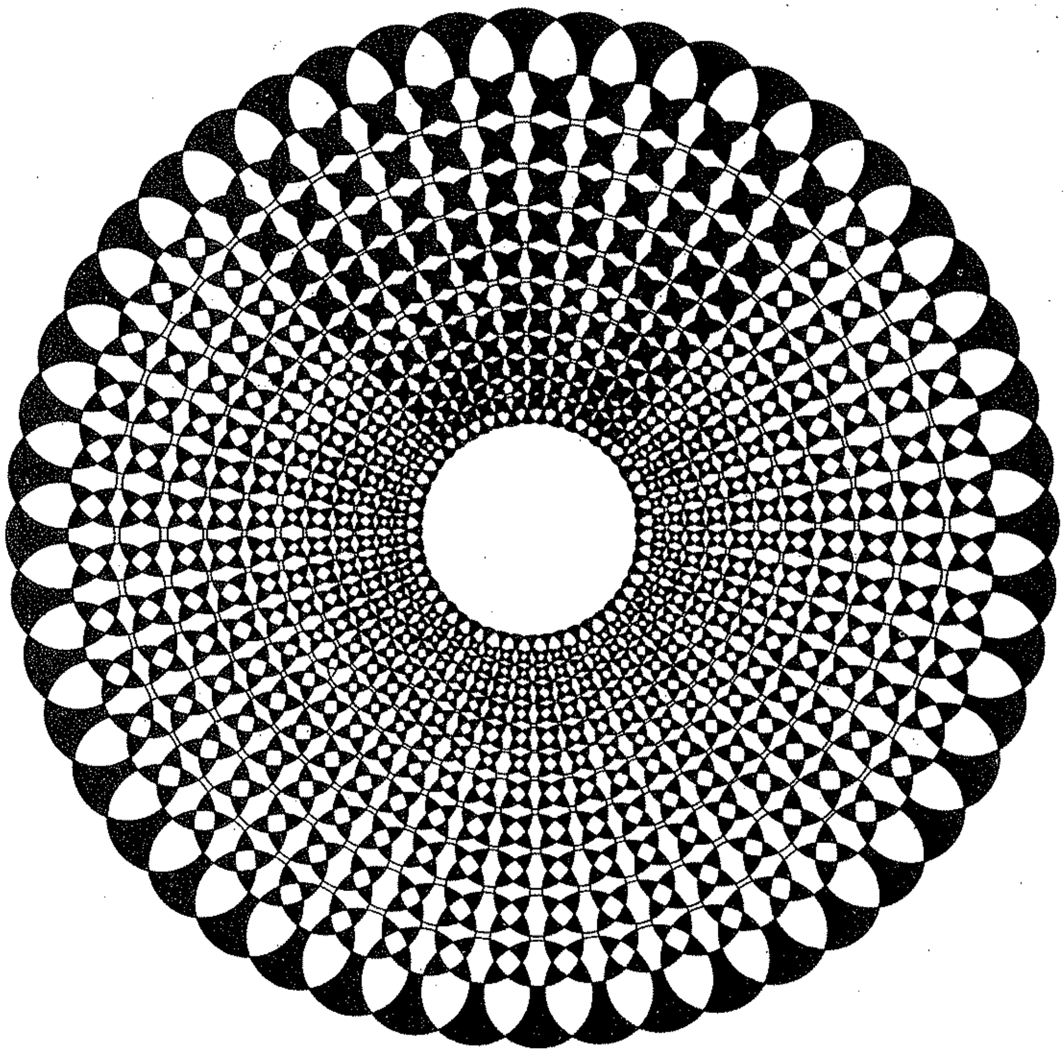
BOSTON COLLEGE MATHEMATICS INSTITUTE

Contemporary Motivated Mathematics



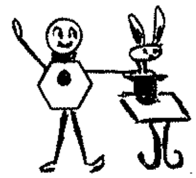
BOOK 1

STANLEY BEZUSZKA

In collaboration with MARY E. FARREY - MARGARET J. KENNEY

BOSTON COLLEGE PRESS

CHESTNUT HILL, MASSACHUSETTS 02167



CHAPTER 1

MATHEMAGIC

1.1 INTRODUCING : LO-SHU AND MAGIC SQUARES.

1. Add the numbers in each row, column and diagonal of the square shown below.

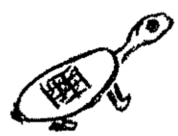
4	9	2
3	5	7
8	1	6

Write in sum of rows

Write in sum of diagonal

Write in sum of columns

Write in sum of diagonal



The above square is called the Lo-Shu magic square.

MAGIC SQUARE

A square array of numbers where the sum of any row, column and diagonal is one and the same number is called a magic square.

MAGIC CONSTANT

The sum of any row, column, or diagonal of a magic square is called the magic constant.

2. What is the magic constant of the Lo-Shu magic square ? ______________

3. How many natural numbers are used in the above magic square ? __________

   The product of 3 x 3 is ________.

ORDER OF A MAGIC SQUARE

A magic square that has 3 x 3 numbers is called a 3 rd order magic square or a 3 by 3 magic square.

4. Complete the list of natural numbers used in the Lo-Shu magic square.

   1, 2, 3, ______________________________

PREFACE

CONTEMPORARY MOTIVATED MATHEMATICS is an attempt to achieve the following objectives.

1. MATHEMATICAL SKILLS

   Students should possess some technical proficiency in the fundamental operations of arithmetic and algebra. But drill, often imposed for the sake of drill, has been one of the most unappealing experiences in mathematics. We are convinced that practice in the techniques of mathematics can be meaningful and functional if properly presented.

   Emphasis in the text is on the basic operations of the real number system. Many of the exercises in the book are open ended. They are presented in such a way that all students, at all levels of ability can do some part of the task. More capable students can solve the text problems completely and often extend the exercises into fruitful projects for participation in science fairs.

   In some problems, the computations may appear rather excessive for paper and pencil work. Students are expected to do a reasonable amount of computation under these circumstances and this is usually sufficient for the attainment of the objective of the problem. We have deliberately inserted such problems for those students who have access to desk calculators or small computers.

2. MATHEMATICAL CONTENT

   Problems in the text deal with number theory and geometry. The exercises develop skills in computation and lead to some interesting or important number patterns and mathematical conclusions. Number pattern recognition and generalizations by induction occur in many problems. Manipulative techniques are stepping stones to mathematical conclusions rather than an end in themselves.

   The element of fun and recreation in some of the problems requires no apology. Mathematics need not be dull and boring. It can be enjoyable.

   An attempt has also been made to present some significant results of mathematics which often do not appear in introductory or advanced work in some specialized fields. Such problems belong to the heritage of mathematics and show the breadth and beauty of man's mathematics. They should be known by students whether or not students specialize later in mathematics.

Many of the problems in the text come from the work of the greatest mathematicians in the world. We hope that the concepts which inspired the famous men of mathematics will turn out to be a glorious adventure in mathematical ideas for the students.

Cover design by Carl Stefani

S. J. B.
Boston College
1972

Use the natural numbers 1 through 16 once and only once. Complete the magic squares so that the magic constant is 34.

17.

13	2	3	16
8		10	5
12	7		9
1	14	15	4

18.

16	5	9	4
	10	6	15
2	11		14
	8	12	1

19.

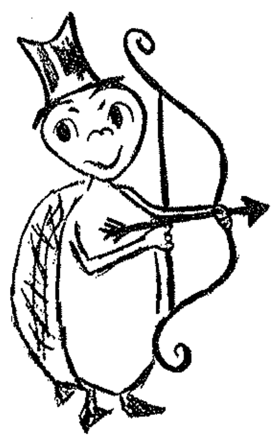
11		14	7
8	13		12
5			9
10	3	15	

20.

10	3	15	6
5			
	13		12
11	2	14	7

5 th ORDER NORMAL MAGIC SQUARE

21.



17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

Write in sum of rows

Write in sum of diagonal

Write in sum of columns

Write in sum of diagonal

22. Magic constant of the above magic square is ____________.

23. How many natural numbers are used in the above magic square ? __________

24. The above magic square has the successive natural numbers 1 through ____.

25. How many times is each of the natural numbers used in the magic square ?

____________.

The number 13 is in the center of the above 5 x 5 magic square. Now the numbers 6 and 20 are symmetric (at the same distance) with respect to 13. The sum 6 + 20 = 26.

26. The following pairs of numbers are also symmetric with respect to 13 in

5. How many times is each natural number 1 through 9 used in the Lo-Shu magic square ? ______________

Use the natural numbers 1 through 9 once and only once . Complete the magic squares so that the magic constant is 15 .

6.

6		2
1	5	9
	3	

7.

	7	6
9		
4	3	

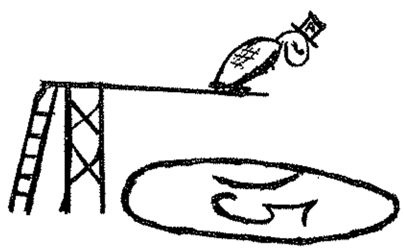
8.

	9	
	5	
6		8

9.

		4
	5	
		2

10. Add the numbers in each row, column and diagonal of the square shown below .



16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Write in sum of rows

Write in sum of diagonal

Write in sum of diagonal

Write in sum of columns

11. Magic constant of the above magic square is ________ .

12. How many natural numbers are used in the above magic square ? __________

13. What is the order of the above magic square ? ______________

14. The above magic square has the successive natural numbers 1 through _____.

15. How many times is each of the natural numbers used in the magic square ?

__________________ .

16. NORMAL MAGIC SQUARE

A normal magic square has the successive natural numbers 1 through 9 , for 3 rd order ,
1 through 16 , for 4 th order ,
1 through ____ , for 5 th order .

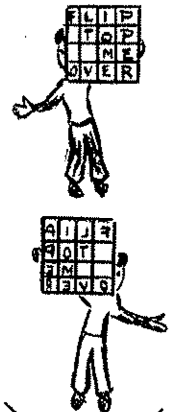
1.2 MAGIC POTPOURRI

FLIP-TOP SQUARE If you flip over the first magic square you get the second one. The individual cells and numbers of the square are reversed .

32.

Write sum of rows

15	96	93	38
94	37	16	95
36	91	98	17
97	18	35	92



Write sum of rows

83	39	69	51
59	61	73	49
71	89	19	63
29	53	81	79

Write sum of diagonals and columns .

33. Magic constant of the above magic squares is ____________ .

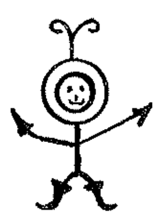
CONCENTRIC MAGIC SQUARES

In each square below use the numbers 1 through 36 only once .

Make the magic constant of the 4 x 4 magic square 74 . / 6 x 6 magic square 111.

34.

1	35		5		6
	11	25		14	
	22			19	29
28		20	21		
	23				27
		3			



35.

	10		8		
2	23			11	
			16	25	34
32	12	21			
		15		14	
36		9			6

36. Consider the dot as the center of each of the above magic squares. What is the sum of pairs of numbers symmetric (at the same distance) with respect to the dot and lying on a straight line through the dot ? ______________ .

the 5 x 5 magic square . Find their sums .

7 + 19 = ______ 5 + 21 = ______

12 + 14 = ______ 1 + 25 = ______

27. Find 4 other pairs of numbers which are symmetric with respect to 13 in the 5 x 5 magic square on page 3 . Find their sums .

______ + ______ = ______ ______ + ______ = ______

______ + ______ = ______ ______ + ______ = ______

Use the natural numbers 1 through 25 once and only once. Complete the magic squares so that the magic constant is 65 .

28.

17	23		10	11
24	5	6		18
1		13	19	25
8	14	20		2
	16	22	3	9

29.

5	23	7	16	14
24		1		8
6	4	13	22	20
18		25		2
12	10	19	3	21

30.

12	10		3	
		25	9	2
6	4	13		20
24			15	8
5	23	7	16	

To compute the magic constant for normal magic squares use the

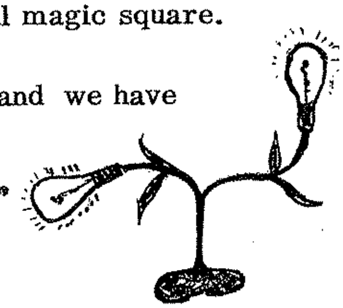
FORMULA

$$\text{MAGIC CONSTANT} = \frac{n(n^2+1)}{2}$$

where n is to be replaced by the order of the normal magic square.

Thus, for a 3 x 3 , or 3 rd order normal magic square , n is 3 and we have

$$\text{MAGIC CONSTANT} = \frac{3(3^2+1)}{2} = \frac{3(9+1)}{2} = \frac{30}{2} = 15 .$$



31. Fill in the chart on the pattern of the second column .

Order of the normal magic square	3 rd 3 x 3	4 th 4 x 4	5 th 5 x 5
Natural numbers used in the normal magic square	1 through 9	___ through ___	___ through ___
Magic constant of the normal magic square	15	______	______

TOP - POPPER

44. What is the sum of each diagonal of a 3 x 3 consecutive natural number square that begins with

16 Ans. ________

19 Ans. ________

22 Ans. ________

25 Ans. ________

45. What is the sum of each diagonal of a 4 x 4 consecutive natural number square that begins with

5 Ans. ________

9 Ans. ________

17 Ans. ________

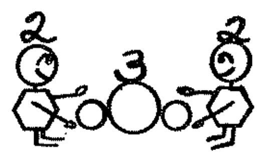
25 Ans. ________

PALINDROMIC NUMBER SQUARES

Make the magic constant of this square 666 .

46.

333		
		323
212		



Make the magic constant of this square 999 .

47.

		525
515		
		323

Each number in each square above reads the same forwards and backwards. Such numbers are called palindromic numbers .

RATIONAL NUMBER SQUARES

In each square below use the numbers

$0 \quad \frac{1}{2} \quad 1 \quad \frac{3}{2} \quad 2 \quad \frac{5}{2} \quad 3 \quad \frac{7}{2} \quad 4$

only once . Make the magic constant of each square 6 .

48.

		$\frac{5}{2}$
	2	
		$\frac{1}{2}$

49.

	4	
	2	
$\frac{7}{2}$		

50.

$\frac{3}{2}$		
	2	
$\frac{1}{2}$		

CONSECUTIVE NUMBER SQUARES

Consecutive natural numbers are used in the squares below.

Write the sum of the diagonals only for each square.

37.

1	2
3	4

38.

1	2	3
4	5	6
7	8	9

39.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

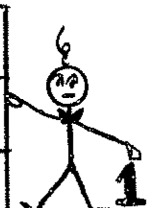
40. Fill in the blanks.

Order of the square.	Consecutive number square. Sum of each diagonal.	Normal magic square. Magic constant.
3 x 3		
4 x 4		
5 x 5		

Write the sum of the diagonals only for each consecutive number square.

41.

4	5	6
7	8	9
10	11	12



Complete

15 + 9 = ______

42.

7	8	9
10	11	12
13	14	15

Complete

15 + 18 = ______

43.

10	11	12
13	14	15
16	17	18

Complete

15 + 27 = ______

Hint: Compare 41, 42, 43 with 38 above. In each case note the number of rows omitted.

Use the numbers 1, 2, 3, 4.

Make Latin squares

61.

1			
	3		
		1	
			3

62.

1			
	1		
		1	
			1

Use the numbers 1, 2, 3, 4.

Make diagonal Latin squares

63.

3			1
	1		
	4		
2			4

64.

4			1
		3	
		1	
2			3

1.4 A 3 x 3 MAGIC SQUARE SECRET REVEALED

You can make up 3 x 3 magic squares by substituting numbers for x, y, z.

<u>Example:</u> Let x = 5
y = 1
z = 3.

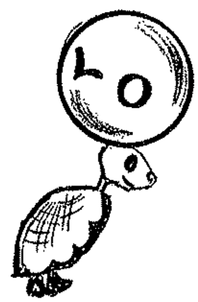
3 x 3

SECRET SQUARE

x − y	x + y − z	x + z
x + y + z	x	x − y − z
x − z	x + z − y	x + y

We have

x − y = 4	x + y − z = 3	x + z = 8
x + y + z = 9	x = 5	x − y − z = 1
x − z = 2	x + z − y = 7	x + y = 6



4	3	8
9	5	1
2	7	6

The magic constant here is <u>15</u>.

Make up magic squares using the 3 x 3 secret square.

65. Let x = 5
y = 3
z = 1

Magic constant is_____

66. Let x = 5
y = 4
z = 1

Magic constant is _____

67. Let x = 9
y = 5
z = 2

Magic constant is_____

51.

0		
$\frac{5}{2}$	3	

52.

		0
	1	$\frac{7}{2}$

53.

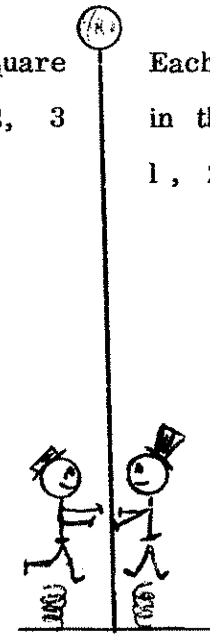
		$\frac{3}{2}$
	2	
	0	

1.3 LATIN SQUARES

Each row and column in the square below contains the numbers 1 , 2, 3 only once .

Magic Latin
3 x 3
square

2	1	3
3	2	1
1	3	2



Each row, each column and each diagonal in the square below contains the numbers 1 , 2, 3, 4 only once .

Magic diagonal Latin
4 x 4 square

2	3	1	4
4	1	3	2
3	2	4	1
1	4	2	3

Magic constant of Latin squares is only for rows and columns.

54. Magic constant of a 3 x 3 Latin square is ________ .

55. Magic constant of a 4 x 4 Latin square is ___________ .

56. What is the magic constant of a 5 x 5 Latin square ? __________
6 x 6 Latin square ? __________

Use the numbers 1 , 2 , 3 in the cells . Make Latin squares.

57.

3		
	3	
		3

58.

		3
	3	
3		

59.

1		
	2	
		3

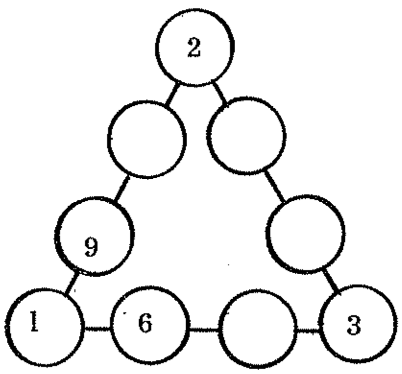
60.

		1
	2	
3		

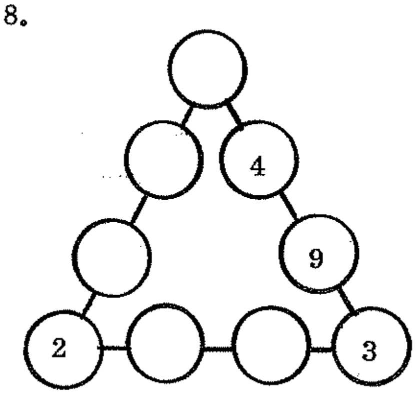
1.5 MAGIC TRIANGLES

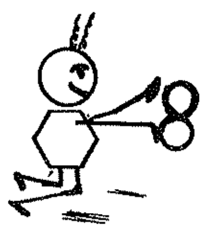
In each triangle use each of the numbers 1 through 9 only once. Make the sum on each side of the triangle 17.

77.

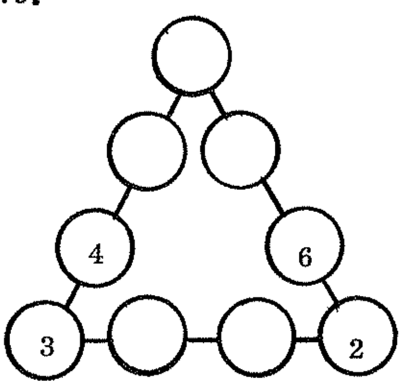


78.

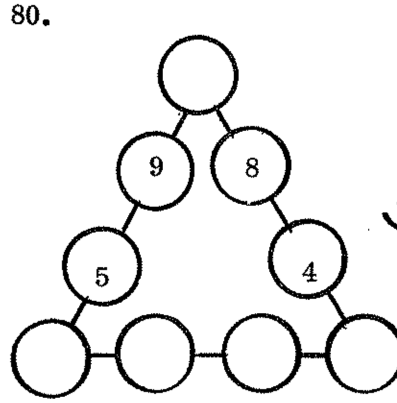




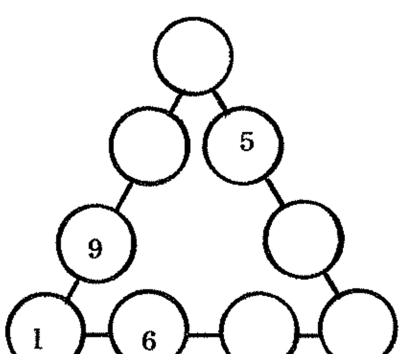
79.



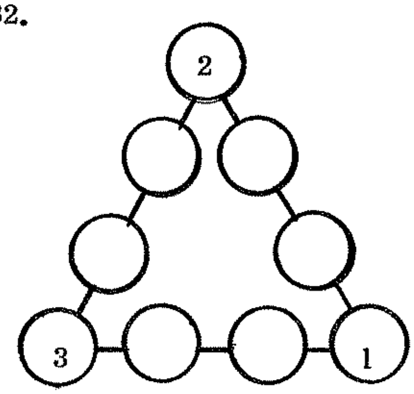
80.



81.



82.



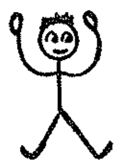
Make up magic squares using the 3 x 3 secret square, page 9.

68. Let x = 8

y = 4

z = 1

Magic constant is ____

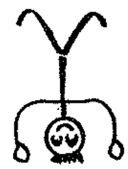


69. Let x = 12

y = 7

z = 3

Magic constant is ____



70. Let x = 20

y = 15

z = 3

Magic constant is ____

71. Let x = 4

y = $\frac{3}{2}$

z = $\frac{1}{2}$

Magic constant is ____

72. Let x = 6

y = 4

z = $\frac{3}{2}$

Magic constant is ____

73. Let x = $\frac{13}{3}$

y = 3

z = 1

Magic constant is ____

74. Let x = 16/3

y = 3

z = 5/3

Magic constant is ____

75. Let x = 13/2

y = 17/4

z = 5/4

Magic constant is ____

76. Let x = 49/6

y = 25/6

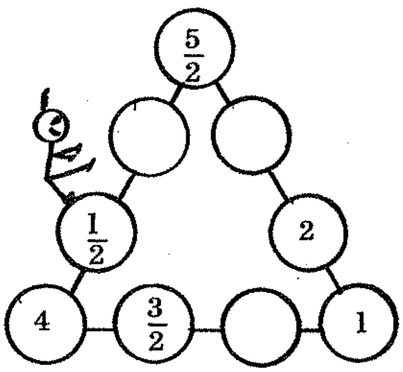
z = 7/6

Magic constant is ____

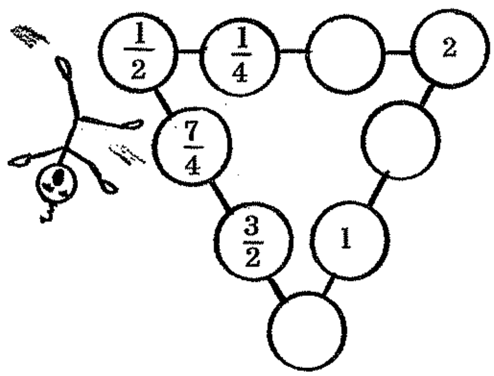
A few more magic triangles.

Use rational numbers of the form $\frac{a}{b}$, where a, b are natural numbers and $b \neq 0$ in the circles of the triangles . Make the sum on each side of the triangle the number shown above each triangle .

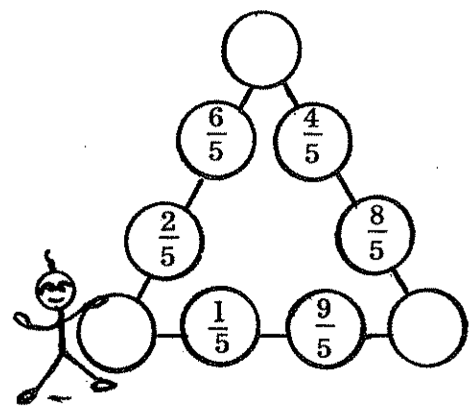
89. Sum is 10



90. Sum is 5



91. Sum is 4



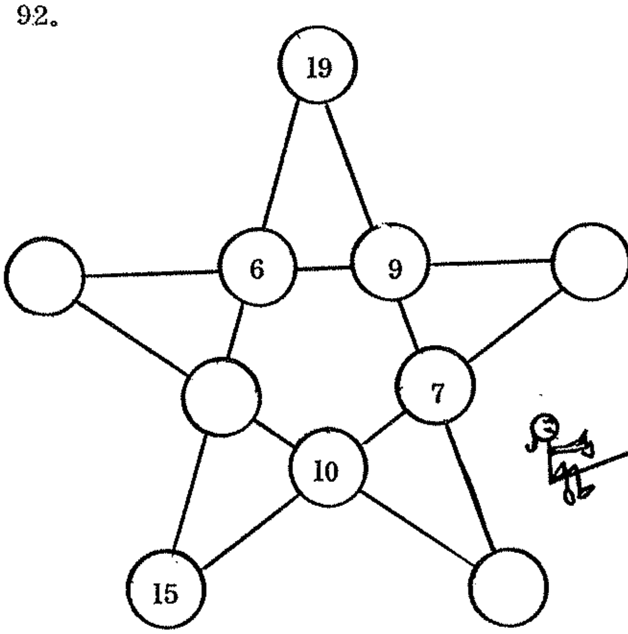
1.6 MAGIC STARS - FIVE POINTED STARS

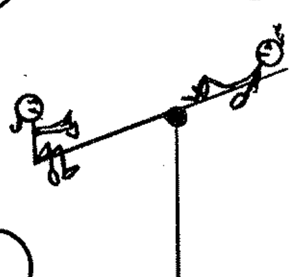
The five pointed star is the smallest magic star . There are 10 points of intersection of the lines of the star . Each line of the star has 4 points .

