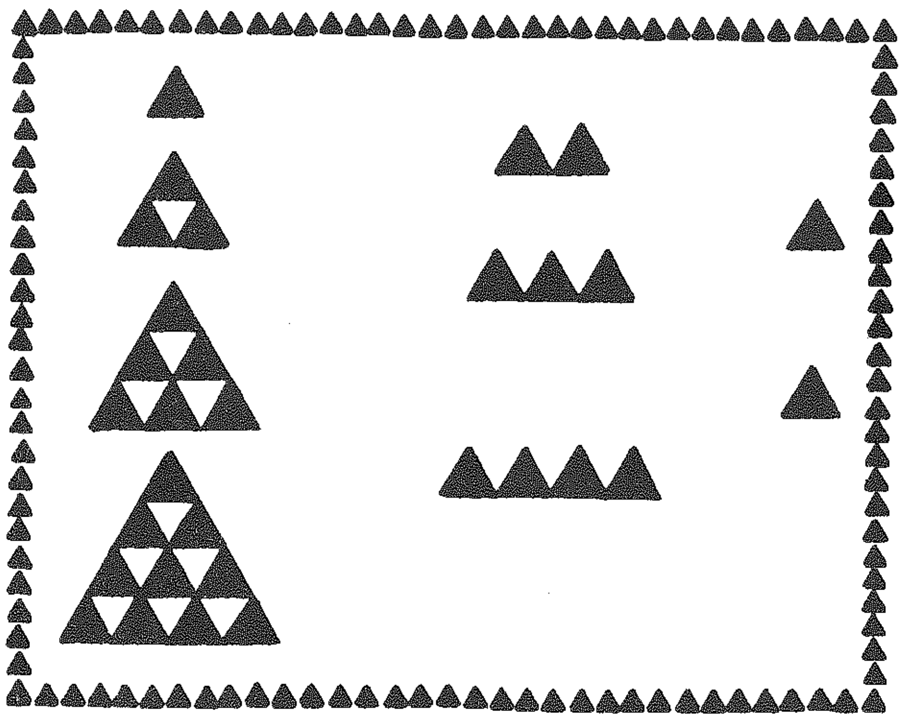
BOSTON COLLEGE MATHEMATICS INSTITUTE

Motivated Math Project Activity

Booklet 9

Applications of Finite Differences



STANLEY BEZUSZKA

In collaboration with

LOU D'ANGELO

MARGARET J. KENNEY

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PREFACE

The series of MOTIVATED MATH PROJECT ACTIVITY BOOKLETS has been written for students and teachers in elementary and secondary schools. Each Booklet treats a topic generally found in the school curriculum or material that is interesting and motivational which may or may not be included in the usual class room textbook. Some topics are treated in two Booklets: one on an elementary level suitable for the middle grades and the other on an advanced level appropriate for junior and senior high schools.

The Booklets can be used in a variety of ways.

Because each Booklet treats a single topic, it is a handy summary and resource unit which can be expanded by both the student and the teacher.

Many of the Booklets, because they summarize and organize a topic in detail, can be used as mini-course modules to supplement standard class instruction or for individualized study. They also provide an invaluable review of a topic and can serve as a criterion for what has been covered on the topic.

The Booklets, unlike a textbook in which a topic may be treated in several nonconsecutive chapters, provide a convenient and readily accessible reference source. The material on a topic can be quickly and easily found.

Each Booklet contains problems which not only reenforce the class room instruction but also provide motivation, interest and challenge. Many problems are open-ended so that all students can achieve some measure of success. These problems are suitable not only for the routine pencil and paper activity but also may be extended by the use of hand electronic calculators or programmed on a computer.

Each Booklet contains solutions to the problems and in many instances comments, explanations and derivations of the key formulas and algorithms. The Booklets are relatively independent of each other and may be studied in any sequence depending on the background and personal preference of the student.

APPLICATIONS
of
FINITE DIFFERENCES

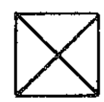
Question 1. How does one find the next few numbers that follow the same pattern as the numbers

2, 5, 8, 11, 14, ___?___, ___?___, ___?___, ___?___ ?

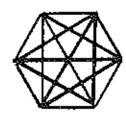
What does one mean by ' the same pattern ' ?
How does one find a formula that will quickly give the 100 th or the 1,000,000 th number in the pattern of the numbers above ?

Question 2. A triangle has 0 diagonals,

a square has 2 diagonals,



a hexagon has 9 diagonals.



How many diagonals does an octagon have ? How many diagonals does an n-gon have ?

Question 3. What is the sum of the first fifty numbers that follow the pattern of the numbers

$2 + 5 + 8 + 11 + 14 + \ldots + (3n - 1) + \ldots,$ where $n = 1, 2, 3, \ldots$.

How does one find a formula that will quickly give the sum of the first 100 or the first 1,000,000 numbers in the above pattern ?

These and other questions similar to the above are answered in this booklet which deals with some elementary aspects of the method of finite differences.

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1. Method of finite differences

The method of finite differences as used here is
a procedure for finding the general or n th
term of certain sequences of numbers.

2. a. Sequence

A sequence is an orderly array of elements in one-to-one correspondence with

1. the elements of a finite proper subset $\{1, 2, 3, \ldots, k\}$, or
2. the elements of the set $\{1, 2, 3, \ldots, n, \ldots\}$

of positive integers.

In (1) we have a finite sequence, and in (2) we have an infinite sequence.

Example 1

Positive integers	1	2	3	4	5
	↕				↕
Sequence	1	4	9	16	25

These are terms of the sequence .

b. Term of a sequence

Each element of a sequence is called a term of the sequence .

c. Sequence: function notation

A sequence is a function which has as its domain the set of positive integers and whose range is a subset of the real numbers .

Example 2

Positive integers	1	2	3	4	5
	↕				↕ (Domain)
Function notation and function values	$f(1) = 1$	$f(2) = 4$	$f(3) = 9$	$f(4) = 16$	$f(5) = 25$ (Range)

The terms of the sequence here are the function values : 1, 4, 9, 16, 25 .

3. General term of a sequence

a. The general or the n th term of a sequence is the generator of the sequence.

The n th term of a sequence can often be expressed in terms of n (where n takes on the values 1, 2, 3, . . .) or it may be expressed verbally (only) .

b. In function language, the general term is the ' rule of correspondence', the 'formula' which describes the function .

Example 3 a. If the n th term of a sequence is n^2 , then the sequence generated is shown below, where $n = 1, 2, 3, \ldots$.

Number of the term	1	2	3	4	. . .	n	. . .
	↕					↕	
Terms of the sequence	1	4	9	16	. . .	n^2	. . .

b. In function notation, the above sequence would be given by $f(n) = n^2$, where $n = 1, 2, 3, \ldots$.

Number of the term	1	2	. . .	n
	↕			↕
Terms of the sequence (function values)	$f(1) = 1$	$f(2) = 4$	. . .	$f(n) = n^2$

Example 4 The sequence of consecutive prime numbers cannot be given by a formula involving n . One can say: list the sequence of consecutive prime numbers .

Number of the term	1	2	3	4	5	. . .
	↕				↕	
Sequence of primes	2	3	5	7	11	. . .

PROBLEMS

Find the first 5 terms for each of the sequences whose n th term (the generator) is given below. Recall $n = 1, 2, 3, \ldots$.

1. $2(n - 1)$ ________________
2. $n(n + 1)$ ________________
3. $2 + (n - 1)3$ ________________
4. $\dfrac{n(3n - 1)}{2}$ ________________
5. $\dfrac{n(n + 1)(2n + 1)}{6}$ ________________
6. $\dfrac{n(n^2 - 3n + 8)}{6}$ ________________

Assume that the pattern for each sequence remains the same. Write the next 3 terms for each sequence.

Challenge ! Find the n th term (the generator) for each of the sequences .

	Terms	n th term
7. 1, 2, 3, 4, 5, 6, . . .	________	________
8. 1, 3, 5, 7, 9, 11, . . .	________	________
9. 3, 5, 7, 9, 11, 13, ...	________	________
10. 1, 3, 6, 10, 15, 21, . . .	________	________
11. 3, 6, 11, 18, 27, 38, . . .	________	________
12. 3, 10, 29, 66, 127, . . .	________	________

4. Differences: first order, D^1

Another common notation for the terms of a sequence is the following :

$$s_1,\ s_2,\ s_3,\ \ldots,\ s_n$$

represent consecutive terms of a sequence, and s_n is the n th term . We use this notation in the Table 1 which displays the 1st order differences D^1 .

Table 1

n	Sequence	1 st order differences D^1
1	s_1	
2	s_2	$s_2 - s_1 = D^1_1$
3	s_3	$s_3 - s_2 = D^1_2$
4	s_4	$s_4 - s_3 = D^1_3$
.	.	. .
.	.	. .
n-1	s_{n-1}	
n	s_n	$s_n - s_{n-1} = D^1_{n-1}$
n+1	s_{n+1}	$s_{n+1} - s_n = D^1_n$

In Table 1, page 3

a. The 1 st order differences are represented by D^1 (read: dee one) .

b. 1 st order differences are differences of successive terms of a sequence.

c. Specific 1st order differences are shown by adding subscripts to D^1 .
Thus,

1 st order difference, first difference	$s_2 - s_1 = D^1_1$
1 st order difference, second difference	$s_3 - s_2 = D^1_2$
⋮	
1 st order difference, n th difference	$s_{n+1} - s_n = D^1_n$

Example 1

n	Sequence	D^1	
1	0		
2	5	5	is D^1_1
3	10	5	is D^1_2
4	15	5	is D^1_3
5	20	5	is D^1_4
⋮	⋮	⋮	

Here the D^1_i , $i = 1, 2, 3, \ldots$, all have the same value .

5 Differences: second order, D^2

Let $s_1, s_2, s_3, \ldots,$ represent the consecutive terms of a sequence, where s_n is the n th term . The following Table 2 displays 1st and 2nd order differences .

Table 2

n	Sequence	D^1	Second order differences D^2
1	s_1		
		D^1_1	
2	s_2		$D^1_2 - D^1_1 = D^2_1$
		D^1_2	
3	s_3		$D^1_3 - D^1_2 = D^2_2$
		D^1_3	
4	s_4		
⋮		⋮	⋮
n-1	s_{n-1}		
		D^1_{n-1}	
n	s_n		$D^1_n - D^1_{n-1} = D^2_{n-1}$
		D^1_n	
n+1	s_{n+1}		

a. The 2 nd order differences are represented by D^2 (read: dee two) .

b. 2 nd order differences are differences of successive 1st order differences.

c. Specific 2 nd order differences are shown by adding subscripts to D^2. Thus,

2 nd order difference, 1 st difference $D_2^1 - D_1^1 = D_1^2$

2 nd order difference, 2 nd difference $D_3^1 - D_2^1 = D_2^2$

⋮

2 nd order difference, n th difference $D_{n+1}^1 - D_n^1 = D_n^2$.

Example 1

n	Sequence	D^1	D^2
1	2		
		3	
2	5		2 is D_1^2
		5	
3	10		2 is D_2^2
		7	
4	17		2 is D_3^2
		9	
5	26		
⋮	⋮		⋮

Here the 1 st order differences, D_i^1, i = 1, 2, 3, ... have different values for different i, but the 2 nd order differences, D_i^2, i = 1, 2, 3, ..., all have the same value.

6. Differences: third order, D^3

Let s_1, s_2, s_3, . . ., represent the consecutive terms of a sequence, where s_n is the n th term. The following Table 3 displays 1 st, 2 nd and 3 rd order differences.

Table 3

n	Sequence	D^1	D^2	Third order differences D^3
1	s_1			
		D_1^1		
2	s_2		D_1^2	
		D_2^1		
3	s_3		D_2^2	$D_2^2 - D_1^2 = D_1^3$
		D_3^1		
4	s_4		D_3^2	$D_3^2 - D_2^2 = D_2^3$
		D_4^1		
5	s_5			
⋮	⋮	⋮	⋮	⋮
n-1	s_{n-1}			
		D_{n-1}^1		
n	s_n		D_{n-1}^2	
		D_n^1		
n+ 1	s_{n+1}		D_n^2	$D_n^2 - D_{n-1}^2 = D_{n-1}^3$
		D_{n+1}^1		
n + 2	s_{n+2}			

a. The 3 rd order differences are represented by D^3 (read : dee three)

b. 3 rd order differences are differences of successive 2 nd order differences.

c. Specific 3 rd order differences are shown by adding subscripts to D^3.

Thus,

3 rd order difference, 1 st difference $D_2^2 - D_1^2 = D_1^3$,

3 rd order difference, 2 nd difference $D_3^2 - D_2^2 = D_2^3$,

⋮

3 rd order difference, n th difference $D_{n+1}^2 - D_n^2 = D_n^3$.

Example 1

n	Sequence	D^1	D^2	D^3
1	1			
		7		
2	8		12	
		19		6 is D_1^3
3	27		18	
		37		6 is D_2^3
4	64		24	
		61		
5	125			
⋮	⋮	⋮	⋮	⋮

Here, the 1 st order differences D_i^1 and 2 nd order differences D_i^2, $i = 1, 2, 3, \ldots$, have different values for different i, but the 3 rd order differences, D_i^3, $i = 1, 2, 3, \ldots$, all have the same value.

7. Differences: higher order, $D^4, D^5, \ldots$,

The higher order differences, namely, 4 th, 5 th, and so on, are defined on the pattern of the 1 st, 2 nd, 3 rd order differences.

a. 4 th order difference, 1 st difference $D_2^3 - D_1^3 = D_1^4$,

4 th order difference, n th difference $D_{n+1}^4 - D_n^4 = D_n^4$.

b. 5 th order difference, 1 st difference $D_2^4 - D_1^4 = D_1^5$,

5 th order difference, n th difference $D_{n+1}^4 - D_n^4 = D_n^5$,

and so on.

Example 2

n	$s_n = n$	D^1
1	1	
		1
2	2	
		1
3	3	
		1
4	4	
		1
5	5	
		1
6	6	
⋮		⋮

Here the D_i^1, $i = 1, 2, 3, \ldots$ all have the same value. All higher order differences are 0.

Example 3

n	$s_n = \frac{n(n+1)}{2}$	D^1	D^2
1	1		
		2	
2	3		1
		3	
3	6		1
		4	
4	10		1
		5	
5	15		1
		6	
6	21		
⋮			

Here only the D_i^2, $i = 1, 2, 3, \ldots$ all have the same value. All higher order differences are 0.

Example 1

n	$s_n = n^3 + 2$	D^1	D^2	D^3
1	3			
		7		
2	10		12	
		19		6
3	29		18	
		37		6
4	66		24	
		61		6
5	127		30	
		91		
6	218			
⋮		⋮	⋮	⋮

Here only the D_i^3 , i = 1, 2, 3, . . . , all have the same value .

PROBLEMS

Determine what differences D^1, D^2, D^3, have one and the same nonzero value for the following sequences (see page 3, problems 1-6) .

