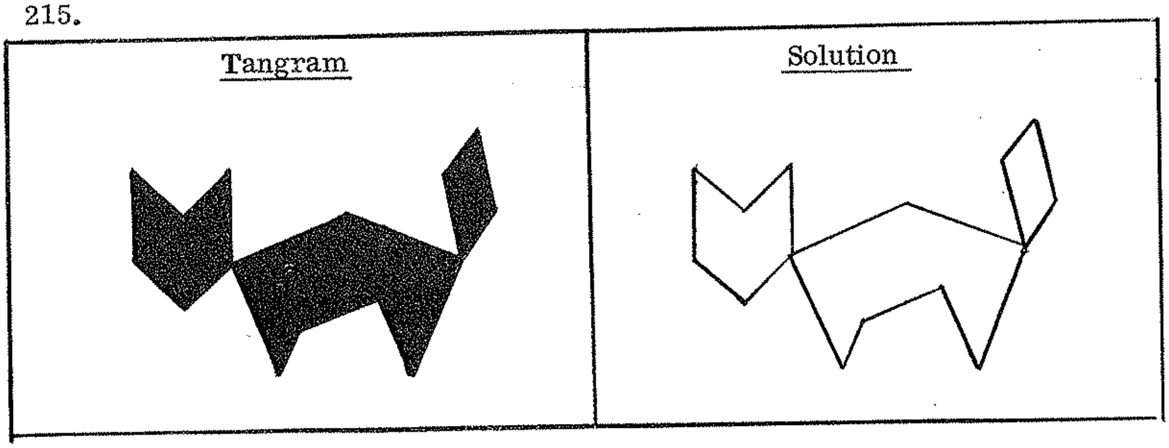
213.



214.



215.

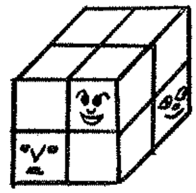


6.3 GEOMETRIC COUNTDOWN - UNIT CUBES

Doozie Qube had a large box of unit cubes. Doozie also had a paint set and brushes.

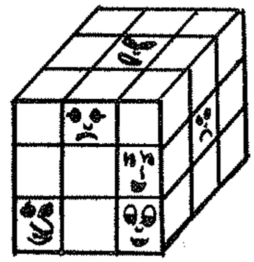
Doozie made a large cube out of $2^3 = 8$ unit cubes. He painted the outside of the large cube. Now

there were $8 \times 6 = 48$ faces in all.
Of the 48 faces, $6 \times 4 = 24$ are painted.
Thus, $48 - 24 = 24$ unpainted faces.



Next, Doozie Qube made a large cube out of $3^3 = 27$ unit cubes. He painted the outside of the large cube.

How many faces on the unit cubes were unpainted ?



216. Fill in the chart on the pattern of the first row.

Number of unit cubes forming the large cube	Total number of faces	Total number of faces painted	Total number of unpainted faces
$2^3 = 8$	$8 \times 6 = 48$	$4 \times 6 = 24$	$48 - 24 = 24$
$3^3 =$			
$4^3 =$			
$5^3 =$			
$6^3 =$			
$7^3 =$			
$8^3 =$			
$9^3 =$			
n^3 where $n = 2, 3, 4, \ldots$			

6.4 POINTS AND UNIT DISTANCES IN A GRID SYSTEM

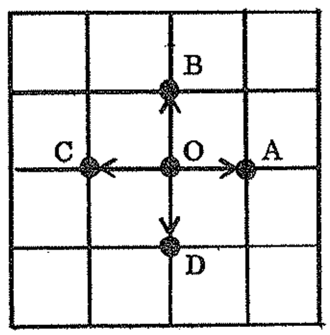
Mini Lattice made a grid system in which the distance between the horizontal lines and the vertical lines was 1 unit. She selected an arbitrary point O in the grid system. Mini was curious about the answer to the question :

> How many points of the grid system are 1, 2, 3, . . . , n units away from the point O ?

Unit distances are measured along horizontal or vertical lines of the grid. Diagonal distances are not considered.

