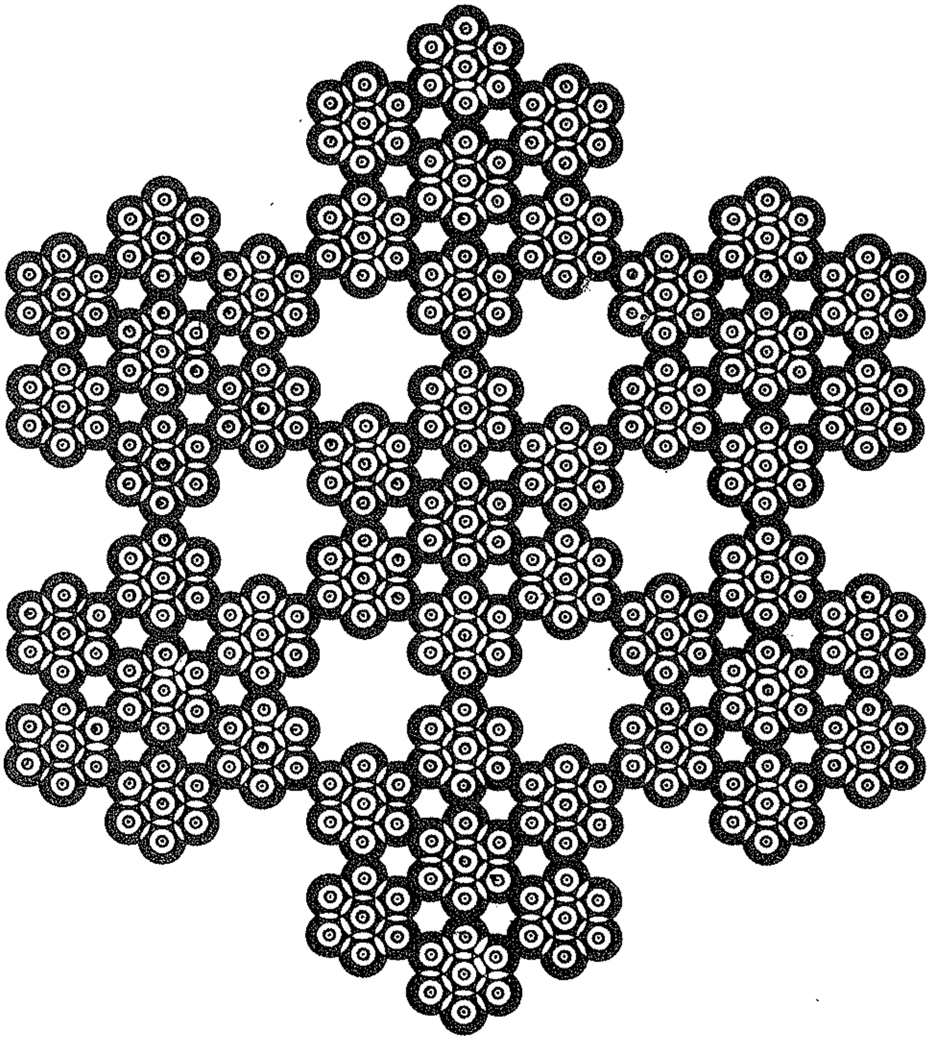
BOSTON COLLEGE MATHEMATICS INSTITUTE

Contemporary Motivated Mathematics



BOOK 2

STANLEY BEZUSZKA

In collaboration with MARY E. FARREY - MARGARET J. KENNEY

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PREFACE

CONTEMPORARY MOTIVATED MATHEMATICS is an attempt to achieve the following objectives .

1. MATHEMATICAL SKILLS

Students should possess some technical proficiency in the fundamental operations of arithmetic and algebra . But drill, often imposed for the sake of drill, has been one of the most unappealing experiences in mathematics . We are convinced that practice in the techniques of mathematics can be meaningful and functional if properly presented .

Emphasis in the text is on the basic operations of the real number system . Many of the exercises in the book are open ended . They are presented in such a way that all students , at all levels of ability can do some part of the task . More capable students can solve the text problems completely and often extend the exercises into fruitful projects for participation in science fairs .

In some problems, the computations may appear rather excessive for paper and pencil work . Students are expected to do a reasonable amount of computation under these circumstances and this is usually sufficient for the attainment of the objective of the problem. We have deliberately inserted such problems for those students who have access to desk calculators or small computers .

2. MATHEMATICAL CONTENT

Problems in the text deal with number theory and geometry . The exercises develop skills in computation and lead to some interesting or important number patterns and mathematical conclusions . Number pattern recognition and generalizations by induction occur in many problems . Manipulative techniques are stepping stones to mathematical conclusions rather than an end in themselves .

The element of fun and recreation in some of the problems requires no apology. Mathematics need not be dull and boring . It can be enjoyable.

3. BOOK 2 of CONTEMPORARY MOTIVATED MATHEMATICS

Many of the problems in Contemporary Motivated Mathematics, Book 1 are extended or generalized in Book 2 . Several new topics have been introduced. Although there is a distinct advantage for students to have completed Book 1 , the present Book 2 is self-contained . Each problem is explained in sufficient detail to facilitate the solution .

Many of the problems in the text come from the work of the greatest mathematicians in the world . We hope that the concepts which inspired the famous men of mathematics will turn out to be a glorious adventure in mathematical ideas for the students.

S. J. B., S. J.
Boston College
November 1969

CHAPTER 1

MATHEMAGIC

1.1 LO-SHU AND THE MAGIC OF SQUARES

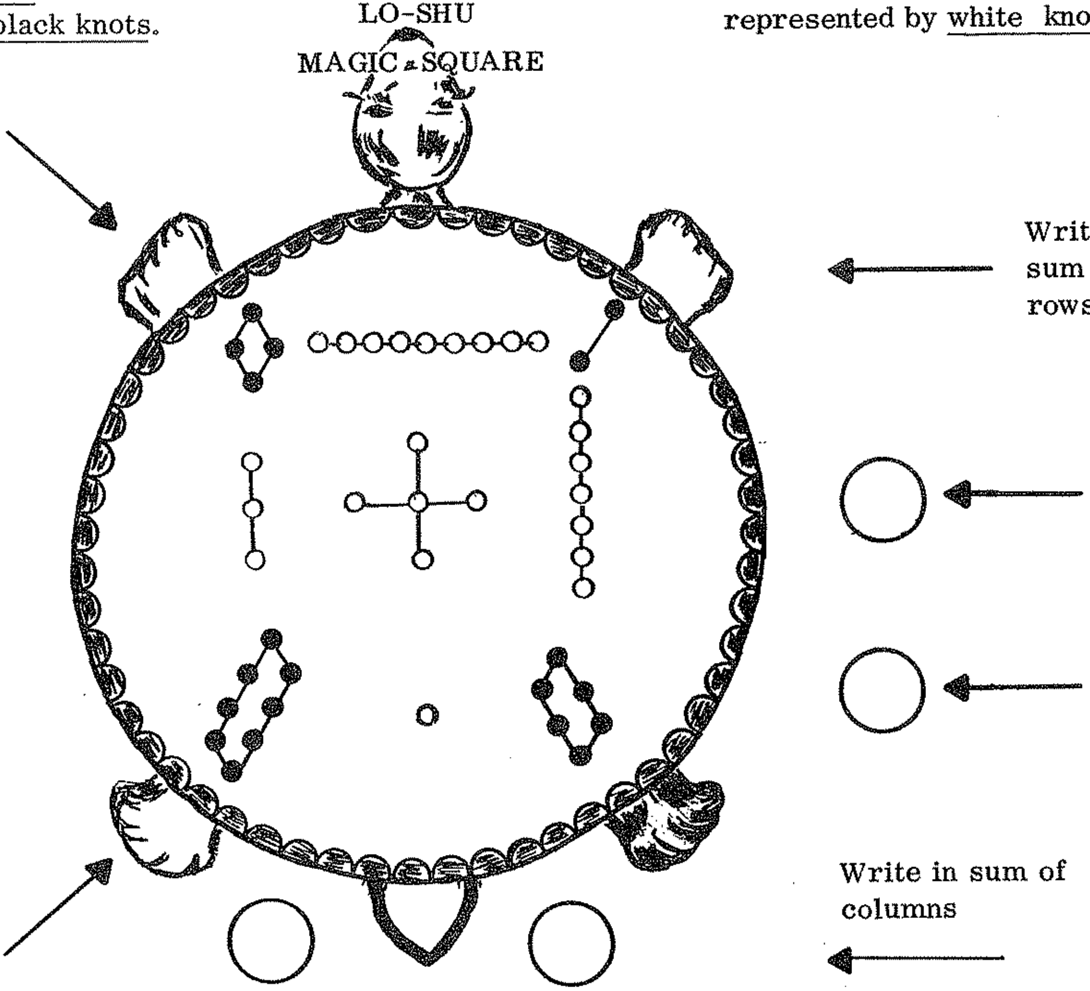
Popular myth claims that the Emperor Yu first saw a magic square upon the back of a tortoise along the bank of the Hwang Ho (the Yellow River). The undisputed appearance of the Lo-Shu magic square was in 1000 A. D. .

The Lo-Shu magic square first appeared as a square array of numbers represented by black and white knots on strings .

The even numbers were represented by black knots.

The odd numbers were represented by white knots.

1.



LO-SHU MAGIC SQUARE

Write in sum of diagonal

Write in sum of rows

Write in sum of diagonal

Write in sum of columns

MAGIC SQUARE

A square array of numbers where the sum of any row , column and diagonal is one and the same number is called a magic square .

MAGIC CONSTANT

The sum of any row , column , or diagonal of a magic square is called the magic constant .

2. The magic constant of the Lo-Shu 3 x 3 normal magic square is ___________ .

Use the natural numbers 1 through 9 once and only once. Complete the 3 x 3 normal magic square so that the magic constant is 15 .

3.

		8
	5	
		4

4.

	9	
3		
	1	

5.

	7	
		1
	3	

Use the natural numbers 1 through 16 once and only once. Complete the 4 x 4 normal magic square so that the magic constant is 34 .

6.

	9	5	16
	6	10	
14		11	
1			13

7.

4	15		1
9		7	12
	10	11	

8.

1			
	7	11	
15	6		
	9	5	16

9.

7			
12		13	8
9	4		5
		3	10

10.

11	8		10
		16	
14		4	
7			6

11.

	15	3	
9			5
12	1		8
		2	11

12.

	5	8	
3			2
	4	1	14
6			7

13.

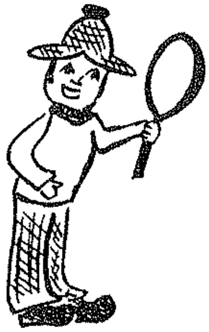
7		9	
			15
2	13	16	
11			10

14. **ORDER OF A MAGIC SQUARE**

A magic square that has
3 x 3 numbers is a 3rd order,
4 x 4 numbers is a 4th order,
5 x 5 numbers is a ____ order,
6 x 6 numbers is a ____ order
magic square

15. **NORMAL MAGIC SQUARE**

A normal magic square has the successive natural numbers
1 through 9 for a 3rd order,
1 through 16 for a 4th order,
1 through ____ for a 5th order,
1 through ____ for a 6th order.



To compute the magic constant for normal magic squares use the

FORMULA

$$\text{MAGIC CONSTANT} = \frac{n\,(n^2 + 1)}{2}$$

where n is the order of the normal magic square.

16. Use the formula. What is the magic constant for a 5 x 5 or 5th order normal magic square? ____________

Use the natural numbers 1 through 25 once and only once. Complete the 5 x 5 normal magic squares.

17.

15		1	24	
16	14	7	5	
22	20	13	6	4
	21			10
9		25	18	11

18.

9		25		11
3	21	19	12	
22	20		6	4
	14	7		23
15	8		24	17

19

9	3	22	16	15
	21	20		8
25	19		7	1
18		6	5	
11	10	4	23	17

20.

17		1	8	
	5	7	14	16
4	6		20	22
10	12	19	21	
11	18	25		9

21.

	1	10	14	
	20	24		7
5		13	17	21
19	23	2		15
8	12		25	4

22. Use the formula for the magic constant on page 3. What is the magic constant for a 6 x 6 or 6th order normal magic square ? ____________

Use the natural numbers 1 through 36 once and only once. Complete the 6 x 6 normal magic squares.

23.

	35	34	3	32	
30		28		11	7
24	23		16	14	19
13		21	22	20	18
12	26	9	10		25
	2		33	5	36

24.

1		4	33	32	6
12			27	11	25
	17	15	16	20	19
13	23	21		14	
30		9		29	7
31	2	34	3		36

25.

1	5		34	32	6
30	8	28	9	11	
18	23		16		19
24		21		17	13
7	26	10		29	12
	35	4	3	2	36

26.

36	25		19	7	
5	29	20		11	32
	10		16	27	3
4	9	21	15	28	
2		17	23	8	35
31	12	13	24		

27.

8	12	28	27	25	
35	1	4	33		
17		15		19	20
23	13		22	18	14
2		34	3	36	5
	30		10	7	29

28.

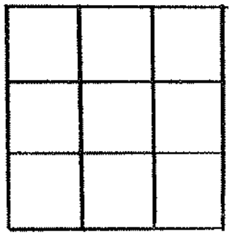
15	18		20	19	16
	1	5	32		34
28	30	8	11		9
10	7	26		12	
4		35	2	36	3
21		14	17	13	

1.2 ODD ORDER NORMAL MAGIC SQUARES - CONSTRUCTIONS

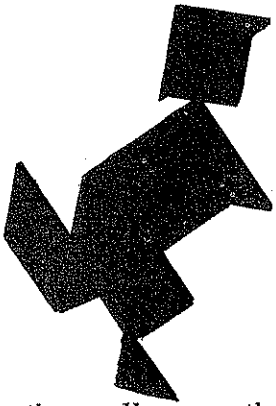
Lo-Shu's magic square cast a spell over mathematicians of all nations in all times. Eventually, the secret of the odd order normal magic squares was solved. When and by whom is a mystery hidden in history.

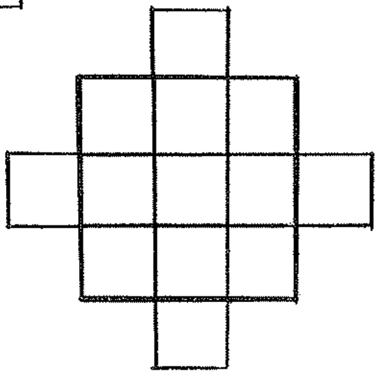
Many years later, the solution was rediscovered and attributed to Bachet de Mezeriac.

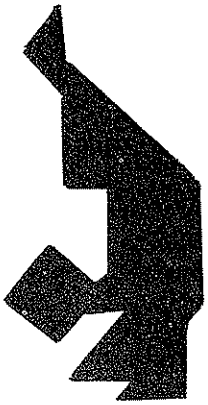
3 x 3 NORMAL MAGIC SQUARE



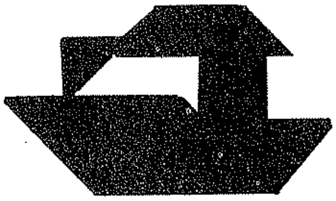
Draw the cells of a 3 x 3 square.

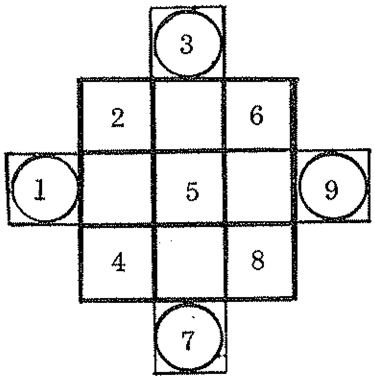




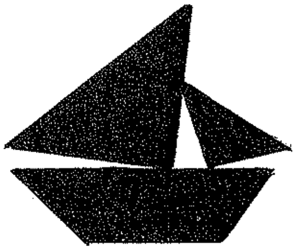


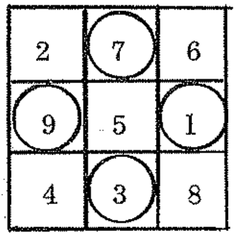
Draw in the cells on the outside of the square as shown.





Write the numbers 1 through 9 in the cells along the diagonal columns. Note the pattern.





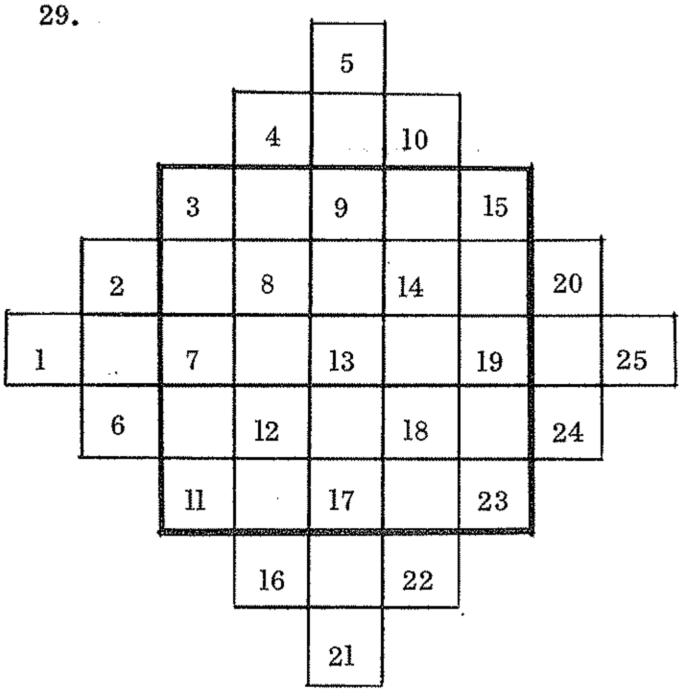
Keep the numbers in the cells of the 3 x 3 square above. Numbers in the cells outside the square are now written in the empty cells of the square but on the opposite side.

The result is a 3 x 3 normal magic square.

5 x 5 NORMAL MAGIC SQUARE

Construct a 5 x 5 normal magic square . Use Bachet 's method shown for the 3 x 3 normal magic square on page 5 .

29.

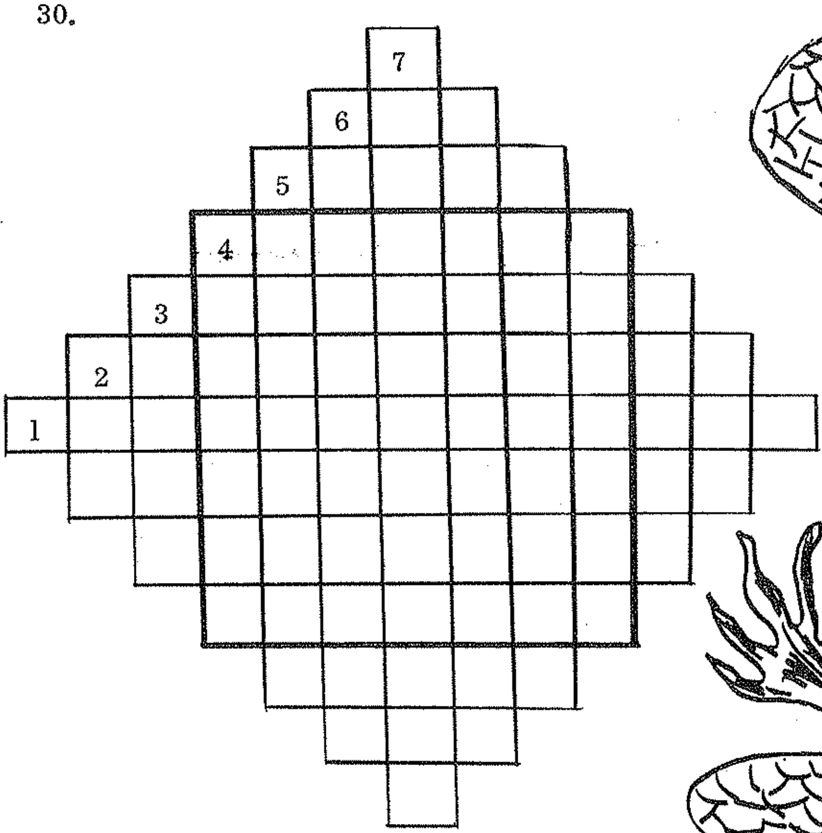


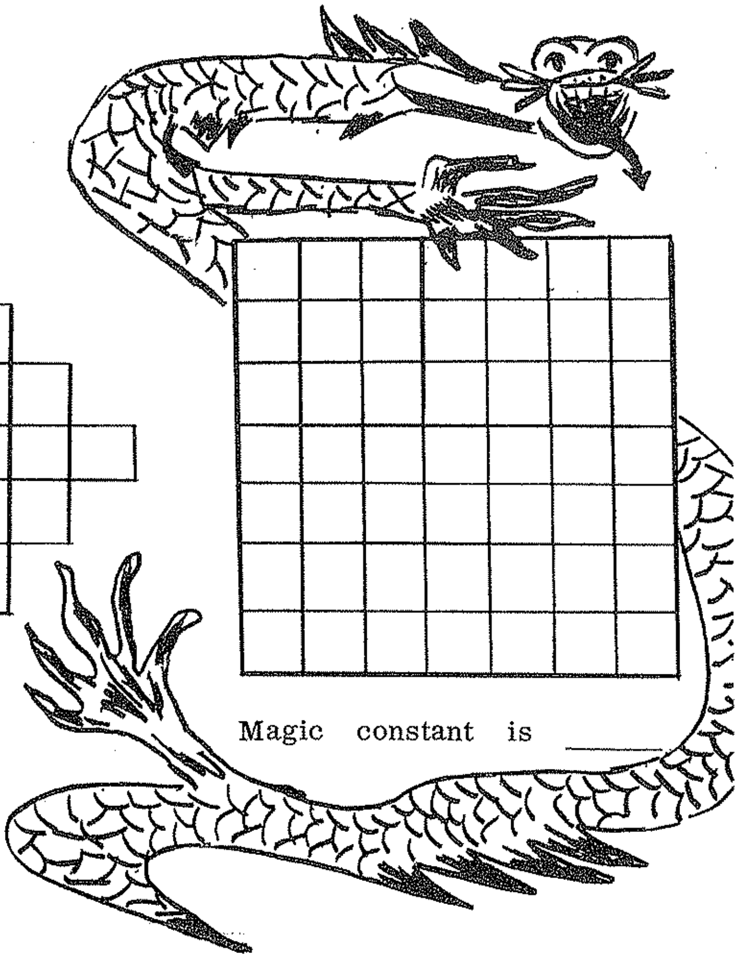
Magic constant is ______

7 x 7 NORMAL MAGIC SQUARE

Use Bachet's method. Construct a 7 x 7 normal magic square.

30.





Magic constant is ______

1. 3 MAGIC POTPOURRI

CONCENTRIC MAGIC SQUARES

A concentric magic square is a series of magic squares. There is a small central magic square and each additional panel makes up another magic square .

31 .

7 x 7 CONCENTRIC MAGIC SQUARE

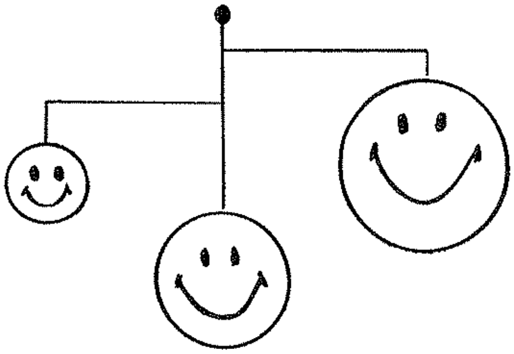
46	1	2			41	40
		13				
				26		6
7			25		33	
11	20			22		
	19		36		15	38
				8	9	

Make the magic constant of the

3 x 3 square 75

5 x 5 square 125

7 x 7 square 175 .



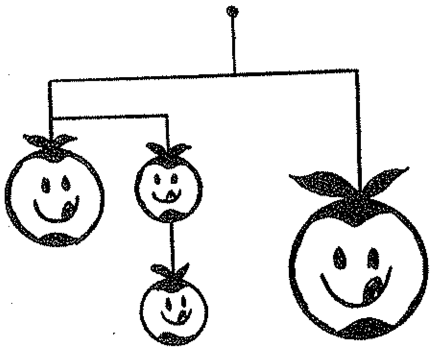
32.

Make the magic constant of the

4 x 4 square 130

6 x 6 square 195

8 x 8 square 260



8 x 8 CONCENTRIC MAGIC SQUARE

	63	62	4				8
		49		19	44	20	9
		25			28	18	10
	22	36	30				
53	42		34				
13		37		26	40		
14						50	
					6	7	64

CONSECUTIVE NUMBER SQUARES

4 x 4 SQUARES

Consecutive natural numbers are used in the cells of the squares below .

33.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

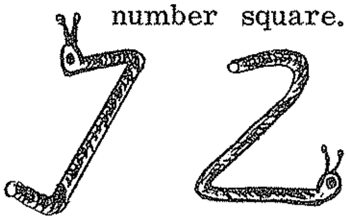
No rows omitted
in this consecutive
number square .

Find the sum of each diagonal only .

34.

5	6	7	8
9	10	11	12
13	14	15	16
17	18	19	20

1st row omitted
in this consecutive
number square.



35.

9	10	11	12
13	14	15	16
17	18	19	20
21	22	23	24

First 2 rows omitted
in this consecutive
number square .

36. What is the magic constant of a 4 x 4 normal magic square ? ______

37. How many cells are there in a 4 x 4 magic square ? ______

38. Complete the additions and compare answers with problems 33, 34, 35 .

34 + 0·(16) = ______

34 + 1·(16) = ______

34 + 2·(16) = ______

TOP POPPER

39. What is the sum of each diagonal of a 4 x 4 consecutive natural number square that begins with

Number	Sum of each diagonal
17	
25	
33	
41	
53	

Hint : Use the results of problem 38

CONSECUTIVE NUMBER SQUARES

5 x 5 SQUARES

40. Write sum of each diagonal

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

No rows omitted in this consecutive number square .

41. Write sum of each diagonal

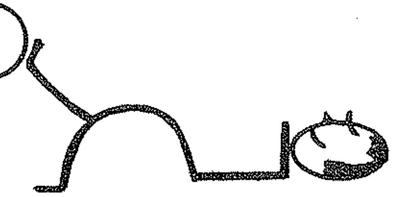
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25
26	27	28	29	30

1st row omitted in this consecutive number square .

42. Write sum of each diagonal

11	12	13	14	15
16	17	18	19	20
21	22	23	24	25
26	27	28	29	30
31	32	33	34	35

First 2 rows omitted in this consecutive number square .



43. What is the magic constant of a 5 x 5 normal magic square ? _____

44. How many cells are there in a 5 x 5 magic square ? _____

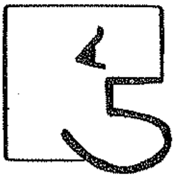
45. Complete the additions and compare answers with problems 40, 41, 42 .

65 + 0•(25) = _____ 65 + 1•(25) = _____

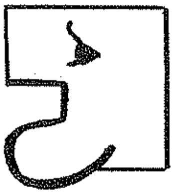
65 + 2•(25) = _____

TOP POPPER

46. What is the sum of each diagonal of a 5 x 5 consecutive natural number square that begins with



Number	Sum of each diagonal
21	
31	
46	
51	
66	



Hint: Use the results of problem 45.

COMMON RATIONAL NUMBER SQUARES

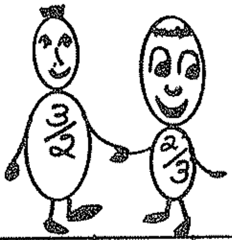
A common rational number is a number that can be written in the form $\frac{x}{y}$ where x, y are natural numbers and $y \neq 0$.