13.

n	$s_n = 2(n-1)$	D^1	D^2	D^3
1	0			
2	2			
3	4			
4	6			
5	8			
⋮	⋮			

14.

n	$s_n = 2 + (n-1)3$	D^1	D^2	D^3
1	2			
2	5			
3	8			
4	11			
5	14			
⋮	⋮			

15.

n	$s_n = n(n+1)$	D^1	D^2	D^3
1	2			
2	6			
3	12			
4	20			
5	30			
⋮	⋮			

16.

n	$s_n = \frac{n(3n-1)}{2}$	D^1	D^2	D^3
1	1			
2	5			
3	12			
4	22			
5	35			
⋮	⋮			

17.

n	$s_n = \frac{n(n+1)(2n+1)}{6}$	D^1	D^2	D^3
1	1			
2	5			
3	14			
4	30			
5	55			
⋮	⋮			

18.

n	$s_n = \frac{n(n^2 - 3n + 8)}{6}$	D^1	D^2	D^3
1	1			
2	2			
3	4			
4	8			
5	15			
⋮	⋮			

8. s_n for sequences where D_i^1 has same value for all $i = 1, 2, 3, ..$

How does one find the formula for the general term of a sequence where all the 1 st order differences have the same value ? Study the following examples.

<u>Example 1</u>

n	$s_n = -3n$	D^1
1	-3	
		-3
2	-6	
		-3
3	-9	
		-3
4	-12	
		-3
5	-15	
⋮	⋮	
n	-3n	

Here, D_i^1 has the same nonzero value for all i = 1, 2, 3, ..., and this constant is the coefficient of n in the general term for the sequence .

<u>Example 2</u>

n	$s_n = 5n$	D^1
1	5	
		5
2	10	
		5
3	15	
		5
4	20	
		5
5	25	
⋮		
n	5n	

Here, D_i^1 has the same nonzero value for all i = 1, 2, 3, ..., and this constant is the coefficient of n in the general term for the sequence .

<u>Example 3</u>

n	$s_n = 2n - 1$	D^1
1	1	
		2
2	3	
		2
3	5	
		2
4	7	
		2
5	9	
⋮	⋮	
n	2n - 1	

Here, D_i^1 has the same nonzero value for all i = 1, 2, 3, ..., and this constant is the coefficient of n in the general term for the sequence .

<u>Example 4</u>

n	$s_n = 3n + 2$	D^1
1	5	
		3
2	8	
		3
3	11	
		3
4	14	
		3
5	17	
⋮	⋮	
n	3n + 2	

Here, D_i^1 has the same nonzero value for all i = 1, 2, 3, ..., and this constant is the coefficient of n in the general term for the sequence .

PROBLEMS

For each sequence, check whether the D_i^1 , i = 1, 2, 3, ..., are all the same nonzero number and if this constant is the coefficient of n in s_n .

19.

n	$s_n = 2n$	D^1
1	2	
		____
2	4	
		____
3	6	
		____
4	8	
⋮	⋮	
n		

Yes _____ No _____

20.

n	$s_n = 4n - 5$	D^1
1		
2		
3		
4		
⋮		
n		

Yes _____ No _____

21.

n	$s_n = 1 - 2n$	D^1
1		
2		
3		
4		
⋮		
n	$1 - 2n$	

Yes _____ No _____

22.

n	$s_n = \frac{3}{4} n + 1$	D^1
1		
2		
3		
4		
⋮		
n	$\frac{3}{4} n + 1$	

Yes _____ No _____

In the examples of Section 8 , page 8, and in the problems above

1. the general term s_n of each sequence was of the form $s_n = a n + b$ where a, b are real numbers,
2. the D_i^1 , $i = 1, 2, 3, \dots ,$ all had the same nonzero value, and
3. this common value was the coefficient of n in $s_n = an + b$.

The above statements can be verified by taking the sequence where the general term is $s_n = an + b$ and using the method of finite differences .

<u>Example 1</u>

n	$s_n = a n + b$	D^1
1	$a + b$	
		a
2	$2a + b$	
		a
3	$3a + b$	
		a
4	$4a + b$	
		a
5	$5a + b$	
⋮	⋮	
n	$a n + b$	

Here, D_i^1 has the same nonzero value a for all $i = 1, 2, 3, \dots ,$ and this constant a is the coefficient of n in the general term for the sequence.

Thus, we now have that

<u>SUMMARY</u>

1. a sequence where the general term is of the form $s_n = an + b$ will have the 1 st order differences D_i^1 , $i = 1, 2, 3, \dots ,$ all the same, and
2. if the 1st order differences D_i^1 , $i = 1, 2, 3, \dots ,$ all have the same value, then the general term of the sequence is of the form $s_n = an + b$,
3. the coefficient of n in the general term $s_n = an + b$ for the sequence is the common value of D_i^1 , $i = 1, 2, 3, \dots$, namely, a .

Consider the example on the following page, where we want to find the general term s_n for the given sequence .

Example 1

n	Sequence	D^1
1	1	
		2
2	3	
		2
3	5	
		2
4	7	
		2
5	9	
⋮	⋮	

1. Here the 1st order differences , D_i^1 , $i = 1, 2, 3, \ldots$, are all the same (if we assume that the pattern continues in the sequence column) , and so the general term is of the form $s_n = an + b$ (see (2) in Summary on page 9).
2. The common value of the D_i^1, $i = 1, 2, 3, \ldots,$ is 2 and this is the coefficient of n in $s_n = an + b$ so that we have $s_n = 2n + b$ (see (3) in the summary on page 9) .
3. How does one find b ? Compare Example 1, page 9 with the Example 1, page 10 . We have

n	Sequence in Example 1, page 10		Sequence in Example 1, page 9
1	1	this is	a + b
2	3	"	2a + b
3	5	"	3a + b
4	7	"	4a + b
5	9	"	5a + b
⋮	⋮		

The first term of the sequence in Example 1, page 10, namely 1 , is equal to the first term , namely a + b , of the sequence in Example 1 , page 9 .

Thus, we have $1 = a + b$.

Since we know from (2) above that a = 2, then substituting for a we get $1 = 2 + b$

so that $b = -1$.

Substitute the value of b in $s_n = 2n + b$ and the general term for the sequence in Example 1, page 10 is $s_n = 2n - 1$.

It should be noted that :

a. We could have found b from the 2 nd terms of the sequences in Examples 1 on pages 9 and 10 , that is $3 = 2a + b$ and since a=2, $3 = 2(2) + b$ so that $b = -1$ as before.

The same is true for the other pairs of corresponding terms of the sequences in the Examples 1, pages 9 and 10 .

b. From the general term $s_n = 2n - 1$, we can get the original five terms of the sequence listed in Example 1, page 10, and actually this is all that the general term $s_n = 2n - 1$ can give.

However, if one were asked to ' give the next few numbers that follow the <u>same pattern</u> as the numbers in the given sequence '

n	1	2	3	4	5	6	7	8
	↕				↕	↕	↕	↕
terms of given sequence	1	3	5	7	9	?	?	?

then from the general term $s_n = 2n - 1$, we have $s_6 = 11$, $s_7 = 13$, $s_8 = 15$.

We can now explain what is meant by ' the same pattern' .

By ' the same pattern ', we mean that if we continue the given sequence of numbers using the general term $s_n = 2n - 1$, then all the D_i^1 , $i = 1, 2, 3, \ldots$, will all have the common value 2 .

c. If we accept (b) above, then we can quickly find the 100 th term (or any other term of the sequence we wish) from the general term $s_n = 2n - 1$ for the sequence in Example 1, page 10. Thus, the 100 th term is

$$s_{100} = 2(100) - 1 = 199 .$$

The following format is rather convenient for solving problems similar to Example 1, page 10 .

<u>Example 1</u> Find the general term s_n for the given sequence .

n	Sequence	D^1
1	4	
		3
2	7	
		3
3	10	
		3
4	13	
		3
5	16	
⋮	⋮	

<u>Note</u> : We will abbreviate some of the terms in the solution for the problems.

<u>Solution</u>

Check: D_i^1 , $i = 1, 2, 3, \ldots$, have common value	$a = 3$
thus, general term has form	$s_n = an + b$
substitute value of D_i^1 in s_n	$s_n = 3n + b$
1 st term of sequence is equal to $a + b$	$4 = 3 + b$
solve for b	$b = 1$
general term of given sequence :	$s_n = 3n + 1$.

Thus, ' Check: D_i^1 , $i = 1, 2, 3, \ldots$, have common value' will be written as :

'Check: D_i^1 have common value '.

' Substitute value of D_i^1 in s_n ' will become : ' substitute value of D_i^1 '

PROBLEMS

Find the general term s_n for each of the following sequences .

23.

n	Sequence	D^1
1	9	
2	14	
3	19	
4	24	
5	29	
⋮	⋮	

Solution

1. Check : D_i^1 have common value $a =$ ______
2. thus, general term has form $s_n =$ ______
3. substitute value of D_i^1 ______
4. 1 st term of sequence = a + b ______
5. solve for b ______
6. general term for given sequence ______

24.

n	Sequence	D^1
1	1	
2	4	
3	7	
4	10	
5	13	
⋮	⋮	

Solution

1. Check: D_i^1 have common value $a =$ ______
2. thus, general term has form $s_n =$ ______
3. substitute value of D_i^1 ______
4. 1 st term of sequence = a + b ______
5. solve for b ______
6. general term for given sequence ______

25.

n	Sequence	D^1
1	9	
2	5	
3	1	
4	- 3	
5	- 7	
⋮	⋮	

Solution

1. Check: D_i^1 have common value $a =$ ______
2. thus, general term has form $s_n =$ ______
3. substitute value of D_i^1 ______
4. 1 st term of sequence = a + b ______
5. solve for b ______
6. general term for given sequence ______

26.

n	Sequence	D^1
1	2	
2	5/2	
3	3	
4	7/2	
5	4	
⋮		

Solution

1. Check: D_i^1 have common value $a =$ ______
2. thus, general term has form $s_n =$ ______
3. substitute value of D_i^1 ______
4. 1 st term of sequence = a + b ______
5. solve for b ______
6. general term for given sequence ______

The procedure and format in the examples and problems on pages 9-12 for finding the general term s_n of a sequence where the D_i^1, i=1, 2, 3, . . . , have one and the same nonzero value are quite standard . However, for many sequences we can shorten considerably the number of steps required by developing now a general formula for getting s_n that involves only the sequence number s_1 and the common value of the D_i^1, i = 1, 2, 3, ...

Study the Table 4 where s_1, s_2, s_3, . . . , are the terms of a given sequence.

Table 4

n	$s_n = an + b$	D^1
1	$s_1 = a + b$	
		$a = D_1^1$
2	$s_2 = 2a + b$	
		a
3	$s_3 = 3a + b$	
		a
4	$s_4 = 4a + b$	
		a
5	$s_5 = 5a + b$	
⋮	⋮	
n	$s_n = an + b$	

From Table 4 , we have $a = s_2 - s_1 = D_1^1$ . . . . . (1)

Thus, in the case where the D_i^1, i = 1, 2, 3, . . . have one and the same nonzero value, we can get the coefficient a quickly by taking the difference of the first two terms of the given sequence .

Substitute for <u>a</u> from (1) in $s_n = a\,n + b$ $\qquad s_n = D_1^1 n + b$ . . . . . (2)

From Table 4, we also have that $s_1 = a + b$ or $b = s_1 - a$, and substitute in the latter for a to get $\qquad b = s_1 - D_1^1$ . . . . . (3)

Replace b in (2) by its equivalent from (3) $\qquad s_n = D_1^1 n + (s_1 - D_1^1)$

so that

$$\boxed{s_n = s_1 + (n - 1)\, D_1^1} \qquad \text{. . . . . (A)}$$

where s_1 is the first term of the given sequence, and D_1^1 is the 1 st order difference, first difference for the sequence .

Formula (A) gives the general term s_n for a sequence where the D_i^1, i=1, 2, 3, . . . , all have one and the same nonzero value .

We illustrate the use of formula (A) by the example on the following page .

Example 1 Given the sequence : 1, 3, 5, 7, 9, ...
Find the general term s_n for this sequence .

Solution:

n	Sequence	D^1
1	1	
		2 = D_1^1
2	3	
		2
3	5	
		2
4	7	
		2
5	9	
⋮	⋮	

Here D_i^1, i = 1, 2, 3, . . . , all have the same nonzero value 2 .
Use the formula (A) from page 13, that is ,

$$s_n = s_1 + (n - 1) D_1^1 .$$

From the table above, $s_1 = 1$ and $D_1^1 = 2$ so that substituting in the formula, we have

$$s_n = 1 + (n - 1)2 = 2n - 1$$

for the general term of the sequence above .

The sequence used in the example here is the same as the sequence used in example 1 , page 10 . Notice how quickly we found the general term s_n using the formula (A), page 13 .

PROBLEMS

Find the general term s_n for each of the following sequences using formula (A), page 13 .

27. 9, 14, 19, 24, 29, 34, . . . s_1 ________ D_1^1 ________

    s_n ________

28. 1, 4, 7, 10, 13, 16, . . . s_1 ________ D_1^1 ________

    s_n ________

29. 9, 5, 1, -3, -7, -11, . . . s_1 ________ D_1^1 ________

    s_n ________

30. 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, . . . s_1 ________ D_1^1 ________

    s_n ________

31. - 1, -3, -5, -7, -9, . . . s_1 ________ D_1^1 ________

    s_n ________

9. **s_n for sequences where D_i^2 has same value for all $i = 1, 2, 3, ..$**

How does one find the general term s_n for a sequence where the 2 nd order differences have the same nonzero value ? We first study the patterns in a few illustrations .

Example 1 Take $s_n = 2n^2$

n	$s_n = 2n^2$	D^1	D^2
1	2		
		6	
2	8		4
		10	
3	18		4
		14	
4	32		4
		18	
5	50		4
		22	
6	72		
⋮			
n	$2n^2$		

Here , D_i^2 has the same nonzero value for all $i = 1, 2, 3 . . .$, and the coefficient of n^2 is 1/2 of this common value .

Example 2 Take $s_n = an^2$

n	$s_n = an^2$	D^1	D^2
1	a		
		3a	
2	4a		2a
		5a	
3	9a		2a
		7a	
4	16a		2a
		9a	
5	25a		2a
		11a	
6	36a		
⋮	⋮		
n	an^2		

Here, D_i^2 has the same nonzero value for all $i = 1, 2, 3, . . .$, and the coefficient of n^2 is 1/2 of this common value .

PROBLEMS

In each problem, verify that the 2 nd order differences D_i^2 all have the same nonzero value and that the coefficient of n^2 is 1/2 of this common value .

32.

n	$s_n = n^2$	D^1	D^2
1			
2			
3			
4			
5			
⋮			
n			

33.

n	$s_n = 4n^2$	D^1	D^2
1			
2			
3			
4			
5			
⋮			
n			

34.

n	$s_n = n^2/2$	D^1	D^2
1			
2			
3			
4			
5			
⋮			
n			

35.

n	$s_n = -n^2$	D^1	D^2
1			
2			
3			
4			
5			
⋮			
n			

Example 1 Take the sequence $s_n = 2n^2 + 3n$

n	$s_n = 2n^2 + 3n$	D^1	D^2
1	5		
		9	
2	14		4
		13	
3	27		4
		17	
4	44		4
		21	
5	65		
⋮	⋮		
n	$2n^2 + 3n$		

Here, D_i^2 has the same nonzero value for all $i = 1, 2, 3, \ldots,$ and the coefficient of n^2 is 1/2 this common value.