MINI'S EXPERIMENTS

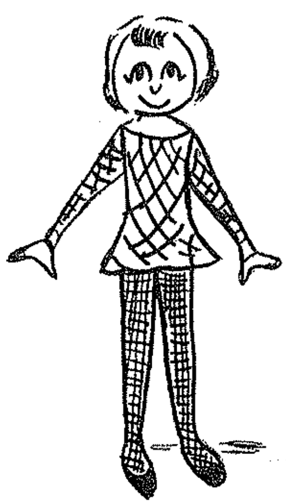
Experiment 1 Points in the grid system which are 1 unit away from O. Points must be 1 unit from point O by any shortest path from O to the points.

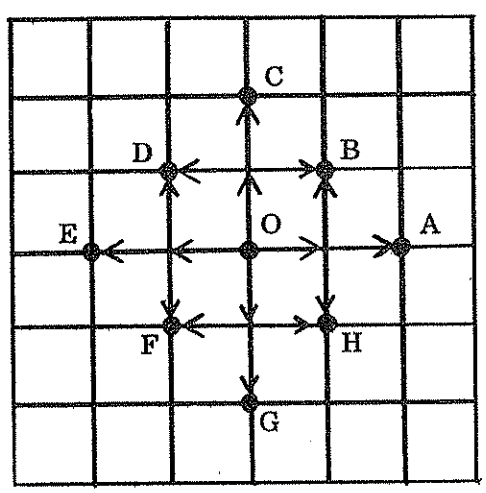


The points A, B, C, D are 1 unit away from O.

Total : 4 points.

Experiment 2 Points in the grid system which are 2 units away from O. Points must be 2 units from point O by any shortest path from O to the points.



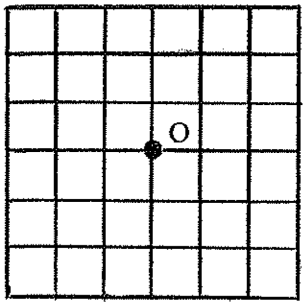


The points A, B, C, D, E, F, G, H are 2 units away from O. In some cases, there are 2 paths which go to a point 2 units away from O.

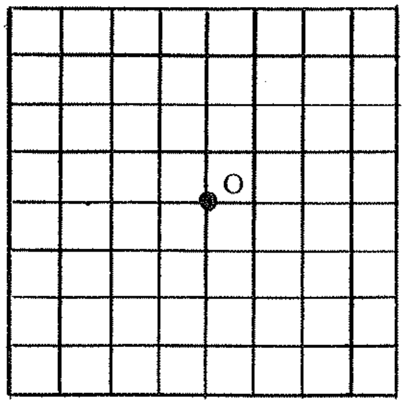
Total : 8 points.