Thus, 3 , $\frac{2}{3}$, 1 , $\frac{5}{2}$ are examples of common rational numbers .

Use common rational numbers. Make the magic constant of each square 6 .

47.

0		
$\frac{5}{2}$	3	

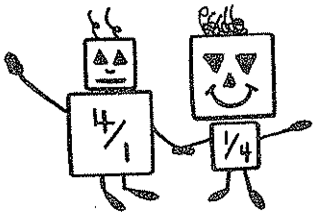


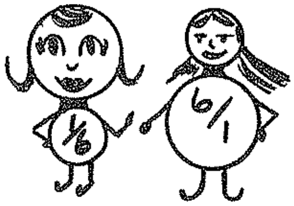
48.

		$\frac{3}{2}$
	2	
	0	

49.

		0
	1	$\frac{7}{2}$

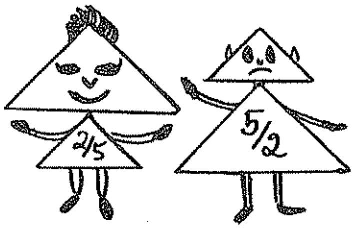




Use common rational numbers . Make the magic constant of each square $8\frac{1}{2}$.

50.

			$\frac{13}{4}$
$\frac{5}{4}$	$\frac{5}{2}$		
$\frac{9}{4}$	$\frac{3}{2}$		
1	$\frac{15}{4}$	$\frac{7}{2}$	



51.

$\frac{5}{2}$	$\frac{5}{4}$		
$\frac{3}{4}$		$\frac{13}{4}$	
	1	$\frac{1}{4}$	$\frac{7}{2}$
$\frac{3}{2}$			$\frac{7}{4}$

52.

	2	$1\frac{1}{4}$	
	$3\frac{1}{4}$	4	
$3\frac{1}{2}$	$\frac{1}{4}$	1	
$1\frac{3}{4}$			

DECIMAL MAGIC SQUARES

Make the magic constant of each square 8.

53.

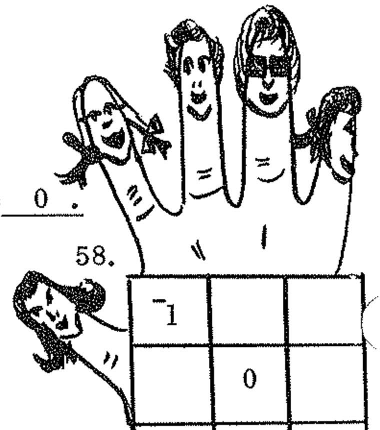
3.8		.4	
1	2.6	2.8	
	1.2	1.4	
.8			

55.

2.6			
		3.2	.4
	.8	.2	
1.2	2.4		1.4

54.

2.8		3.4	
	3.2		3
	3.8		2.4
2.6		3.6	1.2



INTEGER MAGIC SQUARES

Make the magic constant of each square 0.

56.

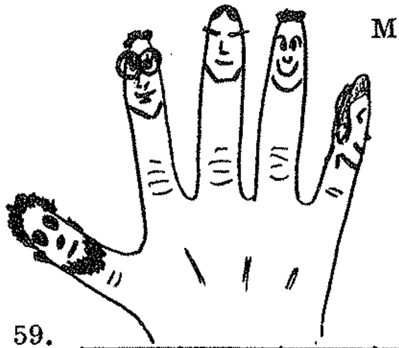
		1
	0	
		¯3

57.

		¯1
2	0	

58.

¯1		
	0	
3		



Make the magic constant of each square ¯2.

60.

		¯6	4
	2		¯7
	¯8	¯3	
¯5	5		

59.

7			¯5
¯7		¯2	
	1		6
		3	¯8

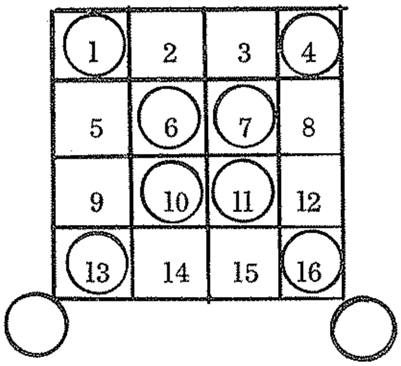
61.

			7
¯1	1		
	¯5	¯2	0
¯8		3	¯3

1.4 EVEN ORDER NORMAL MAGIC SQUARES - CONSTRUCTIONS

In the legends of magic square lovers, we find that very few gained the reputation of the great Isaw. A man of keen insight Isaw achieved lasting fame by finding a solution to the secret 4 x 4 normal magic square.

ISAW ' S 4 x 4 NORMAL MAGIC SQUARE

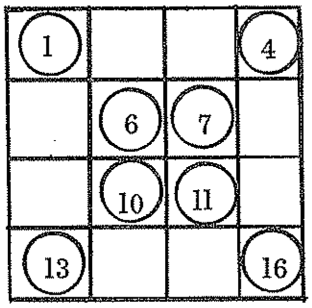


1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

First, Isaw made a 4 x 4 consecutive natural number square.

Write in the sum of each diagonal.

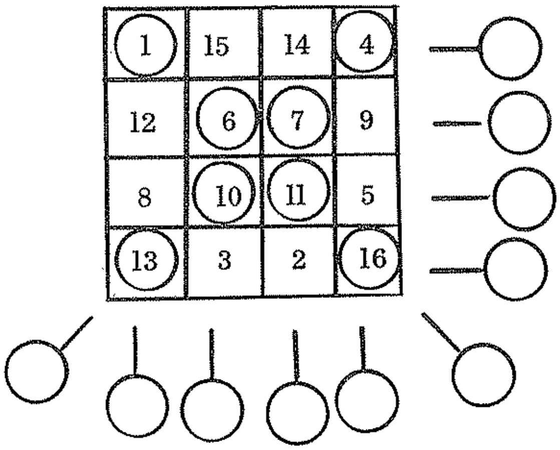
62. Is the sum of each diagonal the same as the magic constant of a 4 x 4 normal magic square ? Yes _____ No ______



1			4
	6	7	
	10	11	
13			16

Isaw copied the diagonals of the above 4 x 4 consecutive natural number square into the cells of a 4 x 4 square.

The numbers 1 and 16, 4 and 13 are at opposite ends of the two diagonals in the square Now 1 + 16 = 17 and 4 + 13 = 17 .



1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

63. Write in the sum of each row, column and diagonal.

Isaw then replaced each number of the consecutive natural number square not on the diagonals by the number which added to it gave 17. Thus, Isaw

replaced 2 by 15, since 2 + 15 = 17,
replaced 3 by 14, since 3 + 14 = 17,
and so on.

15 is called the complement of 2 with respect to 17,

14 is called the complement of 3 with respec to 17, and so on.

Isaw replaced each number of the consecutive natural number square not on the diagonals by its complement with respect to 17.

64. Isaw ' s method requires a few changes to get a 6 x 6 normal magic square. The consecutive number square is changed as shown below. For a 6 x 6 normal magic square, keep the diagonals. Write the complement of other numbers with respect to 37.

1	2	3	34	5	6
7	8	9	10	11	30
13	14	15	16	23	18
24	20	21	22	17	19
25	26	28	27	29	12
31	35	33	4	32	36

6 x 6 NORMAL MAGIC SQUARE

Write sum of each row, column and diagonals.

65. Use Isaw ' s modified method. Keep the diagonals. Write the complement of other numbers with respect to 65.

1	2	3	61	60	6	7	8
9	10	54	12	13	51	15	16
17	47	19	20	21	22	42	24
40	26	27	28	29	30	31	33
32	34	35	36	37	38	39	25
41	23	43	44	45	46	18	48
49	50	14	52	53	11	55	56
57	58	59	5	4	62	63	64

8 x 8 NORMAL MAGIC SQUARE

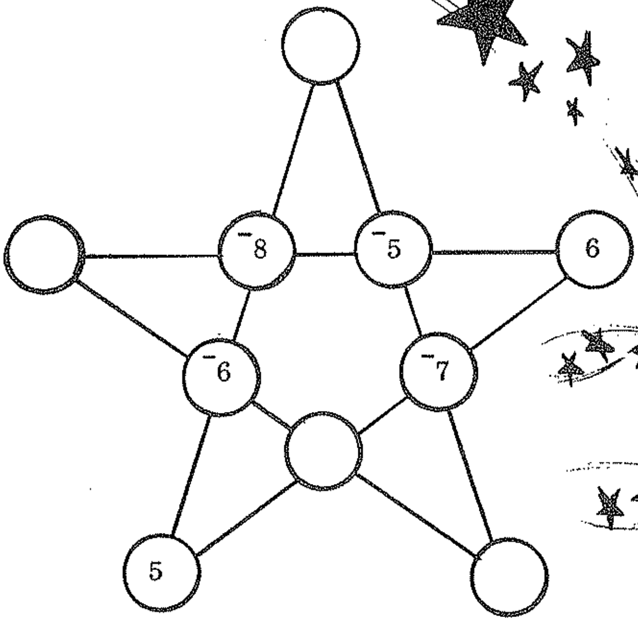
Write sum of each row, column and diagonals.

1. 5 MAGIC STARS - FIVE POINTED STARS

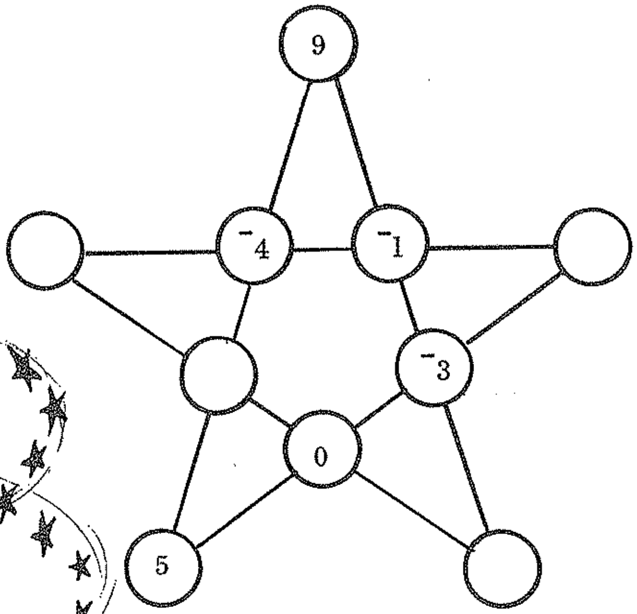
The five pointed star is the smallest magic star. There are 10 points of intersection of the lines of the star. Each line of the star has 4 points.

Place integers at the intersections of the lines of the magic star so that the sum of the integers on each of the 5 lines is that shown above the star.

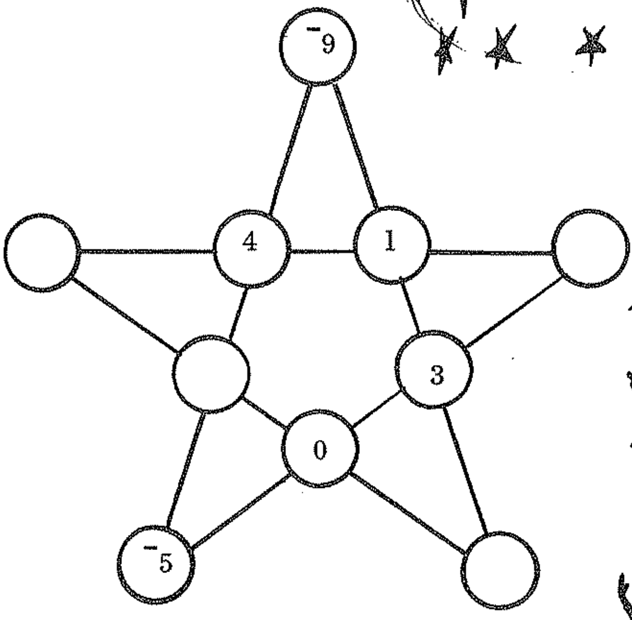
66. Sum is 0



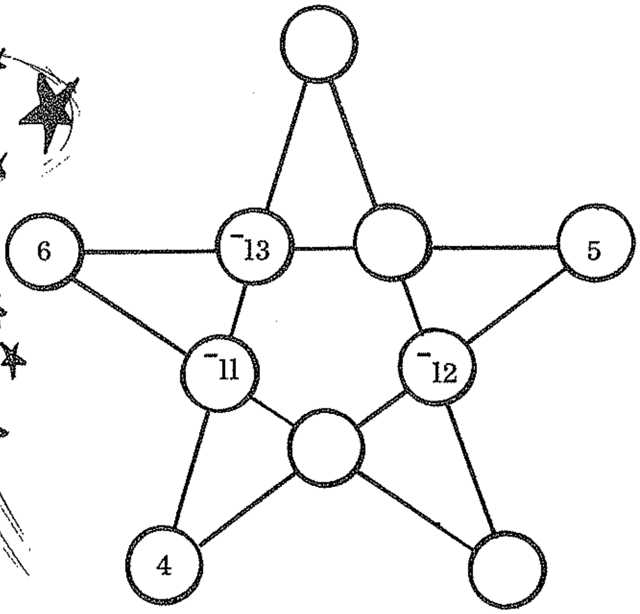
67. Sum is 8

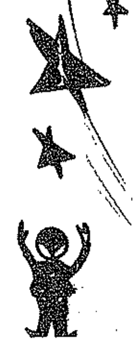


68. Sum is ¯8



69. Sum is ¯12



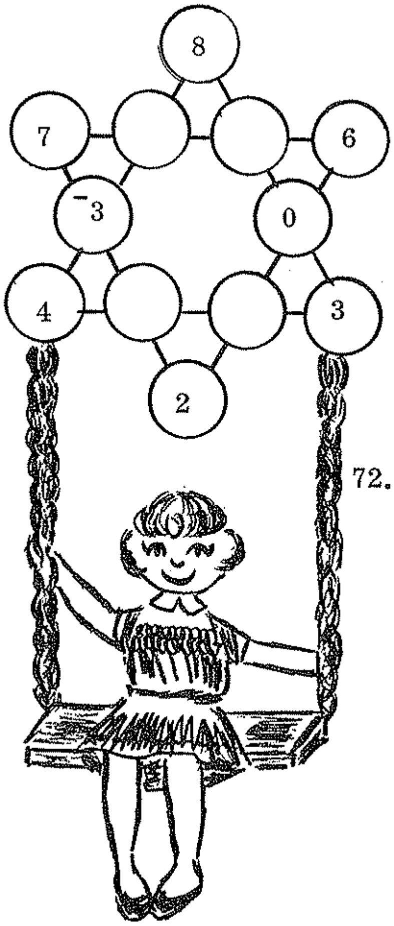


1.6 MAGIC STARS - SIX POINTED STARS

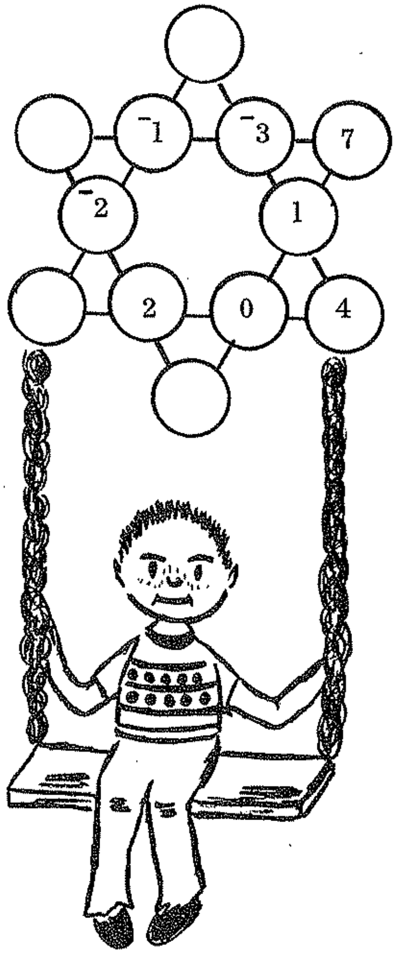
There are 12 points of intersection of the lines of the star. Each line of the star has 4 points of intersection.

Place integers at the intersections of the lines of the magic star so that the sum of the integers on each of the 6 lines is that shown above the star.

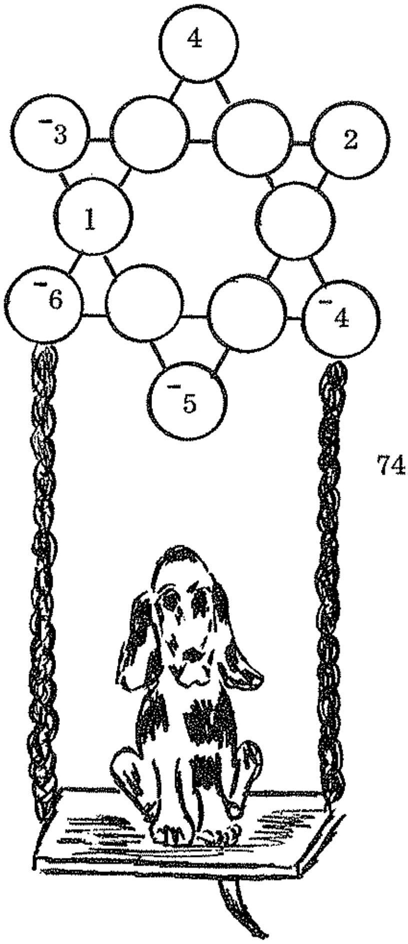
70. Sum is 7



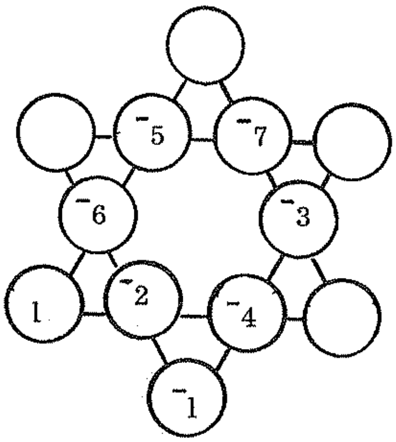
71. Sum is 11



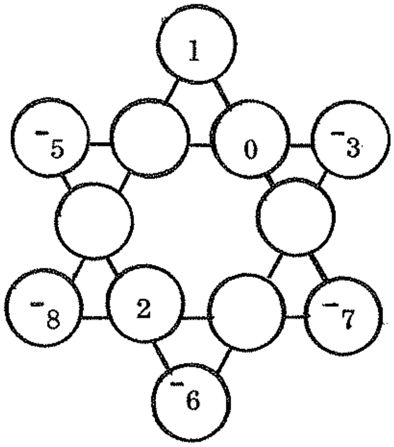
72. Sum is $^{-}2$



73. Sum is $^{-}5$



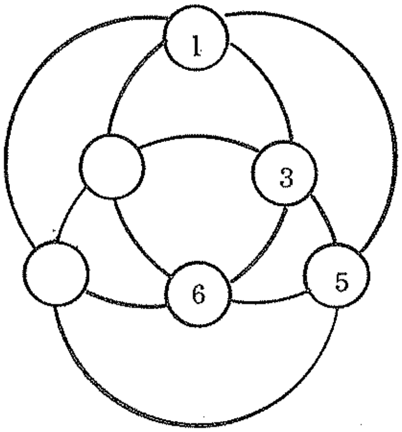
74. Sum is $^{-}10$



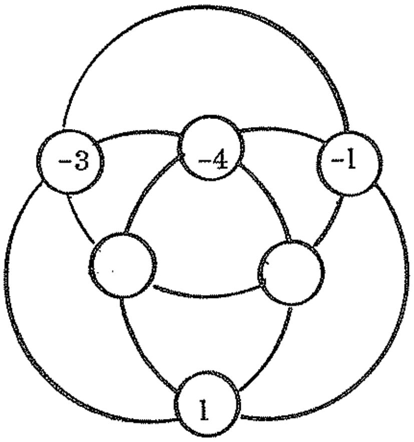
1.7 MAGIC CIRCLES

Each circle below has 4 points of intersection. Place integers at the intersections so that the sum of the numbers for each circle is the number shown above the circles.

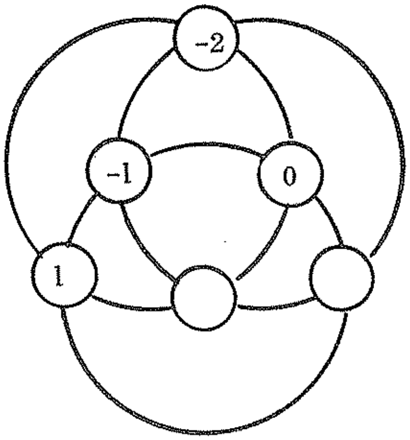
75. Sum is 14

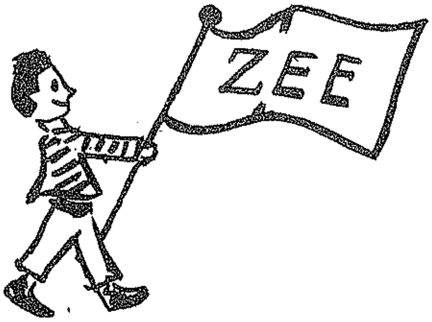


77. Sum is -6



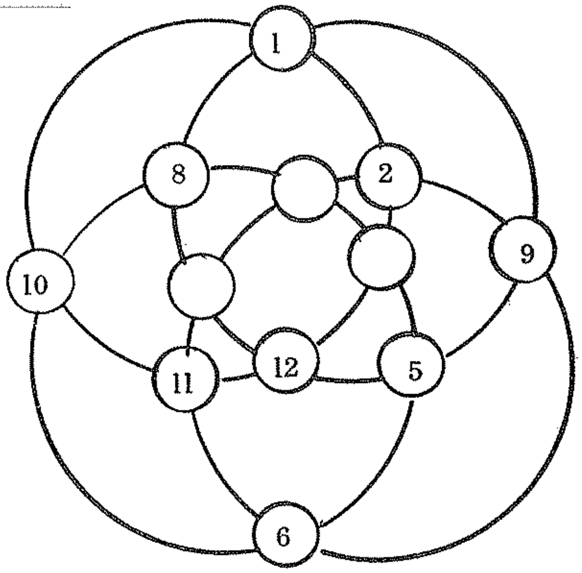
76. Sum is 2



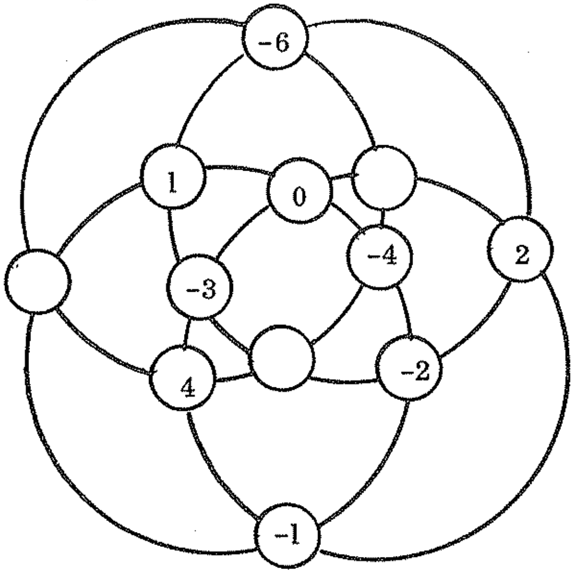


Each circle below has 6 points of intersection. Place integers at the intersections so that the sum of the numbers for each circle is the number shown above the circles.

78. Sum is 39

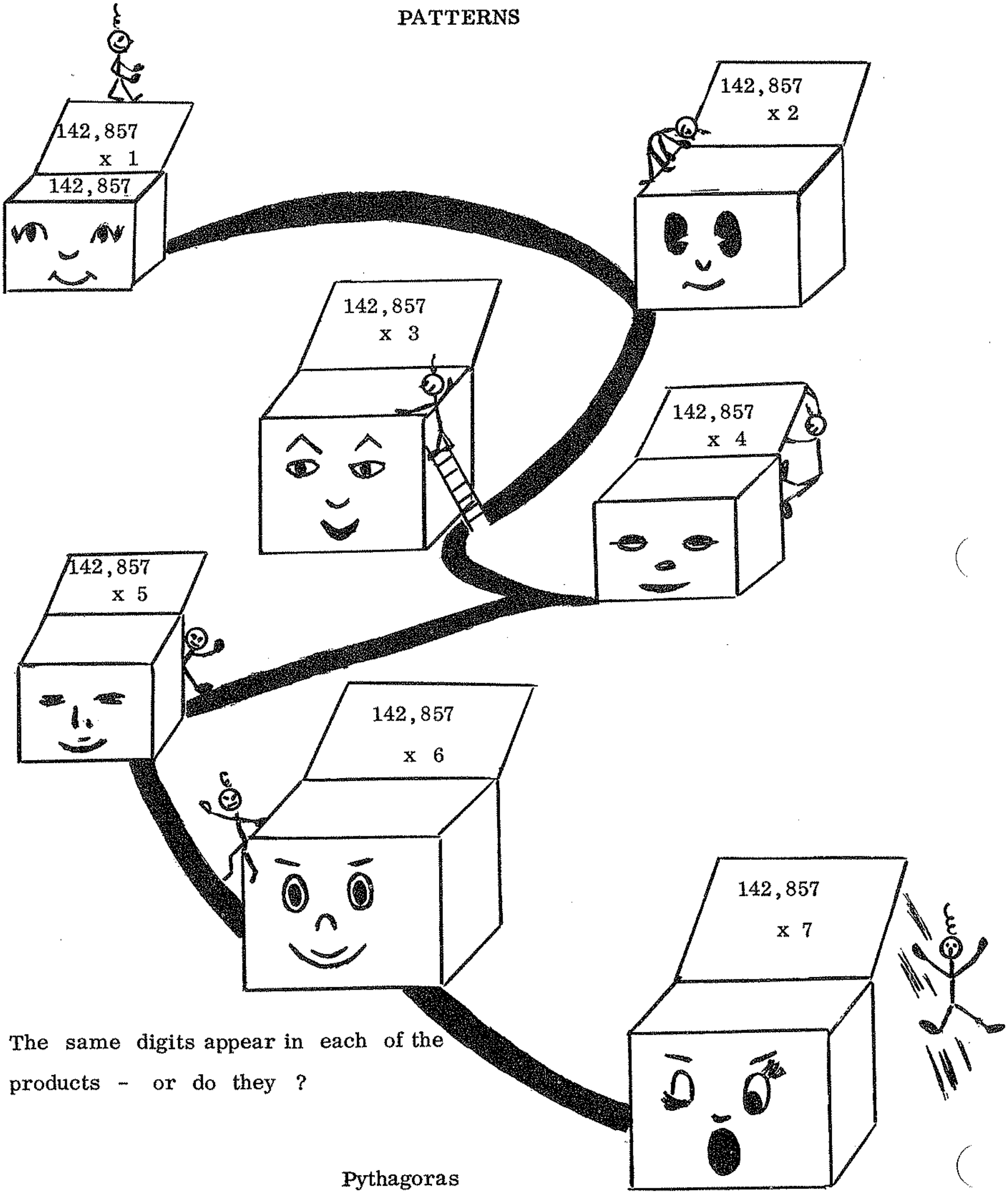


79. Sum is -3



CHAPTER 2

MYSTERIOUS NUMBER PATTERNS



The same digits appear in each of the products - or do they ?

Pythagoras

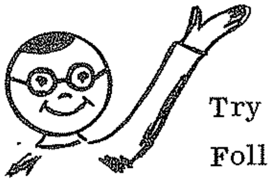
"Number is the origin of all things and the key that unlocks the MYSTERY OF THE UNIVERSE ."

2.1 MAGIC GUESSING - NUMBERS

The number magicians : Dini Dunit, Deedee Duzit and Hugh Duit perform number magic .

80.