Example 2 Take the sequence $s_n = an^2 + bn$

n	$s_n = an^2 + bn$	D^1	D^2
1	$a + b$		
		$3a + b$	
2	$4a + 2b$		$2a$
		$5a + b$	
3	$9a + 3b$		$2a$
		$7a + b$	
4	$16a + 4b$		$2a$
		$9a + b$	
5	$25a + 5b$		
⋮			
n	$an^2 + bn$		

Here, D_i^2 has the same nonzero value for all $i = 1, 2, 3, \ldots,$ and the coefficient of n^2 is 1/2 this common value.

PROBLEMS

In each problem, verify that the 2nd order differences D_i^2 all have the same nonzero value and that the coefficient of n^2 is 1/2 of this common value.

36.

n	$s_n = n^2 - 4n$	D^1	D^2
1			
2			
3			
4			
5			
⋮			
n			

37.

n	$s_n = n^2 + n$	D^1	D^2
1			
2			
3			
4			
5			
⋮			
n			

38.

n	$s_n = -n^2 + 2n$	D^1	D^2
1			
2			
3			
4			
5			
⋮			
n			

39.

n	$s_n = -2n^2 + 3n$	D^1	D^2
1			
2			
3			
4			
5			
⋮			
n			

There is still one more case to consider which follows the pattern of the examples and problems above.

Example 1 Take $s_n = 2n^2 + 3n + 4$

n	$s_n = 2n^2 + 3n + 4$	D^1	D^2
1	9		
		9	
2	18		4
		13	
3	31		4
		17	
4	48		4
		21	
5	69		
.	.		
.	.		
n	$2n^2 + 3n + 4$		

Here, D_i^2 has the same nonzero value for all $i = 1, 2, 3, \ldots$, and the coefficient of n^2 is 1/2 this common value.

Example 2 Take $s_n = an^2 + bn + c$

n	$s_n = an^2 + bn + c$	D^1	D^2
1	$a + b + c$		
		$3a + b$	
2	$4a + 2b + c$		$2a$
		$5a + b$	
3	$9a + 3b + c$		$2a$
		$7a + b$	
4	$16a + 4b + c$		$2a$
		$9a + b$	
5	$25a + 5b + c$		
.			
.			
n	$an^2 + bn + c$		

Here, D_i^2 has the same nonzero value for all $i = 1, 2, 3, \ldots$, and the coefficient of n^2 is 1/2 this common value.

PROBLEMS

In each problem, verify that the 2 nd order differences D_i^2 all have the same nonzero value and that the coefficient of n^2 is 1/2 of this common value.

40.

n	$s_n = -2n^2 + 3n - 4$	D^1	D^2
1			
2			
3			
4			
5			
.			
.			
n			

41.

n	$s_n = \frac{n^2}{2} - n + 1$	D^1	D^2
1			
2			
3			
4			
5			
.			
.			
n			

From the examples and problems, pages 15-17 we have the following

SUMMARY

1. A sequence where the general term is a quadratic in n, namely $s_n = an^2 + bn + c$ has the 2 nd order differences D_i^2, $i = 1, 2, 3, \ldots$, all the same nonzero constant.
2. If the 2 nd order differences D_i^2, $i = 1, 2, 3, \ldots$, all have the same nonzero value, then the general term of the sequence is of the form $an^2 + bn + c$.
3. The coefficient of n^2, namely a, in the general term $s_n = an^2 + bn + c$ for the given sequence is 1/2 the common value of D_i^2, $i = 1, 2, 3, \ldots$.
4. From (3) above, we only have a. Thus, b and c must still be found.

   We illustrate the procedure for finding b and c on the following page.

Example 1 Find the general term s_n for the sequence 9, 18, 31, 48, 69, . . . , (see example 1, page 17) .

Solution: We follow the procedure on page 10 and combine the facts in examples 1 and 2 on page 17 into the table 5 below .

Table 5

n	Sequence	D^1	D^2
1	9 $(a + b + c)$		
		9 $(3a + b)$	
2	18 $(4a + 2b + c)$		
			4 $(2a)$
		13 $(5a + b)$	
3	31 $(9a + 3b + c)$		
			4 $(2a)$
		17 $(7a + b)$	
4	48 $(16a + 4b + c)$		
⋮	⋮	⋮	⋮

a. The D_i^2 , $i = 1, 2, 3, \ldots$, all have the same nonzero value 4. Thus, the general term of the sequence is of the form (see (2) page 17) $$s_n = an^2 + bn + c$$

b. Coefficient a of n^2 is 1/2 the common value of D_i^2 , that is $$a = \frac{1}{2}(4) = 2$$

c. From table 5, 9 and 3a + b represent the same quantity, namely D_1^1 , so that $$9 = 3a + b$$

d. Substitute the value of a from (b) into (c) and solve for b $$9 = 3(2) + b$$ $$b = 3$$

e. From table 5, 9 and a + b + c represent s_1 , the first terms of the sequences, so that $$9 = a + b + c$$

f. Substitute the values found for a and b and solve for c $$9 = 2 + 3 + c$$ $$c = 4 \ .$$

We now have $a = 2$, $b = 3$, $c = 4$ and substituting these values in the general term $s_n = an^2 + bn + c$, we have

$$s_n = 2n^2 + 3n + 4$$

for the general term of the given sequence .

Note that our answer agrees with the sequence that was used in example 1 on page 17 .

PROBLEMS

Find the general term s_n for each of the following sequences assuming that the pattern of the sequence continues .

42.

n	Sequence	D^1	D^2
1	4		
2	12		
3	26		
4	46		
5	72		
⋮	⋮		

a = _____ b = _____ c = _____

s_n = _______________

43.

n	Sequence	D^1	D^2
1	2		
2	9		
3	20		
4	35		
5	54		
⋮	⋮		

a = _____ b = _____ c = _____

s_n = _______________

44.

n	Sequence	D^1	D^2
1	- 2		
2	1		
3	6		
4	13		
5	22		
⋮	⋮		

a = _____ b = _____ c = _____

s_n = _______________

45.

n	Sequence	D^1	D^2
1	- 2		
2	- 3		
3	- 6		
4	- 11		
5	- 18		
⋮	⋮		

a = _____ b = _____ c = _____

s_n = _______________

Read page 13 . We will now develop a general formula (similar to formula (A)) for getting s_n for those sequences where the D_i^2, $i = 1, 2, 3, \ldots$, have the same nonzero value that involves only the value of the sequence number s_1, the value of D_1^1 and D_1^2 . Table 6 contains the essential information .

Table 6

n	$s_n = an^2 + bn + c$	D^1	D^2
1	$s_1 = a + b + c$		
		$s_2 - s_1 = D_1^1 = 3a + b$	
2	$s_2 = 4a + 2b + c$		$D_2^1 - D_1^1 = D_1^2 = 2a$
		$s_3 - s_2 = D_2^1 = 5a + b$	
3	$s_3 = 9a + 3b + c$		$D_3^1 - D_2^1 = D_2^2 = 2a$
		$s_4 - s_3 = D_3^1 = 7a + b$	
4	$s_4 = 16a + 4b + c$		
⋮	⋮		

The D_i^2, $i = 1, 2, 3, \ldots$, all have the same nonzero value $2a$.

a. From table 6, we have $D_1^2 = 2a$ or $a = \frac{1}{2} D_1^2$

b. Also from table 6, we have $D_1^1 = 3a + b$ or $b = D_1^1 - 3a$

and substitute for a $b = D_1^1 - \frac{3}{2} D_1^2$

c. Now from table 6, $s_1 = a + b + c$ or $c = s_1 - a - b$

and substitute for a and b $c = s_1 - \frac{1}{2} D_1^2 - (D_1^1 - \frac{3}{2} D_1^2)$

which simplifies to $c = s_1 - D_1^1 + D_1^2$.

If we substitute the values of a, b, c in the general term $s_n = an^2 + bn + c$, then

$$s_n = \frac{1}{2} D_1^2 n^2 + (D_1^1 - \frac{3}{2} D_1^2) n + (s_1 - D_1^1 + D_1^2)$$

or

$$\boxed{s_n = s_1 + (n-1) D_1^1 + \frac{(n-1)(n-2)}{2} D_1^2} \quad \ldots\ldots (B)$$

where s_1 is the first term of the given sequence,
D_1^1 is the 1st order difference, 1st difference,
D_1^2 is the 2nd order difference, 1st difference .

The formula (B) gives the general term s_n for a sequence where D_i^2, $i = 1, 2, 3, \ldots$, have one and the same nonzero value .

Example 1 Given : the sequence 9, 18, 31, 48, 69, ... (see example 1, page 18).
Find the general term s_n for the sequence assuming that the pattern continues.

Solution

n	Sequence	D^1	D^2
1	$9 = s_1$		
		$9 = D_1^1$	
2	18		$4 = D_1^2$
		13	
3	31		4
		17	
4	48		4
		21	
5	69		
⋮	⋮		

a. Here, the D_i^2, $i = 1, 2, 3, \ldots$, all have the same nonzero value 4. Thus, the general term s_n for the sequence is of the form $s_n = an^2 + bn + c$ and we can use the formula (B) above .

b. From the difference table above : $s_1 = 9$, $D_1^1 = 9$, $D_1^2 = 4$ and substitute these values in formula (B).

$$s_n = 9 + (n-1)9 + \frac{(n-1)(n-2)}{2}4$$

simplifying $s_n = 2n^2 + 3n + 4$, the same result that we found on page 18 .

PROBLEMS

Find the general term s_n for each of the following sequences .

46. 2, 8, 18, 32, 50, 72, . . . $s_1 =$ ____ $D_1^1 =$ ____ $D_1^2 =$ ____

$s_n =$ ________________

47. 5, 14, 27, 44, 65, 90, ... $s_1 =$ ____ $D_1^1 =$ ____ $D_1^2 =$ ____

$s_n =$ ________________

48. 9, 18, 31, 48, 69, 94, ... $s_1 =$ ____ $D_1^1 =$ ____ $D_1^2 =$ ____

$s_n =$ ________________

49. -2, -10, -22, -38, -58, -82,... $s_1 =$ ____ $D_1^1 =$ ____ $D_1^2 =$ ____

$s_n =$ ________________

50. $\frac{1}{2}, \frac{6}{2}, \frac{15}{2}, \frac{28}{2}, \frac{45}{2}, \frac{66}{2}, \ldots$ $s_1 =$ ____ $D_1^1 =$ ____ $D_1^2 =$ ____

$s_n =$ ________________

10. s_n for sequences where D_i^3 has same value for all i = 1, 2, 3, . . .

How does one find the general term s_n for a sequence where the 3 rd order differences have the same nonzero value ?

Recall that when the 1 st order differences were all the same nonzero constant, the general term s_n for the sequence was of the form

$$s_n = an + b, \quad \text{a linear equation in } n .$$

When the 2 nd order differences were all the same nonzero constant, the general term s_n for the sequence was of the form

$$s_n = an^2 + bn + c, \quad \text{a quadratic equation in } n .$$

On the basis of the above, if the 3 rd order differences are all the same nonzero constant, then the general term s_n should perhaps be a cubic equation in n of the form

$$s_n = an^3 + bn^2 + cn + d .$$

We try this conjecture in the example on the next page .

Table 7

Example 1

n	$s_n = an^3 + bn^2 + cn + d$	D^1	D^2	D^3
1	a + b + c + d			
		7a + 3b + c		
2	8a + 4b + 2c + d		12a + 2b	
		19a + 5b + c		6a
3	27a + 9b + 3c + d		18a + 2b	
		37a + 7b + c		6a
4	64a + 16b + 4c + d		24a + 2b	
		61a + 9b + c		6a
5	125a + 25b + 5c + d		30a + 2b	
		91a + 11b + c		
6	216a + 36b + 6c + d			
⋮	⋮	⋮	⋮	⋮
n	$an^3 + bn^2 + cn + d$			

Here, D_i^3 has the same nonzero value for all i = 1, 2, 3, . . . , and the coefficient of n^3 is 1/6 this common value .

Example 1 provides a method for finding the general term s_n for a sequence where the 3 rd order differences are all the same nonzero constant .

Example 2 Find the general term s_n for the sequence : 3, 17, 53, 123, 239, 413 .

Solution: Study example 1, page 18 , since we follow the same procedure here . In table 8, we combine the terms of the given sequence with the facts in table 7 .

Table 8

n	Sequence	D^1	D^2	D^3
1	3 (a + b + c + d)			
		14 (7a + 3b + c)		
2	17 (8a + 4b + 2c + d)		22 (12 a + 2b)	
		36		12 (6a)
3	53		34	
		70		12
4	123		46	
		116		12
5	239		58	
		174		
6	413			
⋮	⋮			

a. The D_i^3 , i = 1, 2, 3, . . . , all have the same nonzero value 12. Thus the general term of the sequence is of the form (see above)
$$s_n = an^3 + bn^2 + cn + d$$

b. Coefficient a of n^3 is 1/6 the common value of D_i^3 , that is
$$a = \frac{1}{6}(12) = 2$$

c. From table 8 , 22 and 12a + 2b represent the same quantity, namely, D_1^2 , so
$$22 = 12a + 2b$$
and substituting a= 2 and solving for b,
$$b = -1$$

d. Also from table 8, 14 and $7a+3b+c$ represent the same quantity, namely D_1^1. Thus $14 = 7a + 3b + c$.
Substitute $a = 2$, $b = -1$ and solve for c $c = 3$.

e. Finally, from table 8, 3 and $a + b + c + d$ both represent the first term of the sequence, s_1 so that $3 = a + b + c + d$
and substitute $a = 2$, $b = - 1$, $c = 3$ so that $d = - 1$.

We now have $a = 2$, $b = -1$
$c = 3$, $d = -1$ and substitute in $s_n = a n^3 + bn^2 + cn + d$
so that $s_n = 2n^3 - n^2 + 3n - 1$
as the general term for the given sequence .

Summary

If D_i^3, $i = 1, 2, 3, \ldots,$ all have one and the same value then the general term of the sequence s_n is of the form

$$s_n = a n^3 + b n^2 + c n + d$$

where

$$a = \frac{1}{6} D_1^3$$

$$12 a + 2 b = D_1^2$$

$$7a + 3b + c = D_1^1$$

$$a + b + c + d = s_1 .$$

PROBLEMS

Find the general term s_n for each of the following sequences .