Use each natural number only once .

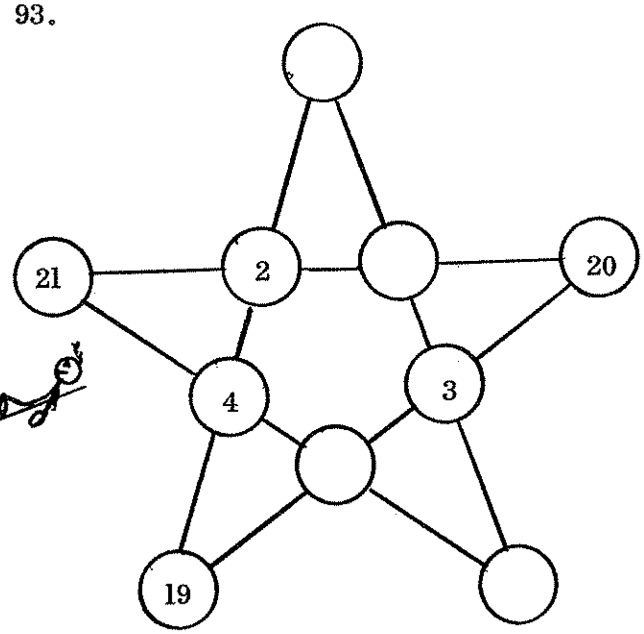
Make the sum on each line of the star 48 .

92.





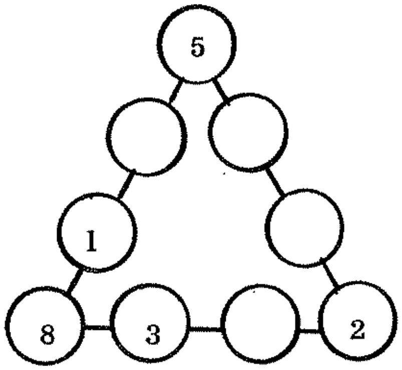
93.

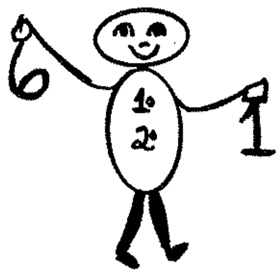


More magic triangles.

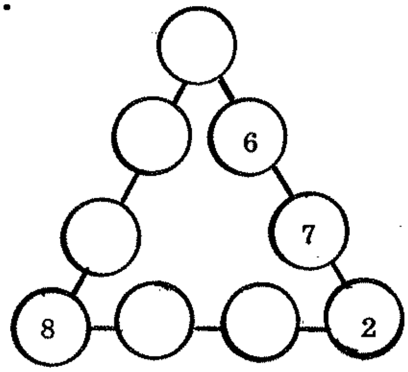
In each triangle use each of the numbers 1 through 9 only once. Make the sum on each side of the triangle 20.

83.

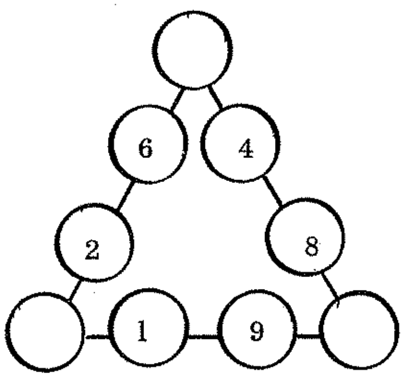




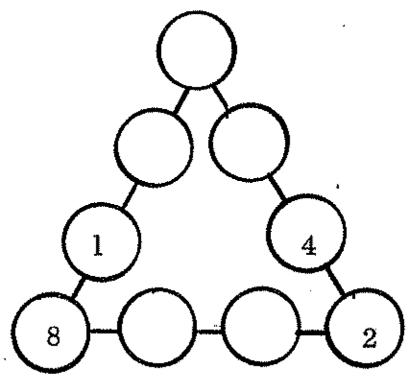
84.



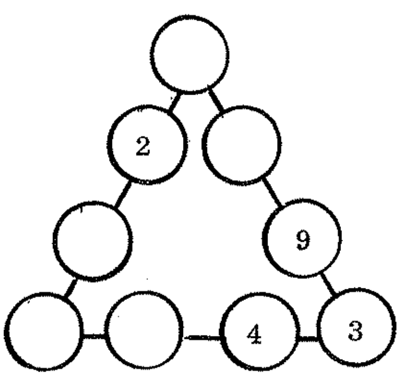
85.



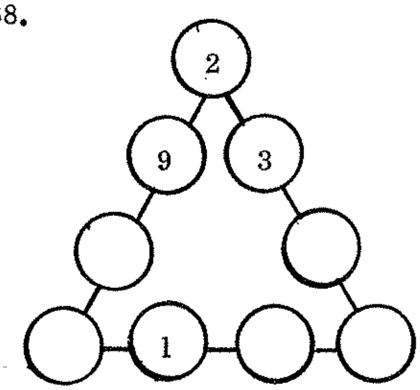
86.



87.



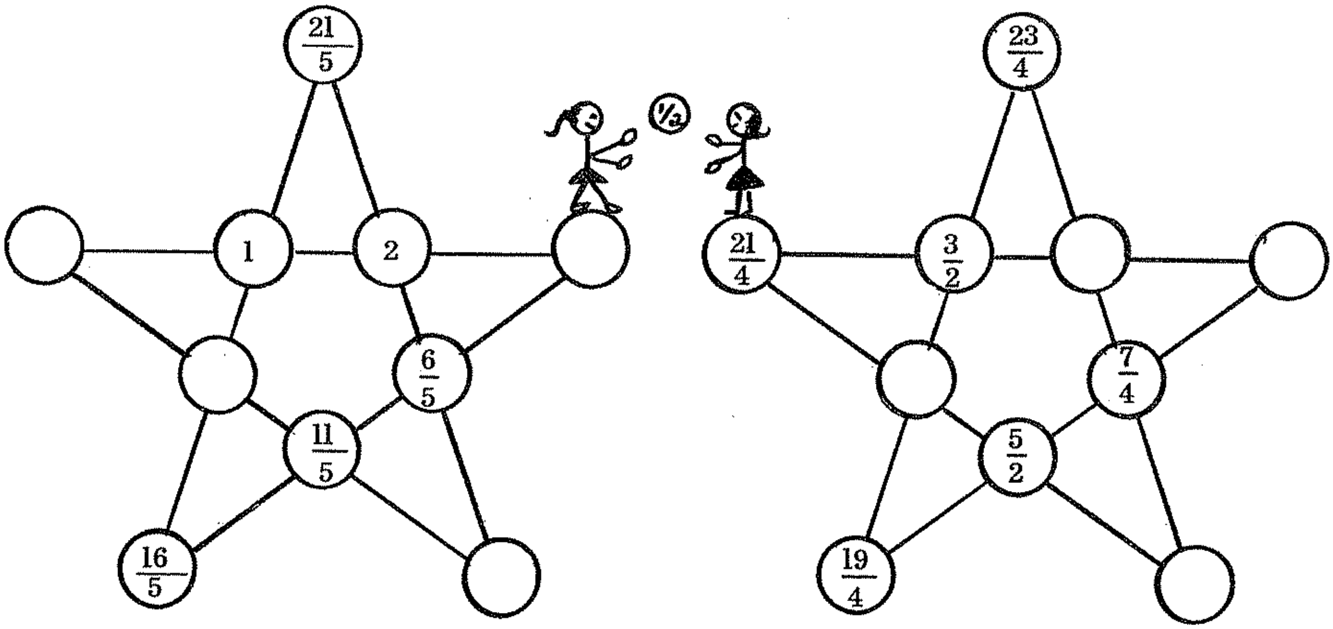
88.



Use rational numbers of the form $\frac{a}{b}$, where a, b are natural numbers and $b \neq 0$ in the circles of the stars. Make the sum on each line of the magic star the number shown above each star.

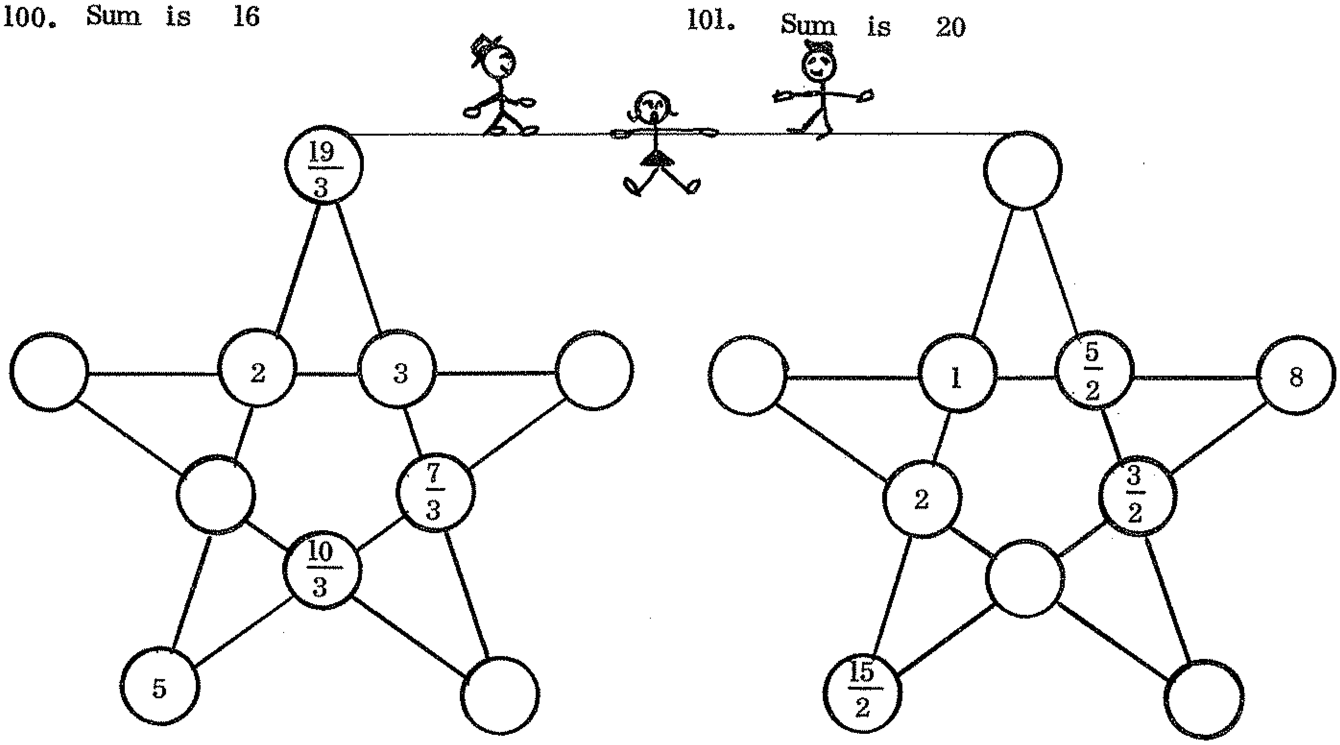
98. Sum is 10

99. Sum is 14



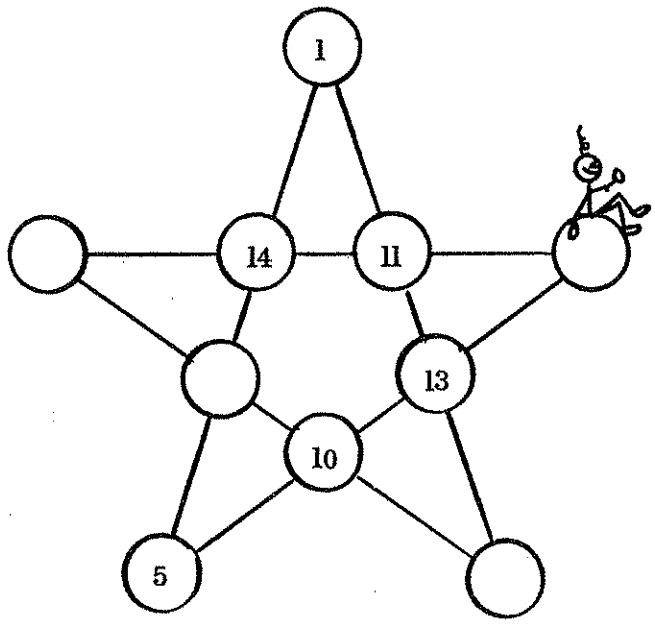
100. Sum is 16

101. Sum is 20

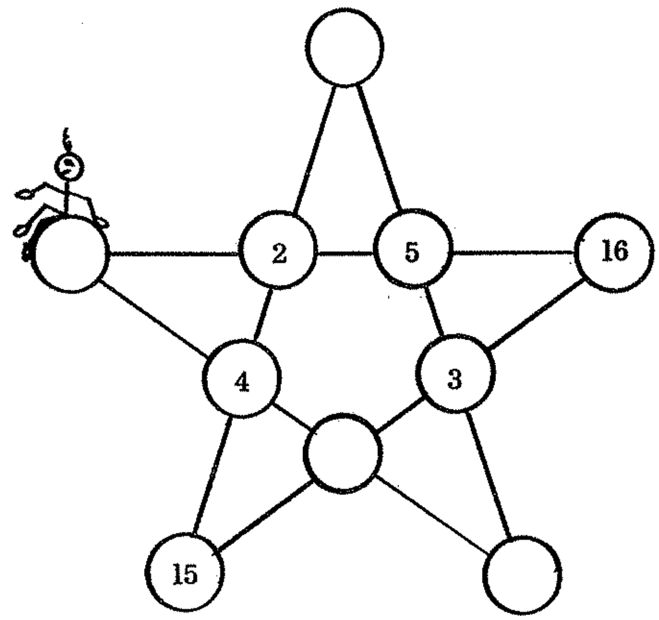


Place natural numbers at the intersections of the lines of the magic stars. Make the sum on each line of the star that shown above the star .

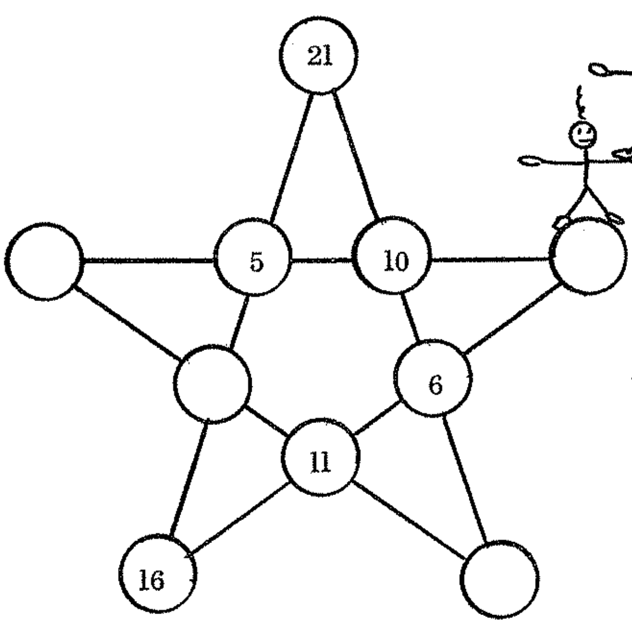
94. Sum is 32



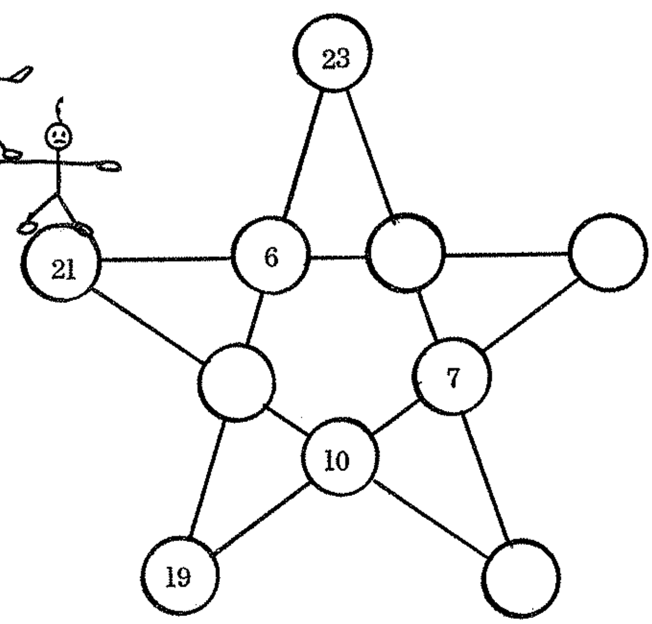
95. Sum is 40



96. Sum is 50



97. Sum is 56



CHAPTER 2

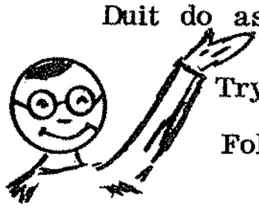
MYSTERIOUS NUMBER PATTERNS

2.1 MAGIC GUESSING - NUMBERS

Dini Dunit does mysterious things with numbers. He shows Deedee Duzit and Hugh Duit some of his magic.

Dini Dunit's Magic 1	Deedee Duzit takes	Hugh Duit takes
1. Take a number from 1 through 100	(6)	(8)
2. Double the number	2 x 6 = 12	
3. Add 10 to the result	12 + 10 = 22	
4. Take half of what you have	22 ÷ 2 = 11	
5. Subtract 5 from what you have	11 - 5 = (6)	

Deedee Duzit always ends up with the number she began with. Did Hugh Duit do as well ?



Try Dini Dunit's Magic 1 on the following numbers.

Follow the pattern in the first column of the chart.

107.

1. Take	(10)	(25)	(37)	(43)	(56)	(68)	(79)	(83)	(99)
2. Double	20								
3. Add 10	30								
4. Take half	15								
5. Subtract 5	(10)								

108. Try a few numbers greater than 100. Did Dini Dunit's Magic 1 pattern work ? Yes ______ No ______

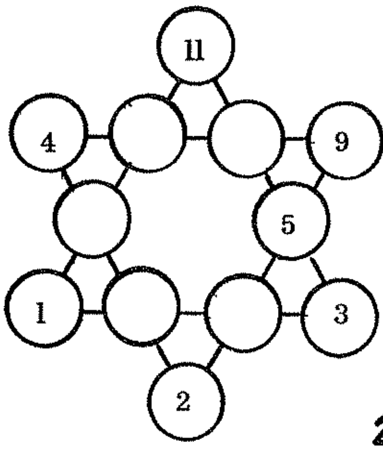
Try Dini Dunit's Magic 1 pattern on your friends.

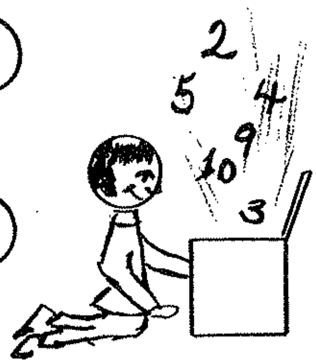
1.7 MAGIC STARS - SIX POINTED STARS

There are 12 points of intersection of the lines of the star. Each line of the star has 4 points.

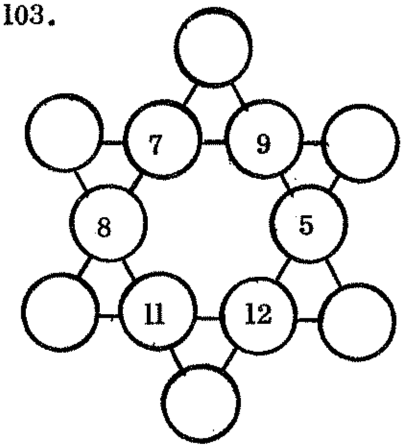
In each star use the natural numbers 1 through 12 . Make the sum on each line of the star 26 .

102.



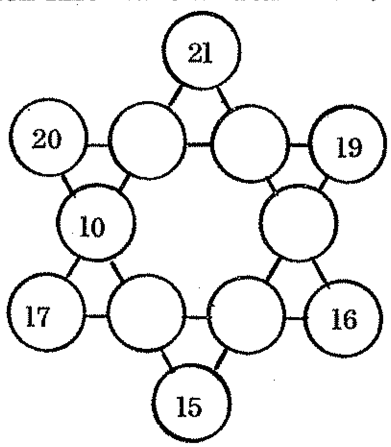


103.



Use natural numbers. Make the sum on each line of the star 59 .

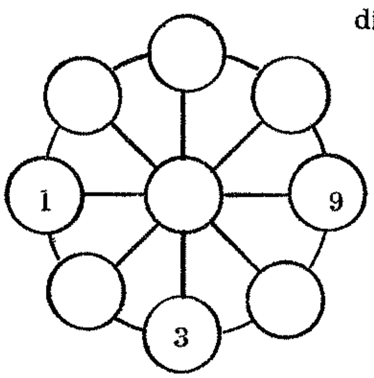
104.

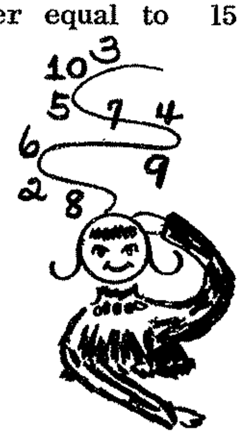


1.8 MAGIC WHEEL

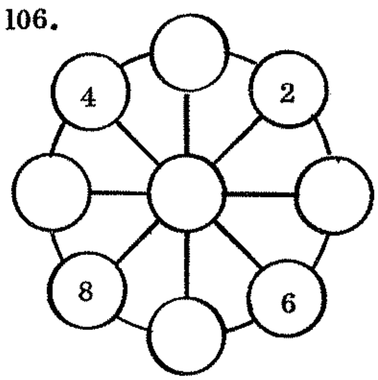
Use the numbers 1 through 9 . Make the sum on each diameter equal to 15 .

105.

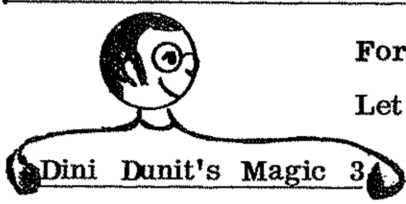




106.



2.2 MAGIC GUESSING - AGES



For the young of heart younger than 100 .
Let January be 1 , February be 2 , and so on .

Dini Dunit's Magic 3

Dini Dunit's Magic 3	Deedee Duzit Born : November , Age 10	Hugh Duit Born: September, Age 11
1. Write number of the month you were born	(11)	(9)
2. Multiply by 10	10 x 11 = 110	________
3. Add 10	110 + 10 = 120	________
4. Multiply result by 10	10 x 120 = 1200	________
5. Add your age in years	1200 + (10) = 1210	________
6. Subtract 100	1210 - 100 = 1 1 1 0	________

First 2-digits give the month you were born in .

Second 2-digits give your age .

Try Dini Dunit's Magic 3 on the given birthdays .
Follow the pattern in the first column of the chart .

112.

	January Age 6	March Age 12	April Age 9	May Age 8	June Age 1	July Age 4	August Age 28	December Age 99
1. Number of month	1							
2. Multiply by 10	10							
3. Add 10	20							
4. Multiply by 10	200							
5. Add age in years	206							
6. Subtract 100	1 06							

113. Try Dini's Magic 3 on Rip Van Winkle , born: January, age 100 . What happened ?

Dini Dunit's Magic 2	Deedee Duzit takes	Hugh Duit tries
1. Take a number from 1 through 100	(8)	(12)
2. Add 10 to it	8 + 10 = 18	______
3. Double the result	2 x 18 = 36	______
4. Add 100	36 + 100 = 136	______
5. Take half of the result	136 ÷ 2 = 68	______
6. Subtract number you started with	68 - 8 = (60)	______

Deedee Duzit always gets the answer 60. Did Hugh Duit get 60 ?

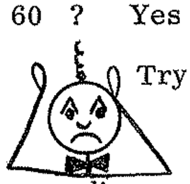
Try Dini Dunit's Magic 2 on the following numbers.

Follow the pattern in the first column of the chart.

109.