Dini Dunit's Magic 1	Deedee Duzit takes	Hugh Duit takes
1. Take a 3 digit number	3 2 1	1 2 3
2. Double 1st digit number	2 x 3 = 6	
3. Add 5 to the result	6 + 5 = 11	
4. Multiply result by 5	5 x 11 = 55	
5. Add 2nd digit number	55 + 2 = 57	
6. Multiply result by 10	10 x 57 = 570	
7. Add 3rd digit number	570 + 1 = 571	
8. Subtract 250	571 - 250 = 3 2 1	



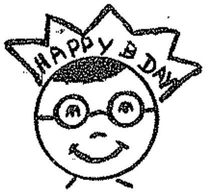
Deedee Duzit always ends up with the number she began with . Did Hugh Duit do as well ?

Try Dini Dunit's Magic 1 on the following numbers . Follow the pattern in the first column of the chart .

81.

1. Take	1 1 1	2 0 7	4 6 9	6 2 5	9 9 9	0 2 3	0 0 0
2. Double 1st digit number	2						
3. Add 5	7						
4. Multiply by 5	35						
5. Add 2nd digit number	36						
6. Multiply by 10	360						
7. Add 3rd digit number	361						
8. Subtract 250	1 1 1						

2.2 MAGIC GUESSING - AGES



Younger than Spring time and younger than 100 .
Let January be 1, February be 2 , and so on .

82.

Dini Dunit's Magic 2	Deedee Duzit Born: November , Age 10	Hugh Duit Born: September, Age 11
1. Write number of the month you were born in	⑪	⑨
2. Multiply by 10	10 x 11 = 110	
3. Add 20	110 + 20 = 130	
4. Multiply by 10	10 x 130 = 1300	
5. Add age in years	1300 + 10 = 1310	
6. Add 165	1310 + 165 = 1475	
7. Subtract 365	1475 - 365 = 1 1 1 0	

First 2-digits give the month you were born in .

Second 2-digits give your age .

Try Dini Dunit's Magic 2 on the birthdays listed below. Follow the pattern in the first column of the chart .

83.

	January Age 6	February Age 9	April Age 12	June Age 8	August Age 13
1. Number of month	1				
2. Multiply by 10	10				
3. Add 20	30				
4. Multiply by 10	300				
5. Add age in years	306				
6. Add 165	471				
7. Subtract 365	0 1 0 6				

84. Try Dini Dunit's Magic 2 on that smooth snoozer Rip Van Winkle who was born in January , age 100 . What happened ?

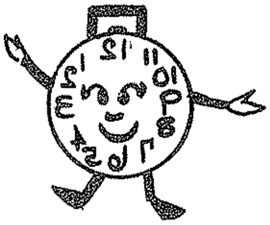
HAPPY BIRTHDAY TO YOU

2.3 MAGIC WITH CALENDARS

Tim Timely the clock maker did calendar magic. Study Tim's timely trick or treat.

JULY 1970

SU	M	T	W	TH	F	S
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28	29	30	31	



85. Tim's calendar magic. Fill in the chart. Follow the pattern in the first column.

1. Select square block of 9 numbers	5 6 7 12 13 14 19 20 21	1 2 3 8 9 10 15 16 17	15 16 17 22 23 24 29 30 31	13 14 15 20 21 22 27 28 29
2. Find smallest number in block	5			
3. Add 8	13			
4. Multiply by 9	117			
5. Sum the block of 9 numbers	117			

Reverse of Tim Timely's calendar magic. Fill in the chart. Follow the pattern of the first row.

86.

Sum of block of 9 numbers is	Divide sum by 9	Subtract 8	The block of 9 numbers on the calendar is
90	10	2	2 3 4 9 10 11 16 17 18
135			
180			

Fred Frosty tried Tim Timely's calendar magic on his favorite month .

DECEMBER 1970

SU	M	T	W	TH	F	S
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30	31		

Fill in the chart. Follow the pattern in problem 85 .

87.

1. Select square block of 9 numbers.	7 8 9 14 15 16 21 22 23	9 10 11 16 17 18 23 24 25	3 4 5 10 11 12 17 18 19	14 15 16 21 22 23 28 29 30
2. Find smallest number in block				
3. Add 8				
4. Multiply by 9				
5. Sum the block of 9 numbers				

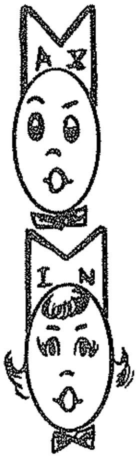
Reverse of the Timely calendar magic. Fill in the chart. Follow the pattern in problem 86 .

88.

Sum of block of 9 numbers is	Divide sum by 9	Subtract 8	The block of 9 numbers on the calendar is
144			
81			
162			

2.4 MAGIC 6 1 7 4

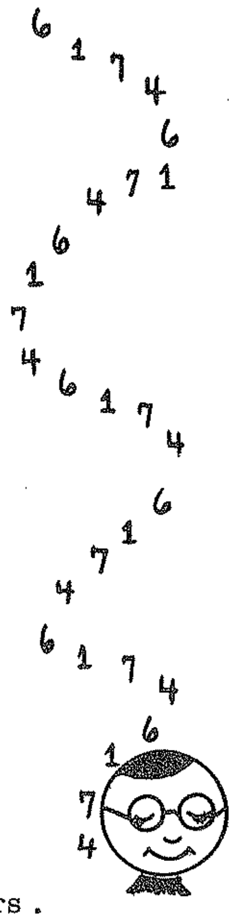
Dini Dunit found a magic number. Here is what he did.



1. Take a 4 digit number, not all digits the same. 3 2 4 1
2. Write the largest number made from the digits. Call this max. max 4 3 2 1
3. Write the smallest number made from the digits. Call this min. - min 1 2 3 4
   Hint: to get min reverse digits in max.
4. Subtract min from max. 3 0 8 7

Repeat steps 2, 3, 4 on 3 0 8 7
max 8 7 3 0
- min 0 3 7 8
8 3 5 2

Repeat steps 2, 3, 4 on 8 3 5 2
max 8 5 3 2
- min 2 3 5 8
6 1 7 4



Dini Dunit's magic number 6 1 7 4 appeared after three subtractions.

89. Check Dini Dunit's magic 6 1 7 4 on the following numbers.

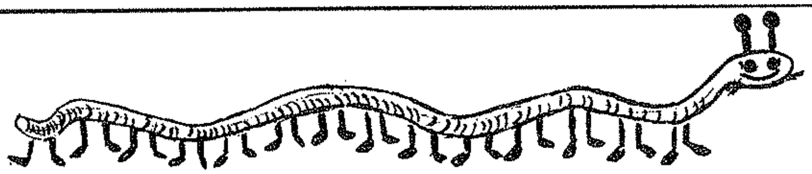
Take 7 1 3 5	Take 4 0 2 6	Take 2 1 9 8	Take 8 9 9 1	Take 3 3 3 4
max - min	max - min	max - min	max - min	max - min
Number of subtractions ____	Number of subtractions ____	Number of subtractions ____	Number of subtractions ____	Number of subtractions ____

90. Dini Dunit's magic 6 1 7 4 led to an interesting conjecture. Try the following numbers.

Take 2 0 8 9	Take 3 5 3 5	Take 3 6 4 1	Take 1 4 7 0	Take 1 7 0 9
max - min ______	max - min ______	max - min ______	max - min ______	max - min ______
Number of subtractions ______	Number of subtractions ______	Number of subtractions ______	Number of subtractions ______	Number of subtractions ______

DINI DUNIT'S CONJECTURE

If Dini Dunit's procedure is used on any 4 digit number where the digits are not all the same, the number 6 1 7 4 will appear after at most ______ subtractions.

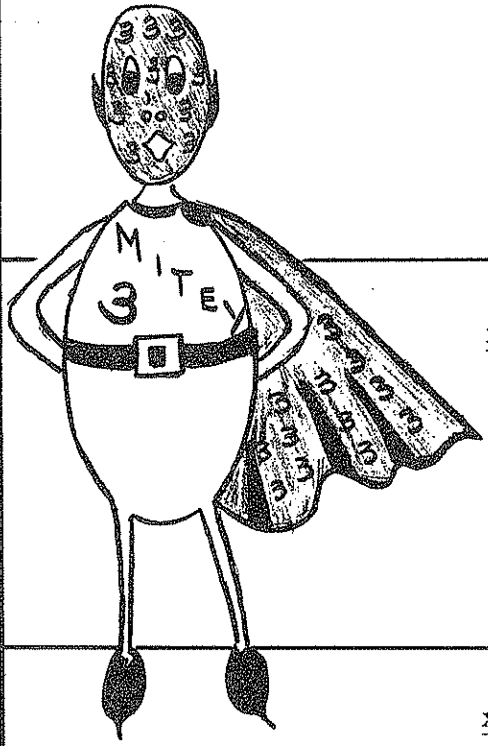


2.5 MAGIC CRYSTAL BALL

MIGHTY THREES SQUARED

91. Find the products. 3^2 = _____ 33^2 = ________ 333^2 = ___________

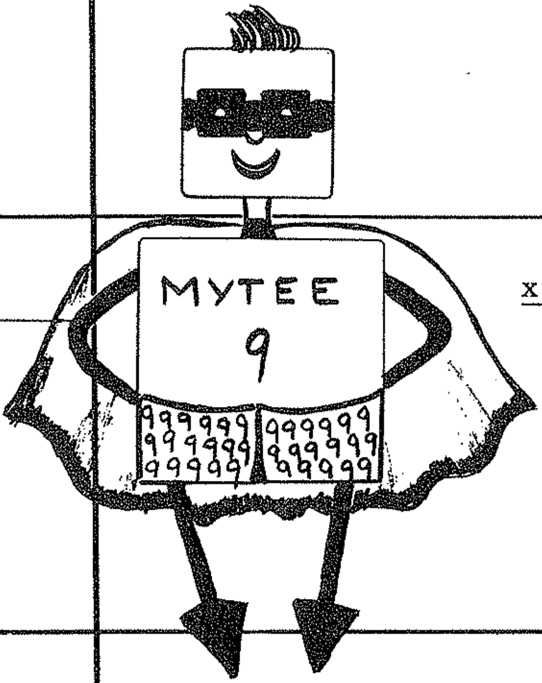
Predict the product at once	Check your prediction
$3{,}333^2$ = _______________	3,333 x 3,333
$33{,}333^2$ = _______________	33,333 x 33,333
$333{,}333^2$ = _______________	333,333 x 333,333
$3{,}333{,}333^2$ = _______________	3,333,333 x 3,333,333



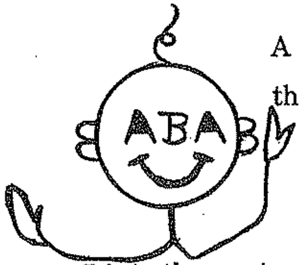
MIGHTY NINES SQUARED

92. Find the products. 9^2 = ______ 99^2 = ______ 999^2 = ______

Predict the product at once	Check your prediction
$9{,}999^2$ = ______	9,999 x 9,999
$99{,}999^2$ = ______	99,999 x 99,999
$999{,}999^2$ = ______	999,999 x 999,999
$9{,}999{,}999^2$ = ______	9,999,999 x 9,999,999



2.6 ABADABA'S NUMBER PALINDROMES



A natural number palindrome has digits which read the same when the digits are taken in reverse order.

Natural number palindromes : 0, 1, 3, 343 , 5445 .

93. List the natural number palindromes in the stated ranges .

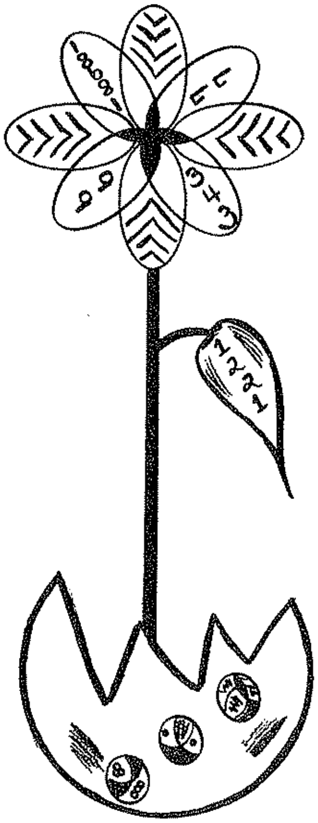
94. List the prime natural number palindromes in the ranges .

From 1 through 100 0 , 1, 2, Total number ○	From 1 through 100 2 , 3 , 5 , Total number ☐
From 101 through 200 Total number ○	From 101 through 200 Total number ☐
From 201 through 300 Total number ○	From 201 through 300 Total number ☐
From 301 through 400 Total number ○	From 301 through 400 Total number ☐
From 401 through 500 Total number ○	From 401 through 500 Total number ☐

Which of the following products of number palindromes are number palindromes ? Fill in the charts. Use the pattern in the first row.

95.

Product	Palindrome	
	Yes	No
11 x 11 = 121	✓	
11 x 22 =		
11 x 33 =		
11 x 44 =		
11 x 55 =		
11 x 66 =		
11 x 77 =		
11 x 88 =		
11 x 99 =		

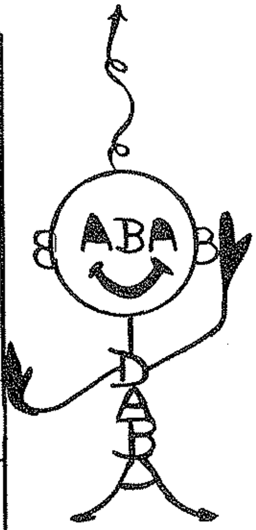


96.

Product	Palindrome	
	Yes	No
22 x 22 =		
22 x 33 =		
22 x 44 =		
22 x 55 =		
22 x 66 =		
22 x 77 =		
22 x 88 =		
22 x 99 =		

97.

Product	Palindrome		Product	Palindrome	
	Yes	No		Yes	No
33 x 33 =			44 x 44 =		
33 x 44 =			44 x 55 =		
33 x 55 =			44 x 66 =		
33 x 66 =			44 x 77 =		
33 x 77 =			44 x 88 =		
33 x 88 =			44 x 99 =		
33 x 99 =					



98.

Product	Palindrome		Product	Palindrome	
	Yes	No		Yes	No
55 x 55 =			66 x 66 =		
55 x 66 =			66 x 77 =		
55 x 77 =			66 x 88 =		
55 x 88 =			66 x 99 =		
55 x 99 =			88 x 88 =		
77 x 77 =			88 x 99 =		
77 x 88 =			99 x 99 =		
77 x 99 =					

THE SYMBOL WEAVERS

Years of Fame and Glory

Abadaba admired the great mathematicians. He wrote the year of birth for . Thales of Miletus

625 B.C.
+ 526 reverse the digits in 625
1151
+ 1511 reverse the digits in 1151
2662 a number palindrome after two reversals

Thales of Miletus was a mathematician, scientist and one of the seven wise men of the ancient world.

Next Abadaba took

Pythagoras of Samos

530 B.C.
+ 035 reverse the digits in 530
565 a number palindrome after one reversal

Pythagoras of Samos was a pupil of Thales. Famous for his work in number theory and the Pythagorean theorem. Organized the Order of the Pythagoreans for the study of mathematics.

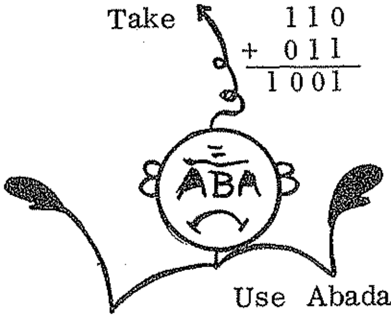
99. Famous mathematicians are listed below and the year they were born. Use Abadaba's addition pattern on page 28. Fill in the chart .

Mathematician	Year born	Palindrome	No. of reversals
Thales of Miletus	625 B. C.	2662	2
Pythagoras of Samos	530 B. C	565	1
Euclid	300 B. C.		
Archimedes	290 B. C.		
Eratosthenes of Cyrene	250 B. C.		
Hero of Alexandria	125 B. C.		
Ptolemy of Alexandria	139 A. D.		
Diophantus	250 A. D.		
Hypatia of Alexandria	375 A. D.		
Al-Khwarizmi	813 A. D.		
Mahavira	850 A. D.		
Abraham Ben Ezra	1097 A. D.		
Omar Khayyam	1100 A. D.		
Bhaskara	1150 A. D.		
Fibonacci	1175 A. D.		
Thomas Bradwardine	1290 A. D.		
Regiomontanus	1436 A. D.		
Leonardo da Vinci	1452 A. D.		
Nicolas Copernicus	1473 A. D.		
YOU 	______ A. D.		

2. 7 PALINDROME ADDITION PATTERNS - DIFFERENT BASES

Abadaba found number palindromes in the addition patterns of natural numbers in various bases .

Example 1 **Base TWO addition**



Take

$$\begin{array}{r} 110 \\ +\ 011 \\ \hline 1001 \end{array}$$

reverse 110

palindrome in base two after 1 reversal

Example 2 **Base TWO addition**

Take

$$\begin{array}{r} 1101 \\ +\ 1011 \\ \hline 11000 \\ +\ 00011 \\ \hline 11011 \end{array}$$

reverse 1101

reverse 11000

palindrome in bas
two after 2 revers

Use Abadaba's addition pattern . Fill in the charts on the pattern of the first rows . All NUMBERS and ADDITIONS are in BASE TWO .

100.

Starting number	Resulting number palindrome	Number of reversals
10	11	1
1100		
1110		
1010		
1011		
11110		
11101		
11011		
10111		

Starting number	Resulting number palindrome	Number revers
100	101	1
111110		
111001		
111011		
110111		
100111		
110010		
101000		
110101		

Abagaba , Abadaba's sister , used her brother's addition pattern to find natural number palindromes in base five .

Example 1 **Base FIVE addition**

Take
$$\begin{array}{r} 2\,1 \\ +\ 1\,2 \\ \hline 3\,3 \end{array}$$
reverse 21

palindrome in base five after 1 reversal

Example 2 **Base FIVE addition**

Take
$$\begin{array}{r} 2\,3 \\ +\ 3\,2 \\ \hline 1\,1\,0 \\ +\ 0\,1\,1 \\ \hline 1\,2\,1 \end{array}$$
reverse 23

reverse 110

palindrome in base five after 2 reversals

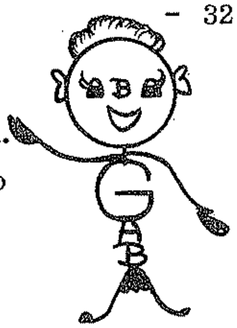
Use Abagaba's addition pattern . Fill in the charts on the pattern of the first rows . All NUMBERS and ADDITIONS are in BASE FIVE .

101.

Starting number	Resulting palindrome	Number of reversals
303		
1011		
24		
111101		
10203		
1041		
304		
4041		
3412		

Starting number	Resulting palindrome	Number of reversals
2024		
2224		
1034		
2123		
3231		
231		
1444		
241		
4234		

COMBINATION OF BASES TEN TWO FIVE



Abadaba and Abagaba worked together on this problem. They wrote one and the same number in base ten, two and five. For example, take twelve

$$12_{ten} = 1100_{two} = 22_{five} .$$

They then used the addition pattern on the numbers in their respective bases and had the following chart. of number palindromes.

		Resulting addition pattern palindrome	Number of reversals
Number in base ten	12	33	1
Same in base two	1100	1111	1
Same in base five	22	44	1

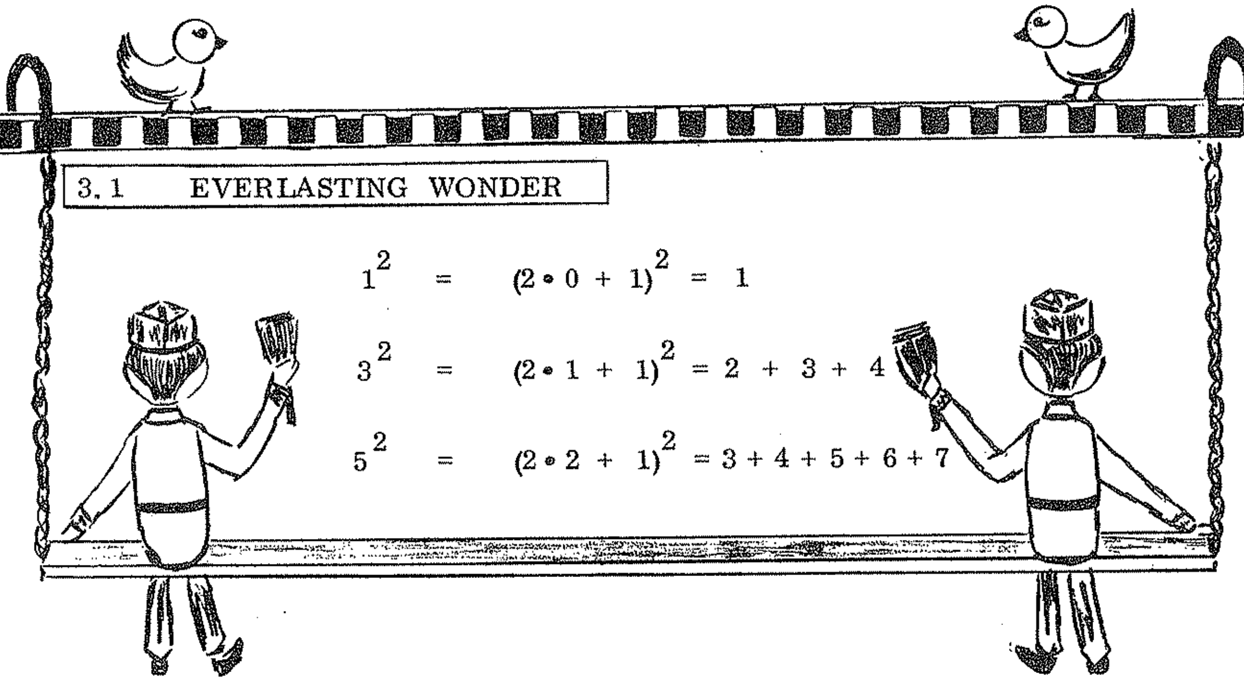
102. Fill in the following chart on the pattern of the first row.

Base ten	Palindrome	Number reversals	Base two	Palindrome	Number reversals	Base five	Palindrome	Nur reve
12	33	1	1100	1111	1	22	44	
19								
27								
39								
17								
31								
9								
72								

CHAPTER 3

CURIOUS NUMBER PATTERNS

3.1 EVERLASTING WONDER



$$1^2 = (2 \cdot 0 + 1)^2 = 1$$

$$3^2 = (2 \cdot 1 + 1)^2 = 2 + 3 + 4$$

$$5^2 = (2 \cdot 2 + 1)^2 = 3 + 4 + 5 + 6 + 7$$

103. **DISCOVER AND CONTINUE**

7^2 = ____________ = ____

9^2 = ____________ = ____

11^2 = ____________ = ____

13^2 = ____________ = ____

15^2 = ____________ = ____

17^2 = ____________ = ____

19^2 = ____________ = ____

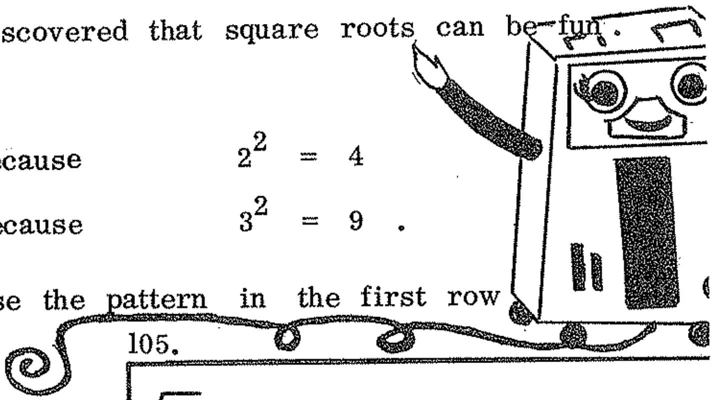
21^2 = ____________ = ____

3.2 SQUARE ROOTS

Ruth Skware enjoyed squares and discovered that square roots can be fun. Work along with Ruth.

$\sqrt{4} = 2$ because $2^2 = 4$

$\sqrt{9} = 3$ because $3^2 = 9$.



Find the square roots. Use the pattern in the first row

104.

$\sqrt{16} = 4$ because $4^2 = 16$

$\sqrt{25}$ = ____ because ______

$\sqrt{36}$ = ____ because ______

$\sqrt{49}$ = ____ because ______

$\sqrt{64}$ = ____ because ______

105.

$\sqrt{81}$ = ____ because ______

$\sqrt{100}$ = ____ because ______

$\sqrt{121}$ = ____ because ______

$\sqrt{144}$ = ____ because ______

$\sqrt{169}$ = ____ because ______

FINDING SQUARE ROOTS

Ruth Skware had two methods for getting square roots. One procedure for finding square roots was to make a chart based on the definition of square root.

Since $2^2 = 4$ then $\sqrt{4} = 2$

$3^2 = 9$ then $\sqrt{9} = 3$

106.

Since $14^2 = 196$ then $\sqrt{196} = 14$

15^2 = ____ then ______

16^2 = ____ then ______

17^2 = ____ then ______

18^2 = ____ then ______

19^2 = ____ then ______

107.

Since 20^2 = ____ then ______

21^2 = ____ then ______

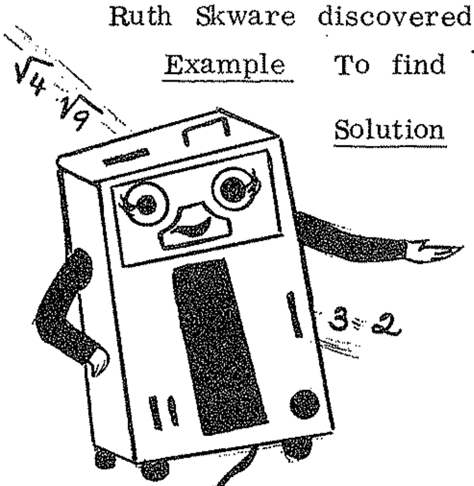
22^2 = ____ then ______

23^2 = ____ then ______

24^2 = ____ then ______

25^2 = then ______

Ruth Skware discovered a simple method for finding square roots.



Example To find $\sqrt{9}$.

Solution Write 9

Subtract consecutive odd natural numbers 1, 3, 5, ... until you get 0

	Number of subtractions
9	
-1	1
8	
-3	2
5	
-5	3
0	

The number of subtractions will be equal to $\sqrt{9}$.

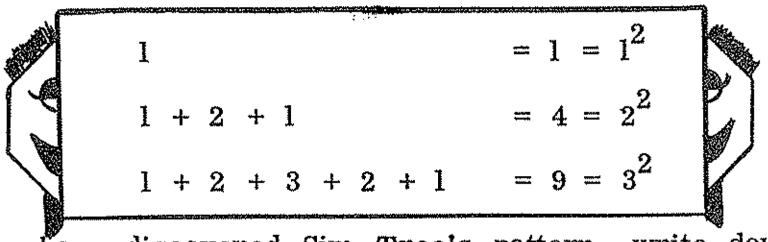
Thus, $\sqrt{9} = 3$

108. Use Ruth Skware's method. Find the following square roots.

$\sqrt{16}$ 16 -1 $\sqrt{16} =$	$\sqrt{25}$ 25 -1 $\sqrt{25} =$	$\sqrt{36}$ 36 -1 $\sqrt{36} =$	$\sqrt{49}$ 49 -1 $\sqrt{49} =$
$\sqrt{64}$ 64 -1 $\sqrt{64} =$	$\sqrt{81}$ 81 -1 $\sqrt{81} =$	$\sqrt{100}$ 100 -1 $\sqrt{100} =$	$\sqrt{121}$ 121 -1 $\sqrt{121} =$

3.3 UP AND DOWN THE SQUARE STAIR

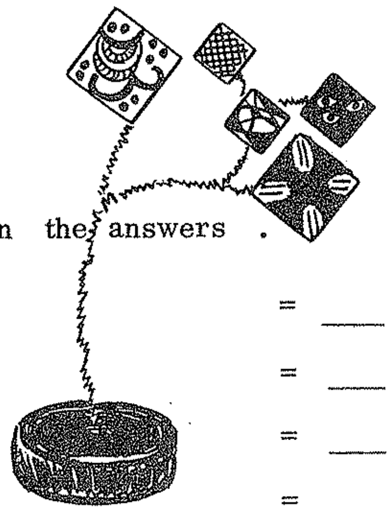
Sim Tree , Ruth Skware's friend, found some interesting square patterns.