51. 1, 8, 27, 64, 125, . . . , $s_n =$ __________
52. 1, 14, 51, 124, 245, . . ., $s_n =$ __________
53. 1, 12, 45, 112, 225, . . . , $s_n =$ __________
54. 2, 14, 48, 116, 230, . . ., $s_n =$ __________
55. -3, 9, 43, 111, 225, . . . , $s_n =$ __________

We will now develop a general formula similar to the formulas (A) page 13 and formula (B) page 20 for getting s_n for those sequences where the D_i^3, $i = 1, 2, 3, \ldots,$ have the same nonzero value that involves only the value of the sequence number s_1 , and the values of D_1^1, D_1^2 , D_1^3 .

a. Use the results in the summary above , thus $a = \frac{1}{6} D_1^3$.

b. Substitute the value of $\underline{a}$ in $12a + 2b = D_1^2$

so that $b = \frac{1}{2} D_1^2 - D_1^3$

c. Substitute for a and b in $7a + 3b + c = D_1^1$

so that $c = D_1^1 + \frac{11}{6} D_1^3 - \frac{3}{2} D_1^2$

d. Substitute for a, b, c in $a + b + c + d = s_1$

so that $d = s_1 - D_1^1 + D_1^2 - D_1^3$.

Now substitute the values of a, b, c, d into $s_n = an^3 + bn^2 + cn + d$ and we have

$$s_n = \frac{1}{6} D_1^3 n^3 + \left(\frac{1}{2} D_1^2 - D_1^3\right) n^2 + \left(D_1^1 + \frac{11}{6} D_1^3 - \frac{3}{2} D_1^2\right) n + \left(s_1 - D_1^1 + D_1^2 - D_1^3\right)$$

hich simplifies to

$$\boxed{s_n = s_1 + (n-1) D_1^1 + \frac{(n-1)(n-2)}{2!} D_1^2 + \frac{(n-1)(n-2)(n-3)}{3!} D_1^3} \quad \ldots (C)$$

where $n! = 1 \cdot 2 \cdot 3 \cdot 4 \ldots \cdot n$.

The above formula (C) gives the general term s_n for a sequence where the D_i^3, $i = 1, 2, 3, \ldots$, have one and the same nonzero value .

PROBLEMS

56. Verify the general term s_n for each of the sequences in Problems 51-55, page 23 using the formula (C) above .

51. same____ not same ____ 52. same ____ not same ____

53. same____ not same ____ 54. same____ not same____

55. same ____ not same ____

We could investigate sequences where the 4 th, 5 th order differences and higher order differences are one and the same nonzero constant . However, we have seen that it is possible to derive formulas for the general term s_n that involve only the first term s_1 of a sequence and D_1^1, D_1^2, D_1^3 (in the formulas (A) page 13, (B) page 20, (C) page 24).

A pattern for a general formula for s_n involving higher order differences than D^3 is rather clear from formula (C) page 24. However, we will now develop a general procedure for getting such a formula . But first, a few preliminaries and a summary .

11. Degree of a polynomial

A polynomial of degree m in n is an expression of the form

$$a n^{m} + b n^{m-1} + c n^{m-2} + \dots + p n + q$$

where a, b, c, . . . , are real numbers, $a \neq 0$, m is a positive integer, $m \geq 1$.

Example 1

a. $an + b$, is the general form of a polynomial of degree 1 in n (or linear in n) .

Particular polynomials of degree 1 in n :

1. $3n + 4$ 2. $n/2$

b. $an^2 + bn + c$ is the general form of a polynomial of degree 2 in n (or quadratic in n).

Particular polynomials of degree 2 in n :

1. $3n^2 - 4n + 5$ 2. $5n^2 - \frac{2}{3}n$ 3. $6n^2$

c. $an^3 + bn^2 + cn + d$ is the general form of a polynomial of degree 3 in n (or cubic in n) .

Particular polynomials of degree 3 in n :

1. $2n^3 - 5n^2 + 3n - \frac{1}{2}$ 2. $3n^3 + 2n$ 3. $\frac{5n^3}{2}$

Polynomials of degree 4, 5, and so on , are defined on the pattern of the polynomials in the example above .

PROBLEMS

What is the degree in n of each of the following polynomials .

57. a. $2n$ degree ______ b. $2 + (n - 1)3$ degree ______

c. $n(n+1)$ degree ____ d. $\frac{n(3n-1)}{2}$ degree ______

e. $\frac{n(n+1)(2n+1)}{6}$ degree ______ f. $\frac{n(n^2 - 3n + 8)}{6}$ degree ______

12. Degree of polynomial and differences

A sequence whose general term is a polynomial of

1. degree 1 in n has 1 st order differences one and the same nonzero constant,

2. degree 2 in n has 2 nd order differences one and the same nonzero constant ,

3. degree 3 in n has 3 rd order differences one and the same nonzero constant ,

⋮

4. degree k in n has k th order differences one and the same nonzero constant .

Conversely, a sequence whose

5. 1 st order differences are one and the same nonzero constant will have the general term s_n of degree 1 in n ,

6. 2 nd order differences are one and the same nonzero constant will have the general term s_n of degree 2 in n ,

⋮

7. k th order differences are one and the same nonzero constant will have the general term s_n of degree k in n .

13. Possible solutions by finite differences

The method of finite differences does not give a solution to all sequence problems.

Sequences that have one and the same nonzero constant for the k th order differences (k = 1, 2, 3, ...) can be solved by the method of finite differences. In this case, the (k + 1) th and all higher order differences will have zeros.

Example 1 Consider the sequence of primes .

n	Primes	D^1	D^2	D^3	D^4	D^5 . . .
1	2					
		1				
2	3		1			
		2		-1		
3	5		0		1	
		2		0		5
4	7		2		-4	
		4		-4		8 . . .
5	11		-2		4	
		2		0		-4
6	13		-2		0	
		4		0		
7	17		-2			
		2				
8	19					
⋮	⋮	⋮	⋮	⋮	⋮	⋮

We do not get one and the same constant for any order difference. Thus, there is no polynomial formula in n that will give the general term for the sequence of primes . We now generalize the procedure that was used in deriving the formulas (A) page 13, (B) page 20 and (C) page 24 .

14. Method of leading differences

Let s_1, s_2, s_3, ..., be the terms of a given sequence. Form the table 9 of differences. We assume that eventually the k th (k = 1, 2, 3,... , m) order differences are all one and the same nonzero constant .

Table 9

Example 2

n	Sequence	D^1	D^2	D^3 . . . . .
1	s_1			
		D_1^1		
2	s_2		D_1^2	
		D_2^1		D_1^3 . . . . .
3	s_3		D_1^2	
		D_3^1		
4	s_4			
⋮	⋮	⋮	⋮	⋮

Call s_1 the lead term, and D_1^1, D_1^2, D_1^3, . . ., and so on, the lead differences. We will now express each term of the sequence, namely s_1, s_2, s_3, . . . , as a sum of the lead term s_1 and the lead differences .

From table 9, page 26, we have

Table 10

$D_1^1 = s_2 - s_1$	$s_1 = 1\, s_1$	$D_1^2 = D_2^1 - D_1^1$
$D_2^1 = s_3 - s_2$	$s_2 = s_1 + D_1^1$	$D_2^2 = D_3^1 - D_2^1$
$\vdots$	$s_3 = s_2 + D_2^1$	$\vdots$
$D_i^1 = s_{i+1} - s_i$	$\vdots$	$D_i^{k+1} = D_{i+1}^k - D_i^k$
$i = 1, 2, 3, \ldots$	$s_{i+1} = s_i + D_i^1$	$k = 1, 2, 3, \ldots$

We derive expressions for the first 3 terms, namely s_1, s_2, s_3 and extend the pattern to s_4 and s_5 .

a. Take s_1 . From table 10, we have $s_1 = 1\, s_1$

b. Take s_2 . From table 10, we have $s_2 = 1\, s_1 + 1\, D_1^1$

c. Take s_3 . From table 10, we have

$s_3 = s_2 + D_2^1$ where D_2^1 is not a lead difference . But from table 10

$D_i^{k+1} = D_{i+1}^k - D_i^k$, and take $k = 1$, $i = 1$, so that

$$D_2^1 = D_1^1 + D_1^2$$

Substitute for s_2 and D_2^1 in s_3 $\quad s_3 = 1\, s_1 + 2\, D_1^1 + 1\, D_1^2$

On the pattern of the derivation above $\quad s_4 = 1\, s_1 + 3\, D_1^1 + 3\, D_1^2 + 1\, D_1^3$

$$s_5 = 1\, s_1 + 4\, D_1^1 + 6\, D_1^2 + 4\, D_1^3 + 1\, D_1^4 .$$

Each term of the sequence can be expressed in terms of s_1 and the lead differences . Moreover, the coefficients of the lead term and lead differences for a given term of the sequence are the numbers in Pascal's triangle .

In s_1	1
s_2	1 1
s_3	1 2 1
s_4	1 3 3 1
s_5	1 4 6 4 1
$\vdots$	$\vdots$

and so on .

The expressions for s_1, s_2, s_3, . . . , in terms of the lead term s_1 and the lead differences D_1^1, D_1^2, D_1^3 . . . , are shown schematically in table 11

Table 11

Terms of sequence ↓	s_1	D_1^1	D_1^2	D_1^3	D_1^4	. . .
$s_1 = 1\, s_1$	1 · s_1					
$s_2 = 1\, s_1 + 1\, D_1^1$	1 · s_1	1 · D_1^1				
$s_3 = 1\, s_1 + 2\, D_1^1 + 1\, D_1^2$	1 · s_1	2 · D_1^1	1 · D_1^2			
$s_4 = 1\, s_1 + 3\, D_1^1 + 3\, D_1^2 + 1\, D_1^3$	1 · s_1	3 · D_1^1	3 · D_1^2	1 · D_1^3		
$s_5 = 1\, s_1 + 4\, D_1^1 + 6\, D_1^2 + 4\, D_1^3 + 1\, D_1^4$	1 · s_1	4 · D_1^1	6 · D_1^2	4 · D_1^3	1 · D_1^4	

Each square in table 11 is divided into two sections. The bottom half has an element of the Pascal triangle. The upper half has the leading term of the sequence or one of the leading differences.

To get a term s_1, s_2, s_3, . . . , of the sequence, form the product of the elements in each square , in a given row, and take their sum (as shown to the left of table 11).

How do we get the general formula for s_n ? The elements in Pascal's triangle (page 27) are the binomial coefficients given by

$$\binom{r}{t} = \frac{r\,!}{t\,!\,(r-t)\,!} \quad , \quad \text{where } r\,! = 1 \cdot 2 \cdot 3 \cdot 4 \ \ldots \cdot r \text{ and } 0\,! = 1 .$$

Number the rows of the Pascal triangle as shown below

row 0	1	coefficients for terms in s_1
row 1	1 1	" s_2
row 2	1 2 1	" s_3
row 3	1 3 3 1	" s_4
row 4	1 4 6 4 1	" s_5
⋮	and so on .	

Rewrite the elements in Pascal's triangle in terms of the binomial coefficients.

Find the binomial coefficients for the row (n - 1) which will be the coefficients for terms in s_n , the general term of the sequence .

Row	Coefficients of		
0	s_1	$r = 0,\ t = 0$	$\binom{0}{0}$
1	s_2	$r = 1,\ t = 0, 1$	$\binom{1}{0}\quad \binom{1}{1}$
2	s_3	$r = 2,\ t = 0, 1, 2$	$\binom{2}{0}\quad \binom{2}{1}\quad \binom{2}{2}$
3	s_4	$r = 3,\ t = 0, 1, 2, 3$	$\binom{3}{0}\quad \binom{3}{1}\quad \binom{3}{2}\quad \binom{3}{3}$
.			
.			
(n-1)	s_n	$r=n-1,\ t= 0, 1, 2, \ldots (n-1)$	$\binom{n-1}{0}\quad \binom{n-1}{1}\quad \binom{n-1}{2}\quad \binom{n-1}{3} \ldots \binom{n-1}{n-1}$

Now the coefficients for the terms in s_n are (see definition on page 28)

$$\binom{n-1}{0} = \frac{(n-1)!}{0!\,((n-1)-0)!} = 1 \qquad \binom{n-1}{1} = \frac{(n-1)!}{1!\,((n-1)-1)!} = n-1$$

$$\binom{n-1}{2} = \frac{(n-1)!}{2!\,((n-1)-2)!} = \frac{(n-1)(n-2)}{2!} \qquad \binom{n-1}{3} = \frac{(n-1)!}{3!\,((n-1)-3)!} = \frac{(n-1)(n-2)(n-3)}{3!}$$

$$\vdots$$

$$\binom{n-1}{n-2} = \frac{(n-1)!}{(n-2)!\,((n-1)-(n-2))!} = n-1 \qquad \binom{n-1}{n-1} = \frac{(n-1)!}{(n-1)!\,((n-1)-(n-1))!} = 1$$

Thus, the general term s_n of the sequence is now given by

$$s_n = s_1 + (n-1)\,D_1^1 + \frac{(n-1)(n-2)}{2!}\,D_1^2 + \frac{(n-1)(n-2)(n-3)}{3!}\,D_1^3 + \frac{(n-1)(n-2)\ldots\ldots(n-4)}{4!}\,D_1^4$$
$$+ \frac{(n-1)(n-2)(n-3)\ldots\ldots(n-5)}{5!}\,D_1^5 + \ldots + (n-1)\,D_1^{n-2} + D_1^{n-1} \qquad \ldots\ldots\ldots (D)$$

where after some D_1^k, the D_1^{k+1} s are all zero,

and s_1 is the first term of the given sequence,

D_1^k, $k = 1, 2, 3, \ldots$ are the lead differences .

Example 1 Find the general term for the sequence: $-3,\ 3,\ 13,\ 27,\ 45, \ldots$.

Solution: First check that some k th order differences have one and the same nonzero value .

n	Sequence	D^1	D^2
1	-3		
		6	
2	3		4
		10	
3	13		4
		14	
4	27		4
		18	
5	45		
⋮	⋮		

Here the 2 nd order differences all have the same nonzero value 4.

From the table, $s_1 = -3$, $D_1^1 = 6$, $D_1^2 = 4$

and use the formula (D) on page 29. Thus,

$$s_n = s_1 + (n-1)\,D_1^1 + \frac{(n-1)(n-2)}{2\,!}\,D_1^2$$

and substitute from above $\quad s_n = -3 + (n-1)\,6 + \frac{(n-1)(n-2)}{2\,!}\,4$

so that $\quad s_n = 2n^2 - 5$, is the general term of the given sequence.

15. Series The techniques of finite differences can be used to find a general expression for the sum of the first n terms of some series.

Given: a sequence $s_1, s_2, s_3, \ldots, s_n$

then the expression $s_1 + s_2 + s_3 + \ldots + s_n$

is called a series.

$s_1, s_2, s_3, \ldots$ are the terms of the series,

s_n is the n th term, the general term or the generator of the series.

Example 1 $\quad 1 + 2 + 3 + \ldots + n$, where $n = 1, 2, 3, \ldots$,

is a series.

1, 2, 3, . . . , are the terms of the series, and

n is the general term or generator of the series, that is, by substituting 1, 2, 3, . . . , for n we get the terms of the series.

Example 2 $\quad 1^2 + 2^2 + 3^2 + \ldots + n^2$, where $n = 1, 2, 3, \ldots$,

is a series.

1, 4, 9, . . . , are the terms of the series, and

n^2 is the general term or generator of the series, that is, by substituting 1, 2, 3, . . . , for n in n^2 we get the terms of the series.

Example 3 Write the first 5 terms of the series whose general term s_n is $2(n-1)$.