1. Take	(10)	(23)	(36)	(41)	(55)	(63)	(78)	(87)	(94)
2. Add 10	20								
3. Double	40								
4. Add 100	140								
5. Take half	70								
6. Subtract starting number	(60)								

110. Try a few numbers greater than 100. Did Dini's Magic 2 pattern always give 60 ? Yes _____ No _____



Try Dini Dunit's Magic 2 Pattern on your friends.

TOP - POPPER

111. You can discover why Dini Dunit's Magic 2 Pattern works by solving the following equation

$$\frac{2(n+10)+100}{2} - n = \underline{\qquad\qquad}$$

□ □ 5	9 □ □	2 □ 8	□ □ 7
+ 2 □ □	+ □ □ 0	+ □ □ 4	+ □ 0 □
□ 6 □	□ □ 9	□ □ □	7 □ 4

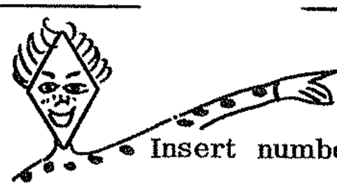
Roe Kolum says his row, column pattern also appears in each of the following four by four arrays. Check him.

119.

3 2 3 9	4 0 2 7	1 2 1 4	1 2 3 6
2 4 7 4	0 9 8 8	2 1 2 5	2 1 3 6
+ 3 7 4 5	+ 2 8 4 6	+ 1 2 3 7	+ 3 3 1 9

2 3 0 6	1 3 1 5	1 2 0 3	1 5 3 9
3 8 2 3	3 1 1 5	2 0 1 3	5 2 1 8
+ 0 2 4 7	+ 1 1 3 6	+ 0 1 2 3	+ 3 1 1 7

4 0 2 6	5 3 1 9	4 1 2 7	1 0 7 8
0 0 1 1	3 3 0 6	1 4 3 8	0 2 6 9
+ 2 1 1 5	+ 1 0 4 7	+ 2 3 1 8	+ 7 6 4 9



ROE KOLUM'S TOP-POPPER

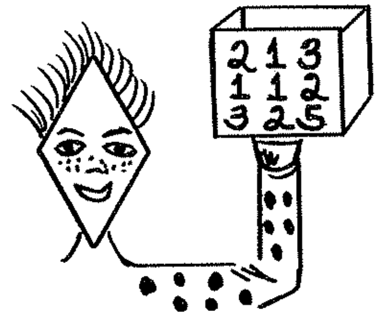
Insert numbers in the boxes. Make each row appear as a column.

120.

1 0 0 1	5 □ 2 □	3 1 2 7	8 □ □ 9
0 0 1 □	0 8 □ 9	□ 2 □ 1	□ 2 4 □
+ □ □ □ 1	+ □ 1 3 8	+ 2 7 3 3	+ 0 □ □ 9
1 □ 1 3	7 □ 8 □	□ □ □ □	□ 7 9 □

4 □ □ □	2 □ □ 9	2 □ 1 □	□ 0 2 □
0 □ □ □	□ 3 9 8	□ 2 □ 5	0 □ □ 2
+ 3 1 2 □	+ 1 9 2 □	+ □ 8 □ 2	+ 2 □ 2 3
7 6 8 1	9 □ □ 1	8 □ 2 □	□ 2 3 4

2.3 MAGIC ARRAYS



Add along with Roe Kolum.

```
  2 1 3
+ 1 4 5
```

Write the sum.

114. The number 213 is in a row. Is it also in a column? Yes ____ No ____

115. The number 145 is in a row. Is it also in a column? Yes ____ No ____

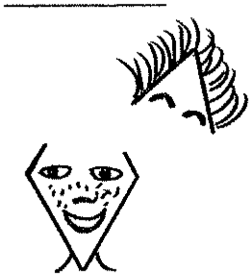
116. Is the sum in a row and in a column? Yes ____ No ____

Roe Kolum says the above pattern appears in each of the following sums. Check him.

117.

```
  3 5 9      2 1 4      1 4 5      1 2 3
+ 5 6 2    + 1 9 0    + 4 1 6    + 2 3 5

  8 0 8      5 2 8      1 3 4      6 1 7
+ 0 1 1    + 2 9 1    + 3 3 7    + 1 7 9
```



ROE KOLUM'S TOP-POPPER

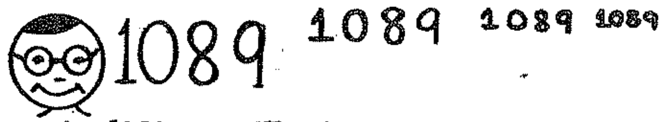
Insert numbers in the boxes. Make each row appear as a column on the pattern shown above.

118.

```
  1 1 2      4 2 □      □ 3 □      1 3 □
+ □ □ □    + 2 5 □    + 3 □ 3    + 3 □ □
  2 4 6      □ 8 4      9 3 2      4 4 8

  □ □ □      1 7 □      □ 2 □      □ □ □
+ □ 2 4    + 7 □ 0    + □ 6 □    + □ 9 □
  8 4 2      □ 0 □      8 9 7      5 □ 7
```

2.4 MAGIC 1089



Dini Dunit's favorite number is 1089. He'll show you why.

Dini Dunit's Magic 1089	Deedee Duzit tries	Hugh Duit takes
1. Take a 3-digit number. 1st digit greater than 3rd.	5 7 2	3 4 1
2. Reverse digits and subtract	− 2 7 5 2 9 7	− ______
3. Reverse digits and add	+ 7 9 2 1 0 8 9	+ ______

Deedee Duzit always gets the number <u>1089</u>. Did Hugh Duit get <u>1089</u>?

Do you get Dini Dunit's magic 1089 for each of the following?
Use the pattern of the first column in the chart.

122.

1. 3-digit number 1st digit greater than 3rd	431	533	675	766	786	814	964
2. Reverse digits and subtract	− 134 297	− ______	− ______	− ______	− ______	− ______	− ______
3. Reverse digits and add	+ 792 1089	+ ______	+ ______	+ ______	+ ______	+ ______	+ ______

2.5 MAGIC CRYSTAL BALL

Find the product $\begin{array}{r} 37 \\ \times\ 3 \\ \hline \end{array}$ Thus 37 x 3 = ________

123.

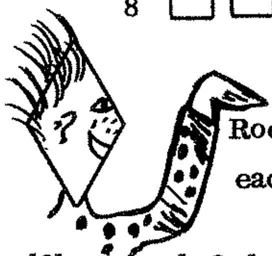
Predict the product at once	Check your prediction
37 x 2 x 3 = ________	$\begin{array}{r} 37 \\ \times\ 6 \\ \hline \end{array}$
37 x 3 x 3 = ________	$\begin{array}{r} 37 \\ \times\ 9 \\ \hline \end{array}$

2 □ 5 □
□ □ □ 6
+ □ □ 8 7
―――――――
8 □ □ □

□ 3 4 9
□ □ □ □
+ □ 2 □ 6
―――――――
□ □ □ 0

3 □ □ 6
□ □ □ □
+ 1 □ 5 □
―――――――
□ 8 9 □

□ □ 3 8
1 □ 6 □
+ □ □ □ 2
―――――――
□ 7 □ □



Roe Kolum says his row, column pattern also appears in each of the following five by five arrays. Check him.

121.

1 3 1 2 8
3 1 7 5 7
1 7 1 3 3
+ 2 5 3 4 6
―――――――

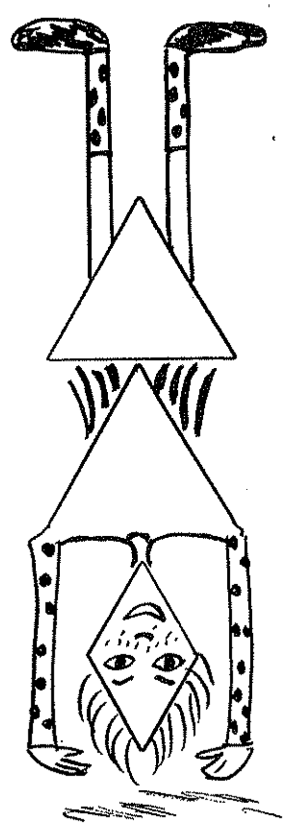
3 4 1 0 8
4 3 1 0 8
1 1 2 1 5
+ 0 0 1 0 3
―――――――

2 4 1 2 9
4 3 1 1 9
1 1 3 2 7
+ 2 1 2 1 9
―――――――

2 3 1 0 7
3 5 3 6 8
1 3 2 4 1
+ 0 6 4 3 5
―――――――

1 1 3 3 9
1 0 0 8 0
3 0 2 4 0
+ 3 8 4 0 6
―――――――

1 1 0 2 5
1 1 0 8 1
0 0 5 4 1
+ 2 8 4 7 1
―――――――



3 1 0 1 6
1 3 5 2 2
0 5 5 3 3
+ 1 2 3 1 8
―――――――

1 2 0 4 8
2 2 5 3 2
0 5 2 1 9
+ 4 3 1 1 0
―――――――

2 1 0 4 9
1 8 5 7 2
0 5 2 3 1
+ 4 7 3 0 5
―――――――

4 0 2 3 9
0 1 2 3 7
2 2 5 1 1
+ 3 3 1 4 2
―――――――

1 1 1 1 6
1 4 8 8 2
1 8 3 2 5
+ 1 8 2 0 2
―――――――

1 2 1 2 6
2 0 5 1 8
1 5 2 0 9
+ 2 1 0 6 1
―――――――

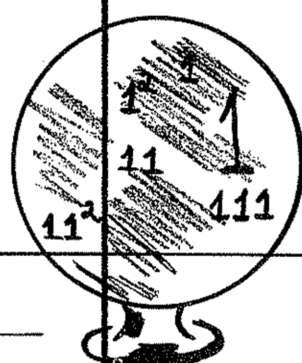
11111 111
Mighty ones squared
111111 111

Find the products.

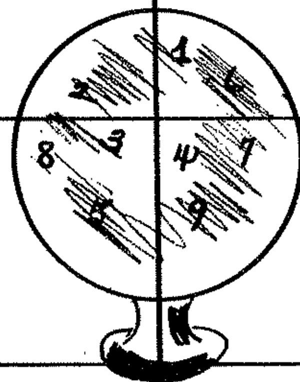
$1^2 =$ ____ $11^2 =$ ____ $111^2 =$ ____

125.

Predict the product at once	Check your prediction
$1,111^2 =$ ________	1,111 x 1,111
$11,111^2 =$ ________	11,111 x 11,111
$111,111^2 =$ ________	111,111 x 111,111
$1,111,111^2 =$ ________	1,111,111 x 1,111,111



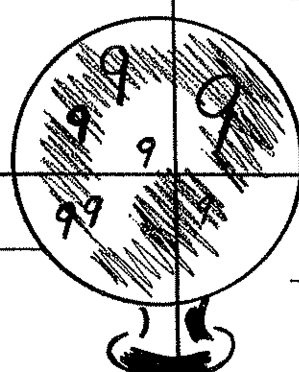
Predict the product at once	Check your prediction
37 x 4 x 3 = ________	37 x 12
37 x 5 x 3 = ________	37 x 15
37 x 9 x 3 = ________	37 x 27



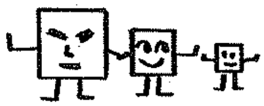
Mighty nines Find the product 12,345,679 x 9 Thus, 12,345,679 x 9 = ________

124.

Predict the product at once	Check your prediction
12,345,679 x 18 = ________	12,345,679 x 18
12,345,679 x 27 = ________	12,345,679 x 27
12,345,679 x 81 = ________	12,345,679 x 81



7 x 8 = ______ - ______ = ______ - ______ = ______



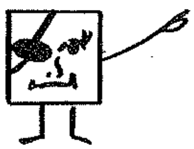
ESAU'S TOP - POPPERS

11 x 12 = ______ - ______ = ______ - ______ = ______

20 x 21 = ______ - ______ = ______ - ______ = ______

31 x 32 = ______ - ______ = ______ - ______ = ______

Middle squares   Isau , Esau's brother, discovered that



$1 \times 3 = 2^2 - 1 = 4 - 1 = 3$

$2 \times 4 = 3^2 - 1 = 9 - 1 = 8$

128. Do you see what Isau saw ? Follow Isau's pattern .

3 x 5 = ______ - ______ = ______ - ______ = ______

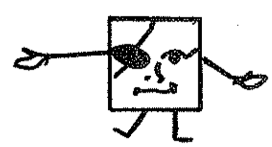
4 x 6 = ______ - ______ = ______ - ______ = ______

5 x 7 = ______ - ______ = ______ - ______ = ______

6 x 8 = ______ - ______ = ______ - ______ = ______

7 x 9 = ______ - ______ = ______ - ______ = ______

ISAU'S TOP - POPPERS

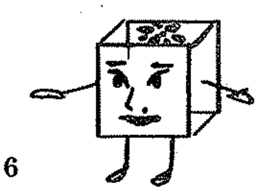


10 x 12 = ______ - ______ = ______ - ______ = ______

17 x 19 = ______ - ______ = ______ - ______ = ______

99 x 101 = ______ - ______ = ______ - ______ = ______

2.7 CONTEMPORARY CUBES



Dee Qube discovered that $1 \times 2 \times 3 = 2^3 - 2 = 8 - 2 = 6$

$2 \times 3 \times 4 = 3^3 - 3 = 27 - 3 = 24$

129. Do you see what Dee Qube found ? Follow his pattern .

3 x 4 x 5 = ______ - ______ = ______ - ______ = ______

2.6 REVIVED SQUARES

Simple squares

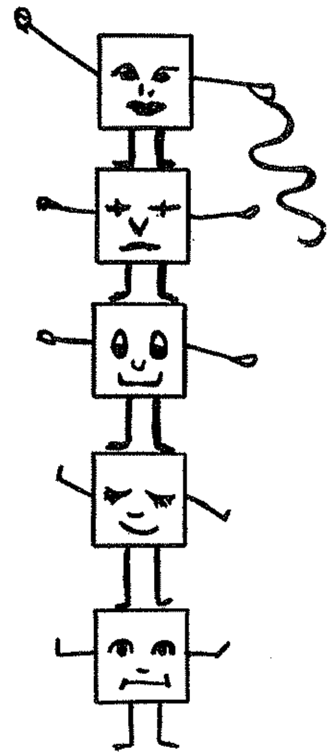
Esau Skware discovered that

$1 \times 2 = 1^2 + 1 = 1 + 1 = 2$

$2 \times 3 = 2^2 + 2 = 4 + 2 = 6$

$3 \times 4 = 3^2 + 3 = 9 + 3 = 12$

126. Do you see what Esau saw ? Follow Esau's pattern.



4 x 5 = ______ + ______ = ______ + ______ = ____________

5 x 6 = ______ + ______ = ______ + ______ = ____________

6 x 7 = ______ + ______ = ______ + ______ = ____________

7 x 8 = ______ + ______ = ______ + ______ = ____________

8 x 9 = ______ + ______ = ______ + ______ = ____________

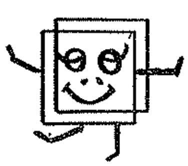
ESAU'S TOP - POPPERS

10 x 11 = ______ + ______ = ______ + ______ = ____________

15 x 16 = ______ + ______ = ______ + ______ = ____________

39 x 40 = ______ + ______ = ______ + ______ = ____________

Equally simple squares



Esau also discovered that

$1 \times 2 = 2^2 - 2 = 4 - 2 = 2$

$2 \times 3 = 3^2 - 3 = 9 - 3 = 6$

127. Do you see what Esau saw ? Follow Esau's pattern .

3 x 4 = ______ - ______ = ______ - ______ = ____________

4 x 5 = ______ - ______ = ______ - ______ = ____________

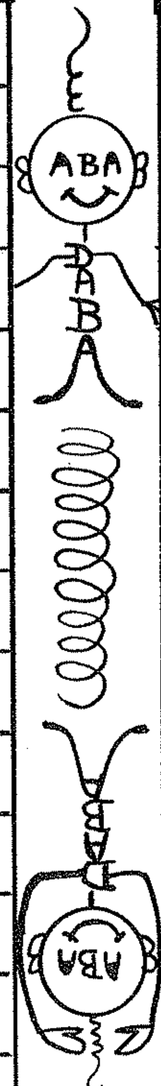
5 x 6 = ______ - ______ = ______ - ______ = ____________

6 x 7 = ______ - ______ = ______ - ______ = ____________

Fill in the charts on the pattern of the first rows.

134.

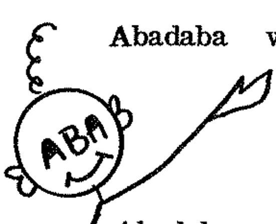
Number	Product	Is it a number palindrome
9^2	81	No
11^2		
15^2		
22^2		
26^2		
69^2		
71^2		
99^2		
101^2		
110^2		
111^2		
115^2		
117^2		
120^2		
121^2		



135.

Number	Product	Is it a number palindrome
3^3	27	No
5^3		
7^3		
9^3		
11^3		
15^3		
55^3		
76^3		
81^3		
94^3		
101^3		
105^3		
110^3		
111^3		
205^3		

2.9 PALINDROME ADDITION PATTERNS – NATURAL NUMBERS



Abadaba wrote

$$\begin{array}{r} 12 \\ +\ 21 \\ \hline 33 \end{array}$$

he reversed the digits in 12 and added .

The result is a number palindrome after one reversal .

Abadaba wrote

$$\begin{array}{r} 37 \\ +\ 73 \\ \hline 110 \\ +\ 011 \\ \hline 121 \end{array}$$

he reversed the digits in 37 and added .

The result is not a number palindrome .

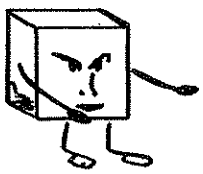
He reversed the digits in 110 and added .

The result is a number palindrome after 2 reversals .

4 x 5 x 6 = ______ - ______ = ______ - ______ = ______

5 x 6 x 7 = ______ - ______ = ______ - ______ = ______

6 x 7 x 8 = ______ - ______ = ______ - ______ = ______



7 x 8 x 9 = ______ - ______ = ______ - ______ = ______

DEE QUBE'S TOP - POPPERS

12 x 13 x 14 = ______ - ______ = ______ - ______ = ______

20 x 21 x 22 = ______ - ______ = ______ - ______ = ______

50 x 51 x 52 = ______ - ______ = ______ - ______ = ______

2.8 PALINDROMES

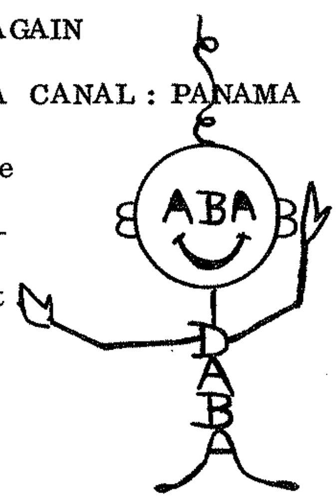
Abadaba wrote : BOB ANNA EVE RADAR SPACECAPS .

130. Does each word read the same when the letters are taken in reverse order ? Yes ______ No ______

Words such as the above are called word palindromes .

Abadaba found sentence palindromes : NIAGARA , O ROAR AGAIN

A MAN , A PLAN , A CANAL : PANAMA



131. Do the sentences read the same when the letters are taken in reverse order ? Yes ______ No ______

Of all palindromes, Abadaba liked number palindromes the best

1 3 3 1	6 1 7 1 6	4 0 8 8 0 4
2 5 4 5 2	8 8 8 8 8	1 2 3 4 3 2 1

Do the numbers remain the same when the digits are taken in reverse order ? Yes ______ No ______

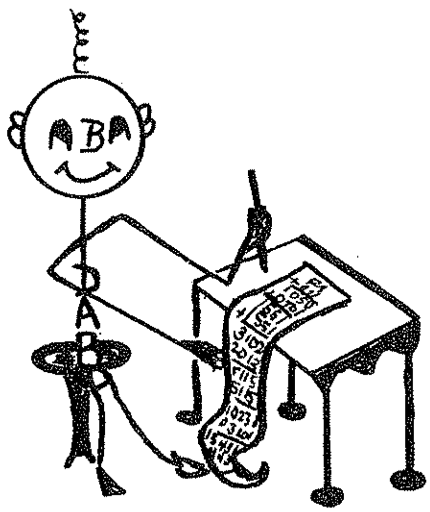
132. Make up four 4-digit number palindromes .

______ ______ ______ ______

133. Make up four 5-digit number palindromes .

______ ______ ______ ______

Here is another addition pattern that Abadaba tried.



Take the number	Number of reversals	Number of number palindromes
129		
+ 921	1	
1050		
+ 0501	2	
1551		1
+ 1551	3	
3102		
+ 2013	4	
5115		2
+ 5115	5	
10230		
+ 03201	6	
13431		3
+ 13431	7	
26862		4
+ 26862	8	
53724		
+ 42735	9	
96459		
+ 95469	10	
191928		

Starting with the number 129, there are 4 palindromic numbers in 10 reversals.

Fill in the charts on the pattern of the first row.

140.

Starting number	Number of number palindromes in 10 reversals
129	4
25	
39	
59	
79	
10	
120	
680	

141.

Starting number	Number of number palindrome in 10 reversals
150	
100	
356	
395	
624	
903	
739	
851	

2.10 PALINDROME ADDITION PATTERNS - DECIMALS

Here are some examples of Abadaba's palindromic decimal numbers:

9.9 12.21 123.321 235.532.

A decimal palindrome names the same number when the digits and the decimal point are taken in reverse order.

Use Abadaba's addition pattern. Fill in the charts on the pattern of the first rows.

136.

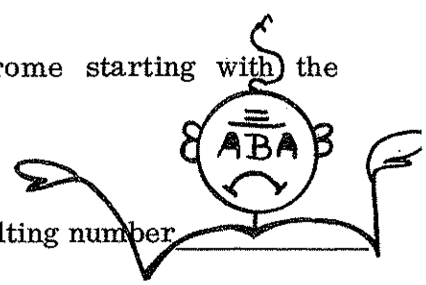
Starting number	Resulting number palindrome	Number of reversals
12	33	1
39		
58		
59		
66		
67		
69		
73		
77		
78		
79		
83		
86		
88		
94		
99		

137.

Starting number	Resulting number palindrome	Number of reversals
100	101	1
119		
174		
199		
198		
182		
688		
197		
589		
662		
829		
464		
849		
771		
672		
110		

138. ABADABA'S PROBLEM

Abadaba has not been successful in getting a number palindrome starting with the number 196 or the number 879.



Why don't you try. Write down your result.

Starting number 196. Number of reversals ________ Resulting number ________

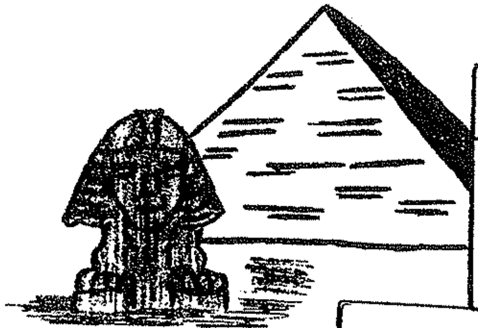
Starting number 879. Number of reversals ________ Resulting number ________

Prime natural numbers. Prime natural numbers are natural numbers whose only exact divisors are 1 and the number itself. Note: 1 is not a prime number.

Thus, 2, 3, 5, 7, are examples of prime numbers.

139. List the prime numbers from 2 through 100 which are number palindromes.

____



CHAPTER 3

99 x 12	999 x 12
99 x 23	999 x 23
99 x 34	999 x 34
99 x 45	999 x 45
99 x 56	999 x 56
99 x 67	999 x 67
99 x 78	999 x 78
99 x 89	999 x 89

CURIOUS NUMBER PATTERNS

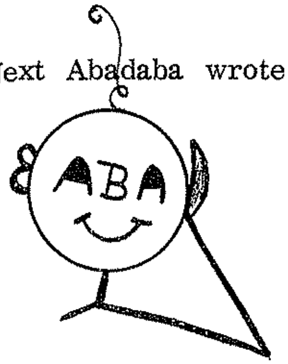
3.1 PEERLESS PYRAMIDS

Cheops The II says :
" More lasting than stones are the discoveries of the mind of man . "

144. Find the product for each block .

Abadaba wrote 1.8
+ 8.1 he reversed the digits in 1.8 and added.
9.9 The result was a palindromic decimal number after one reversal.

Next Abadaba wrote 19.6



+ 6.91 reverse the digits in 19.6 and add
26.51 result is not a palindrome
+ 15.62 reverse the digits in 26.51 and add
42.13 result is not a palindrome
+ 31.24 reverse the digits in 42.13 and add
73.37 result is a palindromic decimal number after 3 reversals.

Use Abadaba's addition pattern for decimals. Fill in the charts on the pattern of the first rows.

142.

Starting number	Resulting number palindrome	Number of reversals
1.8	9.9	1
1.2		
3.8		
4.6		
5.7		
6.9		
7.9		
8.9		
9.31		
16.8		
39.8		
50.9		
65.7		
79.8		
87.6		
99.8		

143.