$1 = 1 = 1^2$

$1 + 2 + 1 = 4 = 2^2$

$1 + 2 + 3 + 2 + 1 = 9 = 3^2$



If you have discovered Sim Tree's pattern, write down the answers .

$1 + 2 + 3 + 4 + 3 + 2 + 1 =$ ____

$1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1 =$ ____

$1 + 2 + 3 + 4 + 5 + 6 + 5 + 4 + 3 + 2 + 1 =$ ____

$1 + 2 + 3 + 4 + 5 + 6 + 7 + 6 + 5 + 4 + 3 + 2 + 1 =$ ____

$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 =$ ____

$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 =$ ____

Sim Tree made use of the above for his second pattern .

Find the quotients .

109.

$$\frac{1 \times 1}{1} = \_\_\_\_$$

$$\frac{22 \times 22}{1 + 2 + 1} = \_\_\_\_$$

$$\frac{333 \times 333}{1 + 2 + 3 + 2 + 1} = \_\_\_\_$$

$$\frac{4444 \times 4444}{1 + 2 + 3 + 4 + 3 + 2 + 1} = \_\_\_\_$$

$$\frac{55,555 \times 55,555}{1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1} = \_\_\_\_$$

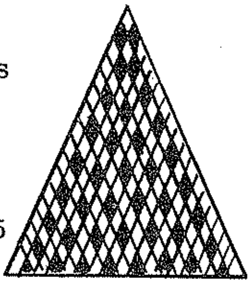
$$\frac{666,666 \times 666,666}{1 + 2 + 3 + 4 + 5 + 6 + 5 + 4 + 3 + 2 + 1} = \_\_\_\_$$

$$\frac{7,777,777 \times 7,777,777}{1 + 2 + 3 + 4 + 5 + 6 + 7 + 6 + 5 + 4 + 3 + 2 + 1} = \_\_\_\_$$

$$\frac{88,888,888 \times 88,888,888}{1+2+3+4+5+6+7+8+7+6+5+4+3+2+1} = \underline{\qquad\qquad}$$

$$\frac{999,999,999 \times 999,999,999}{1+2+3+4+5+6+7+8+9+8+7+6+5+4+3+2+1} = \underline{\qquad\qquad}$$

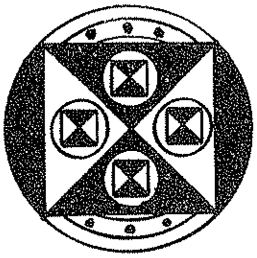
3.4 SYMMETRY PATTERNS



Insert the operations + , x , - , ÷ in the blanks between the numbers so that the result is an equality . Operations may be used more than once in a problem .

Example — Forward pattern: [(1 ____ 2) ____ 3] ____ 4 = 5

Reverse pattern: 5 = [4 ____ (3 ____ 2)] ____ 1



Solution — Forward pattern: [(1 + 2) x 3] - 4 = 5

Reverse pattern: 5 = [4 + (3 - 2)] x 1

Follow the instructions given in the example above . Make equalities for the forward and the reverse patterns .

110.

Forward pattern: [1 + (2 ____ 3) + 4] - 5 = 6

Reverse pattern: 6 = [(5 ____ 4) x (3 + 2)] ____ 1

Forward pattern: 1 + [(2 - 3) ____ (4 - 5)] ____ 6 = 7

Reverse pattern: 7 = [6 ____ (5 - 4)] + [(3 - 2) ____ 1]

Forward pattern: [(1 ____ 2) ____ 3] ____ [(4 ____ 5) ____ 6] ____ 7 = 8

Reverse pattern: 8 = [7 ____ (6 ____ 5)] ____ [(4 ____ 3) ____ (2 ____ 1)]

Forward pattern: [(1 ____ 2) ____ (3 ____ 4)] ____ [(5 ____ 6) ____ (7 ____ 8)] = 9

Reverse pattern: 9 = (8 ____ 7) ____ [6 ____ (5 ____ 4)] ____ [3 ____ (2 ____ 1)]

3.5 QUEST FOR QUOTIENTS

111.

$(1 \div 2) \div 3 = \frac{1}{6}$

$1 \div (2 \div 3) =$ ______

$(1 \div 2) \div (3 \div 4) =$ ______

$\left[1 \div (2 \div 3)\right] \div 4 =$ ______

$\left[(1 \div 2) \div (3 \div 4)\right] \div 5 =$ ______

$\left[1 \div (2 \div 3)\right] \div (4 \div 5) =$ ______

TOP POPPERS

112.

$\left[(1 \div 2) \div 3\right] \div \left[(4 \div 5) \div 6\right] =$ ______

$\left[1 \div (2 \div 3)\right] \div \left[4 \div (5 \div 6)\right] =$ ______

$\left[(1 \div 2) \div (3 \div 4)\right] \div \left[(5 \div 6) \div 7\right] =$ ______

$\left[1 \div (2 \div 3)\right] \div \left[(4 \div 5) \div (6 \div 7)\right] =$ ______

$\left[(1 \div 2) \div (3 \div 4)\right] \div \left[(5 \div 6) \div (7 \div 8)\right] =$ ______

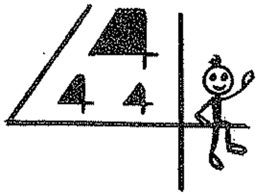
$(1 \div 2) \div \left[\left[(3 \div 4) \div (5 \div 6)\right] \div (7 \div 8)\right] =$ ______

Minnie and Max found all the above quotients. They discovered that grouping made a difference in the quotients. There were two groupings of particular interest.

113. Fill in the blanks in the chart. Use the pattern in the first row

Use parentheses, braces, brackets, and group so that	yields the smallest quotient	yields the larg quotient
$1 \div 2 \div 3$	$(1 \div 2) \div 3 = \frac{1}{6}$	$1 \div (2 \div 3) = \frac{3}{2}$
$1 \div 2 \div 3 \div 4$		
$1 \div 2 \div 3 \div 4 \div 5$		
$1 \div 2 \div 3 \div 4 \div 5 \div 6$		

3.6 REPRESENTATION OF NATURAL NUMBERS



FOUR FOURS

A fascinating challenge !

Natural number	Representation using four 4 s	
0	$0 = (4-4)+(4-4)$	$0 = (4 \times 4) - (4 \times 4)$
1	$1 = \frac{4+4}{4+4}$	$1 = \frac{4}{\sqrt{4}} - \frac{4}{4}$
2	$2 = \frac{4 \times 4}{4+4}$	$2 = \frac{4}{4} + \frac{4}{4}$
3	$3 = \frac{4+4+4}{4}$	$3 = (4 - \sqrt{4}) + \frac{4}{4}$

Many natural numbers can be represented using four 4 s under the following conditions .

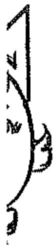
1. Use all four 4 s and only four 4 s in each representation of a natural number .
2. You may use any combination of the following operations
   - a. $+$, $\times$, $-$, $\div$
   - b. $\sqrt{}$ also $\sqrt[4]{}$
   - c. exponents involving 4 s , for example , 4^4 , $4^{\sqrt{4}}$
   - d. factorials involving 4 s .

     Note : $4! = 1 \times 2 \times 3 \times 4$

     $n! = 1 \times 2 \times 3 \times \ldots \times n$

     Also $0! = 1$
3. You may use
   - a. decimals, that is , .4 , .44 , 4.4 and so on
   - b. juxtapositions of the 4 s , that is , 44 , 444 , 4444
   - c. symbols of grouping, that is , ( ) , { } .

For any given natural number more than one representation is often possible using four 4 s .



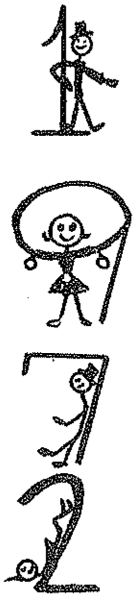
114. Represent each of the natural numbers below using four 4 s .444

Number	Representation	Number	Representation
4	$\frac{4}{4} \times (\sqrt{4} + \sqrt{4})$	19	
5		20	
6		21	
7		22	
8		23	
9		24	
10		25	
11		26	
12		27	
13		28	
14		29	
15		30	
16		31	
17		32	
18		33	$\frac{4 - .4}{.4} + 4!$

four quatre quattro vier четы́ре czter

DIGITS IN 1972

The digits in 1972 may be used to represent natural numbers on the pattern of the four 4 s .



Natural number	Representation using digits in 1972
0	$0 = (1 \times 9) - (7 + 2)$ $0 = (9 - 7) - (1 \times 2)$
1	$1 = (1 + 9) - (7 + 2)$ $1 = 1 + (9 - 7) - 2$
2	$2 = (9 - 7) \times (2 - 1)$
3	$3 = \left[(9 - 7) \times 2\right] - 1$

Use the conditions stated for the four 4 s with proper changes for the digits in 1972 . Represent the following natural numbers .

115.

Number	Representation	Number	Representation
4	$(7 - (\sqrt{9})!) + (2 + 1)$	12	
5		13	
6		14	
7		15	
8		16	
9		17	
10		18	
11		19	$(1 + 9 + 7 + 2)$

3.7 DIGIT REVERSINGS - PRODUCTS

An interesting digit reversing pattern involving products.

A return to magic 1089 .

116. Find the products.

1089
x 9

9 x 1089 =

10989
x 9

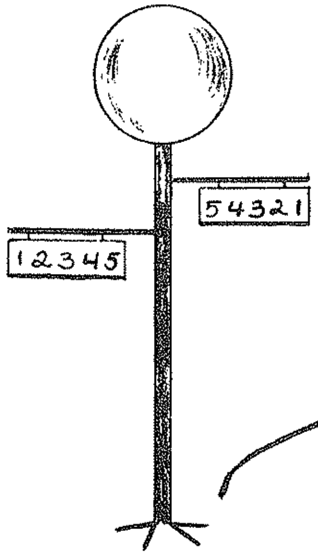
9 x 10989 =

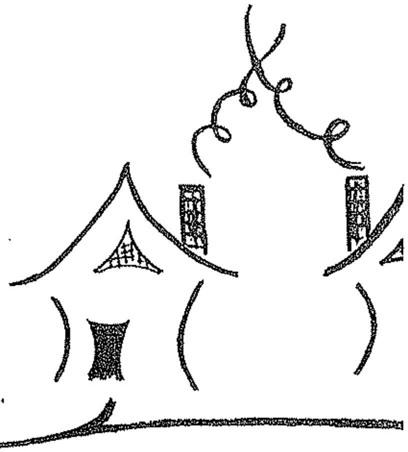
109989
x 9

9 x 109989 =

1099989
x 9

9 x 1099989 =





A new one now : 2178 .

Find the products.

117.

2178
x 4

4 x 2178 =

21978
x 4

4 x 21978 =

219978
x 4

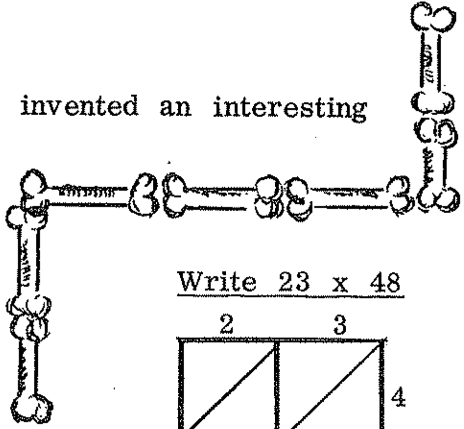
4 x 219978 =

2199978
x 4

4 x 2199978 =

3.8 OPERATION CURIOS. NAPIER, RUSSIAN, EYPTIAN

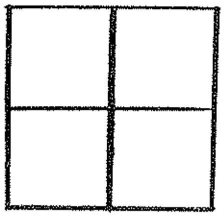
NAPIER'S BONES

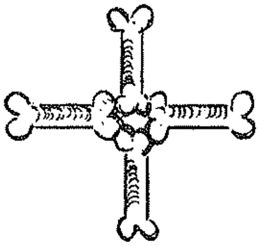


John Napier (1550-1617), a Scottish mathematician invented an interesting method for finding products.

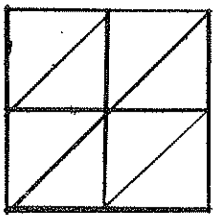
Example Take 23 x 48.

Draw squares

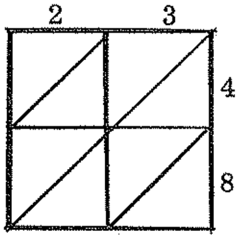




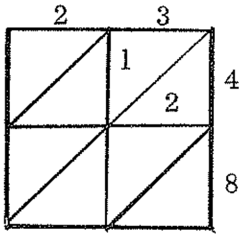
Draw diagonals



Write 23 x 48

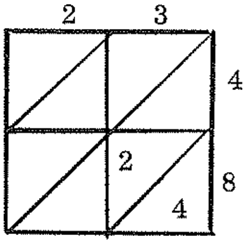


Write partial products.



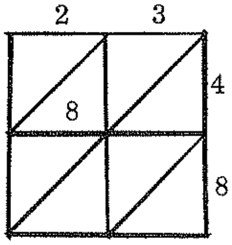
3 x 4 = 12

Use upper and lower divisions of square



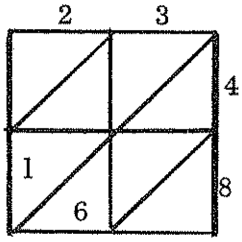
3 x 8 = 24

Use upper and lower divisions of square



2 x 4 = 8

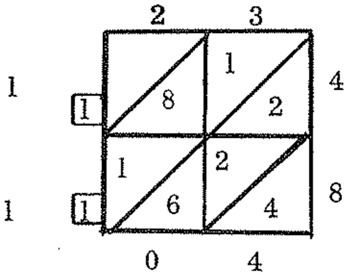
Use lower division only of the square



2 x 8 = 16

Use upper and lower divisions of square

Completed square



Sum along the diagonals

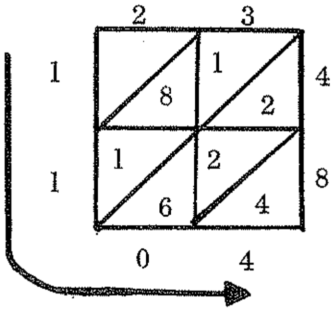
4 = 4

6 + 2 + 2 = 10, write 0 and carry the 1

1 + 1 + 8 + 1 = 11, write 1 and carry a 1

1 = 1

If sum of a diagonal is equal to or greater than 10, write last digit, carry rest to next diagonal.



Read product in direction of arrow.

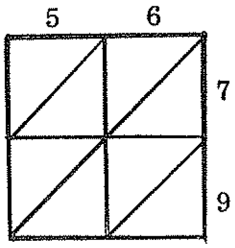
Thus, 23 x 48 = 1104.

Check

$$\begin{array}{r} 48 \\ \times\ 23 \\ \hline 144 \\ 96 \\ \hline 1104 \end{array}$$

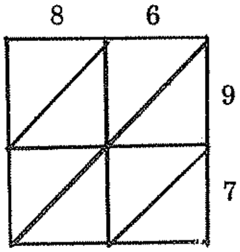
Follow the pattern of the example on page 43. Find the products.

118.



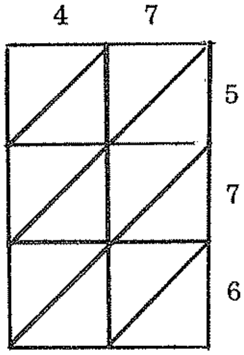
Check

7 9
x 5 6



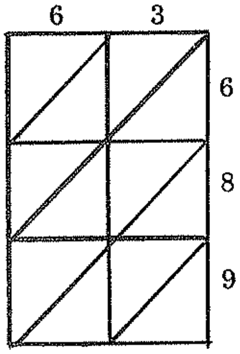
Chec

9 7
x 8 6



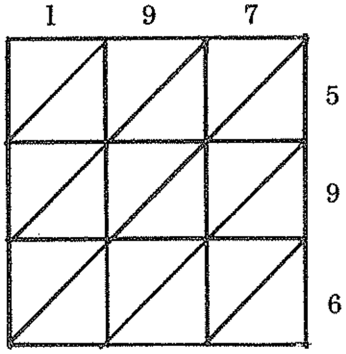
Check

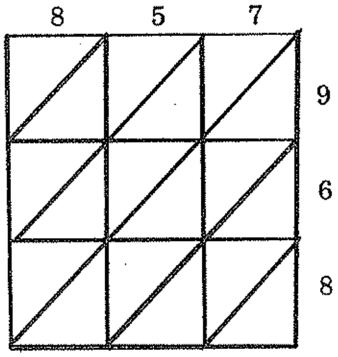
5 7 6
x 4 7

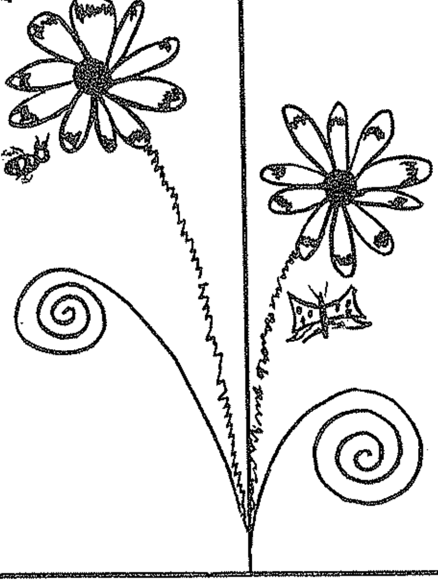


Che

6 8
x 6







Check

5 9 6
x 1 9 7

Check

9 6 8
x 8 5 7

RUSSIAN PEASANT

The following multiplication procedure is said to have been popular with Russian peasants.

Example 1 Take 23 x 48

	Left	Right	Is there an odd number in left column	Number in right column
Write	23	48	→	48
Halve 23, discard remainder, double 48	11	96	→	96
Halve 11 , discard remainder, double 96	5	192	→	192
Halve 5 , discard remainder, double 192	2	384		
Halve 2, double 384	1	768	→	+ 768
			Add	1104

Stop when a 1 appears in the left column

Thus, 23 x 48 = 1104.

Find the sum of the numbers in the right hand column which are opposite odd numbers in the left hand column .

Check

$$\begin{array}{r} 48 \\ \times\ 23 \\ \hline 144 \\ 96 \\ \hline 1104 \end{array}$$

Example 2 Take 48 x 23 Abbreviated form

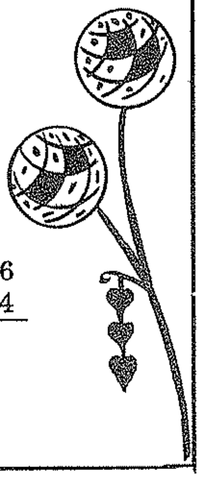
	Left	Right	Odd Left	Number Right
Write	48	23		
Halve and double	24	46		
Halve and double	12	92		
Halve and double	6	184		
Halve and double	3	368	→	368
Halve and double	1	736	→	+ 736
Stop			Add	1104

Thus, 48 x 23 = 1104

Use the Russian peasant multiplication method.

119. Take 24 x 56

Left	Right	Odd Left	Number Right
24	56		



Check 5 6
x 2 4

120. Take 56 x 24

Left	Right	Odd Left	Numb Righ
56	24		

Check 2 4
x 5 6

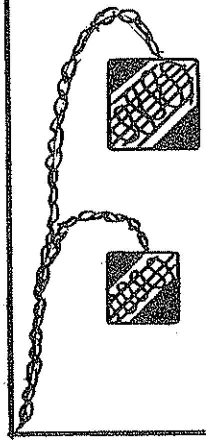
121. Take 142 x 267

Left	Right	Odd Left	Number Right
142	267		

Check 267
x 142

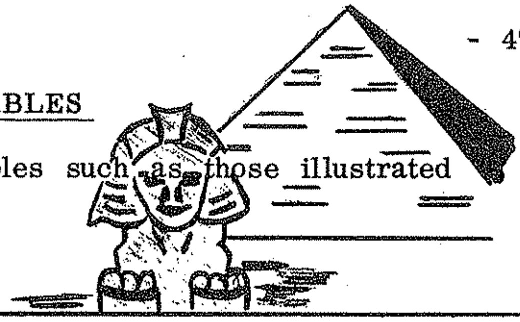
122. Take 365 x 422

Left	Right	Odd Left	Numb Rig
365	422		



Check 422
x 365

EGYPTIAN DOUBLING TABLES



The Egyptians found products using doubling tables such as those illustrated below.

Example 1 Take 28 x 48

	Left	Right	From left column select numbers totaling 28	Take numbers from right column
Write	1	48		
Double 1 and 48	2	96		
Double 2 and 96	4	192	→ 4	192
Continue doubling the numbers	8	384	→ 8	384
	16	768	→ 16	+ 768
			28	Add 1 344

Thus, 28 x 48 = 1344.

From left column select numbers which add up to the multiplier. From right column take the corresponding numbers.

Length of doubling tables was determined by the size of the multiplier.

Check

$$\begin{array}{r} 48 \\ \times\ 28 \\ \hline 384 \\ 96 \\ \hline 1344 \end{array}$$

Example 2 Take 61 x 75 Abbreviated form

	Left	Right	Left column Total 61	Right column numbers
Write	1	75	→ 1	75
Double	2	150		
Double	4	300		
Double	8	600		
Double	16	1200		
Write	10	750		
Double	20	1500	→ 20	1500
Double	40	3000	→ 40	+ 3000
Double	80	6000		
			61	4575

Thus, 61 x 75 = 4575

Check

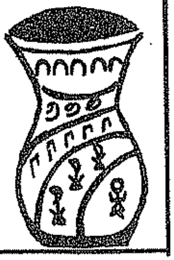
$$\begin{array}{r} 75 \\ \times\ 61 \\ \hline 75 \\ 450 \\ \hline 4575 \end{array}$$

Use Egyptian doubling tables for the following products.

123. Take 13 x 62

Left	Right	Left total	Right number
1	62		

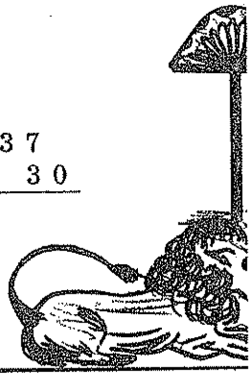
Check 62
x 13



124. Take 30 x 37

Left	Right	Left total	Right number
1	37		

Check 37
x 30



125. Take 104 x 230

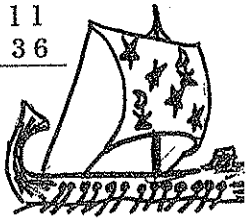
Left	Right	Left total	Right number
1	230		

Check 230
x 104

126. Take 136 x 111

Left	Right	Left total	Right number
1	111		

Check 111
x 136



CHAPTER 4

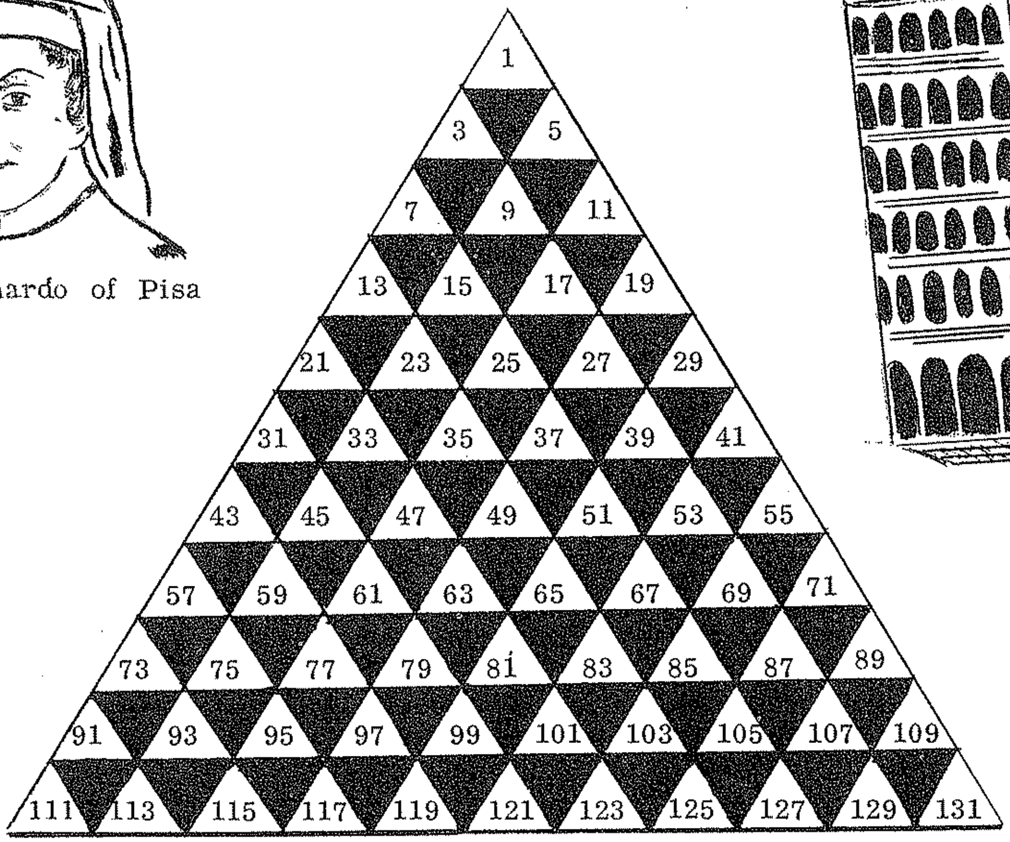
WONDER - FULL WORLD OF NUMBERS

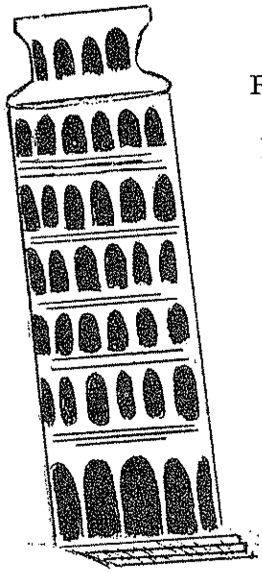
4.1 FIBONACCI REVISITED - GREETINGS LUCAS

Fibonacci's LIBER ABACI fascinated Gigi Luigi. One day Gigi discovered

Leonardo of Pisa

FIBONACCI'S ODD NUMBER TRIANGLE

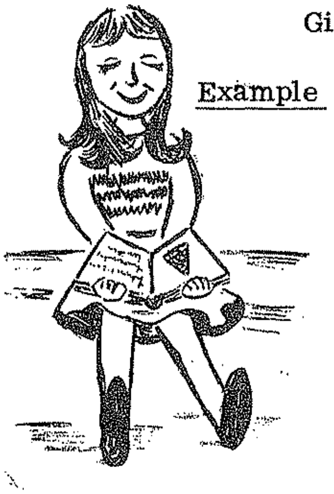




127. Use Fibonacci's triangle. Fill in the blanks.

Row	Sum of odd numbers in each row	Pattern
1	1	= 1 = 1^3
2	3 + 5	= 8 = 2^3
3	7 + 9 + 11	= ______ = ______
4	13 + 15 + 17 + 19	= ______ = ______
5	21 + 23 + 25 + 27 + 29	= ______ = ______

Row	Sum of odd numbers in each row	Patter
6	31 + 33 + 35 + 37 + 39 + 41	= ______ = ___
7	43 + 45 + 47 + 49 + 51 + 53 + 55	= ______ = ___
8	57 + 59 + 61 + 63 + 65 + 67 + 69 + 71	= ______ = ___
9	73 + 75 + 77 + 79 + 81 + 83 + 85 + 87 + 89	= ______ = ___
10	91 + 93 + 95 + 97 + 99 + 101 + 103 + 105 + 107 + 109	= ______ = ___
11	111 + 113 + 115 + 117 + 119 + 121 + 123 + 125 + 127 + 129 + 131	= ______ = ___



Gigi experimented with Fibonacci's triangle.