Solution: Here $s_n = 2(n-1)$. Substitute $n = 1, 2, 3, \ldots$, for n and we have for the first 5 terms of the series:

$$0 + 2 + 4 + 6 + 8 \quad .$$

16. Partial sums for a series

Given: a series $s_1 + s_2 + s_3 + \ldots + s_n$.

The partial sums $S_1, S_2, S_3, \ldots$, of the series are defined as follows:

$S_1 = s_1$, $\qquad S_2 = s_1 + s_2$, $\qquad S_3 = s_1 + s_2 + s_3$,

$S_n = s_1 + s_2 + s_3 + \ldots + s_n$, where S_n is the sum of the first n terms of the series .

As shown in the examples 1, 2, 3 on page 30, the general term s_n of a series will usually be given by a formula involving n or stated verbally.

Our task now is to find a formula for S_n which will give the sum of the first n terms of a series .

It is very important that one clearly understands that

s_n is the general term (generator) of a given series, while

S_n is the sum of the first n terms of a given series .

As in the case of sequences so here , we will be able to find S_n only for some special series, namely series which satisfy the criteria pointed out below.

Example 1 Given: the series $1 + 2 + 3 + \ldots + n$, where $n = 1, 2, 3, \ldots$.

Here the general term s_n is n .

The sum of the first n terms, S_n , is given by the formula

$$S_n = \frac{n(n+1)}{2} , \qquad \text{where } n = 1, 2, 3, \ldots .$$

Study table 12 .

Table 12

n	Partial sums		Substitute for n in $S_n = \frac{n(n+1)}{2}$
1	$S_1 = 1$	$= 1$	$S_1 = 1$
2	$S_2 = 1 + 2$	$= 3$	$S_2 = 3$
3	$S_3 = 1 + 2 + 3$	$= 6$	$S_3 = 6$
4	$S_4 = 1 + 2 + 3 + 4$	$= 10$	$S_4 = 10$
⋮	⋮	⋮	⋮

There are several facts to be noticed in table 12 .

a. The formula $S_n = \frac{n(n+1)}{2}$ quickly gives the sum of the first n terms of the series $1 + 2 + 3 + \ldots + n$.

b. But more relevant to our present discussion is the fact that the partial sums $S_1 = 1$, $S_2 = 3$, $S_3 = 6$, . . . , form a sequence of terms 1, 3, 6, 10, ...

It is precisely this fact that now reduces the problem of finding S_n to a problem of finding the general term for the sequence formed by the partial sums . This reduces the problem of series to the techniques discussed for sequences on the previous pages .

c. Conclusion We can find a formula for S_n which gives the sum of the first n terms of a series provided that the sequence of partial sums S_1, S_2, S_3 , . . . , satisfies the same criteria as did the terms of the sequences in the first 14 sections of this booklet .

Example 1 Given: the series $1 + 2 + 3 + \ldots + n$, where $n = 1, 2, 3, \ldots$.
Find a formula for S_n, the sum of the first n terms of the series .

Solution :

n	Partial sums	Sequence	D^1	D^2
1	$S_1 = 1$	$= 1$		
			2	
2	$S_2 = 1 + 2$	$= 3$		1
			3	
3	$S_3 = 1 + 2 + 3$	$= 6$		1
			4	
4	$S_4 = 1 + 2 + 3 + 4$	$= 10$		1
			5	
5	$S_5 = 1 + 2 + \ldots + 5$	$= 15$		
⋮	⋮	⋮		

a. Here the 2 nd differences are all the same nonzero constant 1 .

b. We can use the general formula (D) on page 29 . From the table above

the first term of the sequence is 1

the lead difference D_1^1 is 2

the lead difference D_1^2 is 1 .

Thus, $S_n = s_1 + (n-1)\, D_1^1 + \frac{(n-1)(n-2)}{2\,!}\, D_1^2$ becomes

$S_n = 1 + (n-1)\, 2 + \frac{(n-1)(n-2)}{2\,!}\, 1 = \frac{n(n+1)}{2}$, which is precisely the formula we used in example 1, page 31 .

The procedure for finding the general term S_n for the series shown in the example 1 above is acceptable, but it required the actual calculation of the first several partial sums .

In practice, we usually know the terms of the series s_n . It would be convenient to develop a method which provides an expression for S_n using the terms of the series directly rather than the partial sums . We now show how to do this .

17. Series: method of finite differences

Given: the series $s_1 + s_2 + s_3 + \ldots + s_n$.

It will prove convenient to introduce the zeroth partial sum S_o , where $S_o = 0$ for each series .

Table 13

n	Partial sums	D^1	D^2	$D^3 \ldots$
0	$S_o = 0$			
		$S_1 - S_o = s_1 = D_1^1$		
1	$S_1 = s_1$		$D_2^1 - D_1^1 = D_1^2$	
		$S_2 - S_1 = s_2 = D_2^1$		.....
2	$S_2 = s_1 + s_2$		$D_3^1 - D_2^1 = D_2^2$	
		$S_3 - S_2 = s_3 = D_3^1$		
3	$S_3 = s_1 + s_2 + s_3$			
⋮	⋮	⋮	⋮	
n-1	$S_{n-1} = s_1 + s_2 + \ldots + s_{n-1}$			
		$S_n - S_{n-1} = s_n = D_n^1$		
n	$S_n = s_1 + s_2 + \ldots + s_n$		$D_{n+1}^1 - D_n^1 = D_n^2$	..
		$S_{n+1} - S_n = s_{n+1} = D_{n+1}^1$		
n+1	$S_{n+1} = s_1 + s_2 + \ldots + s_{n+1}$			
⋮	⋮			

a. In table 13, the 1 st order differences D_i^1 , $i = 1, 2, 3, \ldots$, are simply the terms $s_1, s_2, s_3, \ldots$, of the given series .

b. The pattern in table 13 is summarized below.

Table 14

$s_1 = D_1^1$	$D_1^1 = S_1 - S_o$	$S_1 = S_o + D_1^1$	$D_1^2 = D_2^1 - D_1^1$
$s_2 = D_2^1$	$D_2^1 = S_2 - S_1$	$S_2 = S_1 + D_2^1$	$D_1^3 = D_2^2 - D_1^2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$s_n = D_n^1$	$D_i^1 = S_i - S_{i-1}$	$S_i = S_{i-1} + D_i^1$	$D_i^{k+1} = D_{i+1}^k - D_i^k$
	$i = 1, 2, 3, \ldots$		$k = 1, 2, 3, \ldots$

On the analogy of page 27, we now derive expressions for S_o , S_1, S_2 , S_3, S_4 .

a. Take S_o . From page 32, we have $S_o = 0 = 1 \cdot 0$

b. Take S_1. From table 14, we have $S_1 = S_o + D_1^1 = 1 \cdot 0 + 1 \cdot D_1^1$

c. Take S_2 . From table 14, we have $S_2 = S_1 + D_2^1$, where D_2^1 is not a lead difference. But we have $D_i^{k+1} = D_{i+1}^k - D_i^k$ and set $k = 1$, $i = 1$ to get $D_2^1 = D_1^1 + D_1^2$. Substitute for S_1 and D_2^1 in S_2 , $S_2 = 1 \cdot 0 + 2\,D_1^1 + 1\,D_1^2$

d. Take S_3 . From table 14, we have $S_3 = S_2 + D_3^1$, where D_3^1 is not a lead difference. Two steps are required here to get S_3 in terms of lead differences. (See details in the answer part of this booklet.) $S_3 = 1 \cdot 0 + 3\,D_1^1 + 3\,D_1^2 + 1\,D_1^3$

e. On the pattern of the above , we also have $S_4 = 1 \cdot 0 + 4\,D_1^1 + 6\,D_1^2 + 4\,D_1^3 + 1\,D_1^4$.

Each partial sum can be expressed in terms of the lead differences . Moreover, the coefficients of the lead differences for a given partial sum are the numbers in the rows of the Pascal triangle shown below

in S_o					1				
S_1				1		1			
S_2			1		2		1		
S_3		1		3		3		1	
S_4	1		4		6		4		1
$\vdots$	↑				$\vdots$				

The 1 s in this diagonal are coefficients of S_o .

The expressions for S_o, S_1, S_2, . . . , in terms of the lead differences are shown schematically in table 15 (see page 28 for comparison) .

Table 15

$S_o = 1 \cdot 0 = 0$

$S_1 = 1 \cdot 0 + 1\ D_1^1$

$S_2 = 1 \cdot 0 + 2\ D_1^1 + 1\ D_1^2$

$S_3 = 1 \cdot 0 + 3\ D_1^1 + 3\ D_1^2 + 1\ D_1^3$

$S_4 = 1 \cdot 0 + 4\ D_1^1 + 6\ D_1^2 + 4\ D_1^3 + 1\ D_1^4$

Partial sums ↓	S_o	D_1^1	D_1^2	D_1^3	D_1^4 . . .
S_o	0 / 1				
S_1	0 / 1	D_1^1 / 1			
S_2	0 / 1	D_1^1 / 2	D_1^2 / 1		
S_3	0 / 1	D_1^1 / 3	D_1^2 / 3	D_1^3 / 1	
S_4	0 / 1	D_1^1 / 4	D_1^2 / 6	D_1^3 / 4	D_1^4 / 1

Each square in table 15 is divided into 2 sections. The bottom half has an element of the Pascal triangle. The upper half has a lead difference (except the first column). To get a partial sum S_o, S_1, . . . , of the series, form the product of the elements in each square, in a given row, and take their sum (see to the left of table 15) .

How do we get a general formula for S_n ? The elements in the Pascal triangle, page 33, are the binomial coefficients defined by

$$\binom{r}{t} = \frac{r\,!}{t\,!\,(r-t)\,!} \quad , \qquad \text{where } r\,! = 1 \cdot 2 \cdot 3 \ \ldots \ \cdot r, \quad 0\,! = 1 \ .$$

Number the rows of the Pascal triangle as follows :

Row										Coefficients for terms in
0					1					S_o
1				1		1				S_1
2			1		2		1			S_2
3		1		3		3		1		S_3
4	1		4		6		4		1	S_4
⋮				and so on						⋮

Rewrite the elements in Pascal's triangle in terms of the binomial coefficients . Find the binomial coefficients for the row n which will be the coefficients in S_n , the n th partial sum for a given series .

Row	Coefficients of			
0	S_o	$r = 0,$	$t = 0$	$\binom{0}{0}$
1	S_1	$r = 1,$	$t = 0, 1$	$\binom{1}{0}$ $\binom{1}{1}$
2	S_2	$r = 2,$	$t = 0, 1, 2$	$\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$
3	S_3	$r = 3,$	$t = 0, 1, 2, 3$	$\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$
⋮	⋮			
n	S_n	$r = n,$	$t = 0, 1, \ldots, n$	$\binom{n}{0}$ $\binom{n}{1}$ $\binom{n}{2}$ $\binom{n}{3}$ .... $\binom{n}{n}$

In S_n, $\binom{n}{0}$ is the coefficient of the first term S_o, but since $S_o = 0$, this term will drop out in S_n.

Now the coefficients for the terms in S_n are

$$\binom{n}{0} = \frac{n!}{0!\,(n-0)!} = 1 \qquad \binom{n}{1} = \frac{n!}{1!\,(n-1)!} = n \qquad \binom{n}{2} = \frac{n!}{2!\,(n-2)!} = \frac{n(n-1)}{2!}$$

$$\binom{n}{3} = \frac{n!}{3!\,(n-3)!} = \frac{n(n-1)(n-2)}{3!} \qquad \binom{n}{4} = \frac{n!}{4!\,(n-4)!} = \frac{n(n-1)(n-2)(n-3)}{4!}$$

$$\binom{n}{n-1} = \frac{n!}{(n-1)!\,(n-(n-1))!} = n \qquad \binom{n}{n} = \frac{n!}{n!\,(n-n)!} = 1$$

Thus, the n th partial sum is given by

$$S_n = n\,D_1^1 + \frac{n(n-1)}{2!}\,D_1^2 + \frac{n(n-1)(n-2)}{3!}\,D_1^3 + \frac{n(n-1)(n-2)(n-3)}{4!}\,D_1^4 + \ldots\ldots + n\,D_1^{n-1} + D_1^n \qquad \ldots (E)$$

where D_1^1 is the 1st order difference, first difference and for a series is simply the 1st term s_1 of the series (see page 32 table 13)

and after some D_1^k, the D_1^{k+1} s are all zero .

<u>Example 1</u> Given the series $S_n = 1 + 2 + 3 + \ldots + n$. Find S_n.

<u>Solution :</u> First check that for some $k = 1, 2, 3, \ldots,$ D^k, the kth order differences all have one and the same nonzero value. Then use the facts in table 13 and formula (E) .

n	series D^1	D^2
1	$1 = D_1^1$	
		$1 = D_1^2$
2	2	
		1
3	3	
		1
4	4	
⋮	⋮	⋮

In example 1, page 35, $D_1^1 = 1$ and $D_1^2 = 1$. Use (E) page 35 .

$$S_n = n\,D_1^1 + \frac{n(n-1)}{2\,!}\,D_1^2 = n\,1 + \frac{n(n-1)}{2}\,1 \quad , \text{ so that}$$

$$S_n = \frac{n(n+1)}{2} \quad , \text{ the same result as seen previously on pages 31, 32 .}$$

The method of finite differences does not give a solution for S_n for all series. Only those series where the terms have one and the same nonzero constant for k th order differences can be solved by the method of finite differences .

Example 1 Given: the series $1 + 1/2 + 1/4 + 1/8 + 1/16 + . . . + 1/n^2 = S_n$
Find a formula for the n th partial sum, S_n .

Solution: First check that for some k = 1, 2, 3, . . . , D^k, the k th order differences all have one and the same nonzero value .

n	series D^1	D^2	D^3	D^4	D^5 . . . . .
1	1				
		-1/2			
2	1/2		1/4		
		-1/4		-1/8	
3	1/4		1/8		1/16
		-1/8		-1/16	. . . . .
4	1/8		1/16		1/32
		-1/16		-1/32	
5	1/16		1/32		
		-1/32			
6	1/32				
⋮	⋮	⋮	⋮	⋮	⋮

Here there is no k = 1, 2, 3, . . . , such that D^k , the k th order differences all have one and the same nonzero value . Thus, we cannot use (E) page 35.

Example 2 Given: the series 1 + 2 + 3 + 4 + 5 + . . . + .
Find the generator of the series, s_n , and S_n the n th partial sum for the series .

Solution: Notice particularly the designation of the terms in the tables .

For s_n , the nth general term .

n	Series	D^1
1	$1 = s_1$	
		$1 = D_1^1$
2	2	
		1
3	3	
		1
4	4	
⋮	⋮	⋮

Here use formula (D), page 29 .

$$s_n = s_1 + (n-1)\,D_1^1$$
$$= 1 + (n-1)\,1$$
$s_n = n$, the nth term of the series .

For S_n, the nth partial sum .

n	Series D^1	D^2
1	$1 = D_1^1$	
		$1 = D_1^2$
2	2	
		1
3	3	
		1
4	4	
⋮	⋮	⋮

Here use formula (E), page 35 .

$$S_n = n\,D_1^1 + \frac{n(n-1)}{2\,!}\,D_1^2$$
$$= n \cdot 1 + \frac{n(n-1)}{2\,!}\,1$$
$S_n = \frac{n(n+1)}{2}$, nth partial sum for the series .