Starting number	Resulting number palindrome	Number of reversals
19.6	73.37	3
100.2		
298.7		
565.6		
706.6		
897.3		
20.22		
41.36		
54.54		
88.66		
91.45		
1100.9		
2227.7		
27.466		
50.189		
79.490		

Find the products. Insert = or ≠ in the blanks.

93 x 13 31 x 39 13 x 93 _____ 39 x 31	85 x 21 12 x 58 21 x 85 _____ 58 x 12	64 x 23 32 x 46 23 x 64 _____ 46 x 32
93 x 26 62 x 39 26 x 93 _____ 39 x 62	84 x 36 63 x 48 36 x 84 _____ 48 x 63	92 x 37 73 x 29 37 x 92 _____ 29 x 73

146. Don Nod's 2-digit symmetry pattern also works for some 3-digit numbers.

102 x 402 = 204 x 201

The digits and the x symbols are symmetric with respect to the = symbol.

Find the products. Insert = or ≠ in the blanks.

604 x 203 302 x 406 203 x 604 _____ 406 x 302	603 x 204 402 x 306 204 x 603 _____ 306 x 402	936 x 213 312 x 639 213 x 936 _____ 639 x 312
823 x 215 512 x 328 215 x 823 _____ 328 x 512	933 x 226 622 x 339 226 x 933 _____ 339 x 622	663 x 244 442 x 366 244 x 663 _____ 366 x 442

3.2 NUMBER SYMMETRY PATTERNS

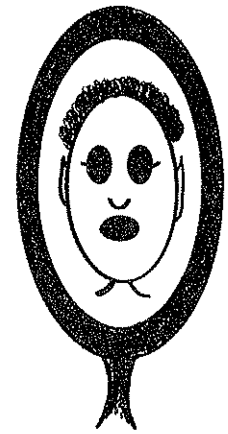
Don Nod, Abadaba's friend, found some surprising number symmetry patterns.

Since

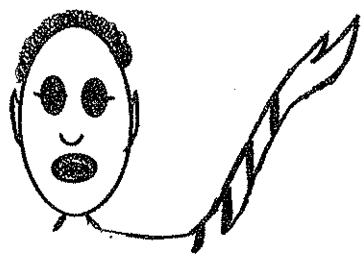
```
     1 2
 x   4 2
     2 4
   4 8
   5 0 4
```

and

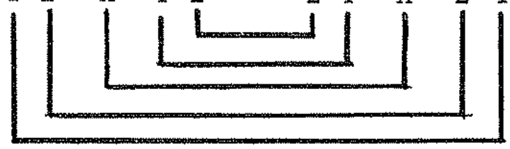
```
     2 4
 x   2 1
     2 4
   4 8
   5 0 4
```



Don Nod wrote



4 2 x 1 2 = 2 1 x 2 4



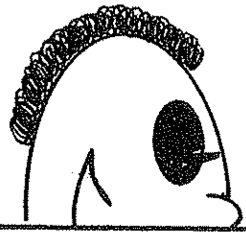
The digits and the x symbols are symmetric with respect to the = symbol.

Abadaba noticed something else. Don's discovery was an example of a very interesting equation palindrome.

Find the products. Insert = or ≠ in the blanks.

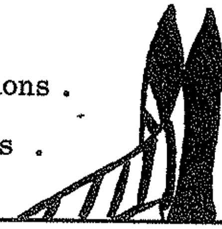
145.

31 x 13 ; 31 x 13 13 x 31 ____ 13 x 31	62 x 13 ; 31 x 26 13 x 62 ____ 26 x 31	56 x 14 ; 41 x 65 14 x 56 ____ 65 x 41
63 x 12 ; 21 x 36 12 x 63 ____ 36 x 21	82 x 14 ; 41 x 28 14 x 82 ____ 28 x 41	84 x 12 ; 21 x 48 12 x 84 ____ 48 x 21



Symmetric patterns for pairs of equations.

Find the products. Fill in the blanks.



```
  114          411
x 114        x 411
```

114 x 114 = ______ | ______ = 411 x 411

Symmetry : Yes ______ No ______

```
  104          401
x 104        x 401
```

104 x 104 = ______ | ______ = 401 x 401

Symmetry : Yes ______ No ______

```
  112          211
x 112        x 211
```

112 x 112 = ______ | ______ = 211 x 211

Symmetry : Yes ______ No ______

```
  113          311
x 113        x 311
```

113 x 113 = ______ | ______ = 311 x 311

Symmetry : Yes ______ No ______

```
  122          221
x 122        x 221
```

122 x 122 = ______ | ______ = 221 x 221

Symmetry : Yes ______ No ______

```
  125          521
x 125        x 521
```

125 x 125 = ______ | ______ = 521 x 521

Symmetry : Yes ______ No ______

Find the products . Insert = or ≠ in the blanks .

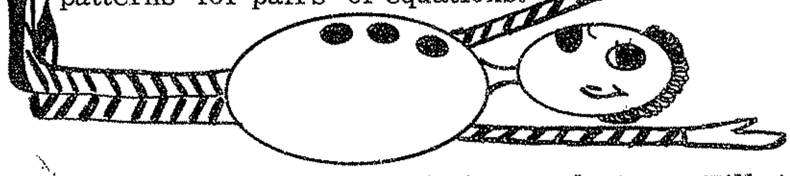
321 x 246 642 x 123	693 x 264 462 x 396	355 x 267 762 x 553
246 x 321 _____ 123 x 642	264 x 693 _____ 396 x 462	267 x 355 _____ 553 x 762
431 x 268 862 x 134	804 x 306 603 x 408	811 x 311 113 x 118
268 x 431 _____ 134 x 862	306 x 804 _____ 408 x 603	311 x 811 _____ 118 x 113

Since 13 x 13 = 169 and 961 = 31 x 31

then

13 x 13 = 169 | 961 = 31 x 31

Don Nod discovered symmetric patterns for pairs of equations.



The digits and symbols are symmetric with respect to the line .

147. Find the products . Fill in the blanks .

102 x 102 201 x 201	103 x 103 301 x 301
102 x 102 = _______ | _______ = 201 x 201	103 x 103 = _______ | _______ = 301 x 301
Symmetry : Yes _____ No _____	Symmetry : Yes _____ No _____

Examples of number tautonyms

1 - digit reduplication	2 - digit reduplication	3 - digit reduplication
22	2424	123123
444	363636	246246246

150 . Find the products . Fill in the blanks on the pattern of the first box .

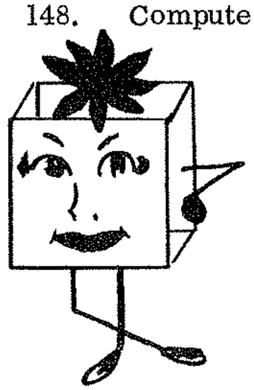
101 x 59 909 505 5959 2 - digit reduplication	101 x 47 ______ - digit reduplication	10101 x 34 ______ - digit reduplication
1001 x 135 ______ - digit reduplication	10101 x 61 ______ - digit reduplication	101 x 78 ______ - digit reduplication
10101 x 83 ______ - digit reduplication	1001 x 346 ______ - digit reduplication	1001 x 819 ______ - digit reduplication

3.3 CUBE SYMMETRIES

Dee Qube found some very interesting cube symmetries. Thus,

$153 = 3^3 + 5^3 + 1^3 =$ 27 + 125 + 1 = 153   True ✓ False ____

148. Compute each cube.



$0^3 =$ ________ $5^3 =$ ________

$1^3 =$ ________ $6^3 =$ ________

$2^3 =$ ________ $7^3 =$ ________

$3^3 =$ ________ $8^3 =$ ________

$4^3 =$ ________ $9^3 =$ ________

149. Which of the following equalities are true. Use the results of problem 148.

$370 = 0^3 + 7^3 + 3^3 =$ ______ + ______ + ______ = ______   True ____ False ____

$435 = 5^3 + 3^3 + 4^3 =$ ______ + ______ + ______ = ______   True ____ False ____

$611 = 1^3 + 1^3 + 6^3 =$ ______ + ______ + ______ = ______   True ____ False ____

$371 = 1^3 + 7^3 + 3^3 =$ ______ + ______ + ______ = ______   True ____ False ____

$238 = 8^3 + 3^3 + 2^3 =$ ______ + ______ + ______ = ______   True ____ False ____

$195 = 5^3 + 9^3 + 1^3 =$ ______ + ______ + ______ = ______   True ____ False ____

$407 = 7^3 + 0^3 + 4^3 =$ ______ + ______ + ______ = ______   True ____ False ____

$516 = 6^3 + 1^3 + 5^3 =$ ______ + ______ + ______ = ______   True ____ False ____

3.4 NUMBER TAUTONYMS

In each of the following words, two or more letters are reduplicated.

MAMA   PAPA   MURMUR   BONBON

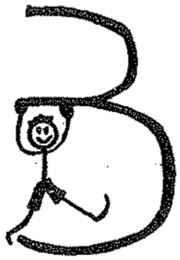
GOGO

Words with such reduplications are called word tautonyms.

Besides word tautonyms, we also have number tautonyms.

HULAHULA

Fill in the blanks



3^0 = ______	3^4 = ______	3^8 = ______
3^1 = ______	3^5 = ______	3^9 = ______
3^2 = ______	3^6 = ______	3^{10} = ______
3^3 = ______	3^7 = ______	3^{11} = ______

Sums and differences of powers of 3

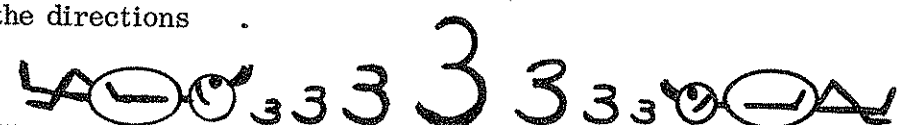
Each natural number greater than 0 can be represented as the sum and / or difference of powers of 3 . No power of 3 is used more than once in any representation of a given natural number .

Examples

$2 = 3 - 1$
$= 3^1 - 3^0$

$5 = 9 - 3 - 1$
$= 3^2 - 3^1 - 3^0$

152. Represent the natural numbers as sums and/or differences of powers of 3 . Follow the directions .



7 = =	31 = =
12 = =	63 = =
14 = =	255 = =
19 = =	127 = =
22 = =	511 = =
24 = =	1023 = =

3.5 REPRESENTATIONS OF NATURAL NUMBERS

Recall $2^0 = 1$ $10^0 = 1$

$3^0 = 1$ $156^0 = 1$

Any number except 0 raised to the zero power is 1 .

Fill in the blanks .

2^0 = ____	2^4 = ____	2^8 = ____
2^1 = ____	2^5 = ____	2^9 = ____
2^2 = ____	2^6 = ____	2^{10} = ____
2^3 = ____	2^7 = ____	2^{11} = ____

Sums of powers of 2

Each natural number greater than 0 can be represented as the sum of powers of 2 . No power of 2 is used more than once in any representation of a given natural number .

Examples

$$3 = 2 + 1 = 2^1 + 2^0$$

$$6 = 4 + 2 = 2^2 + 2^1$$

151. Represent the natural numbers as sums of powers of 2 . Follow the directions .

9 = 8 + 1 $= 2^3 + 2^0$	324 = =
15 = =	353 = =
24 = =	520 = =
49 = =	772 = =
72 = =	1120 = =
160 = =	2110 = =
193 = =	3077 = =

List the first twenty prime natural numbers . (See page 30)

2	3								
							61		71

Even natural numbers . Sum of two odd primes .

Each even natural number greater than 4 can be represented as a sum of 2 odd primes . Primes may be repeated in a sum .

154 . Represent the natural numbers as a sum of 2 odd primes .

18 = 13 + 5	48 =	90 =
26 =	50 =	94 =
30 =	64 =	108 =
36 =	76 =	112 =
40 =	80 =	126 =
42 =	88 =	142 =

Odd natural numbers . Sum of three odd primes .

Each odd natural number greater than 7 can be represented as a sum of 3 odd primes . Primes may be repeated in a sum .

155. Represent the natural numbers as a sum of 3 odd primes .

17 = 11 + 3 + 3	57 =	93 =
25 =	67 =	101 =
31 =	73 =	111 =
39 =	79 =	127 =
43 =	81 =	139 =
51 =	85 =	141 =

Sums of squares of natural numbers

Each natural number can be represented as a sum of no more than 4 squares of natural numbers. Squares of natural numbers may be repeated in the sum.

Examples

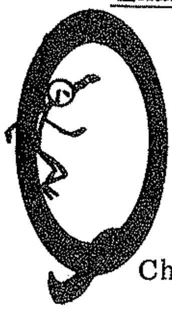
$$2 = 1 + 1 = 1^2 + 1^2 \qquad 10 = 9 + 1 = 3^2 + 1^2$$

153. Represent the natural numbers as a sum of no more than 4 squares of natural numbers. Use the pattern in the examples above.

15 = =	162 = =
16 = =	175 = =
32 = =	183 = =
53 = =	190 = =
63 = =	196 = =
89 = =	218 = =
94 = =	226 = =
137 = =	237 = =
153 = =	311 = =

3.6 MATHEMATICAL ARGOSY

Quickie products : 2-digits a b x a c where b + c = 10



Example

Consider 3 1 x 3 9. Sum of the numbers represented by last digits must always be 10 , here 1 + 9 = 10, and the first digits must be the same.

Product 3 1 x 3 9 = 1 2 0 9

- 0 9: Take product of numbers represented by last digits, here 1 x 9 = 0 9, and always use 2-digits .
- 1 2: Add 1 to number represented by first digit and multiply, (3 + 1) x 3 = 12.

Check

```
   3 9
x  3 1
------
   3 9
 1 1 7
------
1 2 0 9
```

158 . Find the product using the procedure in the example above . Check each product.

2 2 x 2 8 = ______ ______ 2 8 x 2 2	5 2 x 5 8 = ______ ______ 5 8 x 5 2	7 1 x 7 9 = ______ ______ 7 9 x 7 1
4 3 x 4 7 = ______ ______ 4 7 x 4 3	5 3 x 5 7 = ______ ______ 5 7 x 5 3	7 3 x 7 7 = ______ ______ 7 7 x 7 3
4 5 x 45 = ______ ______ 4 5 x 4 5	6 4 x 66 = ______ ______ 6 6 x 6 4	93 x 9 7 = ______ ______ 9 7 x 9 3

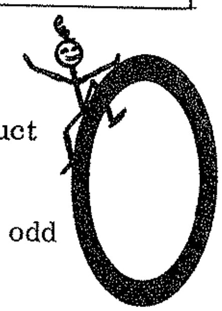
Odd natural numbers. Sum of a prime number and a power of 2 .

Can each odd natural number greater than 1 be represented as a sum of a prime number and a power of 2 ?

156. Try to write the natural numbers as a sum of a prime number and a power of 2

21 = 19 + 2^1 = 19 + 2	75 = + 2 = +	173 = + 2 = +
31 = + 2 = +	93 = + 2 = +	181 = + 2 = +
39 = + 2 = +	119 = + 2 = +	201 = + 2 = +
49 = + 2 = +	125 = + 2 = +	217 = + 2 = +
51 = + 2 = +	131 = + 2 = +	231 = + 2 = +
63 = + 2 = +	157 = + 2 = +	243 = + 2 = +

Natural numbers . Product of a power of 2 and an odd natural number .



Each natural number greater than 0 can be represented as a product of a power of 2 and an odd natural number .

157. Represent the natural numbers as a product of a power of 2 and an odd natural number .

12 = 2^2 x 3 = 4 x 3	56 = 2 x =	347 = 2 x =
15 = 2 x =	73 = 2 x =	458 = 2 x =
16 = 2 x =	96 = 2 x =	550 = 2 x =
23 = 2 x =	138 = 2 x =	674 = 2 x =
24 = 2 x =	176 = 2 x =	717 = 2 x =
28 = 2 x =	291 = 2 x =	1070 = 2 x =
30 = 2 x =	310 = 2 x =	3722 = 2 x =

Tantalizing tautonym division : Magic 13

Take a 3-digit number : 1 2 3 , write the tautonym 1 2 3 1 2 3 . Perform the divisions.

Divide by 7

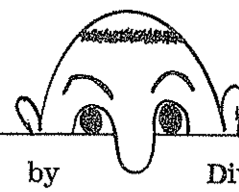
```
    1 7 5 8 9
  -----------
7 | 1 2 3 1 2 3
    7
    ---
    5 3
    4 9
    ---
      4 1
      3 5
      ---
        6 2
        5 6
        ---
          6 3
          6 3
          ---
            0
```

Divide result by original number

```
        1 4 3
      -----------
1 2 3 | 1 7 5 8 9
        1 2 3
        -----
          5 2 8
          4 9 2
          -----
            3 6 9
            3 6 9
            -----
                0
```

Divide result by 1 1

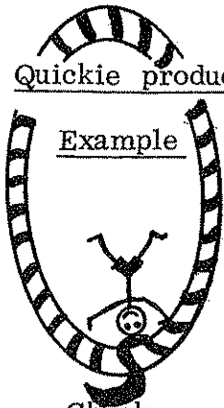
```
      1 3
    -------
1 1 | 1 4 3
      1 1
      ---
        3 3
        3 3
        ---
          0
```



160 . Find the quotients.

Number	Tautonym	Divide by 7	Divide result by	Divide result by
234	234234	7 ⟌ 2 3 4 2 3 4	2 3 4 ⟌	1 1 ⟌
378	378378	7 ⟌ 3 7 8 3 7 8	3 7 8 ⟌	1 1 ⟌

Quickie products : 3-digits   a b c x a b d where c + d = 10



Consider 1 1 3 x 117. Sum of the numbers represented by last digits must always be 10 , here 3 + 7 = 10, and the first two digits must be the same .

Product 1 1 3 x 1 1 7 = 132 21

Take product of numbers represented by last digits, here 3 x 7 = 21 .

Add 1 to number represented by first two digits and multiply, (11 + 1) x 11 = 132 .

Check

```
     1 1 7
   x 1 1 3
     3 5 1
   1 1 7
 1 1 7
 1 3 2 2 1
```

159. Find the product using the procedure in the example above. Check each product.

2 3 2 x 2 3 8 = _____ _____ 2 3 8 x 2 3 2	5 1 1 x 5 1 9 = _____ _____ 5 1 9 x 5 1 1	7 1 3 x 7 1 7 = _____ _____ 7 1 7 x 7 1 3
3 0 4 x 3 0 6 = _____ _____ 3 0 6 x 3 0 4	5 6 5 x 5 6 5 = _____ _____ 5 6 5 x 5 6 5	7 9 2 x 7 9 8 = _____ _____ 7 9 8 x 7 9 2
4 3 3 x 4 3 7 = _____ _____ 4 3 7 x 4 3 3	6 6 4 x 6 6 6 = _____ _____ 6 6 6 x 6 6 4	9 9 1 x 9 9 9 = _____ _____ 9 9 9 x 9 9 1

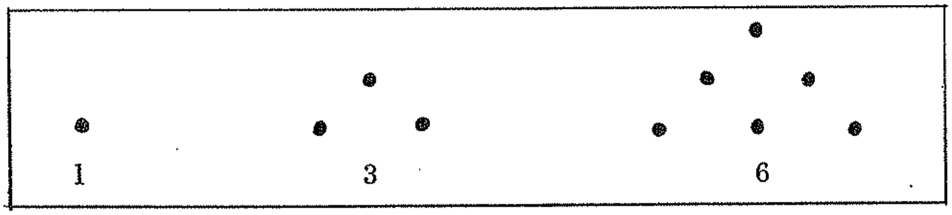
CHAPTER 4

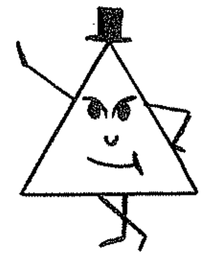
WONDER - FULL WORLD OF NUMBERS

4.1 FIGURATE NUMBERS

Early Greek mathematicians used dots arranged in geometric patterns to represent some sets of natural numbers. Each set of numbers was given a name that corresponded to the geometric figure.

Triangular numbers. Dots are arranged in the form of a triangle.





First 3 triangular numbers are : 1 , 3 , 6 .

161. Study the dot pattern above. Write the next 5 triangular numbers.

______ ______ ______ ______ ______

TRIANGULAR NUMBER FORMULA $\frac{n(n+1)}{2}$

To find 1st triangular number, put 1 for n in the formula : $\frac{1(1+1)}{2} = \frac{1(2)}{2} = 1$

To find the 2nd, 3rd, and so on triangular numbers, put 2, 3...for n in the formula.

162. Use the formula. List the first 20 triangular numbers.

1 3 6 ______ ______ ______ ______ ______ ______ ______

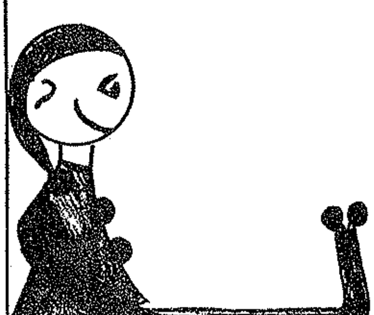
______ ______ ______ ______ ______ ______ ______ ______ ______ ______

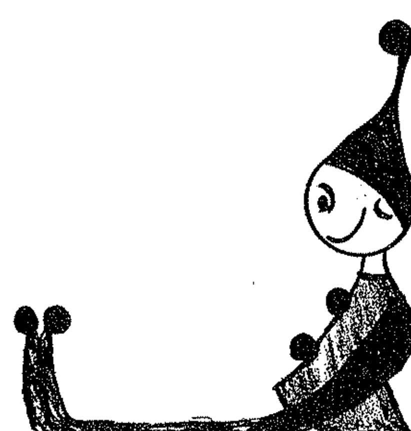
163. Find the sums of the natural numbers on the pattern of the first 3 rows.

1 = 1	1 + 2 + 3 + ... + 7 =	1 + 2 + 3 + ... + 13 =
1 + 2 = 3	1 + 2 + 3 + ... + 8 =	1 + 2 + 3 + ... + 14 =
1 + 2 + 3 = 6	1 + 2 + 3 + ... + 9 =	1 + 2 + 3 + ... + 15 =
1 + 2 + 3 + 4 =	1 + 2 + 3 + ... + 10 =	1 + 2 + 3 + ... + 16 =
1 + 2 + 3 + ... + 5 =	1 + 2 + 3 + ... + 11 =	1 + 2 + 3 + ... + 17 =
1 + 2 + 3 + ... + 6 =	1 + 2 + 3 + ... + 12 =	1 + 2 + 3 + ... + 18 =

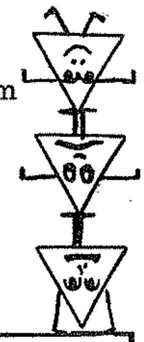
Find the quotients.

Number	Tautonym	Divide by 7	Divide result by	Divide result by
654	654654	7 | 6 5 4 6 5 4	6 5 4 |	11 |
764	764764	7 | 7 6 4 7 6 4	7 6 4 |	11 |
987	987987	7 | 9 8 7 9 8 7	9 8 7 |	11 |





Triangular numbers : sums of triangular numbers

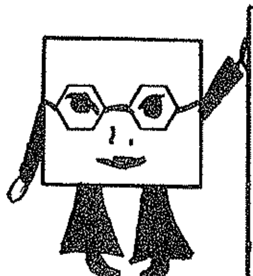


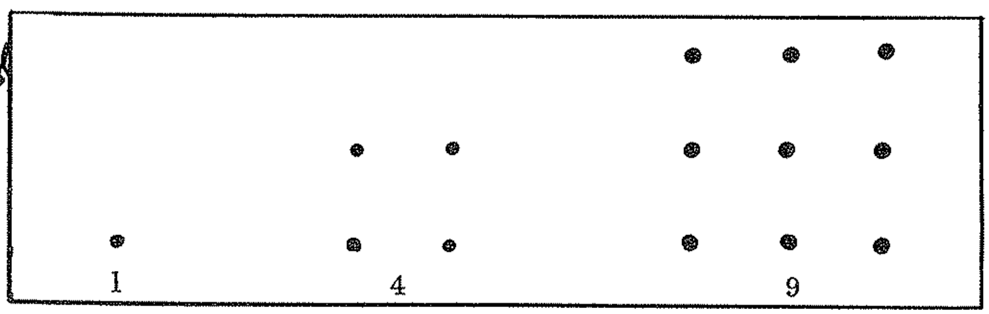
Each triangular number greater than 1 can be represented as the sum of no more than 3 triangular numbers. Triangular numbers may be repeated in the sum.

169. Represent the following triangular numbers as sums of triangular numbers. Follow the directions above.

3 = 1 + 1 + 1	28 =	78 =	153 =
6 =	36 =	91 =	171 =
10 =	45 =	105 =	190 =
15 =	55 =	120 =	210 =
21 =	66 =	136 =	231 =

Square numbers. Dots are arranged in the form of a square.





First 3 square numbers are : 1 , 4 , 9 .

170. Study the dot pattern above. Write the next 5 square numbers.

______ ______ ______ ______ ______

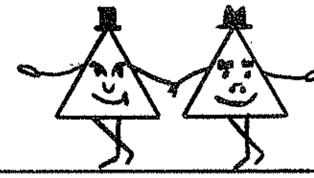
SQUARE NUMBER FORMULA n^2

To find the 1st square number, put 1 for n in the formula: $1^2 = 1$.