Use the grid systems as an aid to fill in the chart given below

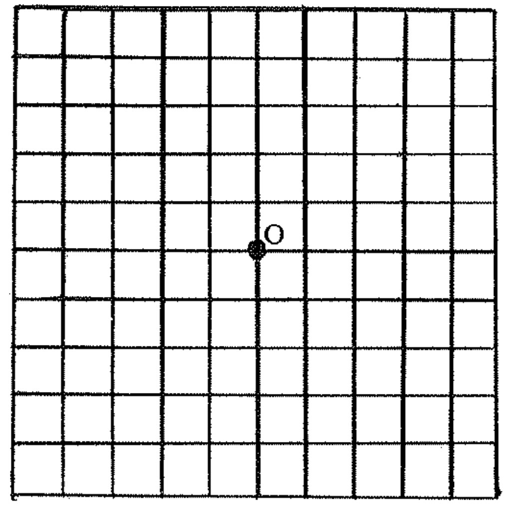
For 3 units



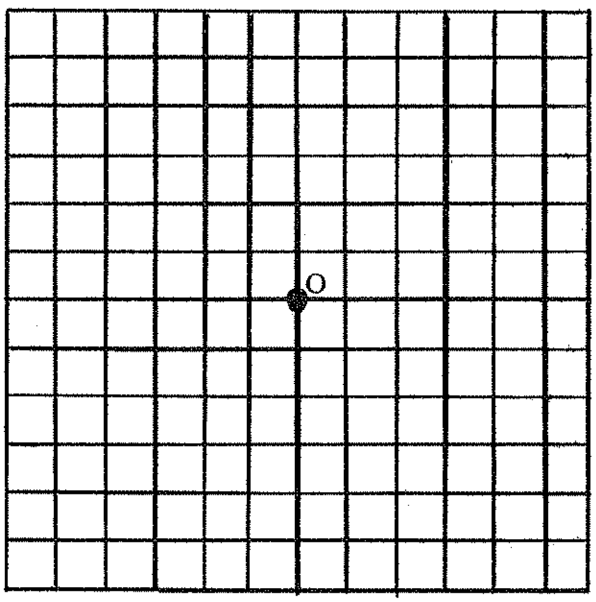
For 4 units

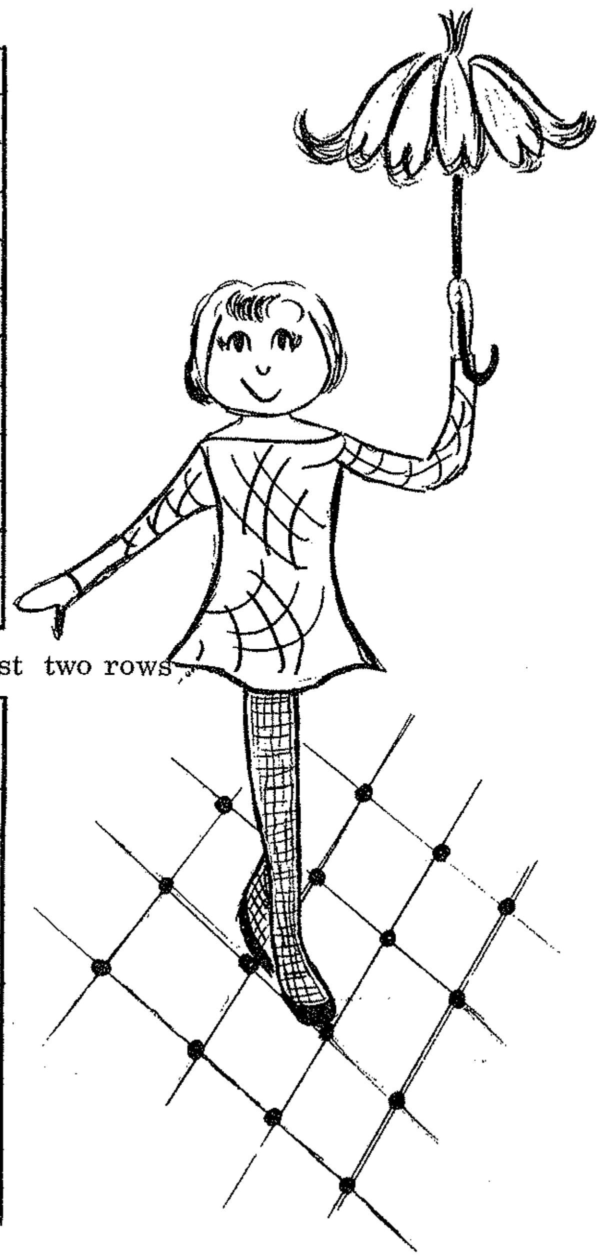


For 5 units



For 6 units





217. Fill in the chart. Follow the pattern in the first two rows.

Points distant from O by	Total number of points
1 unit	4
2 units	8
3 units	
4 units	
5 units	
6 units	
n units, where n = 1, 2, 3, ...	

ARITHMEKNACK

218. Insert any of the operations +, x, –, ÷ in the blanks so that the result is a true statement. Each operation may be used more than once in a problem.