Example

Rows	Sum of numbers in the rows
1	$1^3 = 1$ $= 1^2$
1 , 2	$1^3 + 2^3 = 1 + 3 + 5 = 9 = 3^2$ $= (1 + 2)^2$

The sum of the cubes of 1 and 2 is equal to the square of the sum of 1 and 2 .

Gigi's pattern . Fill in the blanks .

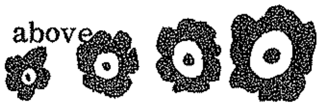
128.

Rows	Sum of the numbers in the rows
1, 2, 3	$1^3 + 2^3 + 3^3 = 1 + 3 + 5 + 7 + 9 + 11 = 36 = 6^2$ $= (1 + 2 + 3)^2$
1, 2, 3, 4	$1^3 + 2^3 + 3^3 + 4^3 =$ ______________________ = ______________

Use the pattern shown on the bottom of page 50.

Rows	Sum of the numbers in the rows
1, 2, 3, 4, 5	$1^3 + 2^3 + 3^3 + 4^3 + 5^3 =$ ______________ ______________ $= (\underline{\qquad\qquad})^2$
1, 2, 3, 4, 5, 6	$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 =$ ______________ ______________ $= (\underline{\qquad\qquad})^2$
1, 2, 3, 4, 5, 6 7	$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 =$ ______________ ______________ $= (\underline{\qquad\qquad})^2$
1, 2, 3 , 4, 5, 6 , 7, 8	$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 =$ ______________ ______________ $= (\underline{\qquad\qquad})^2$

129. Gigi stopped. There was a recognizable pattern above. Now for the generalization. Fill in the blanks.



$1^3 = 1^2$

$1^3 + 2^3 = (1 + 2)^2$

$1^3 + 2^3 + 3^3 = (1 + 2 + 3)^2$

$1^3 + 2^3 + 3^3 + 4^3 =$ ______________

$1^3 + 2^3 + 3^3 + 4^3 + 5^3 =$ ______________

$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 =$ ______________

$1^3 + 2^3 + 3^8 + 4^3 + 5^3 + 6^3 + 7^3 =$ ______________

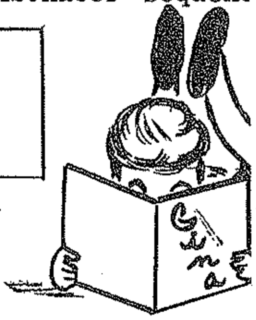
. . . .

$1^3 + 2^3 + 3^3 + 4^3 + \ldots + n^3 =$ ______________

FIBONACCI AND LUCAS SEQUENCES

Gigi Luigi showed her friend Gina Nina a few terms of the Fibonacci sequenc

1	1	2	3	5	8	. . .
F_1	F_2	F_3	F_4	F_5	F_6	. . .



Gina discovered at once the pattern for the formation of the sequence.

Fill in the blanks.

130. Each term of the Fibonacci sequence, except the first two, is the ________ (sum, product of the preceding ________ (two, three) Fibonacci numbers.

Later, Gina Nina found another Fibonacci type sequence. It was called the Lu sequence. She showed Gigi Luigi a few terms of the Lucas sequence.

1	3	4	7	11	18	. . .
L_1	L_2	L_3	L_4	L_5	L_6	. . .

Gigi discovered at once the pattern for the formation of the sequence.

Fill in the blanks.

131. Each term of the Lucas sequence, except the first two, is the ________ (sum , produc of the preceding ________ (two, three) Lucas numbers.

132. List the following terms of the Lucas sequence.

L_1 = 1

L_2 = 3

L_3 = 4

L_4 = ________

L_5 = ________

L_6 = ________

L_7 = ________

L_8 = ________

L_9 = ________

L_{10} = ________

L_{11} = ________

L_{12} = ________

133. Fill in the chart.

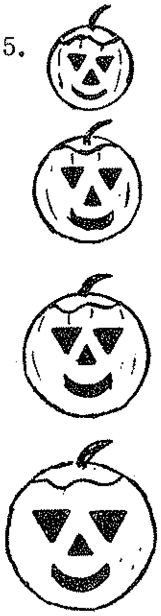
Sum of consecutive Lucas numbers		The sum represented as the difference of Lucas numbers
1	= ______	4 - 3
1 + 3	= ______	7 - 3
1 + 3 + 4	= ______	11 - 3
1 + 3 + 4 + 7	= ______	______
1 + 3 + 4 + 7 + 11	= ______	______
1 + 3 + 4 + 7 + 11 + 18	= ______	______
1 + 3 + 4 + 7 + 11 + 18 + 29	= ______	______
1 + 3 + 4 + 7 + 11 + 18 + 29 + 47	= ______	______
1 + 3 + 4 + 7 + 11 + 18 + 29 + 47 + 76	= ______	______
1 + 3 + 4 + 7 + 11 + 18 + 29 + 47 + 76 + 123	= ______	______

TOP POPPER

134. In general

$1 + 3 + 4 + 7 + 11 + . \;.\; . + L_n =$ ______________

135. Find the following quotients involving consecutive Lucas numbers, Carry out the division to 5 decimal places.



$\frac{L_2}{L_1} = \frac{3}{1} = 3$

$\frac{L_3}{L_2} = \frac{4}{3} = 1.33333$

$\frac{L_4}{L_3} = \frac{7}{4} =$ ______

$\frac{L_5}{L_4} = \frac{11}{7} =$ ______

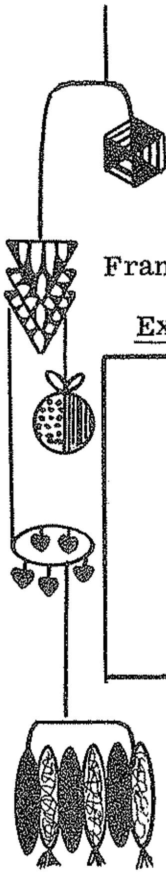
$\frac{L_{18}}{L_{17}} = \frac{5778}{3571} =$ ______

$\frac{L_{19}}{L_{18}} = \frac{9349}{5778} =$ ______

$\frac{L_{20}}{L_{19}} = \frac{15127}{9349} =$ ______

$\frac{L_{25}}{L_{24}} = \frac{167761}{103682} =$ ______

4.2 AMICABLE AND PERFECT NUMBERS



Proper divisors of a natural number

A proper divisor of a natural number n is any exact natural number divisor of n except n itself.

Example Proper divisors of 6 are 1, 2, 3.

Fran Friendly found some AMICABLE NUMBERS in Greek mathematics.

Example

Set of proper divisors of 2 8 4 = { 1, 2, 4, 71, 142 }

Sum of the proper divisors = 1 + 2 + 4 + 71 + 142 = 2 2 0

Set of proper divisors of 2 2 0 = { 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110

Sum of the proper divisors = 1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 +
= 2 8 4

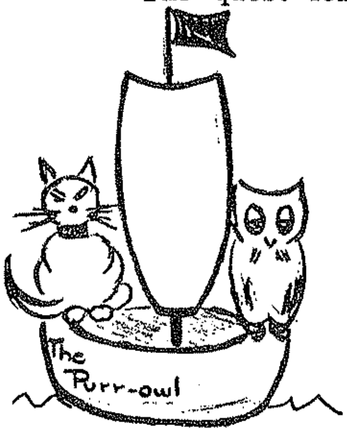
Conclusion: Sum of the proper divisors of 284 is 220.
Sum of the proper divisors of 220 is 284.
The numbers 284 and 220 are called amicable numbers.

136. Fill in the chart. Find the amicable number pairs.

Numbers	Proper divisors	Sum of proper divisors	Amicab number	
			Yes	
56				
74				
120				
140				
2620				
2924				
5020				
5564				

QUEST FOR PERFECT NUMBERS

The quest for perfect numbers began with the ancient Greeks, the Pythagoreans.



Example   Take 6

Set of proper divisors = { 1 , 2 , 3 }

Sum of proper divisors = 1 + 2 + 3 = 6

The number 6 is called a perfect number since it is equal to the sum of its proper divisors .

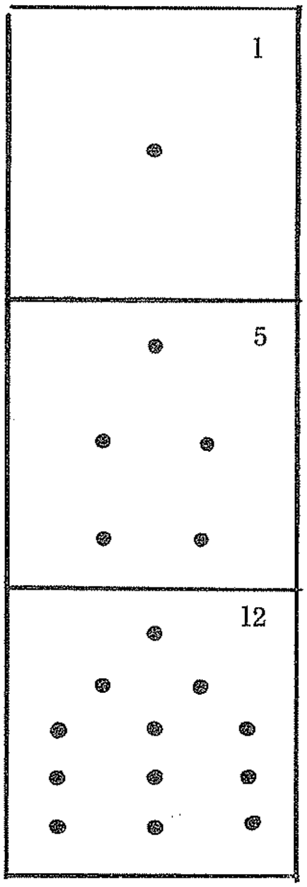
137. Fill in the chart . Find the perfect numbers .

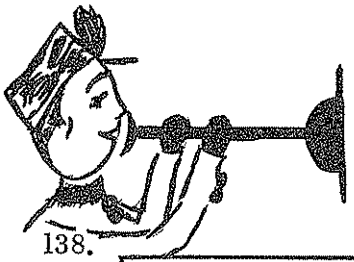
Number	Proper divisors	Sum of proper divisors	Perfect number	
			Yes	No
16	1, 2 , 4 , 8	15		✓
28				
152				
230				
360				
496				
550				
2000				
2525				
4000				
8128				

4.3 FIGURATE NUMBERS - EXTENDED

Greek mathematicians used dots arranged in geometric patterns to represent some sets of natural numbers. The numbers were given names that corresponded to the geometric figures. Figurate numbers are also called polygonal numbers.

PENTAGONAL NUMBERS





Tuto Figaro's challenge

Study the dot pattern of the pentagonal numbers. Write the next 5 pentagonal numbers.

138. _____ _____ _____ _____ _____

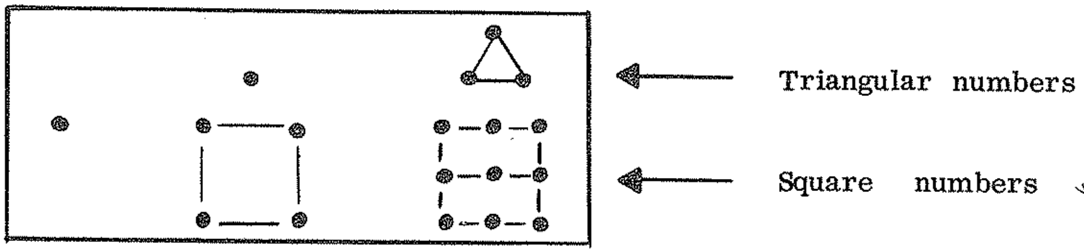
PENTAGONAL NUMBER FORMULA $\frac{n(3n-1)}{2}$

To find the 1 st pentagonal number put 1 for n in tl formula. To find the 2 nd, 3 rd, and so on, pentagon numbers, put 2, 3, . . . for n in the formula.

139. Use the formula. List the first 20 pentagonal numbers.

1	5	12	_____	_____
_____	_____	_____	_____	_____
_____	_____	_____	_____	_____
_____	_____	_____	_____	_____

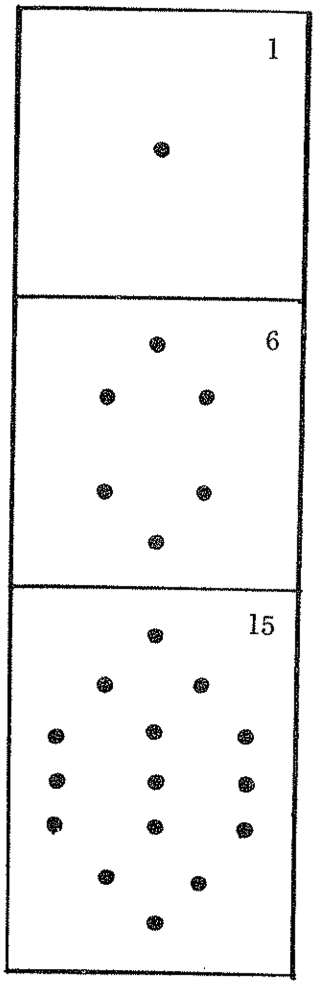
Flora Figaro arranged the pentagonal numbers in a row. Result ? A discovery !

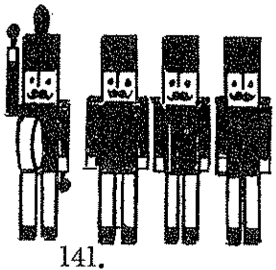


Fill in the blanks.

140. Each pentagonal number except the first is the sum of a __________ number ai a __________________ number.

HEXAGONAL NUMBERS





<u>Another challenge from Tuto Figaro</u>

Study the dot pattern of the hexagonal numbers. Write the next 5 hexagonal numbers.

141.

____ ____ ____ ____ ____

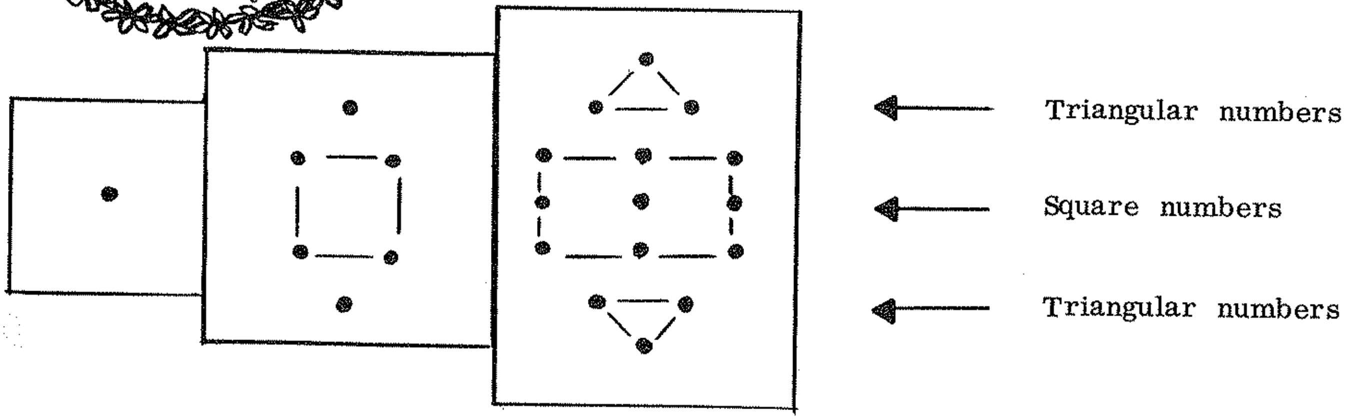
HEXAGONAL NUMBER FORMULA $n(2n-1)$

To find the 1 st hexagonal number put 1 for n in the formula. To find the 2 nd, 3 rd, and so on, hexagonal numbers, put 2, 3, . . . for n in the formula.

142. Use the formula. List the first 20 hexagonal numbers.

1	6	15	____	____
____	____	____	____	____
____	____	____	____	____
____	____	____	____	____

Flora Figaro arranged the hexagonal numbers in a row and made another discovery.



<u>Fill in the blanks</u>.

143. Each hexagonal number except the first is the sum of a ____________ number and twice a ______________ number.

4.4 HAPPY NUMBERS - DAYS AND MONTHS

Take a number	13
Sum the squares of the numbers represented by each digit in 13	$1^2 + 3^2 = 10$
Repeat for 10	$1^2 + 0^2 = 1$

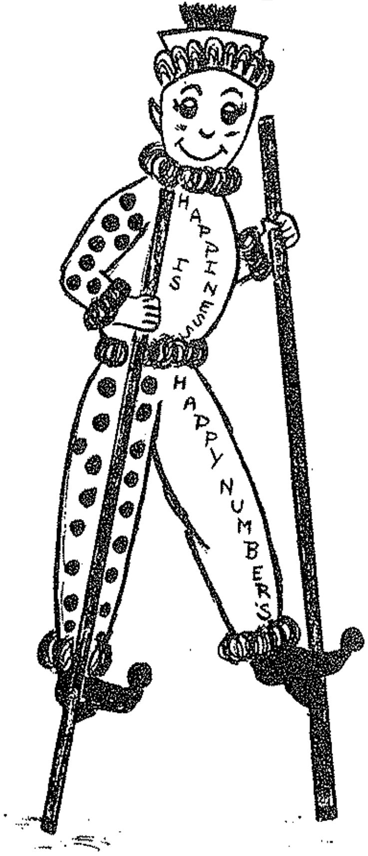
The natural numbers for which the above pattern ends in a 1 are called HAPPY NUMBERS. 13 is a happy number.

Assign consecutive numbers to the letters of the English alphabet as shown.

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

Take	M O N D A Y
Sum the numbers associated with the letters in the word	$13 + 15 + 14 + 4 + 1 + 25 = 72$

Test 72 for a HAPPY NUMBER.



$7^2 + 2^2 = 49 + 4 = 53$

$5^2 + 3^2 = 25 + 9 = 34$

$3^2 + 4^2 = 9 + 16 = 25$

$2^2 + 5^2 = 4 + 25 = 29$

$2^2 + 9^2 = 4 + 81 = 85$

$8^2 + 5^2 = 64 + 25 = 89$

$8^2 + 9^2 = 64 + 81 = 145$

$1^2 + 4^2 + 5^2 = 1 + 16 + 25 = 42$

$4^2 + 2^2 = 16 + 4 = 20$

$2^2 + 0^2 = 4$

$4^2 = 16$

$1^2 + 6^2 = 1 + 36 = 37$

$3^2 + 7^2 = 9 + 49 = 58$

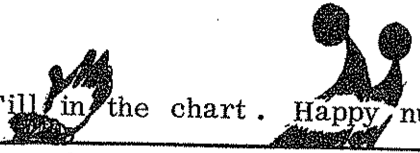
$5^2 + 8^2 = 25 + 64 = 89$

MONDAY'S
number
is
NOT
a
HAPPY
NUMBER

Pattern repeats

144. Fill in the chart . Happy number days .

Day	Sum of numbers associated with the letters in the word	Happy number day	
		Yes	No
MONDAY	13 + 15 + 14 + 4 + 1 + 25 = 72		✓
TUESDAY			
WEDNESDAY			
THURSDAY			
FRIDAY			
SATURDAY			
SUNDAY			

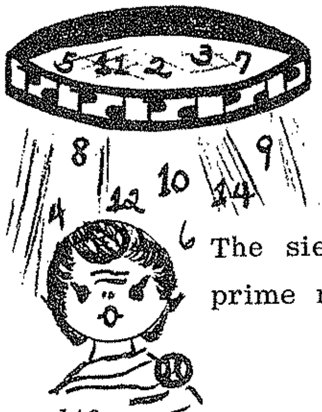


145. Fill in the chart . Happy number months .

Month	Sum of numbers associated with the letters in the word	Happy number month	
		Yes	No
JANUARY			
FEBRUARY			
MARCH			
APRIL			
MAY			
JUNE			
JULY			
AUGUST			
SEPTEMBER			
OCTOBER			
NOVEMBER			
DECEMBER			

4.5 PRIME NUMBERS – FACTS : PLAIN AND FANCY

Prime Natural Numbers



A prime natural number is a natural number other than 0 and 1 which is exactly divisible only by the natural number 1 and itself.

2, 3, and 5 are the first three prime numbers.

The sieve method of Eratosthenes is one procedure for finding the prime numbers. A slightly different method is shown below.

Fill in the chart on the pattern of the first three rows.

146.

L. A. Watts Method

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	2		2		2		2		2		2		2		2		2		2		2		2		2		2
		3			3			3			3			3			3			3			3			3	
–	P	P	C																								

147. Mark with P the columns headed by prime numbers.
Mark with C the columns headed by composite numbers.

Composite Natural Numbers

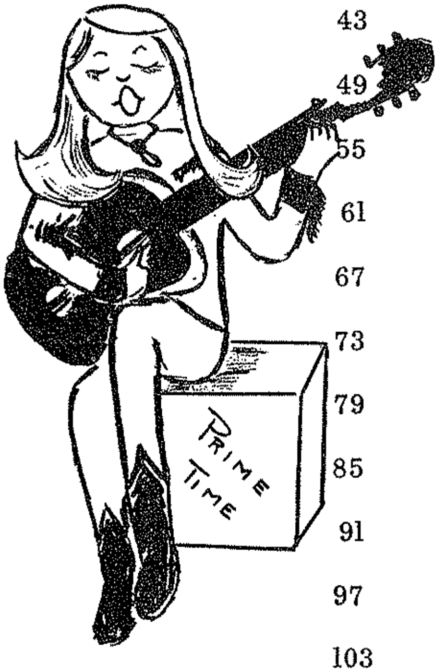
A composite natural number is a natural number which is not 0, 1, or a prime natural number.

148. Fill in the blank.

The entries in the columns marked C are all the ________ divisors of the particular composite number heading the column.

Circle the prime numbers only .

149.



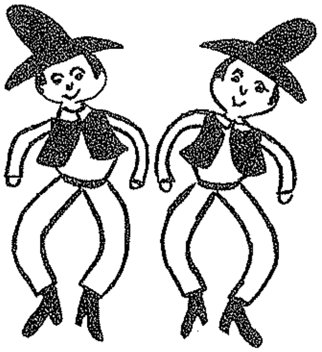
1	②	③	4	⑤	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42
43	44	45	46	47	48
49	50	51	52	53	54
55	56	57	58	59	60
61	62	63	64	65	66
67	68	69	70	71	72
73	74	75	76	77	78
79	80	81	82	83	84
85	86	87	88	89	90
91	92	93	94	95	96
97	98	99	100	101	102
103	104	105	106	107	108
109	110	111	112	113	114
115	116	117	118	119	120
121	122	123	124	125	126

Use the results of the problem above.

150. Except for the prime numbers 2 and 3 , all other prime numbers are in the ______________ column and in the ____________ column .

151. Let __n__ take on the values 1, 2, 3, 4, 5 and so on . Write a formula involving n for the prime numbers in the columns where they occur above.

Each prime number except 2 and 3 can be expressed as ________________ or as _______________ .



TWIN PRIMES

Two prime numbers separated by only one composite number are called twin primes.

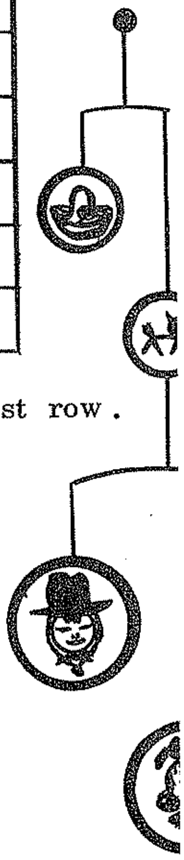
Example 3 and 5 is a pair of twin primes,
5 and 7 is a pair of twin primes.

The sum of twin primes (each prime > 3) has an interesting divisibility property .

List the twin primes less than 126 . Fill in the blanks.
Follow the pattern of the first 2 rows .

152.

Twin primes	Sum	Quotient
3 5	3 + 5 = 8	8 ÷ 12 = 8/12 = 2/3
5 7	5 + 7 = 12	12 ÷ 12 = 1
		÷ 12 =
		÷ 12 =



Twin primes are listed below . Fill in the chart on the pattern of the first row .

153.

Twin primes	Sum	Quotient
137 139	137 + 139 = 276	276 ÷ 12 = 23
179 181		
227 229		
269 271		
281 283		
311 313		

154. Sum of pair of twin primes (each prime > 3) is divisible exactly by ________ .

DISTRIBUTION OF PRIMES

Challenge

Find a prime natural number that is represented by 1 digit ? 2 digits ? More ?

For Bea Prime this was not a difficult problem to solve.

She remembered that

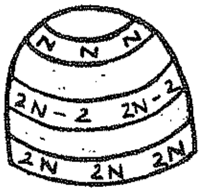
> For any choice of n = 1, 2, 3, . . . either n or 2n is a prime, or there is at least one prime number between n and 2n.

155. Fill in the chart. Use the pattern in the first row.

To find a	Take n	2n is	List one prime
1 digit prime	1 through 4	2 through 8	2
2 digit prime	5 through 49	10 through 98	
3 digit prime	___ through ___	___ through ___	

Bea Prime found another interesting statement on the distribution of primes.

> For any choice of $n \geq 4$, there is at least one prime number equal to n or between n and $2(n-1)$.



Fill in the charts. Use the pattern in the first rows.

156.

n	2 (n - 1)	Prime equal to n or between n and 2 (n - 1)
4	6	5
5		
10		
17		
21		
24		
28		

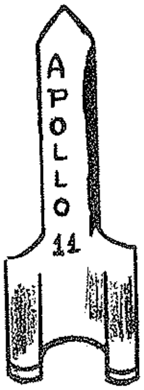
157.

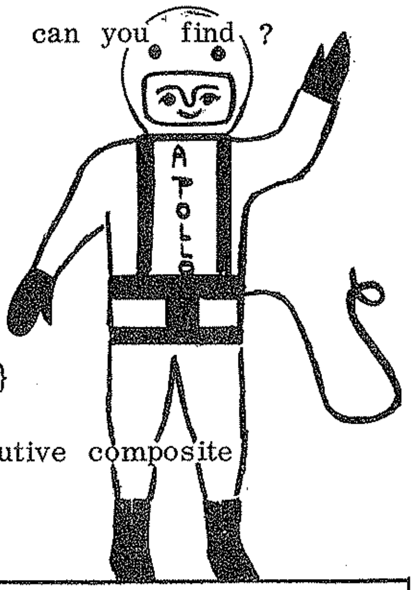
n	2 (n - 1)	Prime equal to n or between n and 2 (n - 1)
35	68	41
39		
41		
45		
50		
58		
63		

CONSECUTIVE COMPOSITE NUMBERS

Challenge

How many consecutive composite natural numbers can you find ?





Example Take $2 \cdot 3 \cdot 4 + 2 = 26$
$2 \cdot 3 \cdot 4 + 3 = 27$
$2 \cdot 3 \cdot 4 + 4 = 28$

Now $D_{26} = \{ 1, 2, 13, 26 \}$
$D_{27} = \{ 1, 3, 9, 27 \}$
$D_{28} = \{ 1, 2, 4, 7, 14, 28 \}$

Thus, 26, 27, 28 are three consecutive composite natural numbers .