To find the 2nd, 3rd, and so on square numbers, put 2, 3, ... for n in the formula.

164. Each sum of consecutive natural numbers beginning with 1 is a ______________ number .

165. The sum of the first n natural numbers beginning with 1 is given by the formula ________________

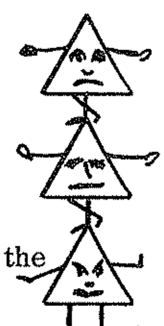


166. Fill in the blanks on the pattern of the first two rows .

Two consecutive triangular numbers		Their sum	Two consecutive triangular numbers		Their sum
1	3	$1 + 3 = 4 = 2^2$	45	55	45 + 55 =
3	6	$3 + 6 = 9 = 3^2$	66	78	66 + 78 =
6	10	6 + 10 =	105	120	105 + 120 =
10	15	10 + 15 =	136	153	136 + 153 =
15	21	15 + 21 =	190	210	190 + 210 =

167. The sum of 2 consecutive triangular numbers is a perfect ______________.

Natural numbers : sums of triangular numbers .

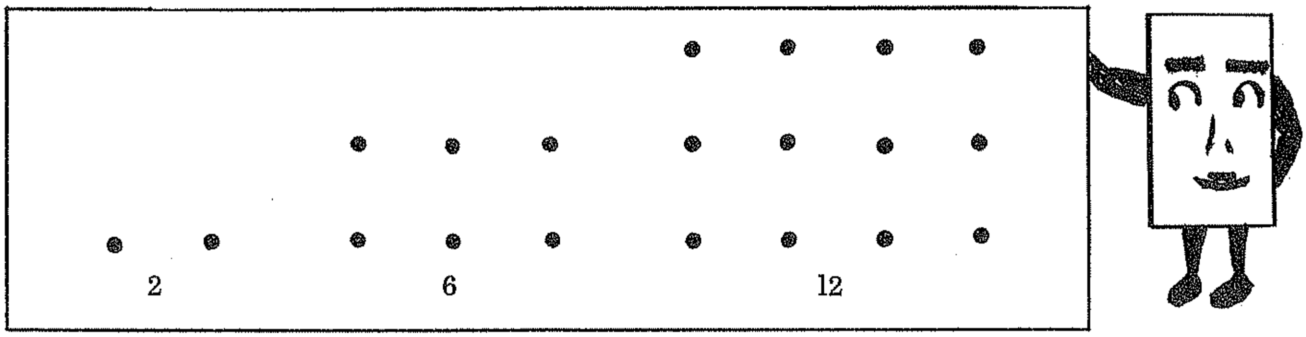


Each natural number greater than 0 is a triangular number or can be represented as the sum of no more than 3 triangular numbers. Triangular numbers may be repeated in a sum .

168. Represent the following natural numbers using triangular numbers. Follow the directions above .

3 = 3	47 =	80 =	145 =
7 = 6 + 1	53 =	94 =	167 =
12 =	60 =	104 =	182 =
19 =	65 =	115 =	196 =
40 =	71 =	133 =	209 =

Rectangular numbers. Dots are arranged in the form of a rectangle.



First 3 rectangular numbers are : 2, 6, 12 .

175. Study the dot pattern above. Write the next 5 rectangular numbers .

______ ______ ______ ______ ______

RECTANGULAR NUMBER FORMULA $n(n+1)$

To find the 1st rectangular number, put 1 for n in the formula ; $1(1+1) = 2$

To find the 2nd , 3rd , and so on rectangular numbers , put 2, 3, ... for n in the formula .

176. Use the formula . List the first 20 rectangular numbers .

2	6	12							

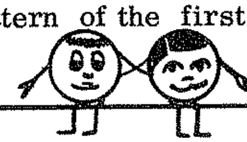
177. Find the sums of the even natural numbers on the pattern of the first 3 rows .

2 = 2	2 + 4 + 6 + . . . + 18 =	2 + 4 + 6 + . . . + 34 =
2 + 4 = 6	2 + 4 + 6 + . . . + 20 =	2 + 4 + 6 + . . . + 36 =
2 + 4 + 6 = 12	2 + 4 + 6 + . . . + 22 =	2 + 4 + 6 + . . . + 38 =
2 + 4 + 6 + 8 =	2 + 4 + 6 + . . . + 24 =	2 + 4 + 6 + . . . + 40 =
2 + 4 + 6 + . . . + 10 =	2 + 4 + 6 + . . . + 26 =	2 + 4 + 6 + . . . + 42 =
2 + 4 + 6 + . . . + 12 =	2 + 4 + 6 + . . . + 28 =	2 + 4 + 6 + . . . + 44 =
2 + 4 + 6 + . . . + 14 =	2 + 4 + 6 + . . . + 30 =	2 + 4 + 6 + . . . + 46 =
2 + 4 + 6 + . . . + 16 =	2 + 4 + 6 + . . . + 32 =	2 + 4 + 6 + . . . + 48 =

171. Use the square number formula. List the first 20 square numbers.

1	4	9	____	____	____	____	____	____	____
____	____	____	____	____	____	____	____	____	____

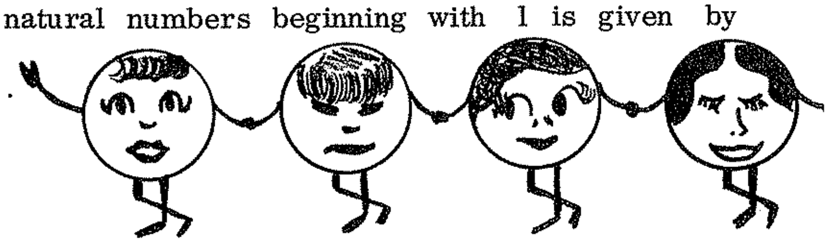
172. Find the sums of consecutive odd natural numbers on the pattern of the first three rows.



$1 = 1 = 1^2$	$1 + 3 + 5 + \ldots + 21 =$
$1 + 3 = 4 = 2^2$	$1 + 3 + 5 + \ldots + 23 =$
$1 + 3 + 5 = 9 = 3^2$	$1 + 3 + 5 + \ldots + 25 =$
$1 + 3 + 5 + 7 =$	$1 + 3 + 5 + \ldots + 27 =$
$1 + 3 + 5 + \ldots + 9 =$	$1 + 3 + 5 + \ldots + 29 =$
$1 + 3 + 5 + \ldots + 11 =$	$1 + 3 + 5 + \ldots + 31 =$
$1 + 3 + 5 + \ldots + 13 =$	$1 + 3 + 5 + \ldots + 33 =$
$1 + 3 + 5 + \ldots + 15 =$	$1 + 3 + 5 + \ldots + 35 =$
$1 + 3 + 5 + \ldots + 17 =$	$1 + 3 + 5 + \ldots + 37 =$
$1 + 3 + 5 + \ldots + 19 =$	$1 + 3 + 5 + \ldots + 39 =$

173. Each sum of consecutive odd natural numbers beginning with 1 is a __________ number.

174. The sum of the first n odd natural numbers beginning with 1 is given by the formula __________________.



182 . Find the happy years . Use the procedure in problem 181 .

Year 1 9 0 0	Year 1 9 6 9	Year 2 0 0 0
Happy _____ Not happy _____	Happy _____ Not happy _____	Happy _____ Not happy _____

4. 3 PRIME NUMBERS: SIEVE OF ERATOSTHENES

A prime natural number is a natural number other than 0 and 1 which is exactly divisible only by the natural number 1 and itself . 2, 3, 5 are prime numbers .

Finding prime numbers List the natural numbers consecutively beginning with 1 .

~~1~~	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~	11	12	13	14	~~15~~
16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40	41	42	43	44	45

1 is not a prime number. Cross it out.

2 is a prime . Do not cross it out but cross out every second number, that is, cross out 4, 6, 8 and so on . DO THIS .

3 is not crossed out . It is a prime . Cross out every third number, that is, cross out 6, 9 , and so on . DO THIS .

4 is crossed out . Go to the next number which is not crossed out .

5 is not crossed out . It is a prime . Cross out every fifth number, that is, cross out 10, 15, and so on . DO THIS .

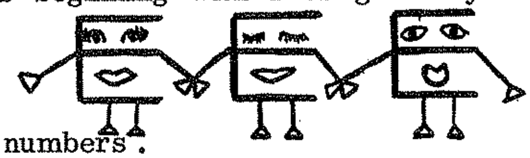
Continue the above process . The numbers that are not crossed out at the end of the process are prime numbers.

183. Write the prime numbers found in the above list of numbers .

____

178. Each sum of consecutive even natural numbers beginning with 2 is a ________________ number .

179. The sum of the first n even natural numbers beginning with 2 is given by the formula ________________ .

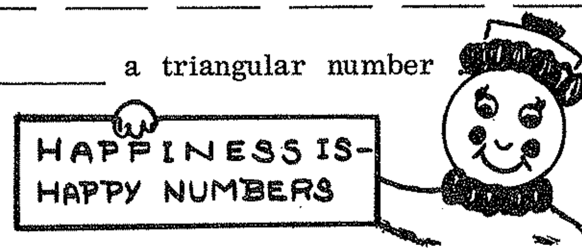


180. List the first 8 triangular and rectangular numbers .

Triangular numbers _____ _____ _____ _____ _____ _____ _____ _____

Rectangular numbers _____ _____ _____ _____ _____ _____ _____ _____

Each rectangular number is __________ a triangular number



4.2 HAPPY NUMBERS

Start with 13

then $1^2 + 3^2 = 10$, sum the squares of the numbers represented by each digit in 13

repeat for 10 $1^2 + 0^2 = 1$.

The starting natural numbers for which the above pattern ends in a 1 are called happy numbers.
Thus, 13 is a happy number .

181. Find the happy numbers.

Start with 7 7 is ______ is not _______ a happy number	Start with 11 11 is _____ is not _____ a happy number
Start with 70 70 is _____ is not ______ a happy number	Start with 83 83 is _____ is not _____ a happy number

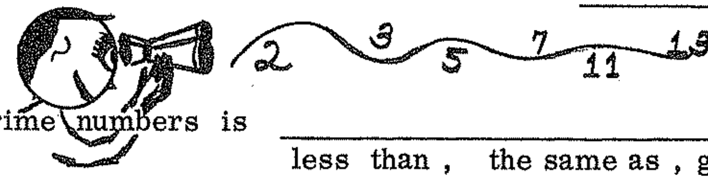
185. Count the prime numbers between 1 and 100. ____________

Count the prime numbers between 100 and 200. ____________

Count the prime numbers between 200 and 300. ____________

<u>Total</u> number of prime numbers between 1 and 300. ____________

186. <u>Insight Top - Popper</u>



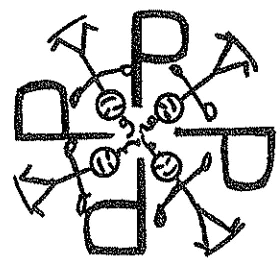
The number of natural prime numbers is ____________
less than, the same as, greater than

the number of natural numbers.

4.4 DISTRIBUTION OF PRIME NUMBERS

Let n be any natural number 1, 2, 3, 4, 5, . . .

Now take 2 n , that is, take twice the natural number.



> For any choice of n , either n or 2 n is a prime number, or there is at least 1 prime number between n and 2 n .

<u>Example</u> Let n = 4.

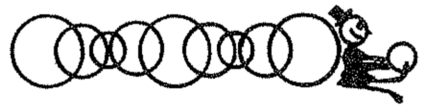
Then 2 n = 8.

There are prime numbers, namely 5, 7, between 4 and 8.

187. Fill in the chart on the pattern of the first row. Use results on page 56.

n	2 n	Primes equal to n or 2 n or in between	n	2 n	Primes equal to n or 2 n or in between
5	10	5 7	41		
13			50		
21			64		
25			79		
30			86		
35			91		

184. Circle only the prime numbers.



1	(2)	(3)	4	(5)	6	7	8	9	10	11	12	13	14	15
16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70	71	72	73	74	75
76	77	78	79	80	81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100	101	102	103	104	105
106	107	108	109	110	111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130	131	132	133	134	135
136	137	138	139	140	141	142	143	144	145	146	147	148	149	150
151	152	153	154	155	156	157	158	159	160	161	162	163	164	165
166	167	168	169	170	171	172	173	174	175	176	177	178	179	180
181	182	183	184	185	186	187	188	189	190	191	192	193	194	195
196	197	198	199	200	201	202	203	204	205	206	207	208	209	210
211	212	213	214	215	216	217	218	219	220	221	222	223	224	225
226	227	228	229	230	231	232	233	234	235	236	237	238	239	240
241	242	243	244	245	246	247	248	249	250	251	252	253	254	255
256	257	258	259	260	261	262	263	264	265	266	267	268	269	270
271	272	273	274	275	276	277	278	279	280	281	282	283	284	285
286	287	288	289	290	291	292	293	294	295	296	297	298	299	300
301	302	303	304	305	306	307	308	309	310	311	312	313	314	315
316	317	318	319	320	321	322	323	324	325	326	327	328	329	330
331	332	333	334	335	336	337	338	339	340	341	342	343	344	345

191 . Use the page 56 . For the prime natural numbers between 1 and 150 , list the <u>consecutive primes</u> separated by <u>exactly</u>

3 non-primes	5 non-primes	7 non-primes	9 non-primes
(7) (11)	(23) (29)		

4.5 REPRESENTATION OF PRIMES

The prime number 5 can be represented as $5 = 6(1) - 1$

7 can be represented as $7 = 6(1) + 1$.

The prime numbers greater than 3 can be represented in the forms

$$6n + 1 \quad \text{or} \quad 6n - 1$$

for a proper choice of n where n = 1, 2, 3, 4, 5, ...

192. Find the proper value for n . Use the pattern of the first two rows .

Prime number	Representation 6 n + 1 or 6 n - 1	n	Prime number	Representation 6 n + 1 or 6 n - 1	n
17	$17 = 6(3) - 1$	3	139		
19	$19 = 6(3) + 1$	3	197		
37			233		
59			277		
71			293		
89			307		
101			313		
127			337		

The numbers $2^2 = 4$ $3^2 = 9$

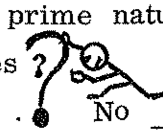
are consecutive squares . There are two prime numbers between 2^2 and 3^2, namely 5 and 7 .

Between two consecutive squares greater than 0 there is at least 1 prime number .

188. Fill in the chart . Follow the pattern of the first row . Use page 56.

Square	Consecutive square	Primes between consecutive squares	Square	Consecutive square	Primes between consecutive square
$3^2 = 9$	$4^2 = 16$	11 13	$9^2 =$	$10^2 =$	
$4^2 =$	$5^2 =$		$10^2 =$	$11^2 =$	
$5^2 =$	$6^2 =$		$11^2 =$	$12^2 =$	
$6^2 =$	$7^2 =$		$12^2 =$	$13^2 =$	
$7^2 =$	$8^2 =$		$13^2 =$	$14^2 =$	
$8^2 =$	$9^2 =$		$17^2 =$	$18^2 =$	

189. On page 56 , can you find two consecutive prime natural numbers separated by only 2 natural numbers which are not primes ?

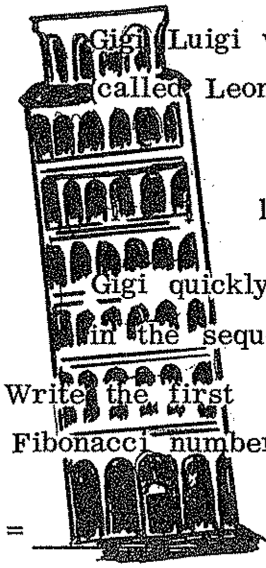


Yes ________ , namely ____________ . No ________

190. On page 56, can you find two consecutive prime natural numbers separated by only an even number of natural numbers which are not primes ?

Yes ________ , namely ____________ . No ________

4. 7 FIBONACCI NUMBERS



Gigi (Luigi was reading the LIBER ABACI written by Fibonacci called Leonardo of Pisa, c. 1200). In the book, Gigi saw the

FIBONACCI SEQUENCE

1 1 2 3 5 8 13 21 34 . . . .

Gigi quickly discovered the pattern of formation of the Fibonacci numbers in the sequence. Try it.

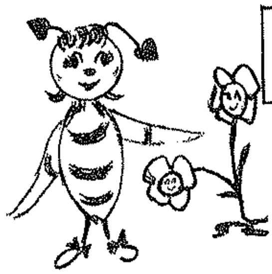
197. Write the first 30 numbers in the Fibonacci sequence. Represent consecutive Fibonacci numbers by F_1, F_2, F_3 and so on.

F_1 = ______	F_2 = ______	F_3 = ______	F_4 = ______
F_5 = ______	F_6 = ______	F_7 = ______	F_8 = ______
F_9 = ______	F_{10} = ______	F_{11} = ______	F_{12} = ______
F_{13} = ______	F_{14} = ______	F_{15} = ______	F_{16} = ______
F_{17} = ______	F_{18} = ______	F_{19} = ______	F_{20} = ______
F_{21} = ______	F_{22} = ______	F_{23} = ______	F_{24} = ______
F_{25} = ______	F_{26} = ______	F_{27} = ______	F_{28} = ______
F_{29} = ______	F_{30} = ______	F_{31} = ______	F_{32} = ______

198. Fill in the chart.

Sum of consecutive Fibonacci numbers			The sum represented as the difference of Fibonacci numbers.
1	=	1	2 - 1
1 + 1	=	2	3 - 1
1 + 1 + 2	=	4	5 - 1
1 + 1 + 2 + 3	=	7	8 - 1
1 + 1 + 2 + 3 + 5	=	______	______
1 + 1 + 2 + 3 + 5 + 8	=	______	______
1 + 1 + 2 + 3 + 5 + 8 + 13	=	______	______
1 + 1 + 2 + 3 + 5 + 8 + 13 + 21	=	______	______

4.6 COMPOSITE NUMBERS



A composite natural number is a natural number which is not 0, 1, or a prime natural number.

A composite natural number has at least one exact natural number divisor besides 1 and itself.

Label the set of exact natural number divisors of 12 D_{12}.

We have $D_{12} = \{ 1, 2, 3, 4, 6, 12 \}$.

12 has at least one exact natural number divisor besides 1 and itself.
12 is a composite number.

Factor   Each of the exact natural number divisors of a natural number is called a factor of the number.

DIVISIBILITY HINTS

193. What natural numbers have a factor 2 ? Those that end in ____________.

194. What natural numbers have a factor 5 ? Those that end in ____________.

Example Take 147. Now $1 + 4 + 7 = 12$, and $12 \div 3 = 4$, zero remainder.
Also, we have $147 \div 3 = 49$, zero remainder.

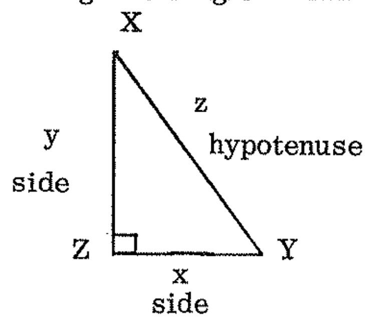
195. Fill in the chart. Use the pattern of the first 2 rows.

Number	Sum	Divisibility test by 3	Result
126	$1 + 2 + 6 = 9$	$9 \div 3 = 3$	$126 \div 3 = 42$
223	$2 + 2 + 3 = 7$	$7 \div 3 = 2\frac{1}{3}$	$223 \div 3 = 74\frac{1}{3}$
348			
501			
733			
834			
973			
1101			

196. Sum the numbers represented by the digits of a number. If the sum is exactly divisible by 3, then the number itself is exactly divisible by ____________.

4. 8 PYTHAGOREAN TRIPLES

Given : the right triangle XYZ



Pythagorean Theorem

For a given unit of length , the sum of the squares of the measures of the sides of a right triangle is equal to the square of the measure of the hypotenuse , that is ,

$$x^2 + y^2 = z^2$$

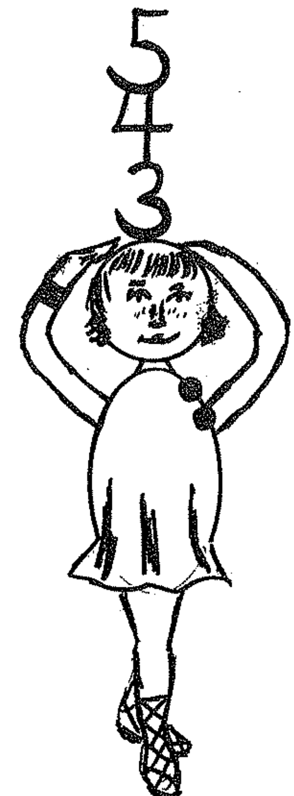
where x, y, z are the measures .

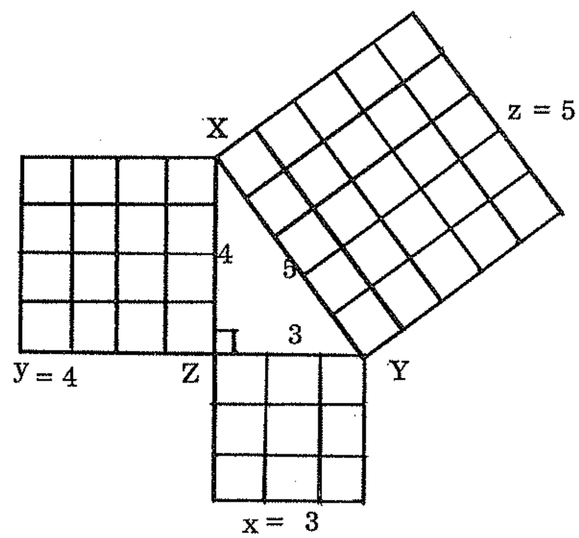
Pythagoras of Samos
530 B.C.

Are there any triples of <u>natural numbers</u> which satisfy the Pythagorean relation

$$x^2 + y^2 = z^2$$

where x and y are not zero ?





In the triangle above $x = 3$ $y = 4$ $z = 5$.

Substitute the values in the relation $x^2 + y^2 = z^2$.

We have $3^2 + 4^2 = 5^2$

or $9 + 16 = 25$.

The latter is a valid relation of equality .

The triple 3, 4, 5 of natural numbers satisfies the Pythagorean relation and is a Pythagorean triple .

The Pythagorean relation is illustrated geometrically in the triangle above. Count the squares on the sides . Count the squares on the hypotenuse .

199. The difference column of problem 198 and the table of Fibonacci numbers will give you an easy method for finding the sum of consecutive Fibonacci numbers. Find the sum of the following consecutive Fibonacci numbers.

Sum of consecutive Fibonacci numbers		The sum as the difference of Fibonacci numbers
$1 + 1 + 2 + 3 + 5 + \ldots + 144$	= ______	______
$1 + 1 + 2 + 3 + 5 + \ldots + 610$	= ______	______
$1 + 1 + 2 + 3 + 5 + \ldots + 2584$	= ______	______
$1 + 1 + 2 + 3 + 5 + \ldots + 6765$	= ______	______
$1 + 1 + 2 + 3 + 5 + \ldots + F_n$	= ______________	

200. Fill in the chart.

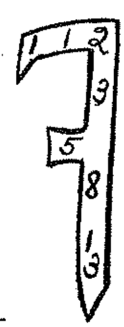
Sum of squares of consecutive Fibonacci numbers		The sum represented as the product of Fibonacci numbers
1^2	$= 1$	1×1
$1^2 + 1^2$	$= 2$	1×2
$1^2 + 1^2 + 2^2$	$= 6$	2×3
$1^2 + 1^2 + 2^2 + 3^2$	= ______	______
$1^2 + 1^2 + 2^2 + 3^2 + 5^2$	= ______	______
$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2$	= ______	______
$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + \ldots + 34^2$	= ______	______
$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + \ldots + 89^2$	= ______	______
$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + \ldots + F_n^2$	= ______________	

201. Find the following quotients involving consecutive Fibonacci numbers. Carry out the division, where necessary, to 5 decimal places.

$\frac{F_2}{F_1} = \frac{1}{1} = 1$ $\qquad \frac{F_5}{F_4} = \frac{5}{3} =$ ______ $\qquad \frac{F_{18}}{F_{17}} = \frac{2584}{1597} =$ ______

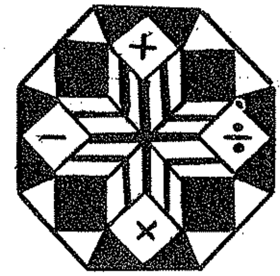
$\frac{F_3}{F_2} = \frac{2}{1} = 2$ $\qquad \frac{F_6}{F_5} = \frac{8}{5} =$ ______ $\qquad \frac{F_{20}}{F_{19}} = \frac{6765}{4181} =$ ______

$\frac{F_4}{F_3} = \frac{3}{2} =$ ______ $\qquad \frac{F_7}{F_6} = \frac{13}{8} =$ ______ $\qquad \frac{F_{25}}{F_{24}} = \frac{75025}{46368} =$ ______



CHAPTER 5

INSPIRED NUMBER PATTERNS



5.1 SETS OF EXACT DIVISORS

Consider $4 \div 2$ and $x \div y$, where $y \neq 0$.