PROBLEMS

Verify the n th partial sum S_n for each of the following series .

58.

1. $1^2 + 2^2 + 3^2 + \dots + n^2 = S_n = \frac{n(n+1)(2n+1)}{6}$
2. $1^3 + 2^3 + 3^3 + \dots + n^3 = S_n = \frac{n^2 (n+1)^2}{4}$
3. $2 + 4 + 6 + \dots + 2n = S_n = n(n+1)$
4. $1 + 3 + 5 + \dots + (2n-1) = S_n = n^2$
5. $2 + 7 + 12 + \dots + (5n-3) = S_n = \frac{n(5n-1)}{2}$
6. $a + (a+d) + (a+2d) + \dots + (a+(n-1)d) = S_n = \frac{n(dn + (2a-d))}{2}$
7. $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \dots + n(n+2) = S_n = \frac{n(n+1)(2n+7)}{6}$
8. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n+1) = S_n = \frac{n(n+1)(n+2)}{3}$
9. $1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = S_n = \frac{n(4n^2-1)}{3}$
10. $2^2 + 4^2 + 6^2 + \dots + (2n)^2 = S_n = \frac{2n(n+1)(2n+1)}{3}$
11. $1^2 + 4^2 + 7^2 + \dots + (3n-2)^2 = S_n = \frac{n(6n^2 - 3n - 1)}{2}$
12. $2 + 5 + 10 + \dots + (n^2+1) = S_n = \frac{n(2n^2 + 3n + 7)}{6}$
13. $1^3 + 3^3 + 5^3 + \dots + (2n-1)^3 = S_n = n^2(2n^2-1)$
14. $1 + 4 + 10 + \dots + \frac{n(n+1)(n+2)}{6} = S_n = \frac{n(n+1)(n+2)(n+3)}{24}$

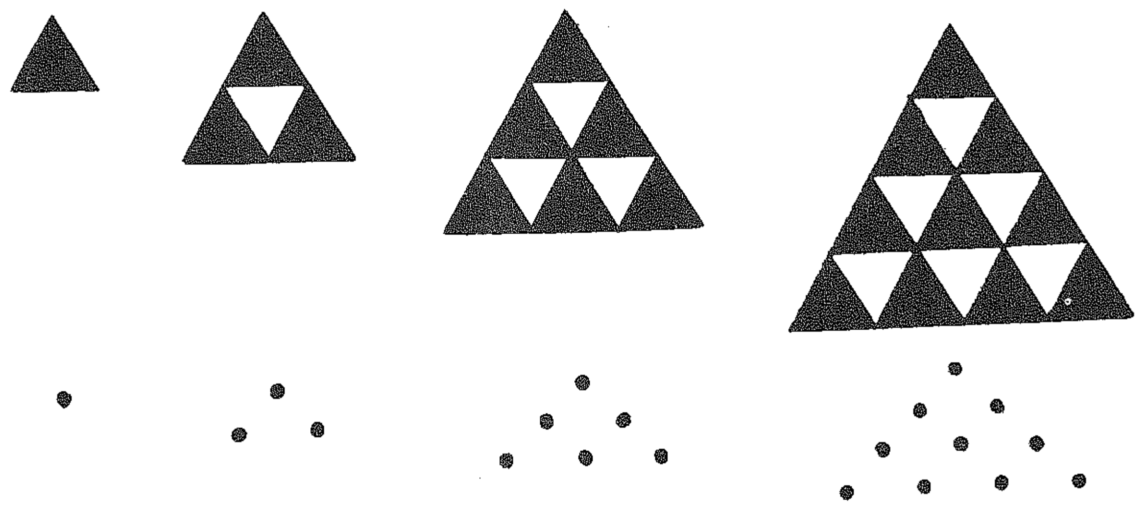
59. Verify s_n , the n th general term of the series and S_n , the n th partial sum for the series .

1. $3(1)(2) + 3(2)(3) + 3(3)(4) + \dots + 3n(n+1) = n(n+1)(n+2) = S_n$
2. $1(2)(3) + 2(3)(4) + 3(4)(5) + \dots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4} = S_n$
3. $3 + 6 + 9 + \dots + 3n = \frac{3n(n+1)}{2} = S_n$
4. $4 + 8 + 12 + \dots + 4n = 2n(n+1) = S_n$
5. $5 + 10 + 15 + \dots + 5n = \frac{5n(n+1)}{2} = S_n$

18. Figurate numbers

The figurate numbers - the representation of numbers by dots in special geometric configurations - originated with the early Pythagorean Brotherhood composed of the followers of the Greek mathematician Pythagoras (c. 530 B.C.). The figurate numbers represent a link between arithmetic and geometry .

a. Triangular numbers



1 3 6 10

Table 16

Number of rows n	Total number of triangles (dots)	D^1	D^2
1	$1 = s_1$		
		$2 = D^1_1$	
2	3		$1 = D^2_1$
		3	
3	6		1
		4	
4	10		
⋮	⋮		
n	s_n		

To get s_n, use formula (D) page 29 .

$$s_n = s_1 + (n-1)D^1_1 + \frac{(n-1)(n-2)}{2!}D^2_1 = 1 + (n-1)2 + \frac{(n-1)(n-2)}{2}1$$

$s_n = \dfrac{n(n+1)}{2}$, the nth term (generator) for the triangular numbers .

If we take the sum of the triangular numbers, $S_n = 1 + 3 + 6 + 10 + \ldots + \dfrac{n(n+1)}{2}$ then by the method on pages 30-35, we have

Table 17

n	Terms of series	D^2	D^3
1	$1 = D^1_1$		
		$2 = D^2_1$	
2	3		$1 = D^3_1$
		3	
3	6		1
		4	
4	10		
⋮	⋮		

Here , to get the n th partial sum, we use formula (E) page 35 .

$$S_n = n\,D_1^1 + \frac{n\,(n-1)}{2\,!}\,D_1^2 + \frac{n(n-1)(n-2)}{3\,!}\,D_1^3 \,,$$ and substitute values from table 17,

$$= n\,1 + \frac{n(n-1)}{2}\,2 + \frac{n(n-1)(n-2)}{6}\,1$$

$$S_n = \frac{n(n+1)(n+2)}{6}\,,$$ the n th partial sum for triangular numbers.

The above formula for S_n is actually a formula for the pyramidal numbers with a triangular base (see Figure 1) .

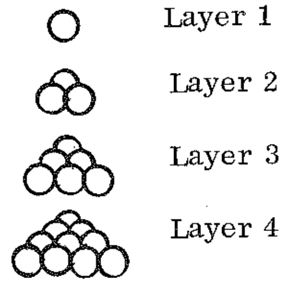
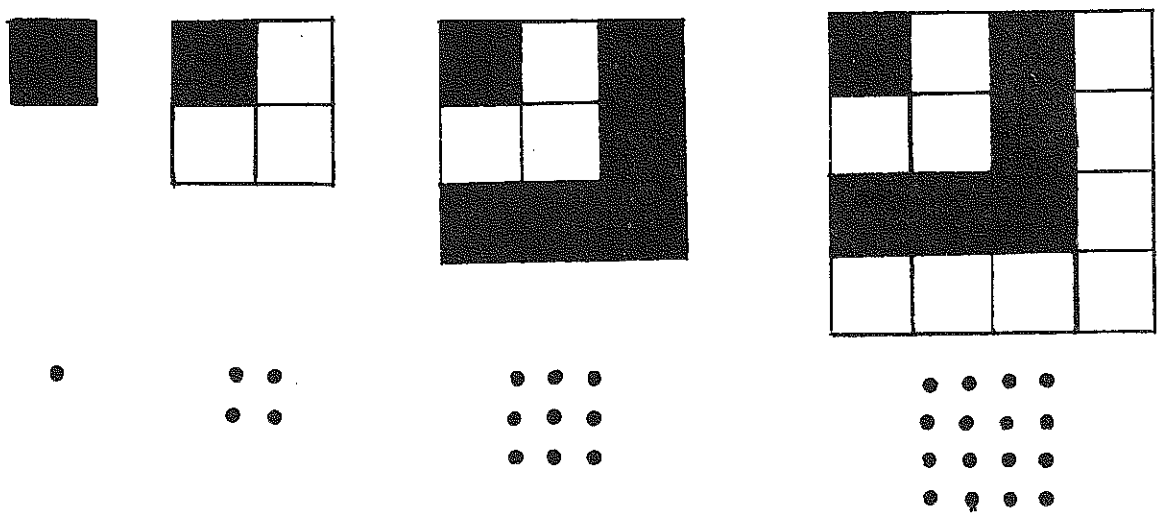


Figure 1

b. Square numbers



1 4 9 16

Table 18

Number of rows n	Total number of squares (dots)	D^1	D^2
1	$1 = s_1$		
		$3 = D_1^1$	
2	4		$2 = D_1^2$
		5	
3	9		2
		7	
4	16		
⋮	⋮	⋮	
n	s_n		

To get s_n , use formula (D) page 29 .

$$s_n = s_1 + (n-1)\,D_1^1 + \frac{(n-1)(n-2)}{2\,!}\,D_1^2$$

$$= 1 + (n-1)\,3 + \frac{(n-1)(n-2)}{2}\,2$$

$s_n = n^2$, the n th term (generator) for the square numbers .

If we take the sum of the square numbers , $S_n = 1^2 + 2^2 + 3^2 + \ldots + n^2$, then by the method on pages 30-35, we have table 19, page 40 .

Table 19

n	Terms of series	D^2	D^3
1	$1 = D_1^1$		
		$3 = D_1^2$	
2	4		$2 = D_1^3$
		5	
3	9		2
		7	
4	16		
⋮	⋮		

Here to get the n th partial sum, we use formula (E), page 35.

$S_n = n D_1^1 + \frac{n(n-1)}{2!} D_1^2 + \frac{n(n-1)(n-2)}{3!} D_1^3$, and substitute values from table 19

$= n\,1 + \frac{n(n-1)}{2} 3 + \frac{n(n-1)(n-2)}{6} 2$

$S_n = \frac{n(n+1)(2n+1)}{6}$, the n th partial sum for square numbers.

The above formula for S_n is actually a formula for the pyramidal numbers with a square base (see Figure 1) .

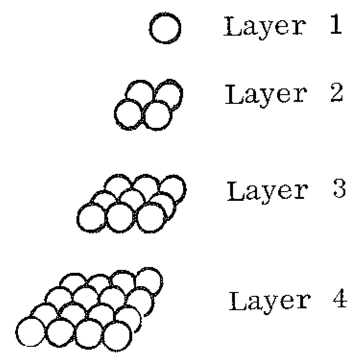


Figure 1

c. Pentagonal numbers

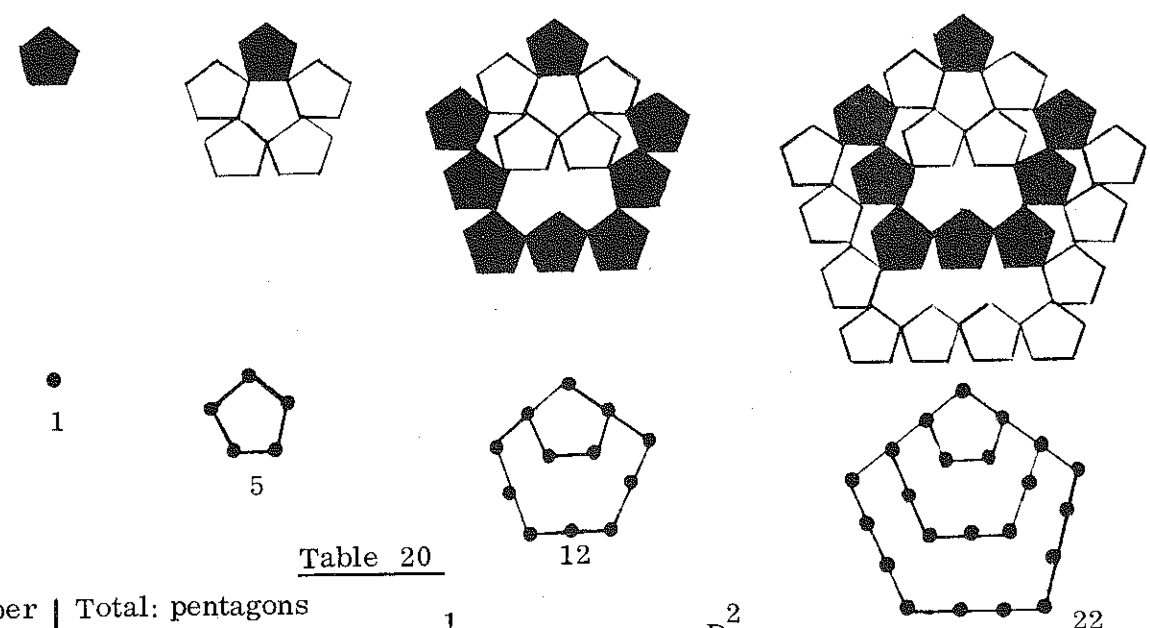


Table 20

Step number n	Total: pentagons (dots)	D^1	D^2
1	$1 = s_1$		
		$4 = D_1^1$	
2	5		$3 = D_1^2$
		7	
3	12		3
		10	
4	22		
⋮	⋮	⋮	⋮
n	s_n		

To get s_n, use formula (D), page 29 .

$$s_n = s_1 + (n-1)\,D_1^1 + \frac{(n-1)(n-2)}{2\,!}\,D_1^2$$

$$= 1 + (n-1)\,4 + \frac{(n-1)(n-2)}{2}\,3$$

$$s_n = \frac{n\,(3n-1)}{2}$$, the n th term (generator) for the pentagonal numbers .

If we take the sum of the pentagonal numbers, $S_n = 1 + 5 + 12 + \ldots + \frac{n\,(3n-1)}{2}$,
then by the method on pages 30-35, we have table 21 .

Table 21

n	Terms of series	D^2	D^3
1	$1 = D_1^1$		
		$4 = D_1^2$	
2	5		$3 = D_1^3$
		7	
3	12		3
		10	
4	22		
⋮	⋮		

Here , to get the n th partial sum, we use formula (E), page 35 .