158. Fill in the blanks.

$3 \cdot 4 \cdot 5 + 3 =$ ______
$3 \cdot 4 \cdot 5 + 4 =$ ______
$3 \cdot 4 \cdot 5 + 5 =$ ______

$4 \cdot 5 \cdot 6 + 4 =$ ______
$4 \cdot 5 \cdot 6 + 5 =$ ______
$4 \cdot 5 \cdot 6 + 6 =$ ______

$2 \cdot 3 \cdot 4 \cdot 5 + 2 =$ ______
$2 \cdot 3 \cdot 4 \cdot 5 + 3 =$ ______
$2 \cdot 3 \cdot 4 \cdot 5 + 4 =$ ______
$2 \cdot 3 \cdot 4 \cdot 5 + 5 =$ ______

$3 \cdot 4 \cdot 5 \cdot 6 + 3 =$ ______
$3 \cdot 4 \cdot 5 \cdot 6 + 4 =$ ______
$3 \cdot 4 \cdot 5 \cdot 6 + 5 =$ ______
$3 \cdot 4 \cdot 5 \cdot 6 + 6 =$ ______

$2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 + 2 =$ ______
$2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 + 3 =$ ______
$2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 + 4 =$ ______
$2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 + 5 =$ ______
$2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 + 6 =$ ______

$3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 + 3 =$ ______
$3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 + 4 =$ ______
$3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 + 5 =$ ______
$3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 + 6 =$ ______
$3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 + 7 =$ ______

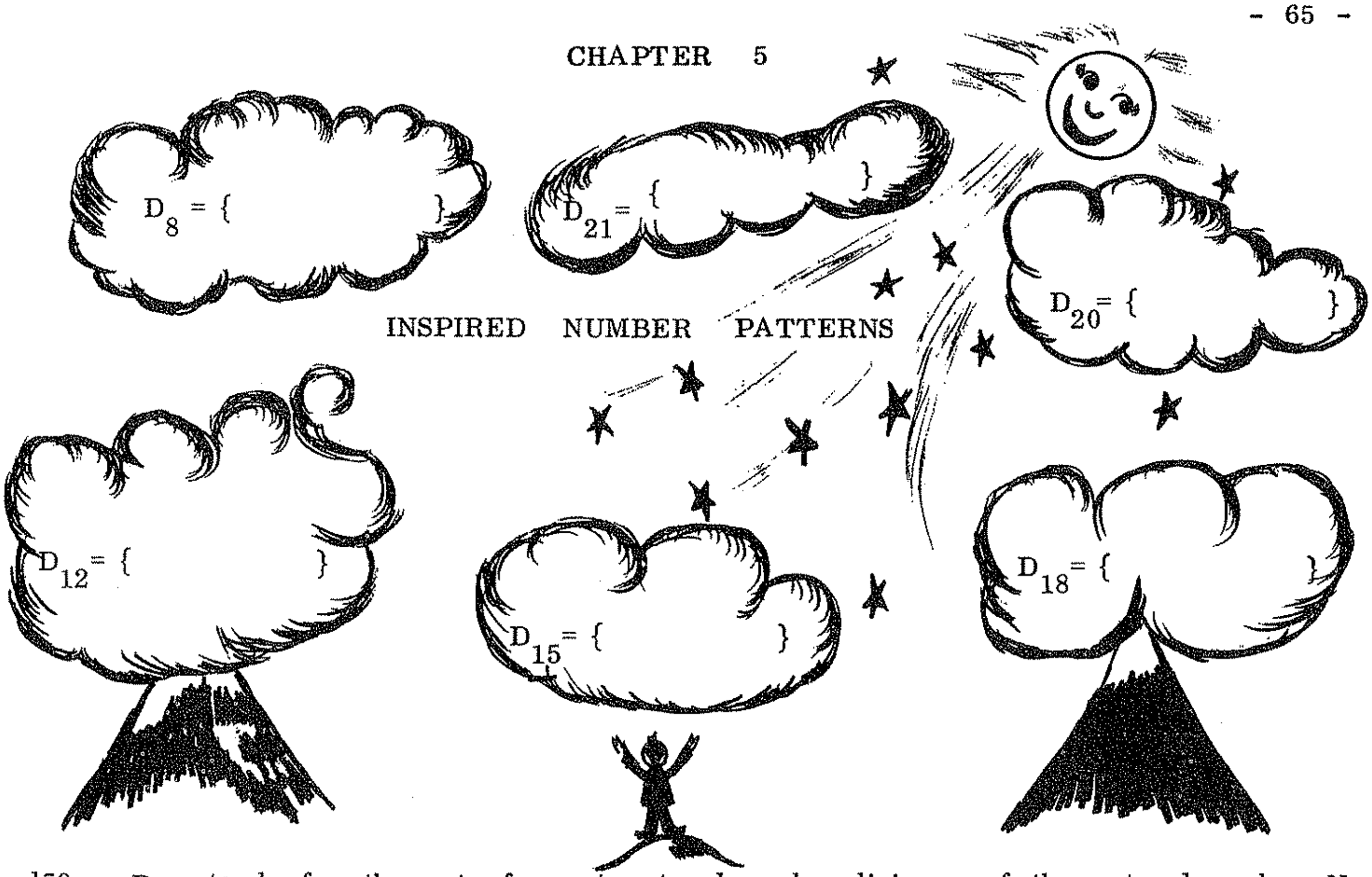
$4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 + 4 =$ ______
$4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 + 5 =$ ______
$4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 + 6 =$ ______
$4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 + 7 =$ ______
$4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 + 8 =$ ______
$4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 + 9 =$ ______

Check one

One can find ______ ______ consecutive composite natural numbers .
only a few , very many

CHAPTER 5

INSPIRED NUMBER PATTERNS

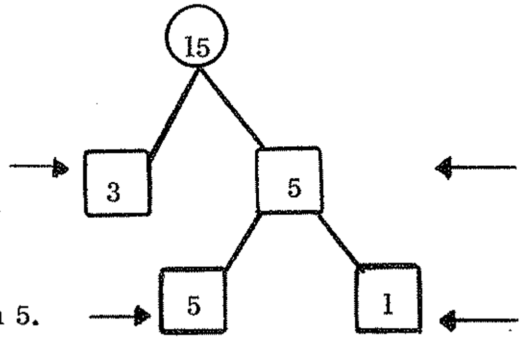


159. D_N stands for the set of exact natural number divisors of the natural number N. Fill in the clouds.

5.1 PRIME FACTORIZATION - TREE DIAGRAM

Bea Prime used a tree diagram for factoring composite numbers into primes.

Example Take 15

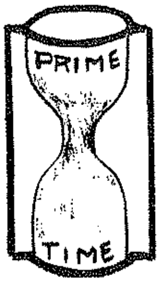


Write the smallest prime that divides 15. Here it is 3.

15 ÷ 3 = 5 Write quotient in this box.

Repeat above step on 5. Smallest prime that divides 5 is 5.

5 ÷ 5 = 1 Write quotient in this box.



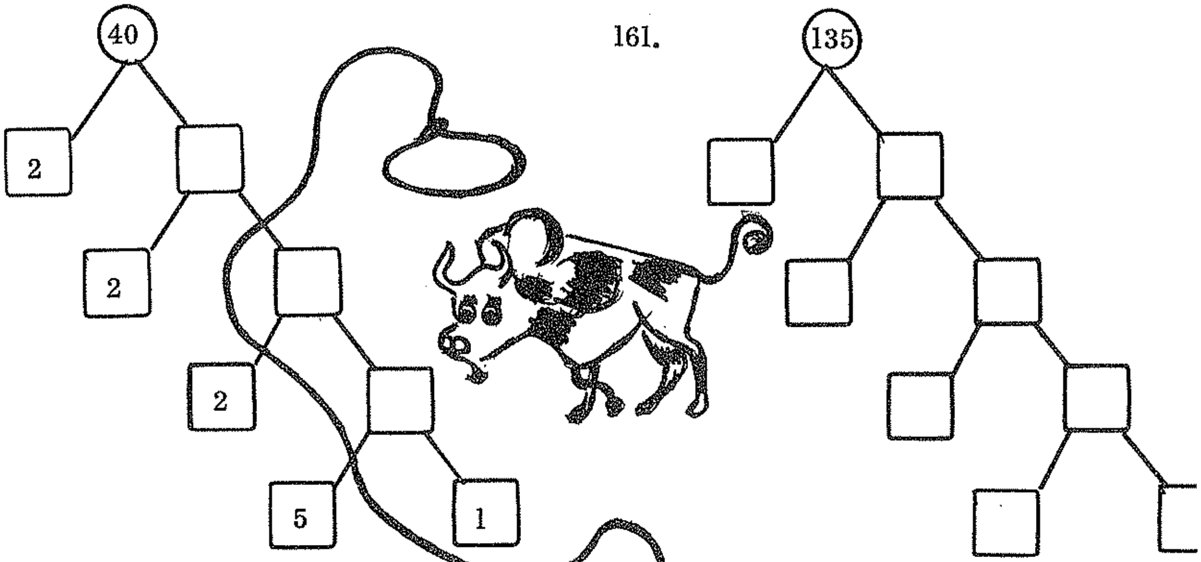
Bea found the distinct primes in the factorization of 15 and the prime factorization of 15 from the left column. Thus,

Set of distinct primes : { 3, 5 }

Prime factorization : 15 = 3 x 5

Fill in the blanks in the tree diagrams. Follow the pattern for 40.

160. 161.



Distinct prime factors : { 2, 5 }

Prime factorization : $40 = 2 \times 2 \times 2 \times 5$

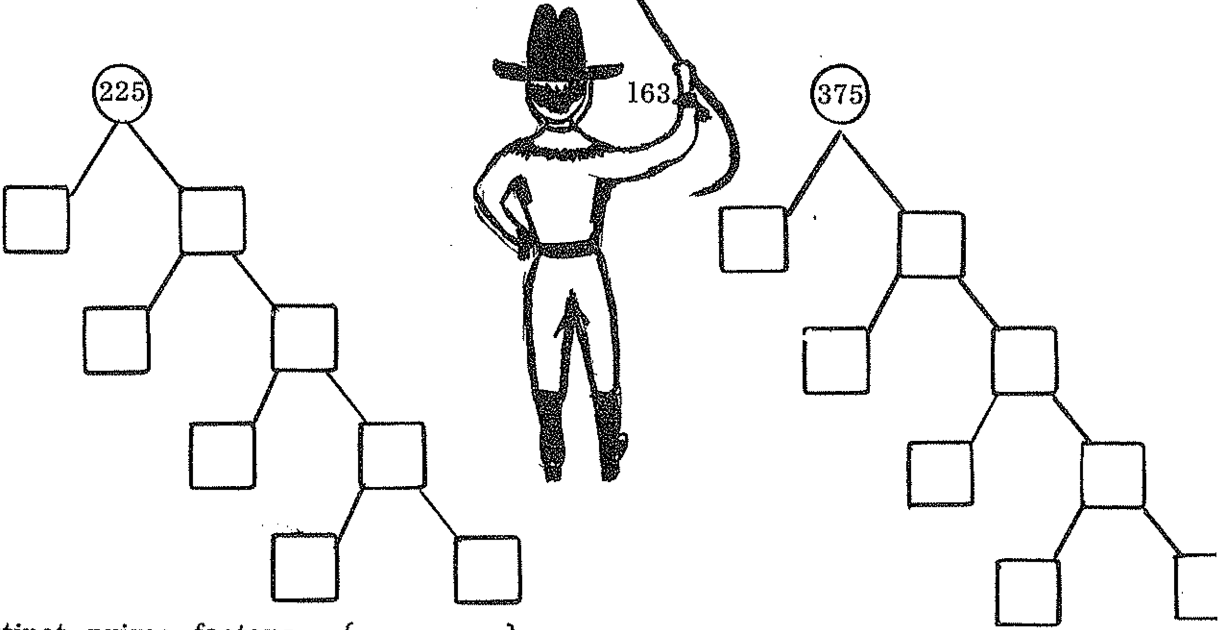
$40 = 2^3 \times 5$

Distinct prime factors : {

Prime factorization : 135 =

135 =

162. 163.



Distinct prime factors : { }

Prime factorization : 225 =

225 =

Distinct prime factors : {

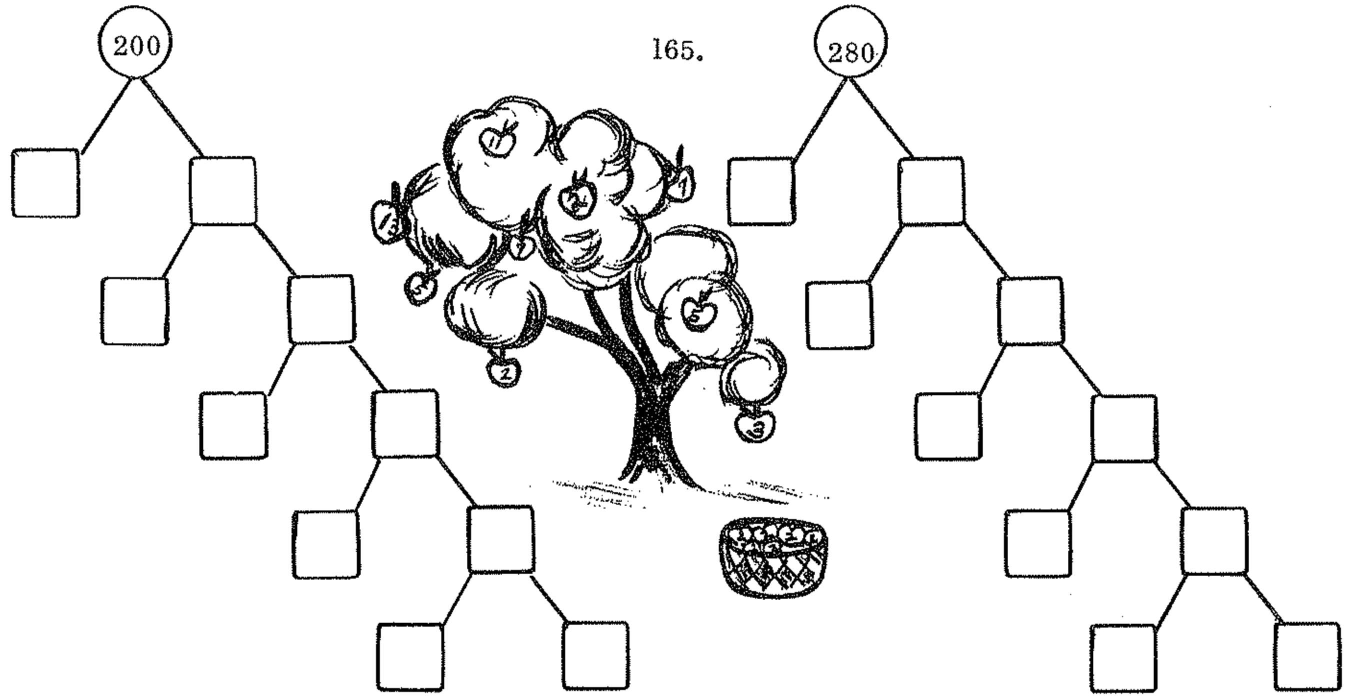
Prime factorization : 375 =

375 =

Fill in the blanks in the tree diagrams .

164.

165.



Distinct prime factors : { }

Prime factorization : 200 =

200 =

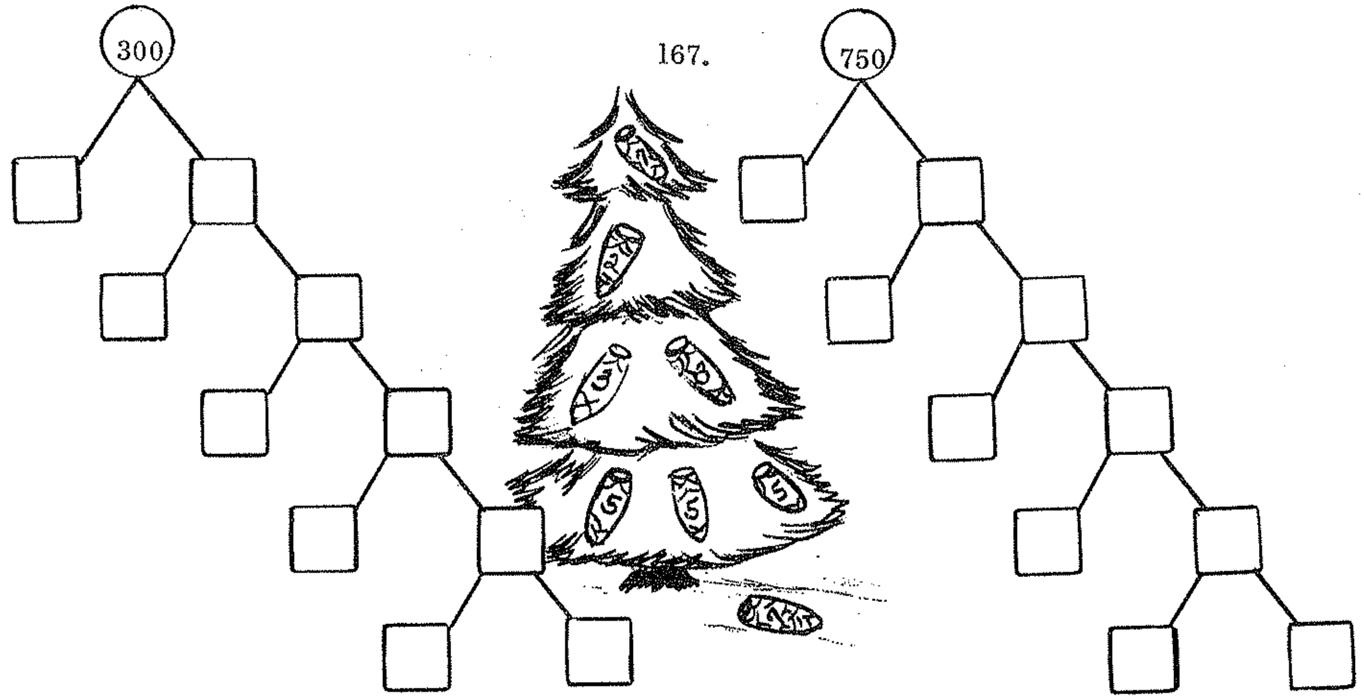
Distinct prime factors :{ }

Prime factorization : 280 =

280 =

166.

167.



Distinct prime factors : { }

Prime factorization : 300 =

300 =

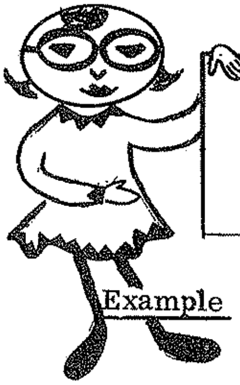
Distinct prime factors : { }

Prime factorization : 750 =

750 =

5.2 EXACT DIVISORS FROM PRIME FACTORIZATION

Zoe Prime used the prime factorization theorem to find the exact natural numbe divisors of composite natural numbers.



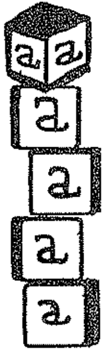
Prime Factorization Theorem

Each composite natural number can be factored into a product of primes, each prime raised to the first or higher power in only 1 way if the order of the primes is not considered.

Example Take 16 where $16 = 2^4$.

Exact divisors: $D_{16} = \{ 2^0, 2^1, 2^2, 2^3, 2^4 \}$

$D_{16} = \{ 1, 2, 4, 8, 16 \}$



General Case

$N = a^r$	1 prime in the prime factorization a is a prime r may be 1, 2, 3, 4, 5, ...

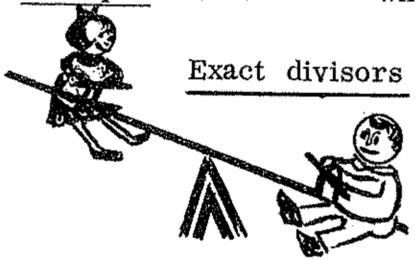
Set of exact divisors: $D_N = \{ a^0, a^1, a^2, a^3, \ldots, a^r \}$

Total number of exact divisors: $(r + 1)$.

168. Fill in the chart. Use the pattern in the first row.

Number	Prime factorization	Prime	Power	Exact divisors						Numb of ex divisc
		a	r	a^0	a^1	a^2	a^3	a^4	a^5 . .	
8	$8 = 2^3$	2	3	1	2	4	8			3 + 1
32										
81										
343										
625										

Exact divisors from the prime factorization of a composite number.



Example Take 6 where $6 = 2 \times 3$

Exact divisors : $D_6 = \{ 2^0 , 2^1 , 3^1 , 2^1 \bullet 3^1 \}$

$D_6 = \{ 1 , 2 , 3 , 6 \}$

General Case

$N = a^1 \bullet b^1$	2 distinct primes in the prime factorization a , b are distinct primes

Set of exact divisors : $D_N = \{ a^0 , a^1 , b^1 , a^1 \bullet b^1 \}$

Total number of exact divisors : $(1 + 1) \bullet (1 + 1) = 4$

169. Fill in the chart. Use the pattern in the first row.

Number	Prime factorization	Primes		Exact divisors			
		a	b	a^0	a^1	b^1	$a^1 \bullet b^1$
15	$15 = 3 \times 5$	3	5	1	3	5	15
22							
38							
46							
65							
95							
133							
143							

5.3 COMPOSITE NUMBERS - DIVISIBILITY HINTS

The following divisibility hints often prove useful for finding the divisors of a composite natural number .

170. A natural number N is divisible exactly by $\boxed{2}$ if it ends in ________

171. A natural number N is divisible exactly by $\boxed{5}$ if it ends in ________

Example 1 Take 132
Now 1 + 3 + 2 = 6
Since 6 ÷ 3 = 2 ,
then 132 is divisible
exactly by 3 ,
and 132 ÷ 3 = 44 .

Example 2 Take 162
Now 1 + 6 + 2 = 9
Since 9 ÷ 9 = 1
then 162 is divisible
exactly by 9 ,
and 162 ÷ 9 = 18 .

Fill in the blanks.

172. A natural number N is divisible exactly by $\boxed{3}$ if the ________ of the numbers represented by the digits in N is exactly divisible by ________ .

173. A natural number N is divisible exactly by $\boxed{9}$ if the ________ of the numbers represented by the digits in N is exactly divisible by ________ .

174. Fill in the chart . Use the pattern shown in the first row .

Number	Sum	Divisibility test by 3	Divisibility test by 9	Quotients
1212	1 + 2 + 1 + 2 = 6	6 ÷ 3 = 2 Test works	6 ÷ 9 = 6/9 Test fails	1212 ÷ 3 = 404 1212 ÷ 9 = 134 .
3582		Test ________	Test ________	
4713		Test ________	Test ________	

DIVISIBILITY BY 4

Example 1

Take 132

Last two digits : 32

Since 32 ÷ 4 = 8

then 132 is divisible exactly by 4 ,

and 132 ÷ 4 = 33 .

Example 2

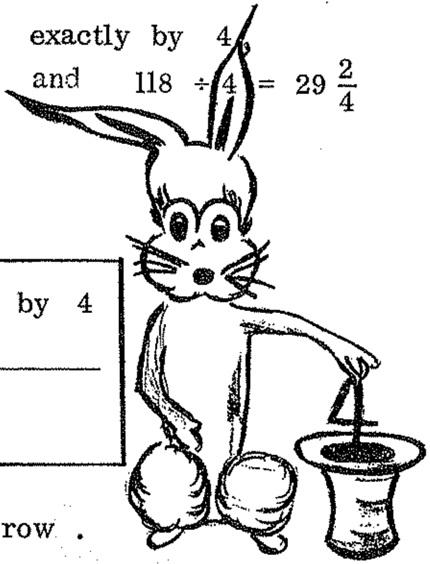
Take 118

Last two digits : 18

Since 18 ÷ 4 = $4\frac{2}{4}$

then 118 is NOT divisible exactly by 4

and 118 ÷ 4 = $29\frac{2}{4}$



Fill in the blank.

175.

A natural number N is divisible exactly by 4 if the number represented by the last ________ digits of N is divisible exactly by 4 .

176. Fill in the chart . Use the pattern shown in the first row .

Number	Last two digits of number	Divisibility test by 4	Quotient
112	12	12 ÷ 4 = 3 Test works	112 ÷ 4 = 28
268		Test ________	
375		Test ________	
482		Test ________	
596		Test ________	
680		Test ________	
1120		Test ________	
2314		Test ________	
2844		Test ________	

5.4 GREATEST COMMON DIVISOR

The greatest common divisor, gcd , of the numbers 12 and 6 is 6. Thus,

gcd (12, 6) = 6 .

U. Klid discovered a routine procedure for finding the gcd of two natural numbers .

Example 1 Consider the numbers 12 and 6 . Find gcd (12 , 6) .

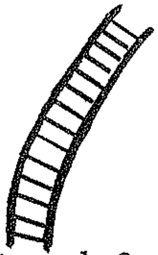
U. Klid used the short division form. Study the pattern .

$$\begin{array}{l|l} 6 & 12 \\ \hline & 2 \;\; R \;\; 0 \end{array}$$

where R stands for remainder

gcd (12 , 6) = 6 .

The divisor for which the remainder R becomes 0 is the gcd of the two given numbers .



Example 2 Consider the numbers 156 and 148 . Find gcd (156 , 148) .

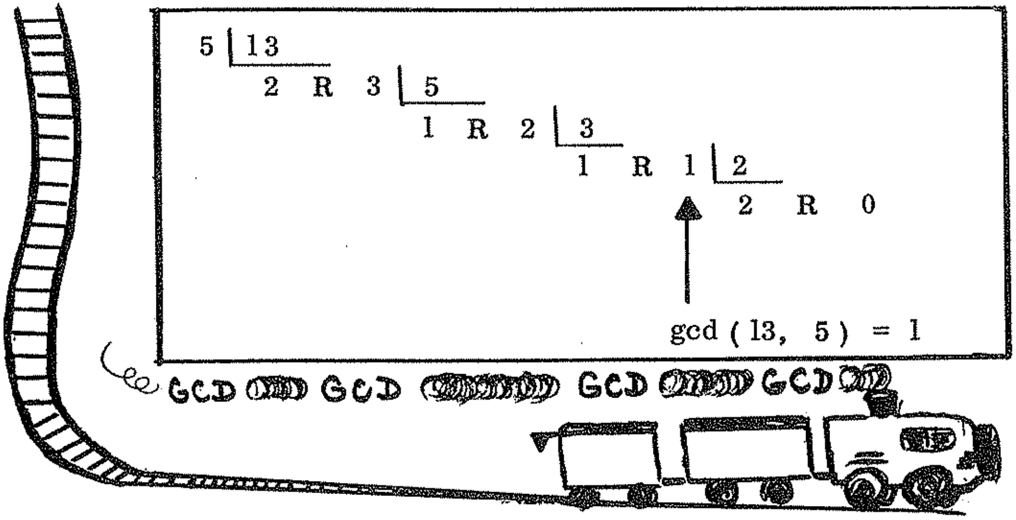
148 | 156
  1 R 8 | 148
    1 8 R 4 | 8
      2 R 0

gcd (156 , 148) = 4

Example 3 Consider the numbers 13 and 5 . Find gcd (13 , 5) .

5 | 13
  2 R 3 | 5
    1 R 2 | 3
      1 R 1 | 2
        2 R 0

gcd (13, 5) = 1

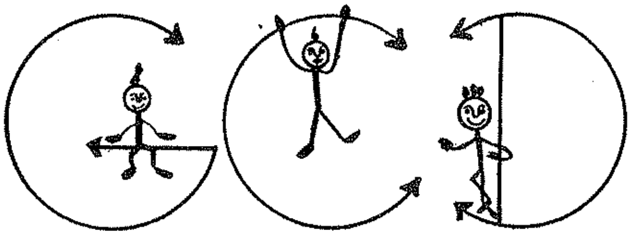


Find the greatest common divisor for each pair of numbers. Use U. Klid's short division routine shown in examples 1 - 3.