Now

$4 \div 2 = 2$	$x \div y = n$
All the numbers here are natural numbers. Thus, 2 is an exact divisor of 4.	If n, x, y are natural numbers with $y \neq 0$, then y is an exact natural number divisor of x.

The set of all exact natural number divisors of 4 is

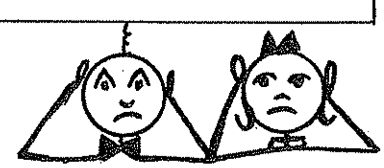
$$D_4 = \{ 1, 2, 4 \} .$$

205. Fill in the chart. Use the pattern of the first row.

$D_3 = \{ 1, 3 \}$	$D_{90} = \{ \quad \}$
$D_{10} = \{ \quad \}$	$D_{98} = \{ \quad \}$
$D_{14} = \{ \quad \}$	$D_{100} = \{ \quad \}$
$D_{16} = \{ \quad \}$	$D_{110} = \{ \quad \}$
$D_{22} = \{ \quad \}$	$D_{125} = \{ \quad \}$
$D_{26} = \{ \quad \}$	$D_{212} = \{ \quad \}$
$D_{29} = \{ \quad \}$	$D_{348} = \{ \quad \}$
$D_{30} = \{ \quad \}$	$D_{400} = \{ \quad \}$
$D_{46} = \{ \quad \}$	$D_{450} = \{ \quad \}$

206. TOP - POPPER

Find $D_0 = \{ \quad \}$



202 . Fill in the chart . Use the pattern of the worked out example .

x	y	z	$x^2 + y^2$	z^2	Pythagorean triple Yes	Pythagorean triple No
1	2	3	$1^2 + 2^2 = 1 + 4 = 5$	9		✓
2	3	4				
12	16	20				
20	99	101				
32	41	58				
33	56	65				
36	77	85				
40	45	55				
48	55	73				
66	112	130				
120	135	165				
131	223	259				

203 . The triple 3 , 4, 5 is a Pythagorean triple .

Is 6, 8, 10 a Pythagorean triple ? Yes ________ No ________

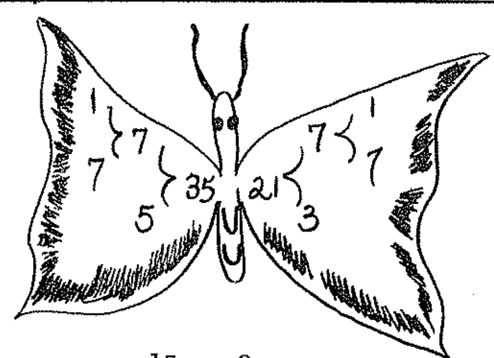
Is 9, 12, 15 a Pythagorean triple ? Yes ________ No ________

Is 30, 40, 50 a Pythagorean triple ? Yes ________ No ________

204 . If x, y, z is a Pythagorean triple, then n x, n y , n z are ______ are not ______ Pythagorean triples, where for n substitute ________________________ .

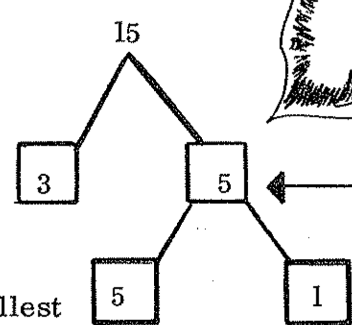
208. Fill in the chart. Use the pattern of the first row.

Set of exact divisors	Set of distinct prime factors	Prime factorization
$D_8 = \{ 1, 2, 4, 8 \}$	$\{ 2 \}$	$8 = 2^3$
$D_{36} = \{ \quad \}$	$\{ \quad \}$	$36 =$
$D_{50} = \{ \quad \}$	$\{ \quad \}$	$50 =$
$D_{54} = \{ \quad \}$	$\{ \quad \}$	$54 =$
$D_{72} = \{ \quad \}$	$\{ \quad \}$	$72 =$
$D_{90} = \{ \quad \}$	$\{ \quad \}$	$90 =$



Tree diagrams for prime factorizations

Take the composite natural number 15



Write the smallest prime that divides 15. Here it is 3.

15 ÷ 3
Divide 15 by 3 and write the quotient.

Repeat above step on 5. → Write the smallest prime that divides 5. Here it is 5.

5 ÷ 5
Divide 5 by 5 and write the quotient.

The procedure stops when a 1 appears in the right hand column.

Left column

The left column will contain the distinct prime factors of a number.

For 15, the set of distinct prime factors is $\{ 3, 5 \}$

The product of all the primes in the left column gives the prime factorization of a number.

For 15, the prime factorization is $15 = 3 \times 5$.

5.2 PRIME FACTORIZATION THEOREM

Take the composite number 6 . The set of all exact natural number divisors or factors of 6 is

$$D_6 = \{ 1, 2, 3, 6 \} .$$

The prime numbers 2 , 3 , are prime factors of 6 .

207 . Fill in the chart . Use the pattern of the first row .

Set of exact divisors	Set of prime factors
$D_{12} = \{ 1, 2, 3, 4, 6, 12 \}$	$\{ 2, 3 \}$
$D_{20} = \{ \qquad \}$	$\{ \qquad \}$
$D_{28} = \{ \qquad \}$	$\{ \qquad \}$
$D_{35} = \{ \qquad \}$	$\{ \qquad \}$
$D_{42} = \{ \qquad \}$	$\{ \qquad \}$

Factorization of composite numbers

Take the composite natural number 6 where $D_6 = \{ 1, 2, 3, 6 \}$.

Composite number		Factorization
6	=	1×6
6	=	$1 \times 1 \times 6$
6	=	2×3

and so on .

Take the composite natural number 12 where $D_{12} = \{ 1, 2, 3, 4, 6, 12 \}$.

Composite number		Factorization
12	=	1×12
12	=	3×4
12	=	$2 \times 2 \times 3 = 2^2 \times 3$

and so on .

A composite natural number is factored when it is expressed as a product of its exact divisors .

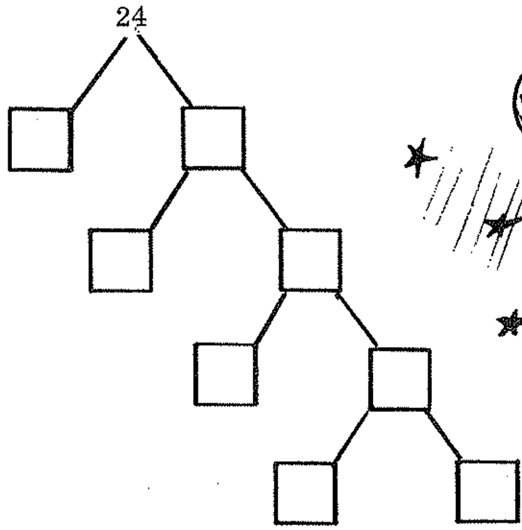
A composite natural number is factored into primes when it is expressed as a product of its distinct prime factors, each prime raised to the 1 st or higher power .

Examples

Prime factorization of 6 : $6 = 2 \times 3$.

Prime factorization of 12 : $12 = 2^2 \times 3$.

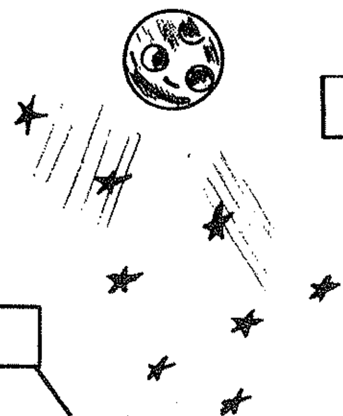
Fill in the blanks in the tree diagrams .

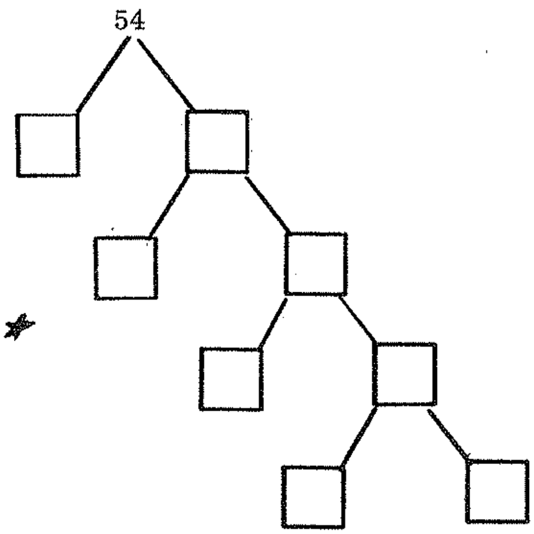


Distinct prime factors : { }

Prime factorization : 24 =

24 =

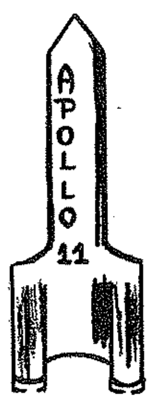


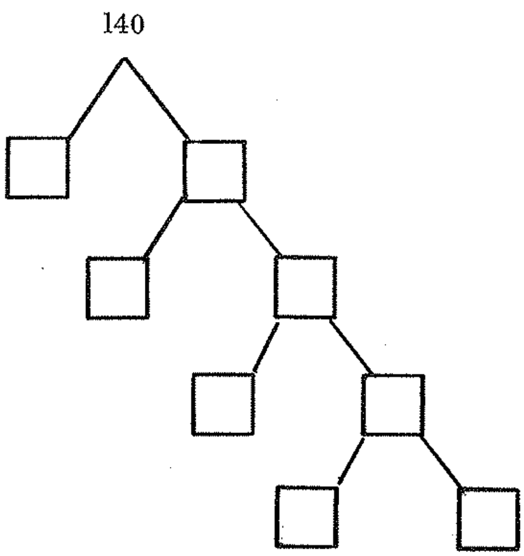


Distinct prime factors : { }

Prime factorization : 54 =

54 =

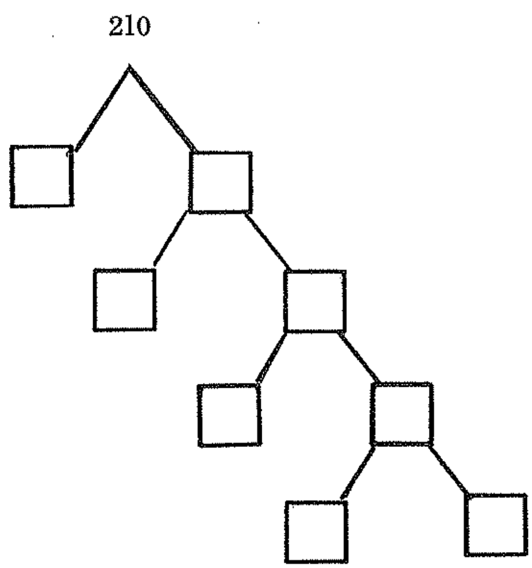




Distinct prime factors : { }

Prime factorization : 140 =

140 =

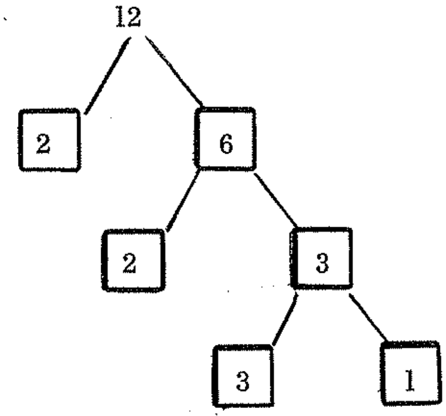


Distinct prime factors : { }

Prime factorization : 210 =

210 =

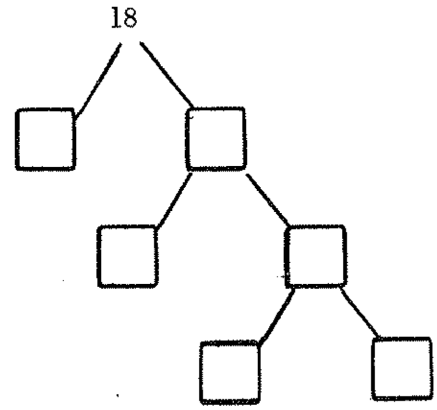
209. Fill in the blanks in the tree diagrams. Follow the example for 12.



Distinct prime factors : { 2, 3 }

Prime factorization : 12 = 2 x 2 x 3

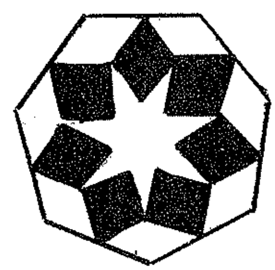
12 = 2^2 x 3

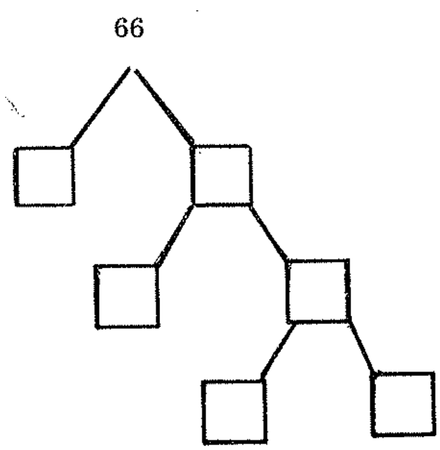


Distinct prime factors : { }

Prime factorization : 18 =

18 =

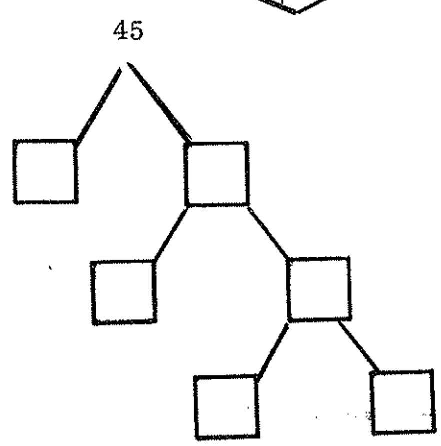




Distinct prime factors : { }

Prime factorization : 66 =

66 =



Distinct prime factors : { }

Prime factorization : 45 =

45 =

Zoe Prime's question

In how many ways can a composite natural number be factored into primes ?

210. Write other prime factorizations of the composite numbers 12 and 70.

$12 = 2 \times 2 \times 3$	$12 = 2 \times 3 \times 2$	12 = ________
$70 = 2 \times 5 \times 7$	$70 = 5 \times 2 \times 7$	70 = ________
70 = ________	70 = ________	70 = ________

Do not consider changes in the order of the primes as a new and different factorization. Now in how many ways can you write the prime factorizations of each of the numbers 12 and 70 ? ________________

211. Zoe Prime's conclusion :

> **Prime Factorization Theorem**
>
> Each composite natural number can be factored into a product of primes, each prime raised to the first or higher power in ________________ way(s) , if the order of the primes is not considered .

5.3 COMPOSITE NATURAL NUMBERS . TOTAL NUMBER OF DIVISORS

Zoe Prime used the prime factorization theorem to predict the total number of exact natural number divisors of a given composite natural number .

<u>Example</u> Take the prime factorization of 12 in exponent form :

$$12 = 2^2 \times 3^1$$

Add 1 to each exponent of the distinct primes :

$$(2 + 1) \qquad (1 + 1)$$

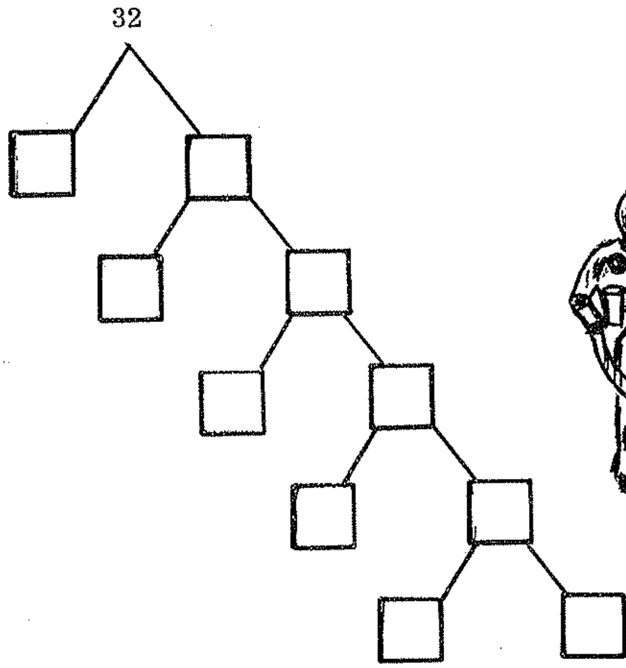
Find the product oi the increased exponents :

$$(2 + 1) \times (1 + 1) = 3 \times 2 = 6$$

The composite natural number 12 has 6 exact natural number divisors .

$$D_{12} = \{ 1, 2, 3, 4, 6, 12 \}$$

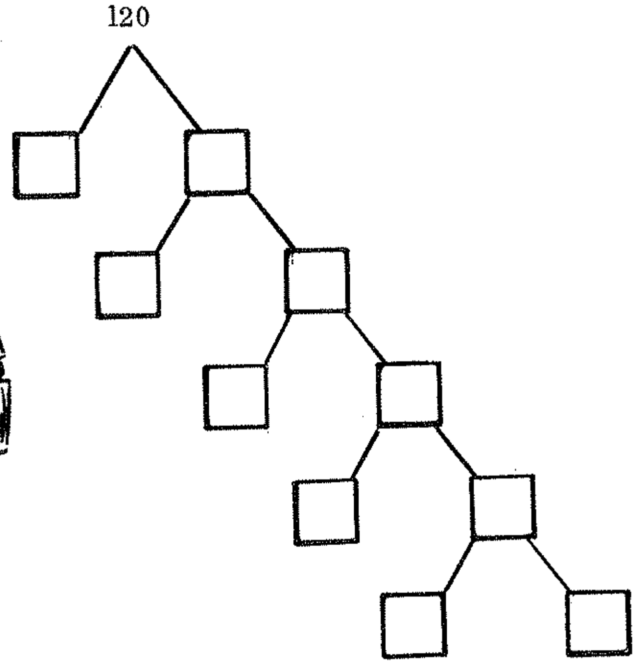
Fill in the blanks in the tree diagrams.



Distinct prime factors : { }

Prime factorization : 32 =

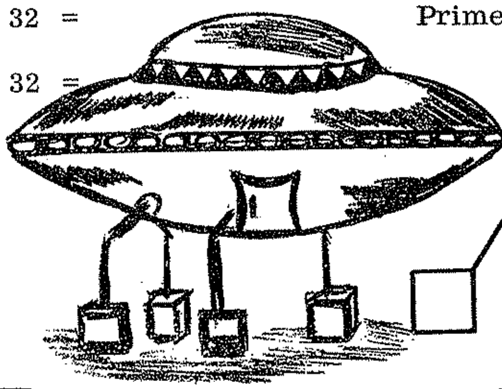
32 =

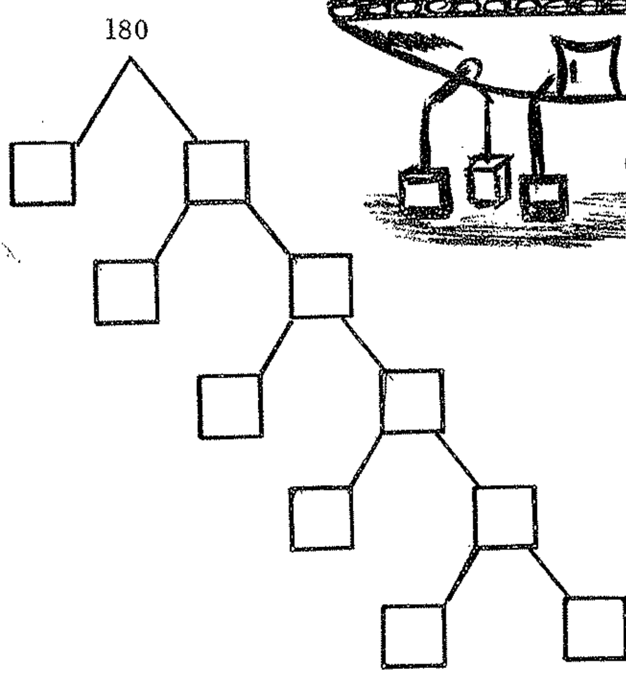


Distinct prime factors : { }

Prime factorization: 120 =

120 =

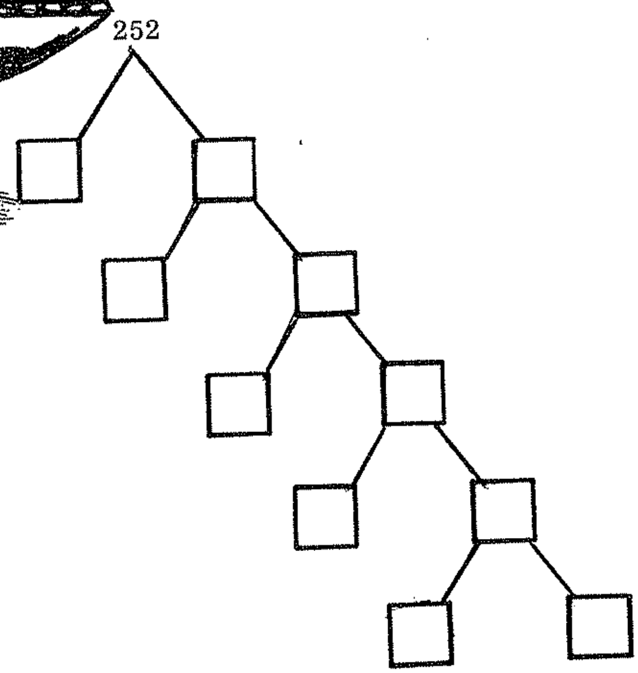




Distinct prime factors : { }

Prime factorization : 180 =

180 =



Distinct prime factors : { }

Prime factorization : 252 =

252 =

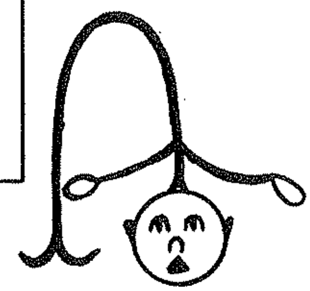
5.4 COMMON DIVISORS . GREATEST COMMON DIVISOR

Take the sets $A = \{ 1, 2, 3, 4, 5, 6 \}$

$B = \{ 0, 2, 4, 6, 8, 10 \}$.

Intersection of two sets

The intersection of two sets, written as $A \cap B$, is the set that contains the elements that are common to both the sets A and B .



For the sets A and B above, we have

$$A \cap B = \{ 2, 4, 6 \}$$

since the elements 2, 4, 6 are common to both sets .

214 . Fill in the chart . Use the pattern in the first row .

Sets	$A \cap B$
$A = \{ 1, 3, 5, 7 \}$ $B = \{ 2, 3, 4, 5 \}$	$\{ 3, 5 \}$
$A = \{ 2, 3, 5, 7, 11, 13, 17 \}$ $B = \{ 1, 2, 3, 5, 8, 13, 21 \}$	
$A = \{ 5, 10, 15, 20, 25 \}$ $B = \{ 10, 20, 30, 40, 50, 60 \}$	
$A = \{ 0, 2, 4, 6, 8, 10, 12 \}$ $B = \{ 3, 6, 9, 12, 15, 18 \}$	
$A = \{ 1, 2, 5, 10 \}$ $B = \{ 1, 2, 4, 5, 10, 20 \}$	
$A = \{ 1, 17 \}$ $B = \{ 1, 19 \}$	
$A = \{ 2, 4, 6, 8, 10 \}$ $B = \{ 1, 3, 5, 7 \}$	

212. Fill in the chart. Use the pattern of the first row.

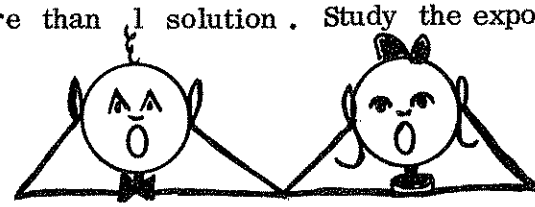
Composite number	Prime factorization	Product of increased exponents	Set of exact divisors
24	$24 = 2^3 \times 3^1$	$4 \times 2 = 8$	$D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$
33			
48			
60			
75			
88			
128			
196			
210			
273			
396			

Find a natural number with <u>exactly</u> 3 exact natural number divisors.

<u>Solution</u>: $2^2 = 4$ and $D_4 = \{1, 2, 4\}$

<u>Hint:</u> There is more than 1 solution. Study the exponent.

213. <u>TOP - POPPER</u>



Find a natural number with <u>exactly</u>

4 exact natural number divisors ________________

5 exact natural number divisors ________________

6 exact natural number divisors ________________

7 exact natural number divisors ________________

5.5 MULTIPLES. LEAST COMMON MULTIPLE

The numbers : 3×0, 3×1, 3×2, 3×3, . . . , $3 \times n$, . . .

where $n = 0, 1, 2, 3, \ldots$ are called multiples of 3.