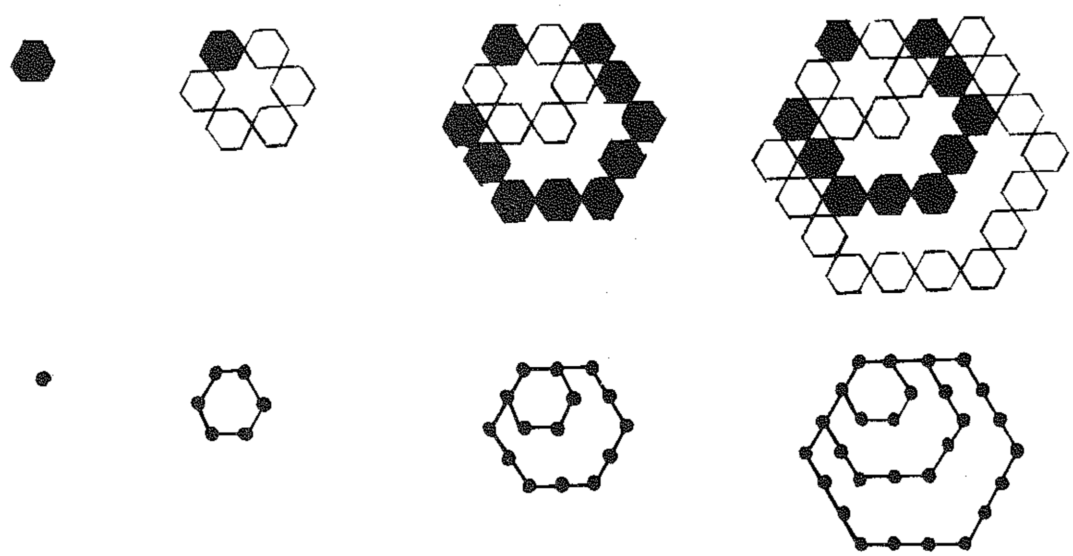
$$S_n = n\,D_1^1 + \frac{n(n-1)}{2\,!}\,D_1^2 + \frac{n\,(n-1)(n-2)}{3\,!}\,D_1^3 ,$$ and substitute values from table 21

$$= n\,1 + \frac{n(n-1)}{2}\,4 + \frac{n(n-1)(n-2)}{6}\,3$$

$$S_n = \frac{n^2\,(n+1)}{2}$$, the n th partial sum for the pentagonal numbers .

S_n above is also a formula for pentagonal pyramidal numbers. From the arrangement of the pentagons or of the dots (page 40), we see that there are gaps in the arrangements . In spite of the name pyramidal, we cannot construct a good geometric model for the pentagonal (or any higher k-gonal) pyramidal numbers . Pyramids can be constructed for triangular and square pyramidal numbers .

d. Hexagonal numbers



1 6 15 28

Table 22

Step number n	Total: hexagons (dots)	D^1	D^2
1	$1 = s_1$		
		$5 = D_1^1$	
2	6		$4 = D_1^2$
		9	
3	15		4
		13	
4	28		
⋮	⋮	⋮	⋮
n	s_n		

To get s_n, use formula (D), page 29 .

$$s_n = s_1 + (n-1) D_1^1 + \frac{(n-1)(n-2)}{2\,!} D_1^2$$
$$= 1 + (n-1)5 + \frac{(n-1)(n-2)}{2} 4$$

$s_n = n(2n-1)$, the n th term (generator) for the hexagonal numbers .

PROBLEMS

60. Find a formula for the n th partial sum, S_n, for the hexagonal numbers.

$S_n =$ ______________________________

61. Verify s_n for each of the following figurate numbers .

a. Heptagonal numbers 1, 7, 18, 34, . . . , $s_n = \frac{n}{2}(5n-3)$

b. Octagonal numbers 1, 8, 21, 40, . . . , $s_n = n(3n-2)$

c. Nonagonal numbers 1, 9, 24, 46, . . . , $s_n = \frac{n}{2}(7n-5)$

d. Decagonal numbers 1, 10, 27, 52, . . . , $s_n = n(4n-3)$

e. k-gonal

Now examine tables 16 (page 38), 18 (page 39), 20 (page 40), 22 (page 42), especially the lead term and the lead differences . We have summarized this information in the table below .

Table 23

Lead term and lead differences	n	Total number of dots		
Triangular numbers	1	$1 = s_1$	$2 = D_1^1$	$1 = D_1^2$
Square numbers	1	$1 = s_1$	$3 = D_1^1$	$2 = D_1^2$
Pentagonal numbers	1	$1 = s_1$	$4 = D_1^1$	$3 = D_1^2$
Hexagonal numbers	1	$1 = s_1$	$5 = D_1^1$	$4 = D_1^2$
⋮	⋮	⋮	⋮	⋮
k-gonal	1	$1 = s_1$	$(k-1) = D_1^1$	$(k-2) = D_1^2$

The last line permits us to find a general formula for s_k, $k = 3, 4, 5, \ldots$, for all plane figurate numbers (assuming that our induction is accepted) . Thus, using formula (D), page 29 where we substitute $s_1 = 1$, $D_1^1 = (k-1)$, $D_1^2 = (k-2)$

$$s_n = s_1 + (n-1)D_1^1 + \frac{(n-1)(n-2)}{2!} D_1^2$$

$$(s_n)_k = 1 + (n-1)(k-1) + \frac{(n-1)(n-2)}{2}(k-2)$$

$$\boxed{(s_n)_k = \frac{n[(k-2)(n-1)+2]}{2}} \quad \ldots\ldots\ldots\ldots \quad (F)$$

where $k = 3, 4, 5, \ldots$, $n = 1, 2, 3, \ldots$ for each k selected ,

$(s_n)_k$ gives the generator for the plane figurate numbers .

Example 1 For the triangular numbers , $k = 3$, and we have from (F)

Triangular numbers $(s_n)_3 = \frac{n[(3-2)(n-1)+2]}{2} = \frac{n(n+1)}{2}$

Example 2 For the square numbers , $k = 4$, and we have from (F)

Square numbers $(s_n)_4 = \frac{n[(4-2)(n-1)+2]}{2} = n^2$

Example 3 Another method for deriving formula (F) above . A study of tables 16 (page 38), 18 (page 39), 20 (page 40) , 22 (page 42) shows that we could have proposed the following questions :

A. Triangular numbers (table 16, page 38) . Find a sequence such that

a. $s_1 = 1$, $s_2 = 3$

b. the first differences D_i^1, $i = 1, 2, 3, \ldots$, form an arithmetic sequence $a + (n-1)d$, where a is the first term, d is the common difference of the terms, and the differences of the D_i^1 s namely, D_i^2, $i = 1, 2, 3, \ldots$, all have the same nonzero value 1 .

B. Square numbers (table 18, page 39) . Find a sequence such that

a. $s_1 = 1$, $s_2 = 4$

b. the first differences D_i^1, $i = 1, 2, 3, \ldots$, form an arithmetic sequence and the differences of the D_i^1 s namely, D_i^2, $i = 1, 2, 3, \ldots$, all have the same nonzero value 2 .

C. k-gonal numbers (table 23, page 42) . Find a sequence such that

a. $s_1 = 1$, $s_2 = k$

b. the first differences D_i^1, $i = 1, 2, 3, \ldots$, form an arithmetic sequence and the differences of the D_i^1 s namely, D_i^2, $i = 1, 2, 3, \ldots$, all have the same nonzero value $(k-2)$.

PROBLEMS

62. Take example 3, part C, page 43 .
  a. Show in a table the first 6 terms of the sequence, and
  b. derive a general formula $(s_n)_k$ for the sequence .

Hint:

n	Sequence	D^1	D^2
1	$s_1 = 1$		
		$(k-1) = D^1_1$	
2	$s_2 = k$		$(k-2) = D^2_1$
		$? = D^1_2$	
3	$s_3 = ?$		
⋮			

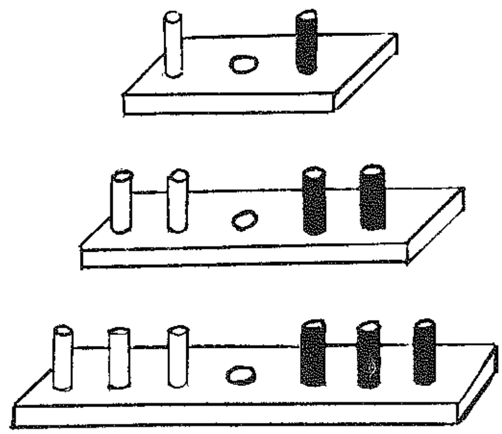
where $D^2_1 = D^1_2 - D^1_1$ and substituting, solve for D^1_2 ,

then $D^1_2 = s_3 - s_2$, and substituting, solve for s_3 , and so on.

63. Find the general term S_n , the n th partial sum, for each of the following series whose terms form an arithmetic sequence, that is , $s_n = a + (n-1)d$.

  a. $s_n = a + (n-1)d$ and $a = 1$, $d = 1$
  b. $s_n = a + (n-1)d$ and $a = 1$, $d = 2$
  c. $s_n = a + (n-1)d$ and $a = 1$, $d = 3$
  d. $s_n = a + (n-1)d$ and $a = 1$, $d = k - 2$, k a positive integer, $k \geq 3$.

64. The Peg Swap



Pairs of pegs (one white, one black) are placed on a peg board , with 1 extra hole in the middle of the board (see illustration) .

Interchange the positions of the black pegs with the white pegs according to the following rules :
  a. white pegs move to the right, black pegs to the left,
  b. only one peg can be moved at a time,
  c. a peg can move to an adjacent hole, or
  d. a peg can jump over one peg of the opposite color to an empty hole .

Find: 1. the smallest number of moves needed to interchange 1, 2, 3, 4, 5 pairs of pegs,
  2. a general formula for the smallest number of moves needed to interchange n pairs of pegs . Fill in the table below.

1.

Pairs of pegs n	Smallest number of moves	D^1	D^2	D^3	D^4
1	3				
2	___				
3	___				
4	___				
5	35				

2.

$s_n =$ ______________________

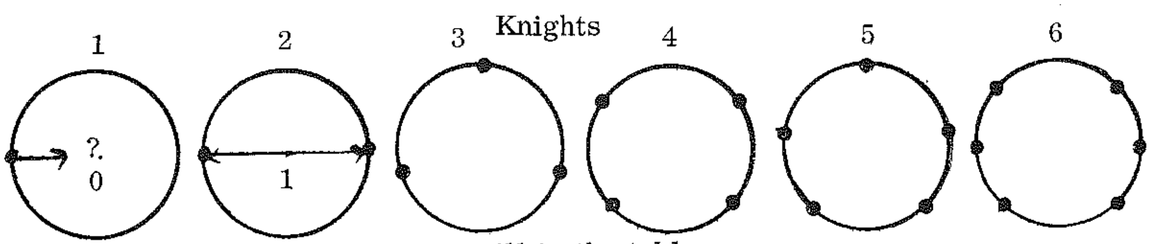
65. Knights of the Round Table and handshakes

A. King Arthur's knights had a tradition that whenever a group of them gathered at the Round Table, each knight would shake hands with every knight at the Table. No knight shook hands with himself.

1. Find the total number of handshakes exchanged for 1, 2, 3, 4, 5, 6 knights present at the Round Table .

2. Find a general formula for the number of handshakes if n knights are at the Table .

Let the dots in the figure below represent the knights. Draw lines to show the number of handshakes .



Fill in the table

1.

Number of knights n	Number of handshakes	D^1	D^2	D^3	D^4
1	0				
2	1				
3	___				
4	___				
5	___				
6	___				
⋮					

2. s_n = ____________________

B. Repeat the problem above but with the change that no knight shakes hands with his immediate neighbors to the right or to the left .

3. Find the total number of handshakes exchanged for 1, 2, 3, 4, 5, 6 knights.

4. Find a general formula for the number of handshakes if n knights are at the table . Use a drawing similar to the circles and dots above .

Fill in the table

3.

Number of knights n	Number of handshakes	D^1	D^2	D^3	D^4
1	0				
2	0				
3	___				
4	___				
5	___				
6	___				
.					

4. s_n = ____________________

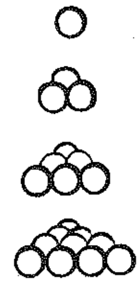
66. The Fair Grapefruit pyramids

1. The fruit stand owner at the Brockton Fair displayed the grapefruit in a pyramid .

   Each layer of the pile is triangular so that each grapefruit is supported by 3 others on the layer below it .

   a. Find the number of grapefruit needed for piles of 1, 2, 3, 4, 5, 6 layers .

   b. Find a general formula for the number of grapefruit needed for a pile of n layers .



Layer 1

Layer 2

Layer 3

Layer 4

⋮

Fill in the table

a.

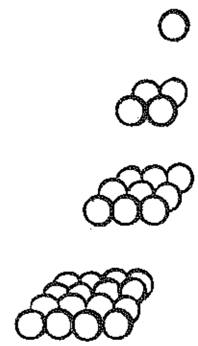
Number of layers n	Total number of grapefruit	D^1	D^2	D^3
1	1			
2	___			
3	___			
4	___			
5	___			
6	56			

b. s_n = ________________

2. The peaches were also displayed in a pyramid . Each layer is arranged in a square so that each peach is supported by 4 peaches on the layer below it .

   a. Find the number of peaches needed for piles of 1, 2, 3, 4, 5, 6 layers .

   b. Find a general formula for the number of peaches needed for a pile of n layers .



Layer 1

Layer 2

Layer 3

Layer 4

⋮

Fill in the table

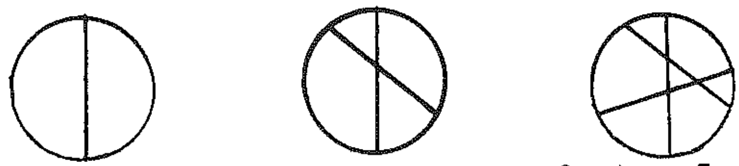
a.

Number of layers n	Total number of peaches	D^1	D^2	D^3
1	1			
2	___			
3	___			
4	___			
5	55			
6	91			

b. s_n = ________________

67. Pie Patter

1.



1 cut 2   2 cuts 4   3 cuts 7

A pie can be cut into 2 pieces by 1 straight cut . With 2 straight cuts, the greatest number of pieces you can get is 4 . If the size and shape of the pieces do not matter, what is

a. the greatest number of pieces into which a pie can be divided by 3, 4, 5, 6 straight cuts , and

b. find a general formula for the greatest number of pieces that you can get from n cuts.

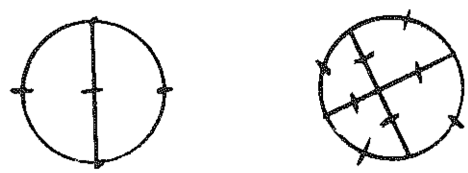
Fill in the table

a.

Number of cuts n	Number of pieces	D^1	D^2	D^3	D^4
1	2				
2	4				
3	7				
4	___				
5	___				
6	22				

b. s_n = ______________________

2.



1 cut 3   2 cuts 8

Slice the pie as above . Find the number of edges for the pieces, either straight or curved for 1, 2, 3, 4, 5 cuts . A straight edge common to 2 pieces should be counted only once . For example, after 1 cut there are 3 edges, after 2 cuts there are 8 edges , and so on .

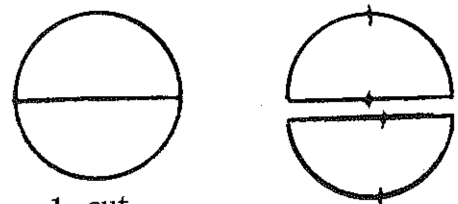
Also find a general formula for the number of edges after n cuts .

Fill in the table

Number of cuts n	Number of edges	D^1	D^2	D^3	D^4
1	3				
2	8				
3	___				
4	___				
5	35				

s_n = ______________________

3. Slice the pie as in part 1 , page 47 . Now take the pieces apart and count every edge, curved or straight . For example , 1 cut gives 4 edges .