177. Find gcd (18 , 12).

12 | 18

gcd (18, 12) =



178. Find gcd (48 , 27).

27 | 48

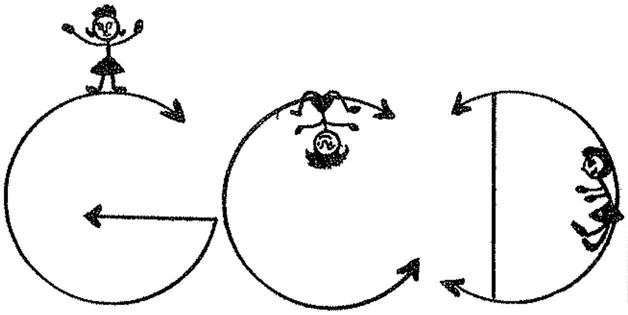
gcd (48, 27) =

179. Find gcd (158 , 144).

144 | 158

gcd (158 , 144) =

180. Find gcd (60 , 45).



45 | 60

gcd (60 , 45) =

Find the greatest common divisor for each pair of numbers.
Use U. Klid's short division routine shown in examples 1 - 3.

181. Find gcd (57 , 13).

13 | 57

gcd(57,13) =

182. Find gcd (79 , 37).

37 | 79

gcd (79 , 37) =

183. Find gcd (151 , 67).

67 | 151

gcd (151, 67) =

184. Find gcd (289 , 221).

221 | 289

gcd (289 , 221) =

185. Find gcd (75, 51).

51 | 75

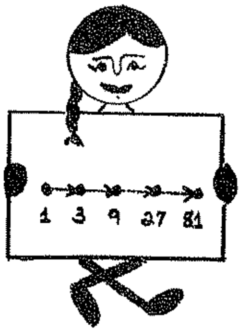
gcd (75 , 51) =

5.5 FACTOR LATTICES - EXTENSION

Mazie Dots used geometric figures such as line segments, squares, cubes, and so on, to represent the exact natural number divisors of a natural number N. Mazie called these figures **FACTOR LATTICES.**

Example 1 1 prime

$N = a^r$ a is a prime
$r = 1, 2, 3, \ldots$

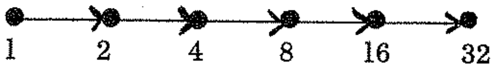


Take

32 where $32 = 2^5$

$D_{32} = \{ 1, 2, 4, 8, 16, 32 \}$

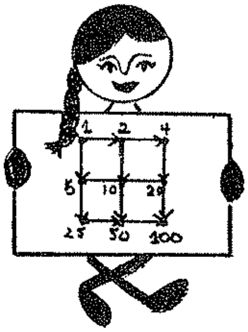
Factor lattice



Mazie's factor lattice in this case is a straight line segment with 5 congruent subsegments. Read: 1 divides 2, 2 divides 4, 4 divides 8, and so on.

Example 2 2 primes

$N = a^r \bullet b^s$ a, b are primes
$r, s = 1, 2, 3, \ldots$

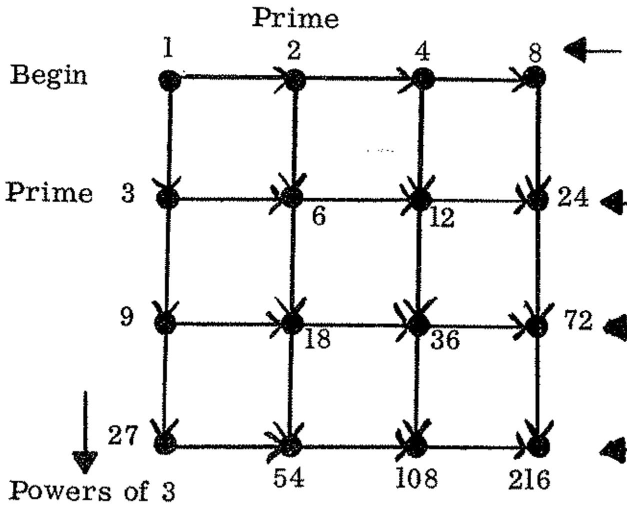


Take

216 where $216 = 2^3 \bullet 3^3$

$D_{216} = \{ 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 27, 36, 54, 72, 108, 216 \}$

Factor Lattice



lcm (2, 3) = 6
lcm (6, 4) = 12
lcm (12, 8) = 24

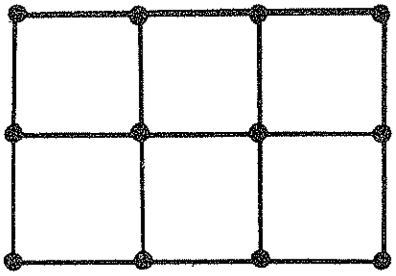
lcm (9, 6) = 18
lcm (18, 12) = 36
lcm (36, 24) = 72

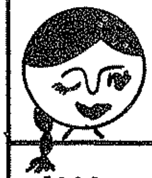
lcm (27, 18) = 54
lcm (54 , 36) = 108
lcm (108, 72) = 216

186. Work along with Mazie Dots. Draw a factor lattice for each number .

72 = $2^3 \cdot 3^2$

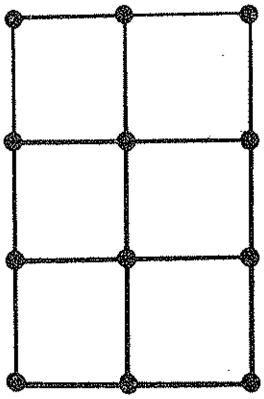
D_{72} = { }





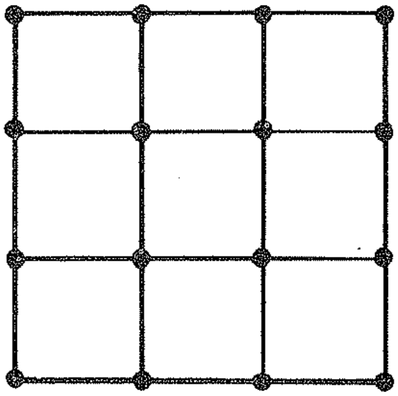
108 =

D_{108} = {



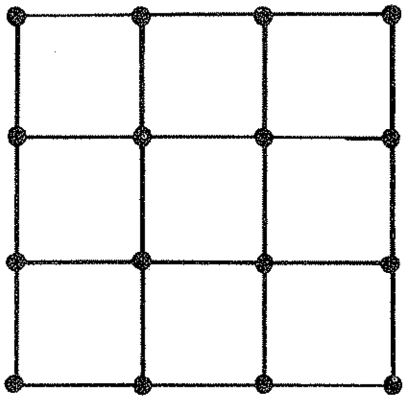
1000 =

D_{1000} = { }



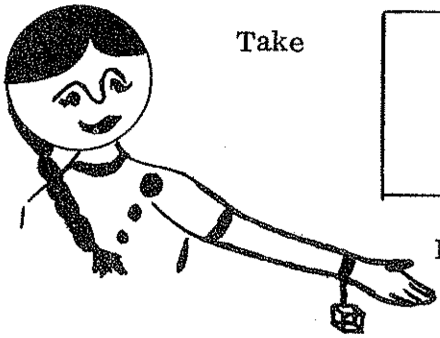
3375 =

D_{3375} = {



Example **3 primes**

$N = a^1 \bullet b^1 \bullet c^1$ a, b, c are primes



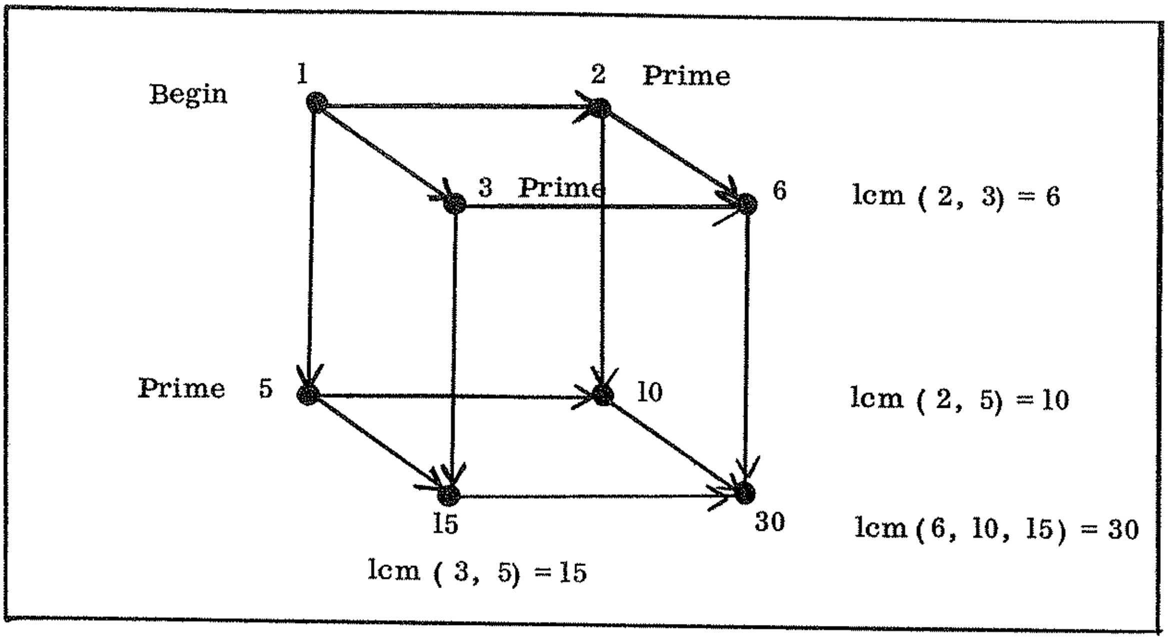
Take

$30 = 2 \bullet 3 \bullet 5$

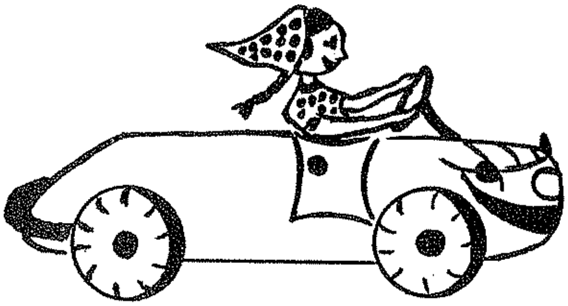
$D_{30} = \{ 1, 2, 3, 5, 6, 10, 15, 30 \}$

Mazie Dots used a cube for the

Factor Lattice



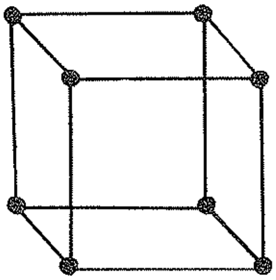
1. Begin at the upper left hand vertex of the cube. Write 1.
2. The distinct primes in the divisors of 30, namely, 2, 3, and 5 are associated with the end points of the three line segments coming from 1.
3. The relation " is the least common multiple of " is used to associate the divisors of 30 with the remaining vertices of the cube.

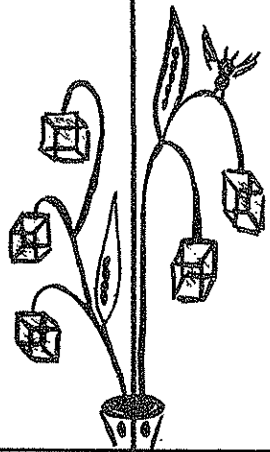


187. Work along with Mazie Dots. Draw a factor lattice for each number.

$70 = 2 \cdot 5 \cdot 7$

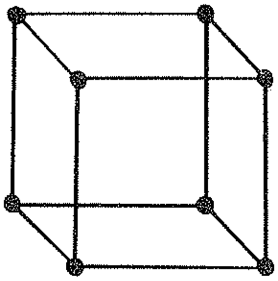
$D_{70} = \{ \qquad \}$





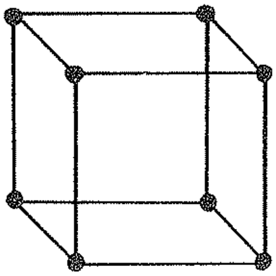
$105 =$

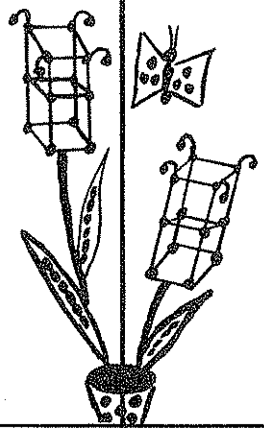
$D_{105} = \{$



$110 =$

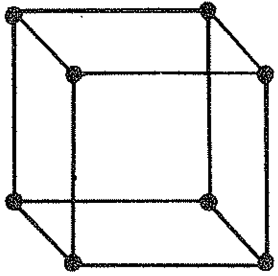
$D_{110} = \{ \qquad \}$





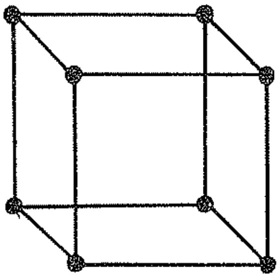
$231 =$

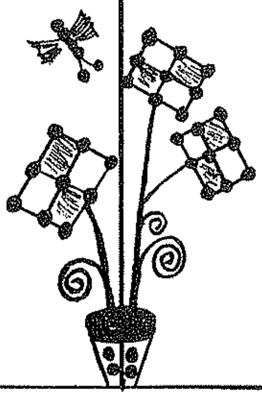
$D_{231} = \{$



$385 =$

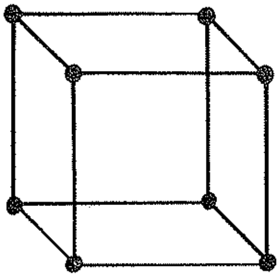
$D_{385} = \{ \qquad \}$





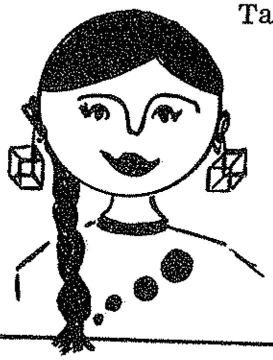
$715 =$

$D_{715} = \{$



Example 3 primes

$N = a^2 \bullet b^1 \bullet c^1$ a, b, c are primes



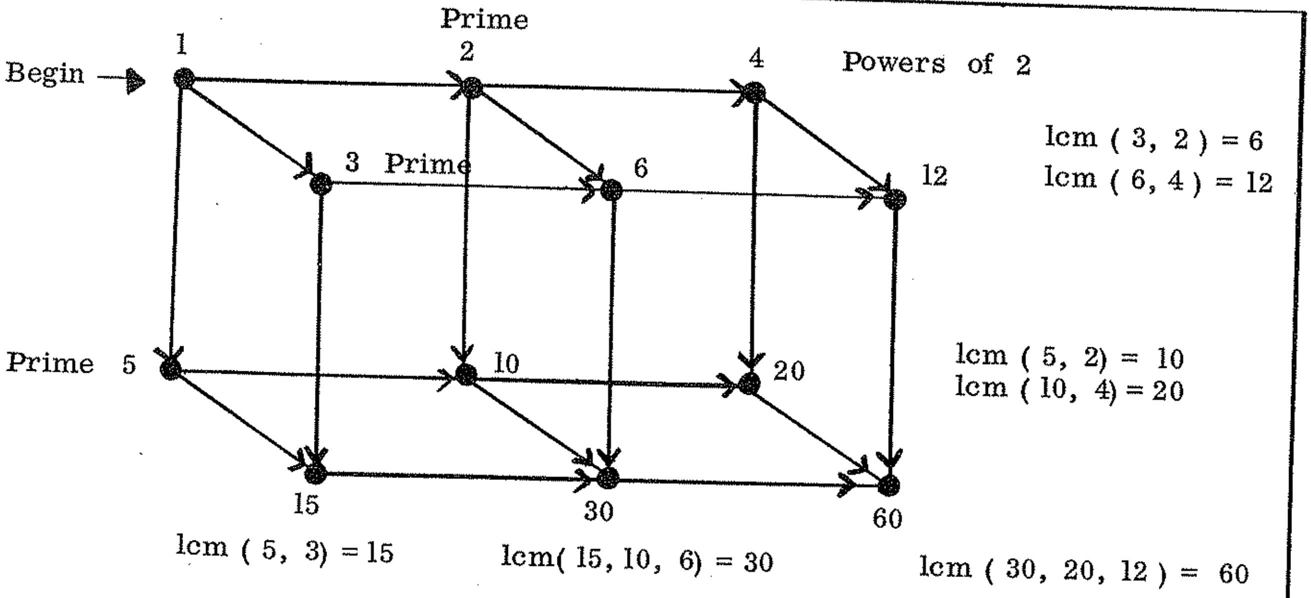
Take

$60 = 2^2 \bullet 3 \bullet 5$

$D_{60} = \{ 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60 \}$

Mazie Dots used two cubes for the

Factor Lattice

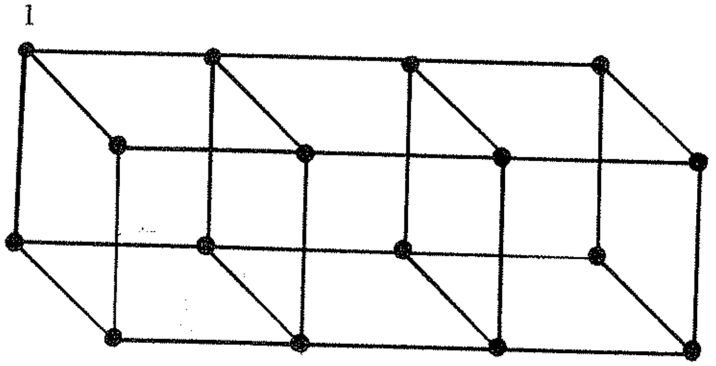


$N = a^3 \bullet b^1 \bullet c^1$ a, b, c are primes

188. Draw the factor lattice for Mazie Dots.

$120 = 2^3 \bullet 3 \bullet 5$

$D_{120} = \{ \qquad \}$

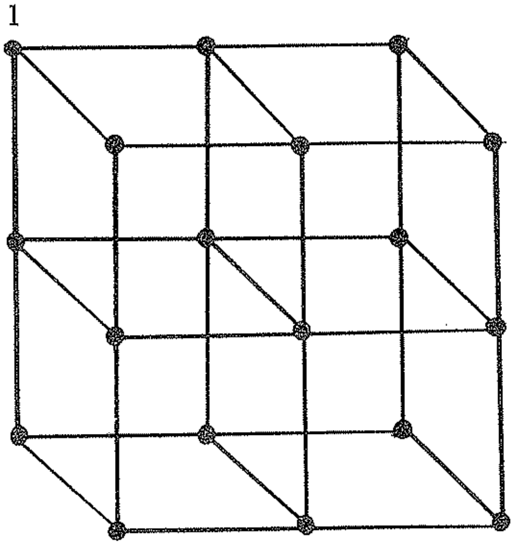


189. Draw the factor lattices for Mazie Dots .

180 = $2^2 \cdot 3^2 \cdot 5$

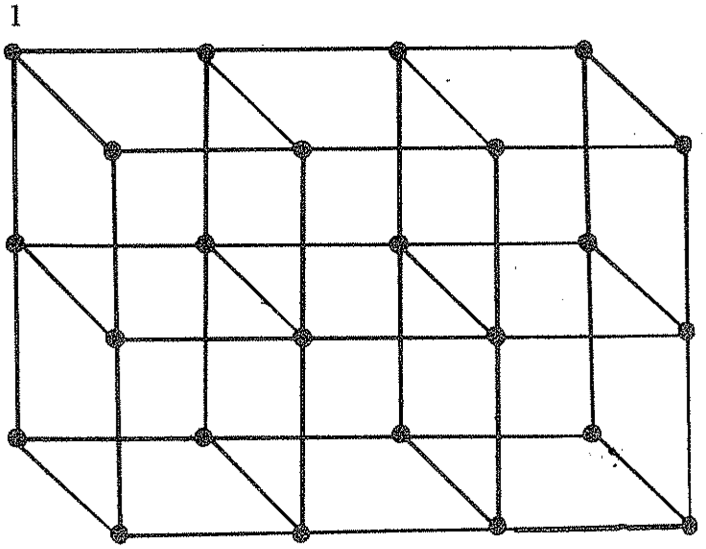
D_{180} = {

}



360 = $2^3 \cdot 3^2 \cdot 5$

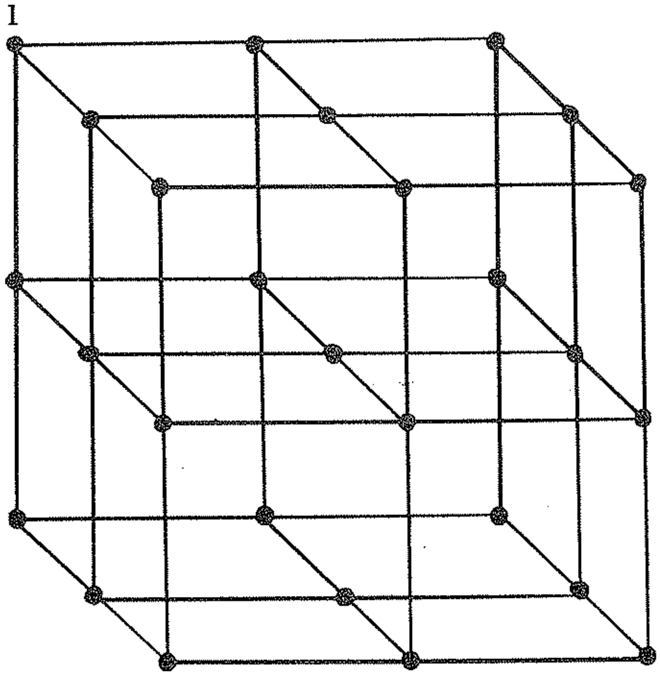
D_{360} = {



900 = $2^2 \cdot 3^2 \cdot 5^2$

D_{900} = {

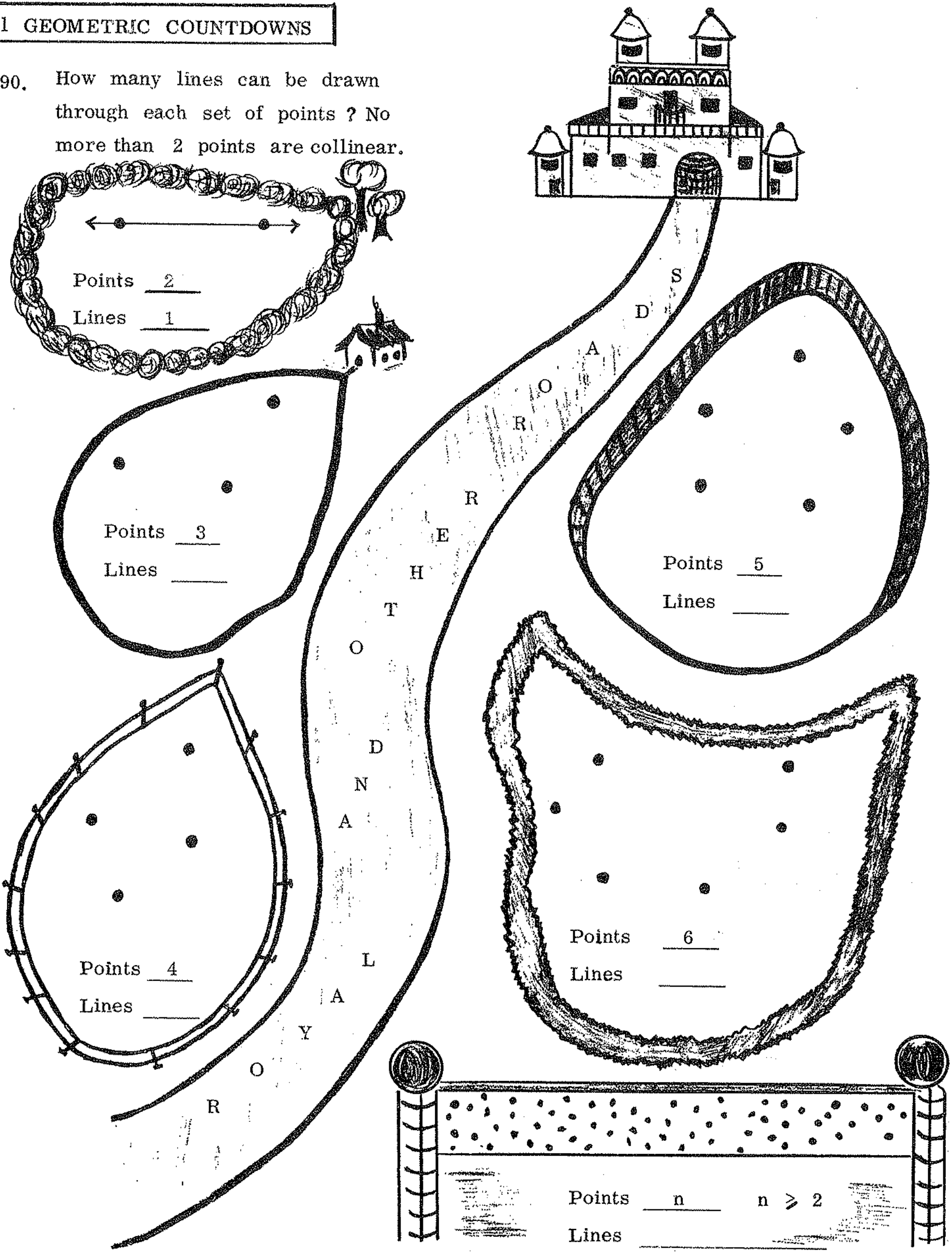
}



CHAPTER 6

6.1 GEOMETRIC COUNTDOWNS

190. How many lines can be drawn through each set of points ? No more than 2 points are collinear.



Points 2
Lines 1

Points 3
Lines ______

Points 5
Lines ______

Points 4
Lines ______

Points 6
Lines ______

Points n   n $\geqslant$ 2
Lines ______

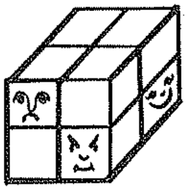
SPACE PAINT - HOW

Doozie Qube, Suzie Qube's brother, had a large box of unit cubes.

Doozie made a large cube out of $2^3 = 8$ unit cubes. He painted the outside of the large cube. Now

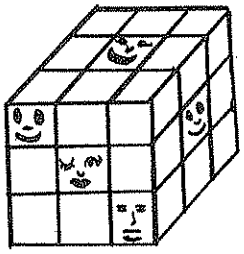
8 unit cubes had 3 faces painted,

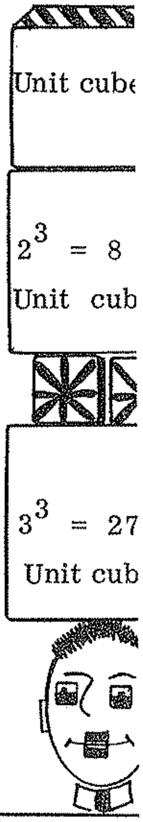
0 unit cubes had 2, 1, or 0 faces painted.



Next, Doozie Qube made a large cube out of $3^3 = 27$ unit cubes. He painted the outside of the large cube.

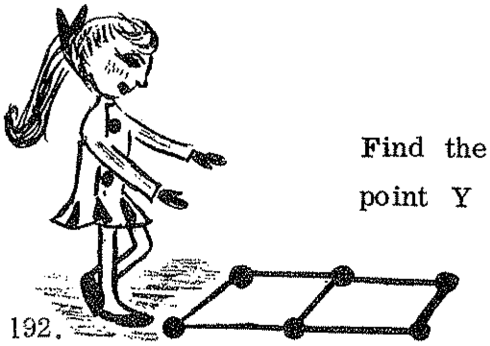
How many unit cubes had 3, 2, 1 or 0 faces painted ?