The set of all multiples of 3 is represented as

$$M_3 = \{ 0, 3, 6, 9, 12, 15, \ldots, 3n, \ldots \}.$$

In the set of multiples of 3, <u>3 n</u> is called the <u>general term</u>.

216. Write the first 8 multiples and the general term. Follow the example above.

$M_5 = \{$	$\}$
$M_7 = \{$	$\}$
$M_{18} = \{$	$\}$

Consider $M_2 = \{ 0, 2, 4, 6, 8, \ldots, 2n, \ldots \}$

$M_4 = \{ 0, 4, 8, 12, 16, \ldots, 4n, \ldots \}$

Common multiples Least common multiple

The set of common multiples of 2 and 4 is the intersection

$$M_2 \cap M_4 = \{ 0, 4, 8, 12, 16, \ldots, 4n, \ldots \}.$$

The smallest <u>nonzero</u> number in the set of common multiples is called the least common multiple (LCM or lcm). Thus, lcm (2, 4) = 4 .

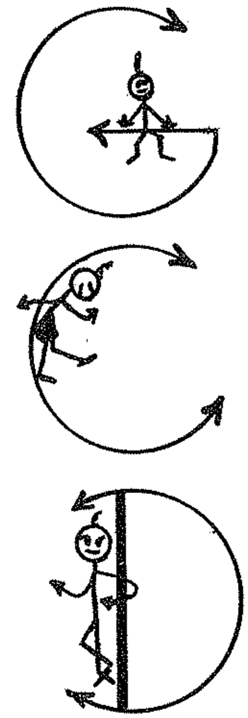
217. Fill in the chart. Use the pattern of the first row .

Multiples	Common multiples	lcm
$M_3 = \{ 0, 3, 6, 9, \ldots, 3n, \ldots \}$ $M_4 = \{ 0, 4, 8, 12, \ldots, 4n, \ldots \}$	$\{ 0, 12, 24, \ldots, 12n, \ldots \}$	12
$M_3 = \{ \quad \}$ $M_7 = \{ \quad \}$	$\{ \quad \}$	
$M_3 = \{ \quad \}$ $M_5 = \{ \quad \}$	$\{ \quad \}$	
$M_2 = \{ \quad \}$ $M_6 = \{ \quad \}$	$\{ \quad \}$	

Consider the numbers 6 and 12 where

$$D_6 = \{ 1, 2, 3, 6 \}$$

$$D_{12} = \{ 1, 2, 3, 4, 6, 12 \}$$



Common divisors   Greatest common divisor

The set of common divisors of 6 and 12 is the intersection of the sets D_6 and D_{12}, namely,

$$D_6 \cap D_{12} = \{ 1, 2, 3, 6 \} .$$

The largest number in $D_6 \cap D_{12}$ is called the greatest common divisor (GCD or gcd) of 6 and 12 . Thus , gcd (6, 12) = 6 .

215 . Fill in the chart . Use the pattern of the first row .

Divisors	Common divisors	gcd
$D_7 = \{ 1, 7 \}$ $D_{14} = \{ 1, 2, 7, 14 \}$	{ 1, 7 }	7
$D_{12} = \{ \quad \}$ $D_{18} = \{ \quad \}$	{ }	
$D_{27} = \{ \quad \}$ $D_{36} = \{ \quad \}$	{ }	
$D_{26} = \{ \quad \}$ $D_{39} = \{ \quad \}$	{ }	
$D_{45} = \{ \quad \}$ $D_{60} = \{ \quad \}$	{ }	
$D_{50} = \{ \quad \}$ $D_{100} = \{ \quad \}$	{ }	
$D_{75} = \{ \quad \}$ $D_{150} = \{ \quad \}$	{ }	
$D_{144} = \{ \quad \}$ $D_{156} = \{ \quad \}$	{ }	

5.6 FACTOR LATTICES

The exact natural number divisors of a natural number N can be represented by geometric figures such as lines, squares, cubes and so on. The geometric figures when used for this purpose are called <u>factor lattices</u>.

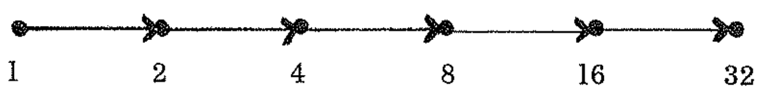
A. Natural number N with 1 prime in its prime factorization.

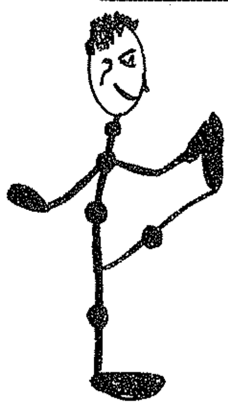
<u>Example</u> Take the number $32 = 2^5$. One prime in the factorization.

Write the divisors $D_{32} = \{ 1, 2, 4, 8, 16, 32 \}$.

The factor lattice in this case is a straight line segment consisting of 5 congruent (same length) subsegments. Notice that 5 is the exponent in the prime factorization $32 = 2^5$.

<u>Factor lattice</u>



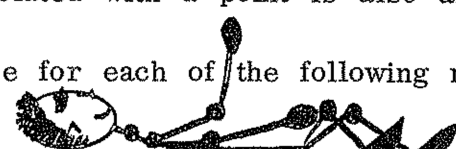


Read : 1 divides 2, 2 divides 4, 4 divides 8 and so on.

The exact divisors of 32 are associated from left to right with the endpoints on the straight line segment. Start with the smallest divisor. Sense arrows have been added to the subsegments.

Endpoints of line segments are darkened to illustrate the fact that each number associated with a point is also an exact divisor of itself.

220. Draw a factor lattice for each of the following numbers. Use the pattern in the first row.

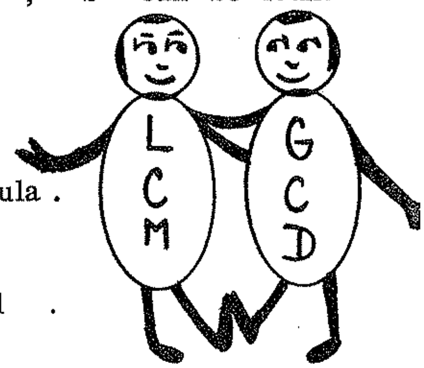


N	Prime factorization	Divisors	Factor lattice
4 =	2^2	$D_4 = \{ 1, 2, 4 \}$	
27 =		$D_{27} = \{ \quad \}$	
81 =		$D_{81} = \{ \quad \}$	
125 =		$D_{125} = \{ \quad \}$	

Formulas for gcd (a, b) and lcm (a, b)

The greatest common divisor of two natural numbers a , b can be found from the formula

$$gcd(a, b) = \frac{a \times b}{lcm(a, b)}$$



Example Find gcd (3, 4).

Solution Substitute 3 for a , 4 for b in the formula. Substitute lcm (3, 4) = 12. Thus ,

$$gcd(3, 4) = \frac{3 \times 4}{12} = \frac{12}{12} = 1 .$$

218. Fill in the blanks. Use the formula gcd (a, b). Follow the pattern in the first row.

a	b	lcm (a , b)	gcd (a, b)
10	15	lcm (10 , 15) = 30	$gcd(10, 15) = \frac{10 \times 15}{30} = \frac{150}{30} = 5$
8	12		
9	30		
10	18		

The least common multiple of two natural numbers a, b can be found from the formula

$$lcm(a, b) = \frac{a \times b}{gcd(a, b)}$$

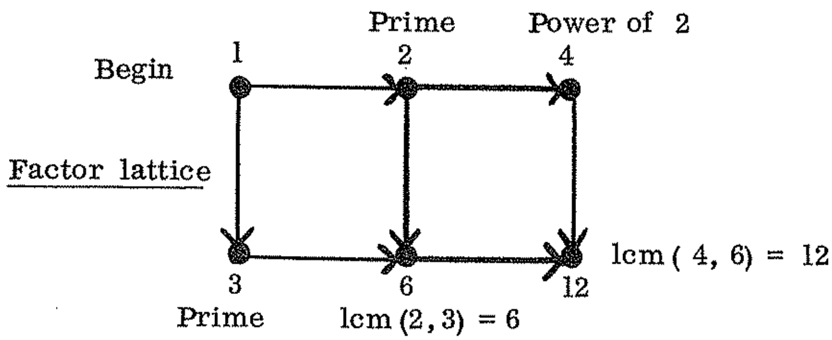
219. Fill in the blanks. Use the formula lcm (a, b). Follow the pattern in the first row

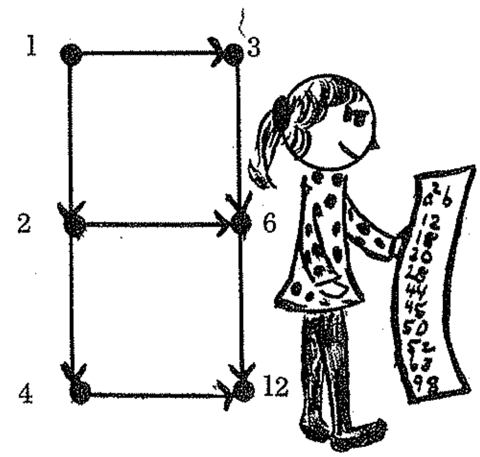
a	b	gcd (a , b)	lcm (a, b)
10	15	gcd (10 , 15) = 5	$lcm(10, 15) = \frac{10 \times 15}{5} = \frac{150}{5} = 30$
16	24		
20	28		
27	48		

<u>Example</u> Take the number $12 = 2^2 \times 3$. Two distinct primes in the factorization, one prime raised to the 2 nd power.

Write the divisors $D_{12} = \{ 1, 2, 3, 4, 6, 12 \}$.

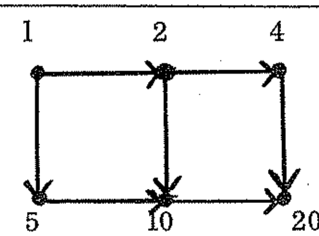
The factor lattice in this case is 2 adjoining squares drawn horizontally or vertically.





Notice that the relation " is a power of " is used to associate divisors with vertices along one side of the squares.

222 . Draw a factor lattice for each of the following numbers. Use the pattern in the first row.

N Prime factorization	Divisors	Factor lattice
$20 = 2^2 \times 5$	$D_{20} = \{ 1, 2, 4, 5, 10, 20 \}$	
136 =	$D_{136} = \{ \qquad \}$	
208 =	$D_{208} = \{ \qquad \}$	
250 =	$D_{250} = \{ \qquad \}$	

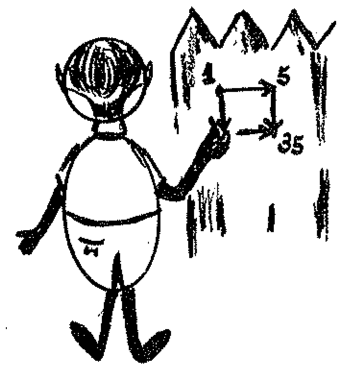
B. Natural number N with 2 distinct primes in its prime factorization.

There are several types of factor lattices here depending on the exponents of the primes.

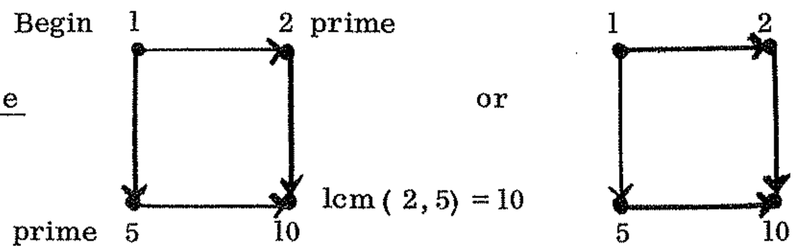
Example Take the number $10 = 2 \times 5$. Two distinct primes in the factorization, each prime raised to the 1 st power.

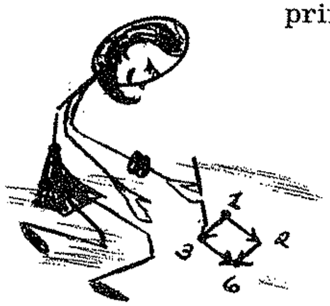
Write the divisors $D_{10} = \{ 1, 2, 5, 10 \}$.

The factor lattice in this case is a square.



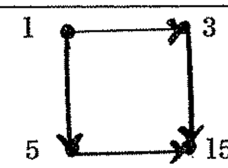
Factor lattice

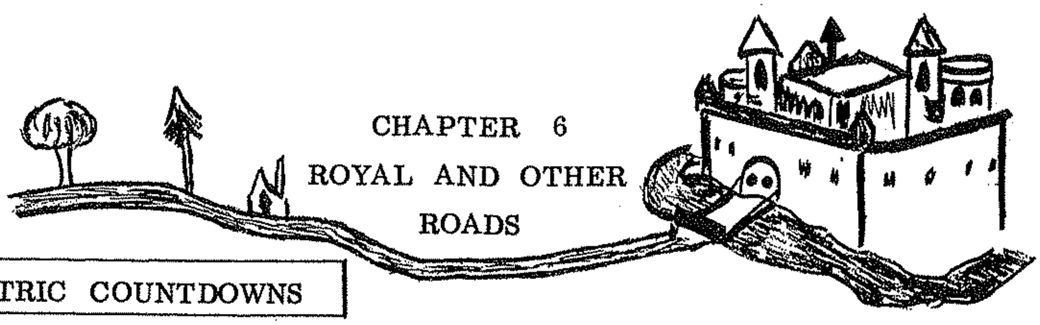




Associate 1 with the topmost left vertex of the square.
The two distinct primes, 2 and 5, are associated with the endpoints of the two line segments coming from 1.
The relation " is the least common multiple of " is used to associate the last number, namely 10, with the remaining vertex of the square.

221. Draw a factor lattice for each of the following numbers. Use the pattern in the first row.

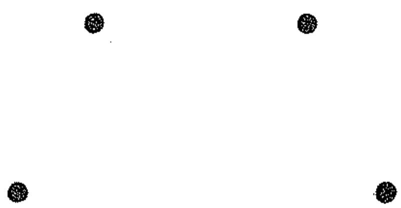
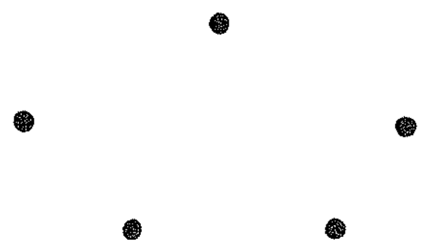
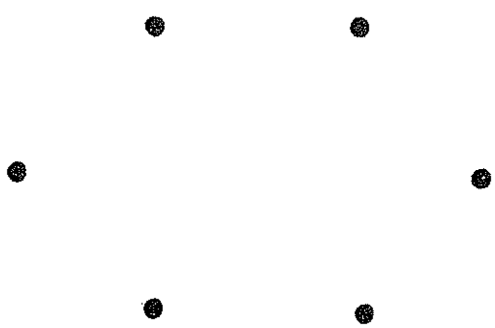
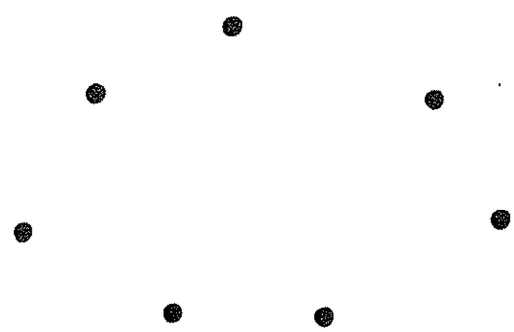
N Prime factorization	Divisors	Factor lattice
$15 = 3 \times 5$	$D_{15} = \{ 1, 3, 5, 15 \}$	
85 =	$D_{85} = \{ \quad \}$	
323 =	$D_{323} = \{ \quad \}$	
407 =	$D_{407} = \{ \quad \}$	



CHAPTER 6
ROYAL AND OTHER ROADS

6.1 GEOMETRIC COUNTDOWNS

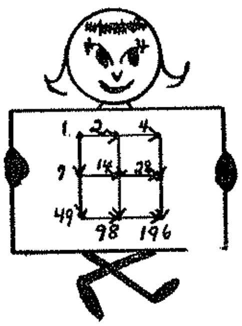
224. How many lines can be drawn through the following sets of points ?

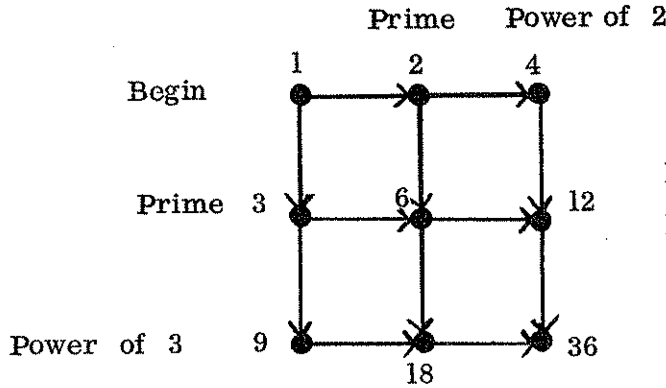
 Points 2 Lines 1	 Points 3 Lines ______
 Points 4 Lines ______	 Points 5 Lines ______
 Points 6 Lines ______	 Points 7 Lines ______

<u>Example</u> Take the number $36 = 2^2 \times 3^2$. Two distinct primes in the factorization, each prime raised to the 2 nd power.

Write the divisors $D_{36} = \{ 1, 2, 3, 4, 6, 9, 12, 18, 36 \}$

The factor lattice in this case is 4 adjoining squares which form the square shown below.





lcm (2, 3) = 6

lcm (4, 6) = 12

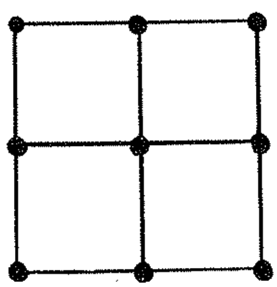
lcm (6, 9) = 18

lcm (12 , 18) = 36

223 . Draw a factor lattice for each of the following numbers .

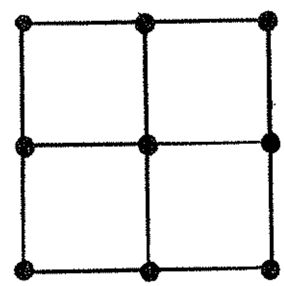
100 =

$D_{100} = \{ \quad \}$



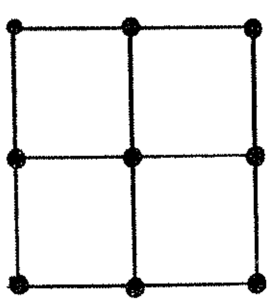
484 =

$D_{484} = \{ \quad \}$



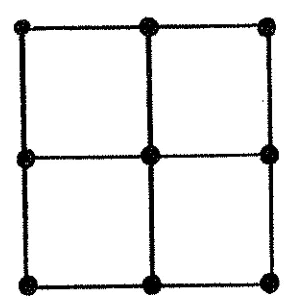
676 =

$D_{676} = \{ \quad \}$



1089 =

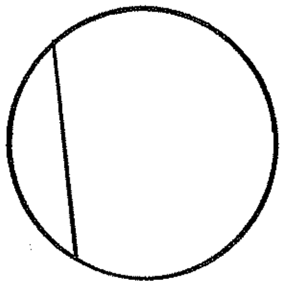
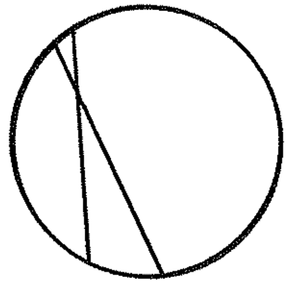
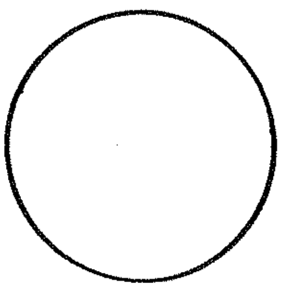
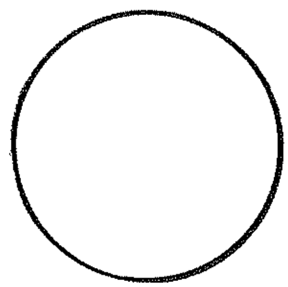
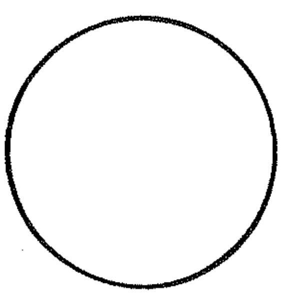
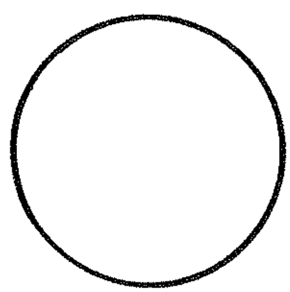
$D_{1089} = \{ \quad \}$



226 . Pie in the sky . 

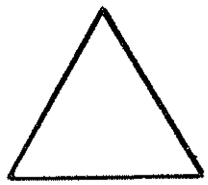
What is the largest number of pieces of pie that you can get by straight cuts across the whole pie ? The pieces do not have to be the same size.

Fill in the chart . Use the examples and hint .

 Number of cuts 1 Number of pieces 2	 Number of cuts 2 Number of pieces 4
 Number of cuts 3 Number of pieces 7	 Number of cuts 4 Number of pieces ____
 Number of cuts 5 Number of pieces ____	 Number of cuts 6 Number of pieces ____

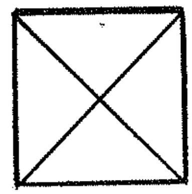
225 . Fill in the chart on the pattern of the first row .

> A diagonal of a polygon is any straight line connecting two nonconsecutive vertices .



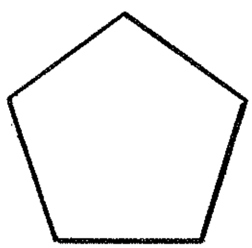
Number of sides 3

Number of diagonals 0



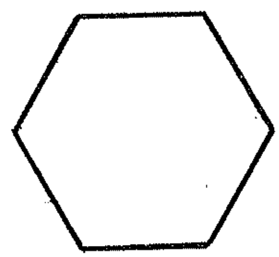
Number of sides 4

Number of diagonals 2



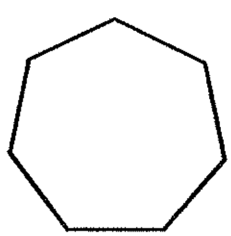
Number of sides 5

Number of diagonals ______



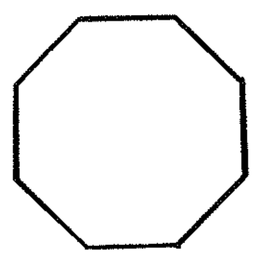
Number of sides 6

Number of diagonals ______



Number of sides 7

Number of diagonals ______

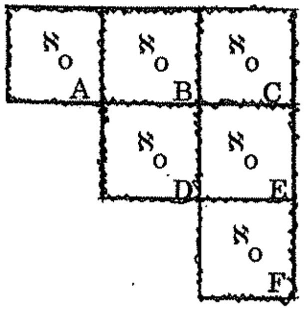


Number of sides 8

Number of diagonals ______

228 . <u>Counting skill and imagination</u> .

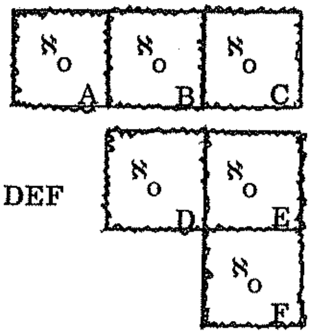
In how many ways can you select a block of 3 stamps connected on at least 1 side?

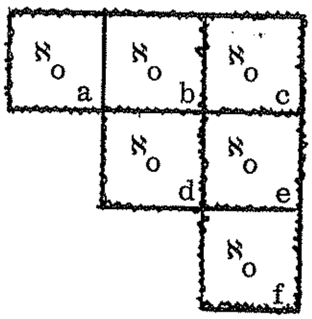
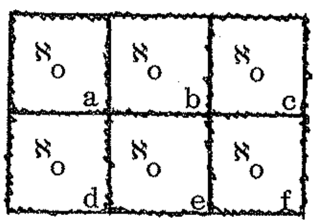
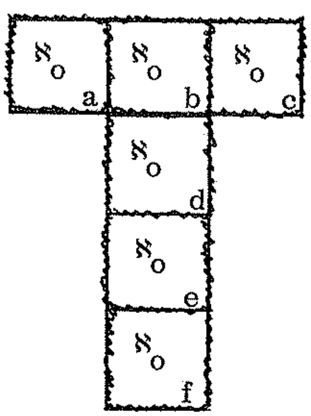


Hint: Example of two selections.