1. { [(3 _+_ 4) _-_ 5] _+_ 6 } _-_ 7 = 1

2. (5 ___ 4) ___ (3 ___ 2) ___ 1 = 8

3. [(5 ___ 8) ___ 17] ___ (13 ___ 17) = 1

4. (4 ___ 4) ___ (3 ___ 3) ___ 2 = 19

5. (1 ___ 3) ___ 5 ___ (7 ___ 9) = 71

6. [(24 ___ 8) ___ (35 ___ 7)] ___ 1 = 14

7. [(7 ___ 9) ___ (12 ___ 3)] ___ 5 = 20

8. [{ (12 ___ 2) ___ 10 } ___ 3] ___ 7 = 15

9. [(28 ___ 21) ___ 14] ___ (6 ___ 1) = 9

10. { [(9 ___ 13) ___ 11] ___ 14 } ___ 3 = 25

11. { [(10 ___ 8) ___ 6] ___ 4 } ___ 2 = 1

12. [(3 ___ 6) ___ 12] ___ (9 ___ 2) = 24

13. { [(4 ___ 8) ___ 4] ___ 8 } ___ 4 = 8

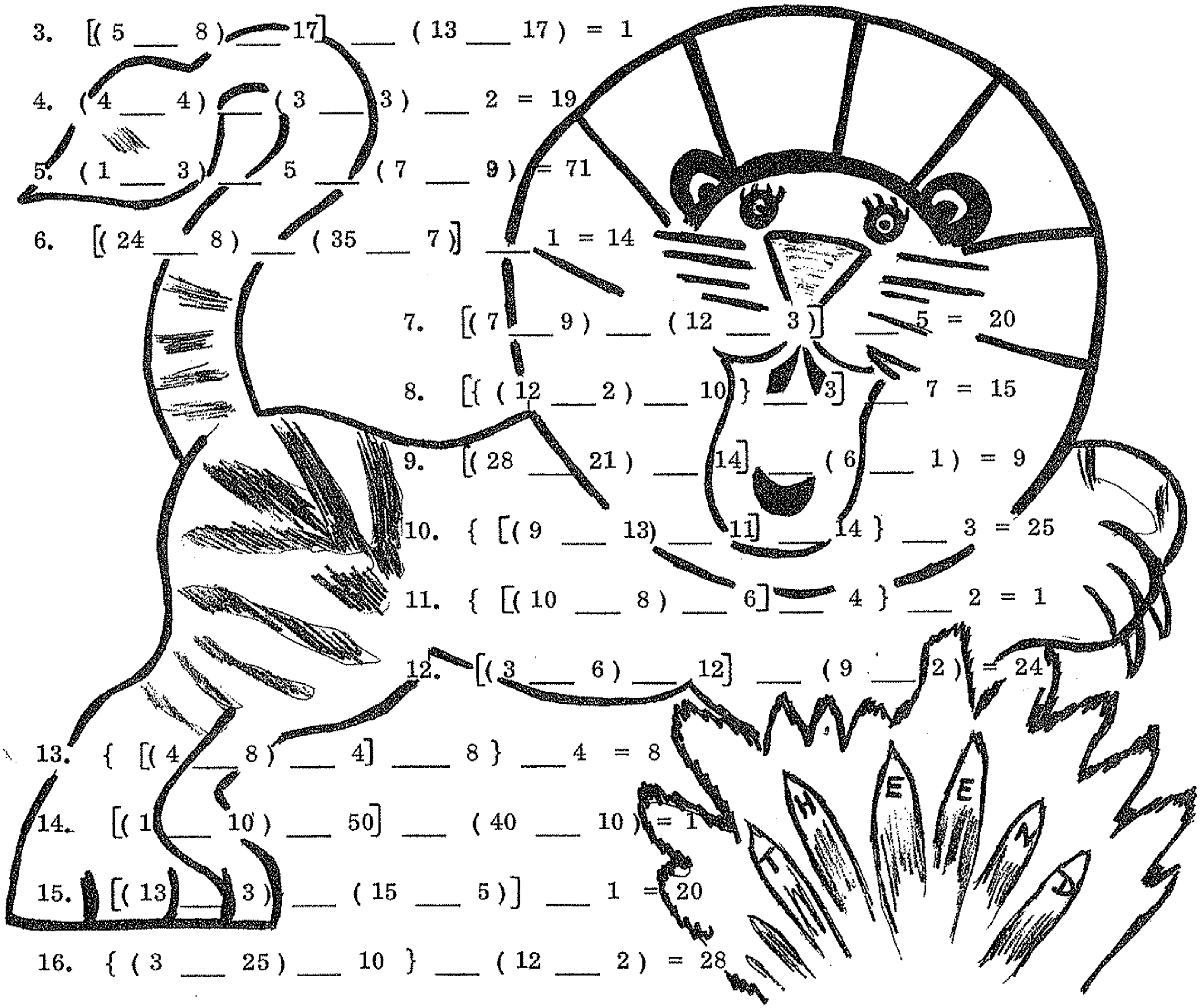
14. [(1 ___ 10) ___ 50] ___ (40 ___ 10) = 1

15. [(13 ___ 3) ___ (15 ___ 5)] ___ 1 = 20

16. { (3 ___ 25) ___ 10 } ___ (12 ___ 2) = 28

17. { (7 ___ 1) ___ (11 ___ 1) } ___ 13 = 47

18. { 9 ___ [(11 ___ 3) ___ 5] } ___ 6 = 18



CONTEMPORARY MOTIVATED MATHEMATICS