191. Fill in the chart on the pattern of the first row .

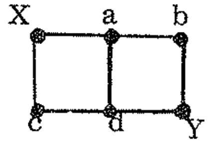
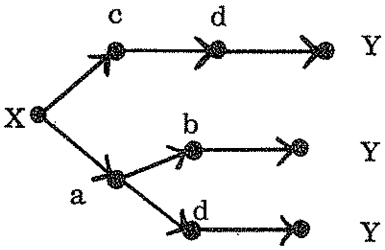
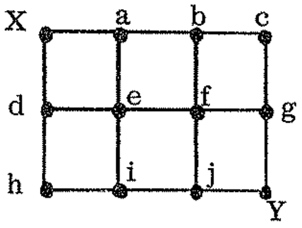
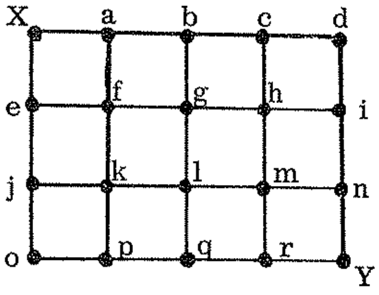
Number of unit cubes forming the large cube	Unit cubes with 3 faces painted	Unit cubes with 2 faces painted	Unit cubes with 1 face painted	Unit cub faces pa
$2^3 = 8$	8	0	0	0
$3^3 = 27$				
$4^3 = 64$				
$5^3 = 125$				
$6^3 = 216$				
n^3 $n = 2, 3, 4, \ldots$				



192.

A WALK WITH PATTY PATH

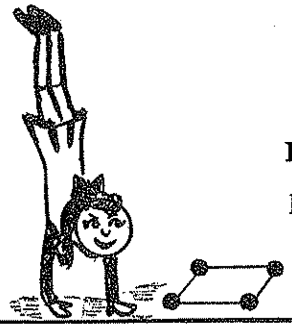
Find the number of ways of getting from point X to point Y . Patty may only go to the right or down at any intersection point .

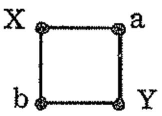
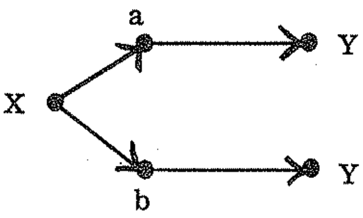
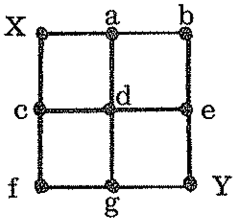
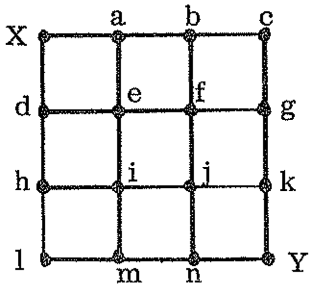
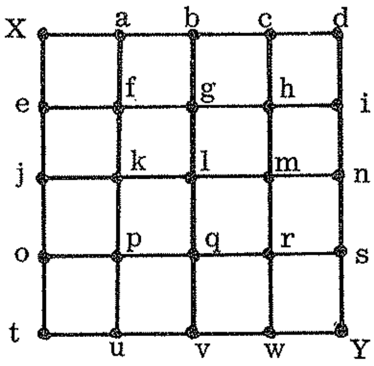
Rectangles	Tree diagram solution
1 x 2 	 XcdY XabY XadY Total 3
2 x 3 	Total
3 x 4 	Total

PATTY PATH'S SQUARE WALK

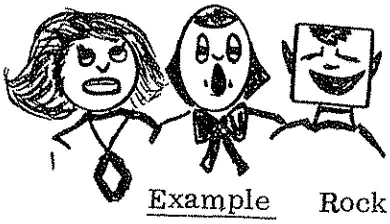
Find the number of ways of getting from point X to point Y . Patty may only go to the right or down at any intersection point .

193.



Squares	Tree diagram solution
1 x 1 	 XaY XbY Tot 2
2 x 2 	Tota
3 x 3 	Tot
4 x 4 	Tot

6.2 PERMUTATIONS - STRAIGHT AND SPIRAL



Permutation

A permutation is an ordered arrangement of all or some of the objects in a given set.

Example Rock singers : Will Roehr , Hooten Holler , Isle Bellow
Rock contest : at Dew Scream Stadium

In how many ways can a single winner be selected ?

First Prize
Will Roehr
or Hooten Holler
or Isle Bellow

There are 3 ways of selecting a single winner .

Formula $\frac{n!}{(n-r)!}$

Here $n = 3$, $r = 1$.

$$\frac{n!}{(n-r)!} = \frac{3!}{(3-1)!} = 3$$

In how many ways can two winners be selected ?

First Prize	Second Prize
Will Roehr	Hooten Holler
Will Roehr	Isle Bellow
Hooten Holler	Will Roehr
Hooten Holler	Isle Bellow
Isle Bellow	Will Roehr
Isle Bellow	Hooten Holler

There are 6 ways of selecting two winners .

Formula $\frac{n!}{(n-r)!}$

Here $n = 3$, $r = 2$.

$$\frac{n!}{(n-r)!} = \frac{3!}{(3-2)!} = 6$$

In how many ways can three winners be selected ?

First Prize	Second Prize	Third Prize
Will Roehr	Hooten Holler	Isle Bellow
Will Roehr	Isle Bellow	Hooten Holler
Hooten Holler	Will Roehr	Isle Bellow
Hooten Holler	Isle Bellow	Will Roehr
Isle Bellow	Will Roehr	Hooten Holler
Isle Bellow	Hooten Holler	Will Roehr

There are 6 ways of selecting three winners .

Formula $\frac{n!}{(n-r)!}$

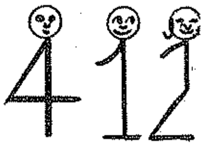
Here $n = 3$, $r = 3$.

$$\frac{n!}{(n-r)!} = \frac{3!}{(3-3)!} = 6$$

Note : $0! = 1$
If $r = n$, then $\frac{n!}{(n-n)!}$ becomes $n!$.

194. 

Given : the digits 1 , 2 , 3, 4 .
Fill in the chart . Use the pattern in the first row .



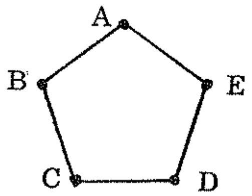
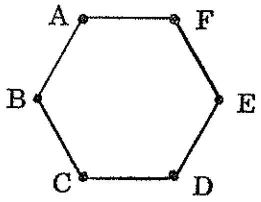
How many single digit numbers can be made from the given digits ? No digits are to be repeated . Solution 1 2 3 4 There are 4 single digit numbers .	Formula $\frac{n!}{(n-r)!}$ Here n = 4 , r = 1 . $\frac{n!}{(n-r)!} = \frac{4!}{(4-1)!} = \frac{4!}{3!} = 4$
How many 2-digit numbers can be made from the given digits ? No digits are to be repeated . Solution 1 2 1 3 1 4 There are ______ 2-digit numbers .	
How many 3-digit numbers can be made from the given digits ? No digits are to be repeated . Solution 1 2 3 1 2 4 1 3 4 There are ______ 3-digit numbers .	
How many 4-digit numbers can be made from the given digits ? No digits are to be repeated . Solution 1 2 3 4 1 2 4 3 1 3 2 4 There are ______ 4-digit numbers .	

Permutation Formula

$$\frac{n!}{(n-r)!}$$

n distinct objects, taken r at a time, $0 < r \leq n$

195.

How many different ways are there to label the vertices of a	Use the permutation formula. Find the number of different ways.
Pentagon with A , B, C, D, E 	
Hexagon with A, B, C, D, E, F 	

TOP POPPER

Use each of the digits 1, 4, 5 , 8 , 9 only once in representing a number .

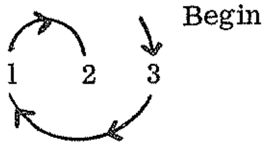
196.

How many different 5 - digit numbers can you form that	Number of different numbers
begin with 1 4 5 _ _	
begin with 1 4 _ _ _	
begin with 1 _ _ _ _	
end with _ _ _ _ 1	
end with _ _ _ 4 1	
end with _ _ 5 4 1	
end with _ 8 5 4 1	
like this _ 6 8 1 _	
or this _ 4 5 _ _	
or this _ 5 _ _ _	
or this _ _ 9 _ _	

SPIRAL PERMUTATIONS

Ronda Spiro's spiral permutation of 3 distinct objects .

1. Ronda took the three numbers 1 , 2 , 3 . She wrote them in a row 1 2 3 .
2. She began with the last number in the row and drew a spiral as shown below .



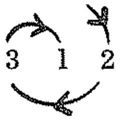
Write the numbers in the order in which the spiral passes through the numbers . Ronda wrote the following spiral permutation of 1 2 3 :

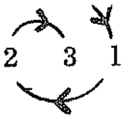
3 1 2

3. Each spiral permutation is called an <u>ORBIT</u> .

Example 1

Take 1 2 3 , 1 st orbit  yields 3 1 2,

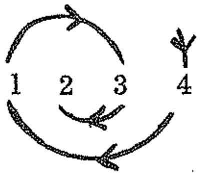
take 3 1 2 , 2 nd orbit  yields 2 3 1,

take 2 3 1 , 3 rd orbit  yields 1 2 3.

It takes 3 orbits to return 1 2 3 to its original order .

Example 2

Take 1 2 3 4

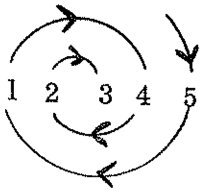


1 st orbit	4 1 3 2
2 nd orbit	2 4 3 1
3 rd orbit	1 2 3 4

It takes 3 orbits to return 1 2 3 4 to its original order.

197. Find the number of orbits to return each of the following numbers to the original order .

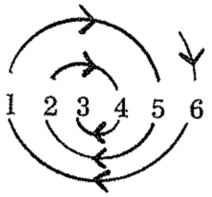
Take 1 2 3 4 5



	1	2	3	4	5
1st orbit	5	1	4	2	3

Orbits

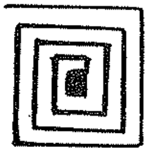
Take 1 2 3 4 5 6



	1	2	3	4	5	6
1st orbit	6	1	5	2	4	3

Orbits

Take	1	2	3	4	5	6	7
1st orbit	7	1	6	2	5	3	4

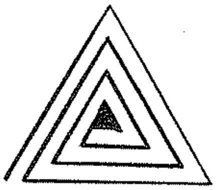


Orbits

Take	1	2	3	4	5	6	7	8
1st orbit	8	1	7	2	6	3	5	4

Orbits

Take	1	2	3	4	5	6	7	8	9
1st orbit									



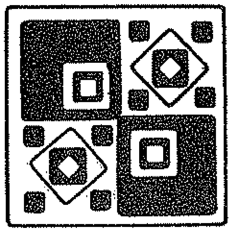
Orbits

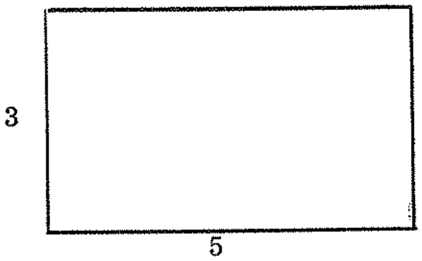
Take	1	2	3	4	5	6	7	8	9	10
1st orbit										

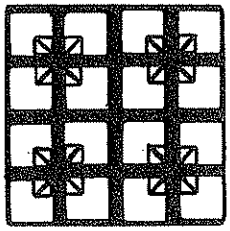
Orbits

6.3 PARTITION OF RECTANGLES

Example Consider the 3 by 5 rectangle



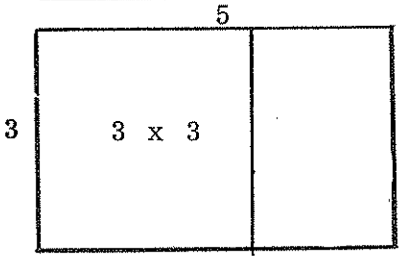




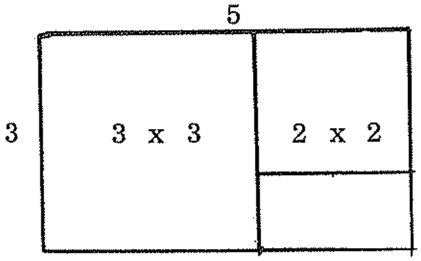
Partition the rectangle into squares by

1. finding the largest square or squares in the 3 x 5 rectangle,
2. then finding the largest square or squares in the remaining portion,
3. then finding the largest square or square in the remaining portion , and so on .

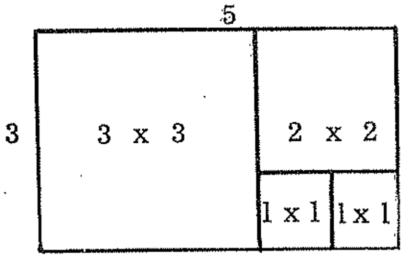
Solution : Geometric



One 3 x 3 square



One 3 x 3 square
One 2 x 2 square



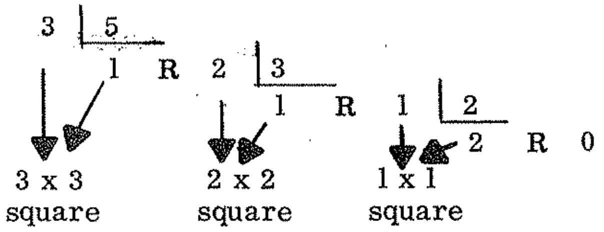
One 3 x 3 square
One 2 x 2 square
Two 1 x 1 squares

The 3 x 5 rectangle has been partitioned into : one 3 x 3 square
one 2 x 2 square
two 1 x 1 squares .

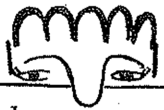
Solution : Arithmetic

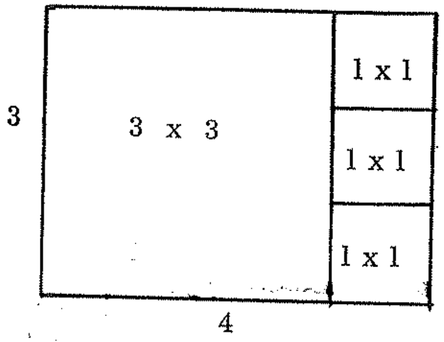
Use U. Klid's short division procedure shown on page 72 .

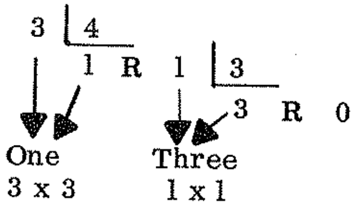
For the 3 x 5 rectangle , the divisor is 3 and the dividend is 5 .



Thus, one 3 x 3 square , one 2 x 2 square and two 1 x 1 squares.
In each division, the divisor gives the size of the square and the quotient gives the number of such squares .

198.  Fill in the chart. Use the pattern in the first row.

<table>
<tr><th>Rectangle : geometric solution</th><th>Rectangle :arithmetic solution</th></tr>
<tr><td>

One 3 x 3
Three 1 x 1</td><td>
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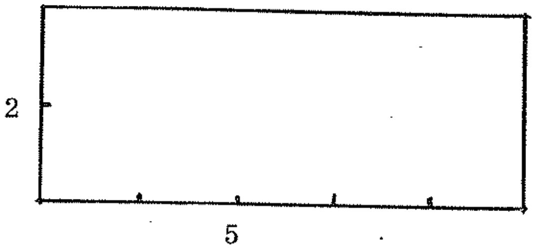
<tr><td>
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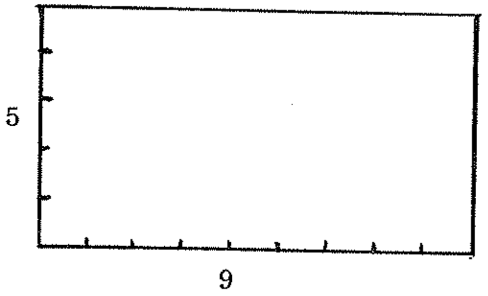
<tr><td>
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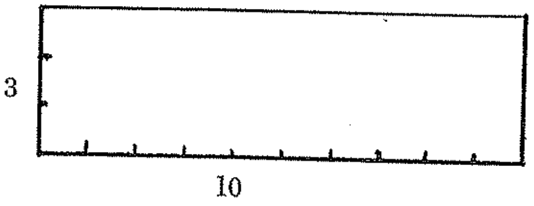
<tr><td>
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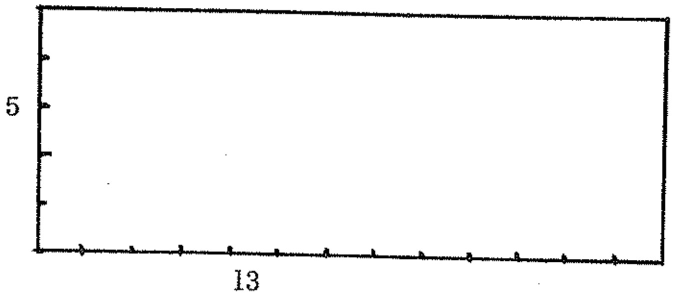
<tr><td>
</td><td></td></tr>
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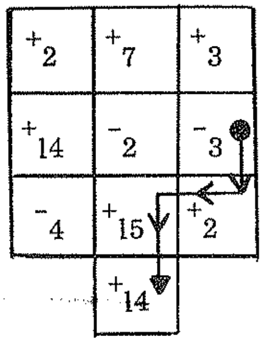
6.4 MATHEMATICAL FUN HOUSE

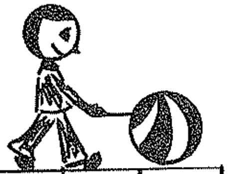
ADD-A-TRAILS



+2	+7	+3
+14	-2	-3
-4	+15	+2
	+14	

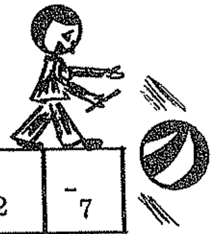
Begin at any of the 9 boxes. Draw a continuous trail to the end-of-trail box. A trail can only go vertically or horizontally. A trail cannot retrace or cross itself.

The sum of all the numbers on the trail must be equal to the number in the ~~end~~-of-trail box.



199.

+13	+15	+17
+19	-13	-3
-5	+7	+7
	+21	



200.

+4	-2	-7
-3	+2	+14
+10	+13	-15
	+12	

201.

+36	+2	-16
+8	-10	+18
-4	+7	+2
	+61	

202.

-6	+7	+4	-9	+7
-7	+5	-4	-2	-5
+8	+9	+10	+1	-7
+2	+3	-8	+8	-3
+6	-7	+10	+7	+1
		+11		

203.

-7	+1	+9	-1	+10
+30	-15	-8	-3	-4
+4	-7	+8	-30	+4
+15	+9	-2	+7	+6
+3	+2	+10	-9	-6
		+25		

Add along with Eddie.

204.

$^{-}2$	$^{+}9$	$^{-}8$	$^{+}4$	$^{-}22$
$^{-}15$	$^{-}10$	$^{+}6$	$^{+}2$	$^{-}21$
$^{-}4$	$^{-}4$	$^{-}4$	$^{+}16$	$^{+}25$
$^{+}24$	$^{+}14$	$^{-}1$	$^{-}18$	$^{+}26$
$^{-}12$	$^{-}15$	$^{-}1$	$^{+}20$	$^{+}28$
		$^{-}29$		

205.

$^{+}18$	$^{-}15$	$^{-}9$	$^{+}16$	$^{-}20$
$^{-}20$	$^{+}31$	$^{+}10$	$^{-}6$	$^{+}1$
$^{+}15$	$^{+}4$	$^{-}4$	$^{+}20$	$^{-}18$
$^{-}2$	$^{+}6$	$^{-}3$	$^{+}9$	$^{+}31$
$^{-}16$	$^{+}2$	$^{+}8$	$^{-}10$	$^{-}1$
		$^{+}32$		

206.

$^{-}1$	$^{-}2$	$^{-}3$	$^{-}4$	$^{-}5$	$^{-}6$	$^{-}7$
$^{+}7$	$^{+}6$	$^{+}5$	$^{+}4$	$^{+}3$	$^{+}2$	$^{+}1$
$^{+}6$	$^{-}7$	$^{+}4$	$^{-}5$	$^{+}2$	$^{-}3$	$^{+}1$
$^{+}1$	$^{+}7$	$^{-}2$	$^{-}3$	$^{-}4$	$^{+}6$	$^{+}5$
$^{+}3$	$^{+}1$	$^{+}4$	$^{+}2$	$^{-}5$	$^{-}6$	$^{-}7$
$^{+}7$	$^{-}2$	$^{+}5$	$^{-}4$	$^{-}3$	$^{-}1$	$^{+}6$
$^{+}2$	$^{-}5$	$^{+}4$	$^{-}3$	$^{-}7$	$^{+}1$	$^{+}6$
			$^{+}4$			

207.

$^{+}5$	$^{+}6$	$^{+}7$	$^{-}8$	$^{+}18$	$^{+}16$	$^{+}14$
$^{-}8$	$^{-}10$	$^{+}1$	$^{-}3$	$^{+}9$	$^{+}11$	$^{-}2$
$^{+}12$	$^{-}15$	$^{+}17$	$^{-}8$	$^{+}12$	$^{+}10$	$^{-}5$
$^{+}13$	$^{+}5$	$^{-}1$	$^{+}3$	$^{+}6$	$^{+}7$	$^{-}8$
$^{+}19$	$^{-}9$	$^{-}10$	$^{-}11$	$^{+}13$	$^{-}13$	$^{+}14$
$^{+}4$	$^{-}15$	$^{+}16$	$^{-}17$	$^{-}18$	$^{+}15$	$^{+}50$
$^{-}7$	$^{+}27$	$^{-}19$	$^{-}12$	$^{-}19$	$^{-}24$	$^{+}25$
			$^{+}34$			

ALPHAMETICS

Alphametics is the name given to mathematical puzzles in which some or all of the digits of a number are replaced by letters of the alphabet. The letters usually spell out words that have a meaning.

Example Subtraction

Alphametic

L E G
- O R
A R M

Identify the digits represented by the letters. Same letters must be replaced by the same digi Different letters must be replaced by different digits. The operation here is subtraction.

Solution

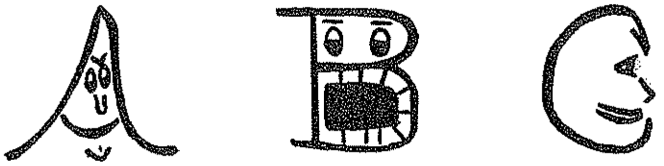
4 2 8
- 5 7
3 7 1

Replacements for letters

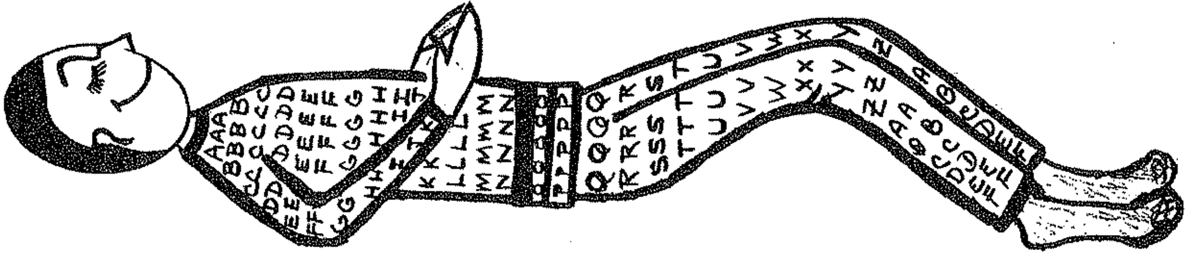
G	8	R	7	M	1
E	2	O	5	A	3
L	4				

208. Find digit replacements for the letters in each of the following alphametics.

Alphametic	Solution	Replacements
W I L L - Y O U T R Y	1 0 8 8 - 6 5 2 4 3 6	L - 8 R - 3 U - 2 I - 0 Y - 6 T - 4 O - 5 W - 1
J U S T - V E R Y E A S Y		T ___ U ___ Y ___ E ___ S ___ A ___ R ___ J ___ V ___
F I V E - F O U R O N E		E ___ N ___ R ___ I ___ V ___ O ___ U ___ F ___



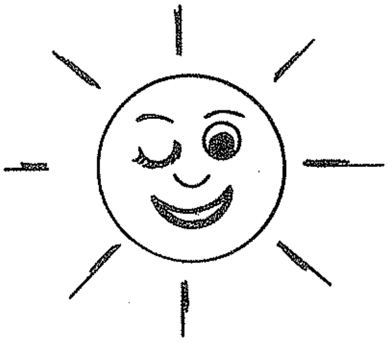
MUCH - TOO SOON		H ___ U ___ O ___ T ___ N ___ M ___ C ___ S ___
RAIN - THEN SUN		N ___ A ___ I ___ H ___ E ___ S ___ U ___ R ___ T ___
TEAM - CANT WIN		M ___ I ___ T ___ E ___ N ___ W ___ A ___ C ___
MOON - RACE SET		N ___ C ___ E ___ A ___ T ___ S ___ O ___ M ___ R ___
QUART - PINT PINT		T ___ I ___ R ___ U ___ N ___ P ___ A ___ Q ___
HURRY - WERE LATE		Y ___ A ___ E ___ U ___ R ___ W ___ T ___ L ___ H ___



ARITHMEKNACK

Insert any of the operations + , x , − , ÷ in the blanks. Each operation can be used exactly once in each equation. Mark the resulting largest value with an L and the smallest value with an S.

209. (1 + 2) − 3 = 0
(1 − 2) + 3 = 2
(1 ____ 2) ____ 3 = ____
(1 ____ 2) ____ 3 = ____
(1 ____ 2) ____ 3 = ____
(1 ____ 2) ____ 3 = ____



(1 ____ 2) ____ 3 = __
(1 ____ 2) ____ 3 = __
(1 ____ 2) ____ 3 = __
(1 ____ 2) ____ 3 = __
(1 ____ 2) ____ 3 = __
(1 ____ 2) ____ 3 = __

210. [(1 ____ 2) ____ 3] ____ 4 = ____
[(1 ____ 2) ____ 3] ____ 4 = ____
[(1 ____ 2) ____ 3] ____ 4 = ____

[(1 ____ 2) ____ 3] ____ 4 = __
[(1 ____ 2) ____ 3] ____ 4 = __
[(1 ____ 2) ____ 3] ____ 4 = __

[(1 ____ 2) ____ 3] ____ 4 = ____
[(1 ____ 2) ____ 3] ____ 4 = ____
[(1 ____ 2) ____ 3] ____ 4 = ____

[(1 ____ 2) ____ 3] ____ 4 = __
[(1 ____ 2) ____ 3] ____ 4 = __
[(1 ____ 2) ____ 3] ____ 4 = __

[(1 ____ 2) ____ 3] ____ 4 = ____
[(1 ____ 2) ____ 3] ____ 4 = ____
[(1 ____ 2) ____ 3] ____ 4 = ____

[(1 ____ 2) ____ 3] ____ 4 = __
[(1 ____ 2) ____ 3] ____ 4 = __
[(1 ____ 2) ____ 3] ____ 4 = __

[(1 ____ 2) ____ 3] ____ 4 = ____
[(1 ____ 2) ____ 3] ____ 4 = ____
[(1 ____ 2) ____ 3] ____ 4 = ____

[(1 ____ 2) ____ 3] ____ 4 = _
[(1 ____ 2) ____ 3] ____ 4 = _
[(1 ____ 2) ____ 3] ____ 4 = _

CONTEMPORARY MOTIVATED MATHEMATICS