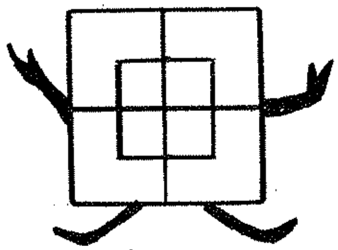
ABC

DEF

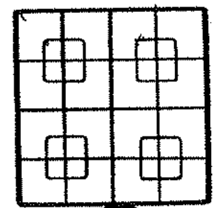


Block of 6 stamps	Number of ways of selecting a block of 3 stamps connected on at least one side.
	a b c Total
	a b c Total
	a b c Total

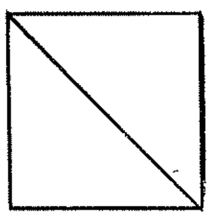
227 . <u>Finders Keepers</u> . How many squares and triangles can you find ?



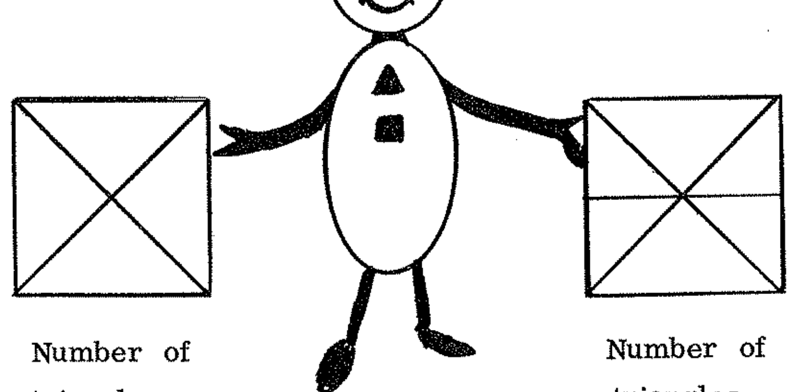
Number of squares _____



Number of squares ______

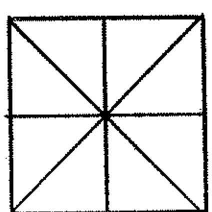


Number of triangles _____

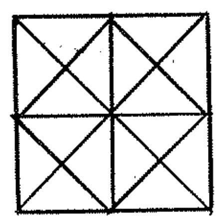


Number of triangles _____

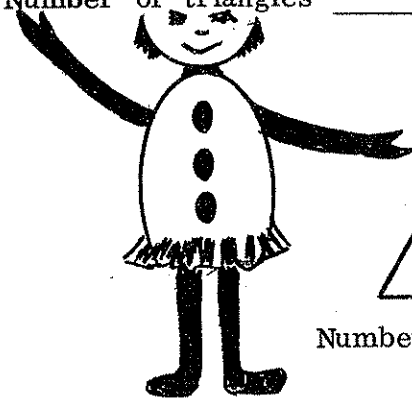
Number of triangles _____

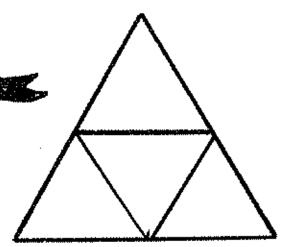


Number of triangles _____

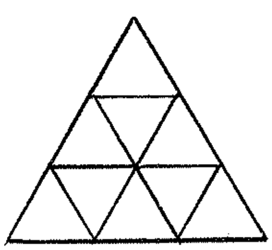


Number of triangles _____

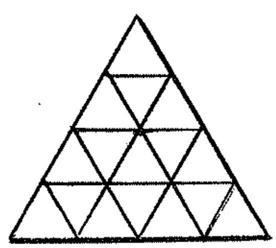




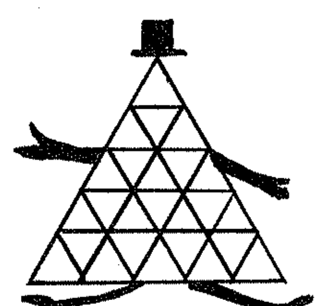
Number of triangles _____



Number of triangles _______



Number of triangles _______

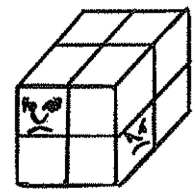


Number of triangles _______

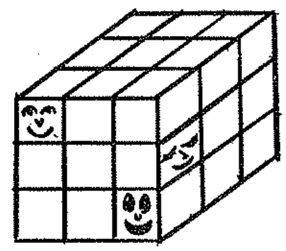
Space paint - how.


Unit cube

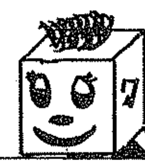
Doozie Qube , Suzie Qube's brother , had a large box of unit cubes .


$2^3 = 8$ unit cubes

Doozie made a large cube out of $2^3 = 8$ unit cubes . He painted the outside of the large cube. Now
8 unit cubes had 3 faces painted ,
no unit cube had only 2 or 1 face painted .


$3^3 = 27$ unit cubes

Next, Doozie Qube made a larger cube out of $3^3 = 27$ unit cubes . He painted the large cube. Now, how many unit cubes had 3 , 2, 1 or no faces painted ? Do the following problem .



230 . Fill in the chart on the pattern of the first row .

Number of unit cubes forming the large cube	Unit cubes with 3 faces painted	Unit cubes with 2 faces painted	Unit cubes with 1 face painted	Unit cubes with 0 faces painted
$2^3 = 8$	8	0	0	0
$3^3 = 27$				
$4^3 = 64$				
$5^3 = 125$				
$6^3 = 216$				

Space know - how. Suzie Qube drew the figures 1 and 2 on some heavy paper .

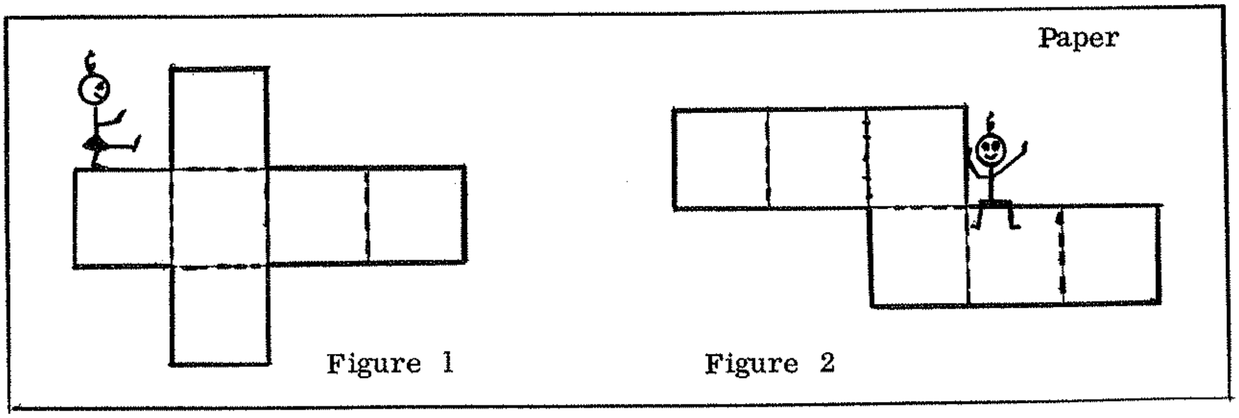


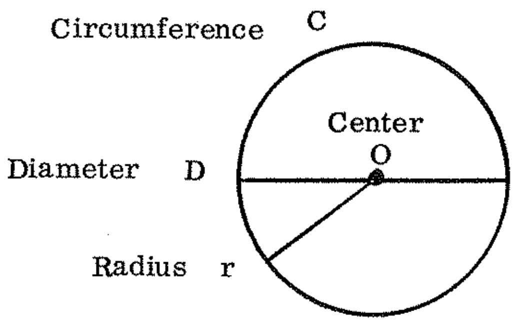
Figure 1 Figure 2

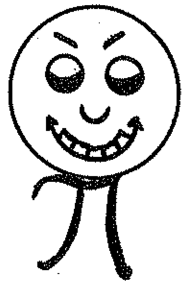
Suzie Qube cut out the figures 1 and 2 along the heavy lines and fol ed the paper along the dotted lines . The result was a cube .

229. Suzie Qube found 9 other figures , different from the two above , from which she could make a cube . Can you draw them ? Put 1 figure in each of the regions below .

6.2 GEOMETRIC KNOWNS AND UNKNOWNS

Circles within circles





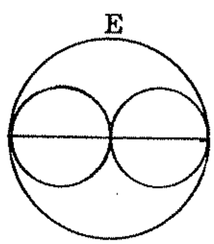
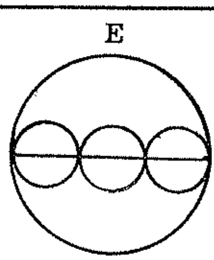
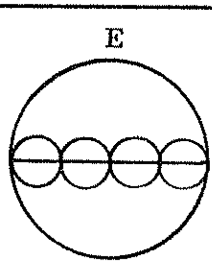
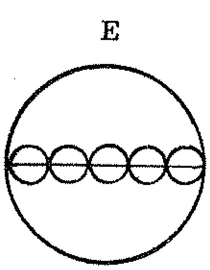
Relations

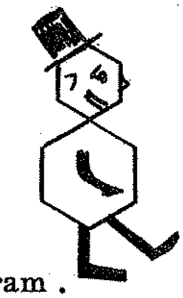
$C = 2\pi r$, where $\pi = \frac{22}{7}$

or $C = \pi D$, where $D = 2r$.

232 . Fill in the chart on the pattern of the first row .

The centers of the smaller tangent circles lie on the diameter of circle E .

Diameter of E is 6 in .	Small circles	Their diameter	Their circumference	Sum of their circumferences	Circumference of big circle E
	2	3 in	3π in	$2 \times 3\pi$ in = 6π in	6π in
	3				
	4				
	5				

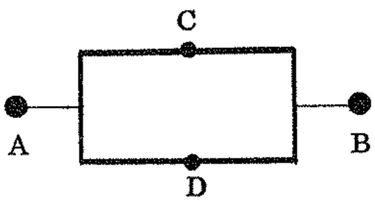
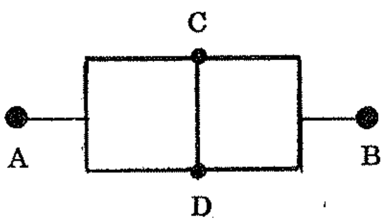
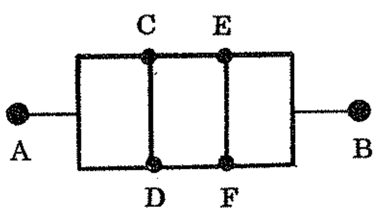
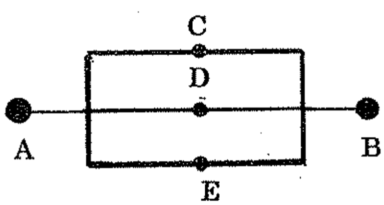
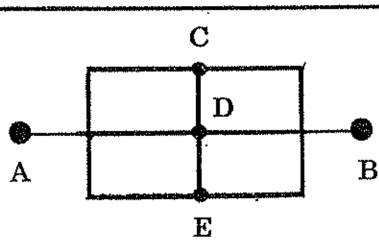
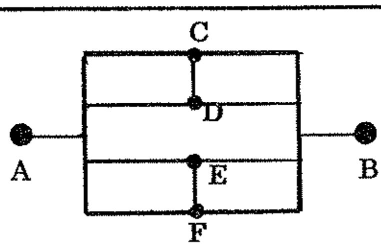


A - walking and A - counting.

The paths connecting points A and B are shown in the first column. You may walk from left to right and at a point of intersection you may go down or up.

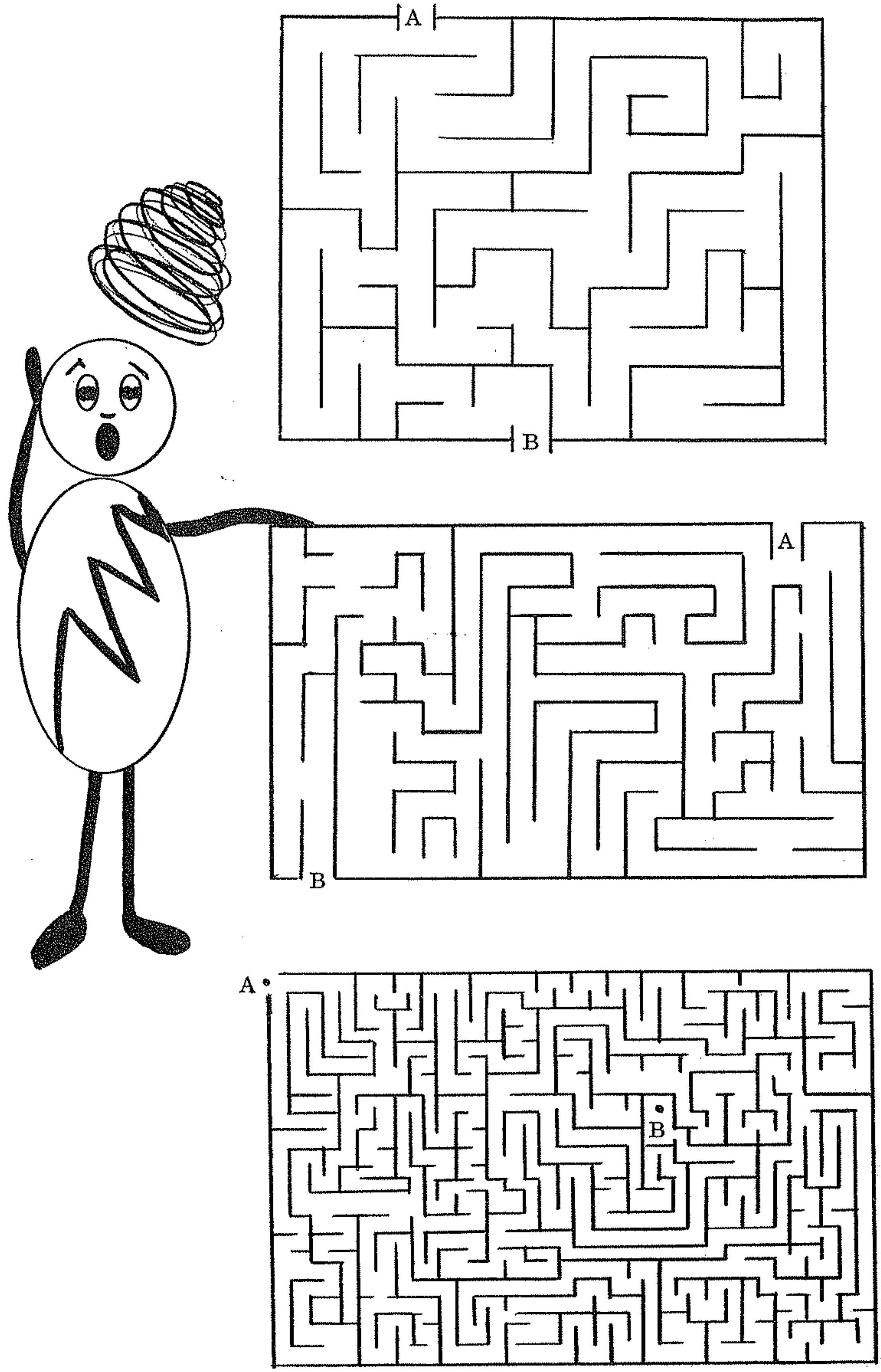
At no point of intersection may you go from right to left.

231 . Fill in the chart on the pattern of the first row. Draw the tree diagram.

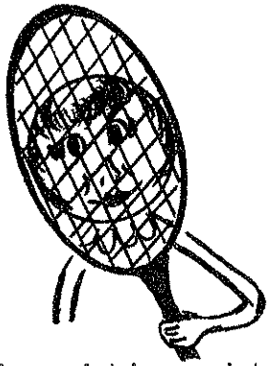
Path diagram	Number of ways in which you can go from point A to point B
	A → C → B is ACB A → D → B is ADB Total 2
	Total
	Total
	Total
	Total
	Total

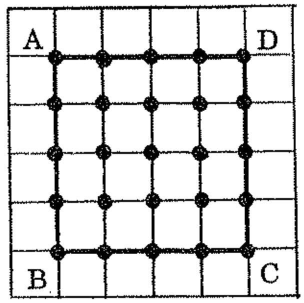
Mazes

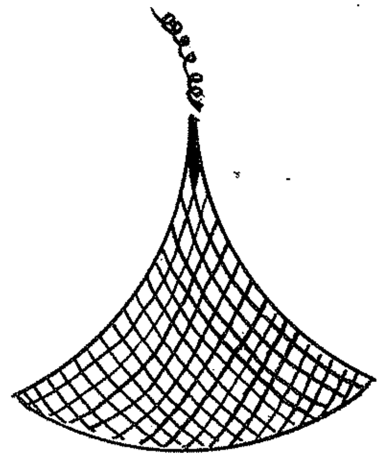
234. For each of the mazes , find a path from point A to point B .



Lattice points and areas







A lattice point is a point of intersection of a horizontal and vertical line in a grid system.

The square ABCD is drawn in the grid system in such a way that each of the vertices is a lattice point.

Let B represent the number of lattice points on the sides, here 16.

I represent the number of lattice points in the interior, here 9.

then

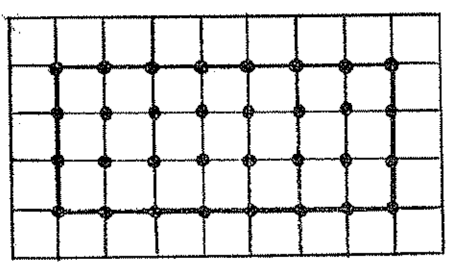
LATTICE POINT FORMULA

$$\text{Area} = I + \frac{B}{2} - 1, \text{ square units}$$

For the square above $\quad \text{Area} = 9 + \frac{16}{2} - 1 = 9 + 8 - 1 = 16$ square units

Check $\quad \text{Area} = L \times W = 4 \times 4 = 16$ square units

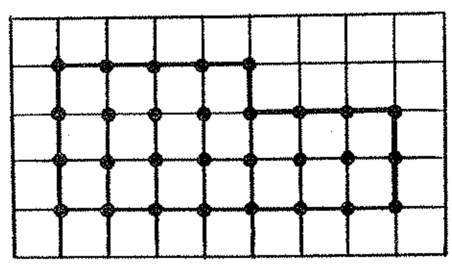
233. Fill in the blanks. Use the lattice point formula.



I =

B =

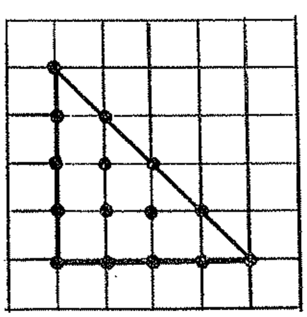
Area =



I =

B =

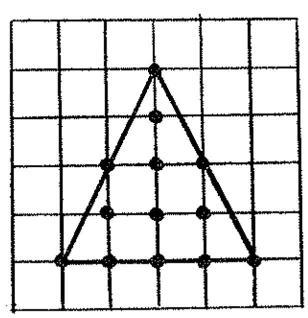
Area =



I =

B =

Area =



I =

B =

Area =

Add - a - trails

$\frac{1}{2}$	$\frac{7}{16}$	$\frac{3}{4}$
$\frac{3}{16}$	$\frac{5}{8}$	$\frac{5}{16}$
$\frac{7}{8}$	$\frac{3}{8}$	$\frac{1}{4}$
	3	

$\frac{1}{5}$	$\frac{3}{5}$	$\frac{3}{10}$
$\frac{3}{4}$	$\frac{7}{10}$	$\frac{1}{20}$
$\frac{5}{6}$	$\frac{2}{5}$	$\frac{4}{5}$
	$2\frac{1}{4}$	

$\frac{1}{7}$	$\frac{13}{14}$	$\frac{1}{2}$
$\frac{3}{5}$	$\frac{5}{6}$	$\frac{2}{7}$
$\frac{7}{8}$	$\frac{4}{7}$	$\frac{1}{14}$
	$2\frac{1}{2}$	

$\frac{1}{16}$	$\frac{1}{3}$	$\frac{1}{2}$
$\frac{1}{8}$	$\frac{15}{16}$	$\frac{7}{8}$
$\frac{5}{6}$	$\frac{1}{4}$	$\frac{3}{4}$
	$3\frac{1}{3}$	

$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{1}{2}$
$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{6}{8}$
$\frac{1}{2}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{7}{8}$	$\frac{2}{8}$
$\frac{5}{8}$	$\frac{1}{8}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{2}{4}$
$\frac{7}{8}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{5}{8}$	$\frac{1}{4}$
		6		

$\frac{1}{3}$	$\frac{19}{20}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{5}$
$\frac{7}{8}$	$\frac{2}{5}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{1}{4}$
$\frac{1}{6}$	$\frac{3}{10}$	$\frac{3}{5}$	$\frac{5}{8}$	$\frac{5}{6}$
$\frac{2}{3}$	$\frac{3}{8}$	$\frac{7}{10}$	$\frac{4}{5}$	$\frac{3}{8}$
$\frac{4}{5}$	$\frac{1}{3}$	$\frac{4}{5}$	$\frac{2}{3}$	$\frac{1}{6}$
		$4\frac{1}{10}$		

6.3 MATHEMATICAL FUN HOUSE

Add - a - trails

2	9	11
20	3	17
13	8	5
	55 End	

13 + 20 + 2 + 9 + 3 + 8 = 55

Begin at any of the 9 boxes. Draw a continuous trail to the end-of-trail box. A trail can only go vertically or horizontally. A trail cannot retrace or cross itself. The sum of all the numbers on the trail must be equal to the number in the end-of-trail box.

235.

7	8	5
4	3	6
9	2	10
	40	

12	6	8
15	21	23
10	17	30
	111	

24	15	65
55	38	29
75	85	16
	247	

3	5	7	6	8
3	6	4	5	9
1	2	4	7	6
5	7	3	5	8
4	6	8	4	2
		114		

10	11	12	13	14
20	19	18	17	16
15	21	22	24	23
25	28	30	32	34
26	29	31	33	27
		490		

4 3 2 F 2 2 I + 3 4 N 4 8 8 D		F ___ I ___ N ___ D ___
1 3 5 7 3 6 4 1 2 3 9 8 + 1 0 0 5 W H A T		T ___ A ___ H ___ W ___
9 6 4 2 2 4 O H + N O 9 1 1 9 4 6 8		H ___ O ___ N ___
2 5 I S 2 5 I T + E A S Y 1 4 7 4 6		S ___ A ___ T ___ E ___ Y ___ I ___
T 0 1 1 3 3 H 4 1 0 2 1 I 6 5 + 5 0 2 N 7 1 1 4 1 3 K		K ___ N ___ I ___ H ___ T ___
9 H O W H I G H 8 I S + 8 U P 1 4 1 0 9		W ___ O ___ H ___ G ___ S ___ I ___ P ___ U ___

Alphametics

Alphametics is the name given to mathematical puzzles in which some or all of the digits of a number are replaced by letters of the alphabet. The letters usually spell out words that have a meaning.

Addition problem

	7	D	T
	5	O	H
+	2	6	I
1	5	5	S

Solution

	7	7	3
	5	1	4
+	2	6	8
1	5	5	5

D = 7 H = 4

O = 1 I = 8

T = 3 S = 5

Identify the digits represented by the letters. Same letters must be replaced by the same digit. Different letters must be replaced by different digits. The replacement of the letters by digits must then satisfy the sum given.

Hint: In some alphametics, there are several solutions.

236.

Alphametic	Solution	Replacements
O N E 2 6 4 + 3 0 3 1 2 9 9		E __ N __ O __
4 8 7 T W O + 4 9 1 1 5 8 1		O __ W __ T __
9 5 4 C 8 2 6 A + J A N 1 8 0 0 0		C __ A __ N __ J __

CONTEMPORARY MOTIVATED MATHEMATICS

Arithmeknack

Each of the numbers 1 through 9 must be placed exactly once in the 9 blanks of each block of 3 equations so that the result is a true statement.

Problem	Solution
(___ + ___) - ___ = 1	(3 + 7) - 9 = 1
(___ - ___) x ___ = 1	(5 - 4) x 1 = 1
(___ - ___) ÷ ___ = 1	(8 - 6) ÷ 2 = 1

237.

(9 + ___) ÷ ___ = 2
(___ - 5) + ___ = 2
(___ x ___) - 4 = 2

___ - (___ - 5) = 6
___ x (___ ÷ 4) = 6
___ - (___ + 1) = 6

(___ + 7) ÷ ___ = 3
(8 - ___) - ___ = 3
(___ - 6) ÷ ___ = 3

4 + (___ ÷ ___) = 7
(___ + 6) - ___ = 7
___ + (___ x 2) = 7

(9 + ___) ÷ ___ = 4
5 - (___ - ___) = 4
(8 x ___) ÷ ___ = 4

___ + (___ ÷ 2) = 8
(___ + 3) x ___ = 8
(9 - ___) + ___ = 8

(___ + 9) - ___ = 5
(___ x 6) - ___ = 5
(___ ÷ 4) + ___ = 5

(2 x ___) - ___ = 9
(___ ÷ ___) + ___ = 9
___ + (___ x 1) = 9

(___ + ___) - ___ = 10
___ + (9 ÷ ___) = 10
___ + (___ - ___) = 10