1 cut 4 edges

a. Find the total number of edges for 1, 2, 3, 4, 5 cuts .
b. Find a general formula for the number of edges for n cuts .

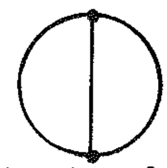
<u>Fill in the table</u>

a.

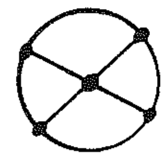
Number of cuts n	Number of edges	D^1	D^2	D^3	D^4
1	4				
2	___				
3	___				
4	___				
5	60				

b. s_n = ____________________

4. Slice the pie as in part 1, page 47 . We will now count the number of points on the cut pieces of pie. A point occurs where two cuts cross or where a cut reaches the edge of the pie . Each point is counted once, no matter how many pieces meet at the point . For example, there are 2 points after 1 cut, 5 points after 2 cuts, and so on .



1 cut 2



2 cuts 5

a. Find the number of points for 1, 2, 3, 4, 5 cuts.
b. Find a general formula for the number of points after n cuts .
c. Form the expression:
(number of pieces) + (number of points) - (number of edges) and find its value for n cuts .

<u>Fill in the table</u>

a.

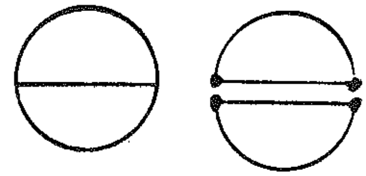
Number of cuts n	Number of points	D^1	D^2	D^3	D^4
1	2				
2	5				
3	___				
4	___				
5	20				

b. s_n = ____________________

c. ____

5. Slice the pie as in part 1, page 47. Read part 4, page 48.

We now take the pieces of pie apart and count the total number of points . For example , from 1 cut we get 4 points .



a. Find the total number of points for 1, 2, 3, 4, 5 cuts.

b. Find a general formula for the total number of points for n cuts .

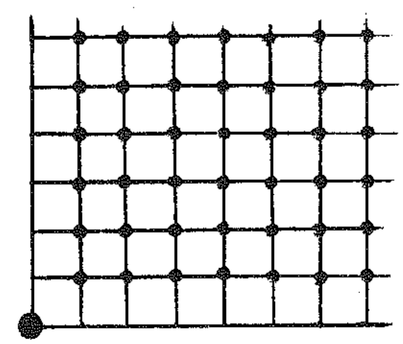
Fill in the table

a.

Number of cuts n	Number of points	D^1	D^2	D^3	D^4
1	4				
2	___				
3	___				
4	___				
5	60				

b. s_n = ______________________

68. Square City travel



A. The streets of Square City are laid out in a square tiling pattern . The Square City cab driver has a cab stand at a corner marked A .

He will drive by the shortest route to any corner not on the outskirts of the city, that is, he will drive only to the corners marked with dots .

Note: Since a destination 2 blocks to the right and 1 block up is different from a destination 2 blocks up and 1 block to the right, the problem here is the same as looking for the different ways to write n as a sum of two positive integers, counting order (1 + 2 and 2 + 1 are counted as different destinations of 3 blocks) .

a. How many destinations lie 1, 2, 3, 4, 5 blocks from point A ?

b. Find a general formula for the number of destinations which are n blocks from A .

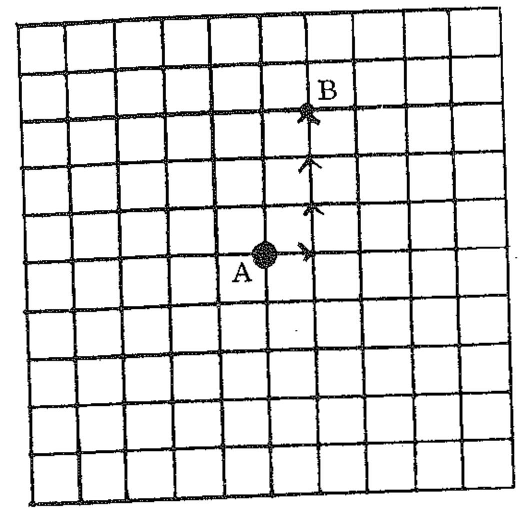
Fill in the table

a.

Number of blocks n	Number of destinations	D^1	D^2	D^3	D^4
1	0				
2	1				
3	___				
4	___				
5	4				

b. s_n = ______________________

B. In Square City, the cab stand was changed to the point marked A. The cab drivers take the shortest route possible. Since they cannot cut across blocks, the distance is computed in terms of the actual number of blocks travelled. The figure shows a route from A to B, a distance of 4 blocks. Passengers are always dropped off at the intersection of two blocks.



a. Find the number of destinations 1, 2, 3, 4, 5 blocks from point A.

b. Find a general formula for the number of destinations which are n blocks from A.

c. Find a general formula for the total number of destinations which are n or fewer blocks from A.

<u>Fill in the table</u>

a.

Number of blocks n	Number of destinations	D^1	D^2	D^3	D^4
1	4				
2	___				
3	___				
4	___				
5	20				

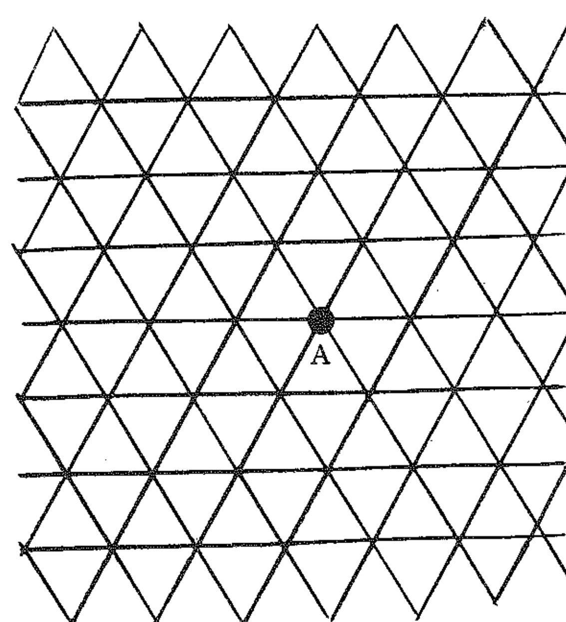
b. s_n = ______________________________

c. S_n = ______________________________

69. <u>Triville City travel</u>

The cab drivers of Triville follow the same rules as those of Square City (they take the shortest path to a given destination), but the streets of their city are laid out in a triangular tiling pattern.

Passengers are always dropped off at the intersection of two streets.



a. Find the number of destinations (shortest route) which are 1, 2, 3, 4, 5 blocks from A.

b. Find a general formula for the number of destinations which are n blocks from A.

c. Find a general formula for the total number of destinations which are n or fewer blocks from A.

Fill in the table

a.

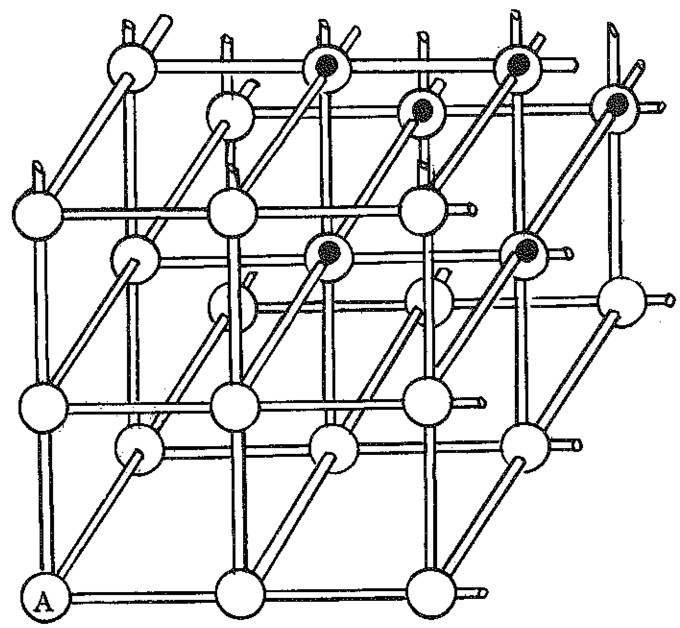
Number of blocks n	Number of destinations	D^1	D^2	D^3	D^4
1	6				
2	___				
3	___				
4	___				
5	30				

b. s_n = ______________________

c. S_n = ______________________

70. Space shuttle service

A. Consider a space station shuttle driver with a shuttle stand at a corner A of the space station. He takes passengers to their rooms which are laid out in a cubical gridwork shown in the illustration.
The shuttle craft must travel in the connecting tubes which all have the same length. Always take the shortest route. The shuttle driver does not go to any room on the outer surface of the space station (he goes only to the rooms marked with a dot). Read problem 68, A page 49.



a. How many rooms lie 1, 2, 3, 4, 5, 6 tube lengths from A.

Note: This part is equivalent to looking for the different ways of writing n as a sum of 3 positive integers, counting order (that is, 1 + 1 + 2, 1 + 2 + 1, 2 + 1 +1 are counted as three distinct sums of four tube lengths).

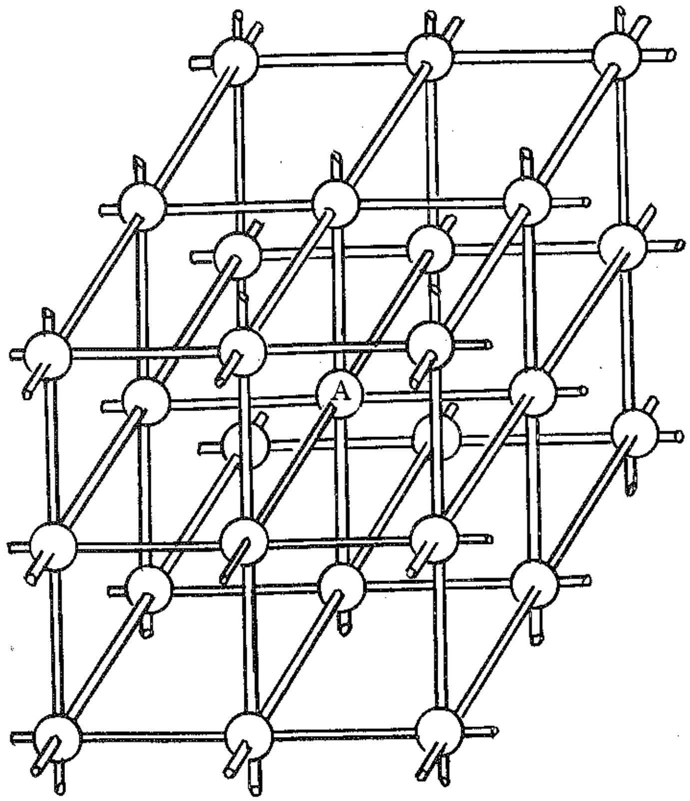
b. Find a general formula for the number of rooms (shortest route) which are n tube lengths from the shuttle stand A.

Fill in the table

a.

Number of tube lengths n	Number of rooms	D^1	D^2	D^3	D^4
1	0				
2	0				
3	1				
4	___				
5	___				
6	10				

b. s_n = ______________________



B. The shuttle stand is changed to point A.

The shuttle craft must travel to the rooms in the connecting tubes which all have the same length .

The shuttle craft takes the shortest route from the shuttle stand A .
Read problem 68 B, page 50 .

a. Find the number of rooms which are 1, 2, 3, 4, 5 tube lengths from the shuttle stand A .

b. Find a general formula for the number of rooms which are n tube lengths from the shuttle stand A .

c. Find a general formula for the total number of rooms which are less than or equal to n tube lengths from the shuttle stand A .

<u>Fill in the table</u>

a.

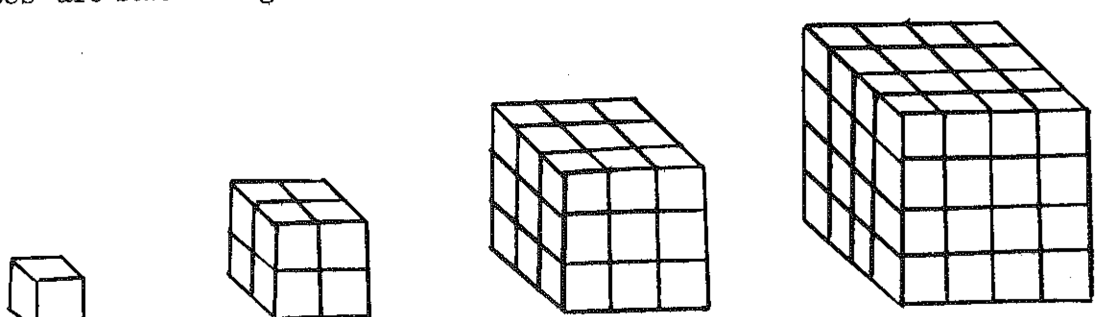
Number of tube lengths n	Number of rooms	D^1	D^2	D^3	D^4
1	6				
2	___				
3	___				
4	66				
5	102				

b. $s_n =$ ______________________________

c. $S_n =$ ______________________________

71. Painting cubes

Unit cubes are stacked together to form larger cubes as shown below .



Edge length	1 unit	2 units	3 units	4 units
Number of unit cubes	$1^3 = 1$	$2^3 = 8$	$3^3 = 27$	$4^3 = 64$

Suppose now that the 6 outer surfaces of each of the stacked cubes are painted. The unit cubes on the outside in each stacking will have 1, 2, or 3 surfaces painted, while the unit cubes in the interior of the stacked cubes may have 0 surfaces painted .

a. In a cube stacking measuring 1, 2, 3, 4, 5, 6 units on an edge, find the number of unit cubes that have 1, 2, or 3 surfaces painted.

The case where 0 surfaces are painted is worked out below .

b. Find general formulas which give the number of unit cubes with 1, 2, 3 surfaces painted in terms of the length of the edge in each stacking .

Fill in the table

Length of edge n	Unit cubes with 0 faces painted	1 face painted	2 faces painted	3 faces painted
1	0	0	0	0
2	0	0	0	___
3	1	___	___	___
4	8	___	___	___
5	27	___	___	___
6	64	96	48	___

b. Zero faces painted . See table above .

Length of edge n	Unit cubes with 0 faces painted	D^1	D^2	D^3
1	0			
		0		
2	0			
			1	
		1		
3	1			5
			6	
		7		
4	8			6
			12	
		19		
5	27			6
			18	
		37		
6	64			
⋮	⋮	⋮	⋮	⋮
n	s_n			

Other than $D_1^3 = 5$, we have $D_i^3 = 6$ for $i = 2, 3, 4, \ldots,$. We now form a new sequence by dropping s_1 . Call the new sequence t_N , where $t_1 = 0$, $t_2 = 1, \ldots$.

N	Sequence t_N	D^1	D^2	D^3
1	0			
		1		
2	1			
			6	
		7		
3	8			6
			12	
		19		
4	27			6
			18	
		37		
5	64			
⋮	⋮			

By formula (D), page 29, we have $t_1 = 0$, $D_1^1 = 1$, $D_1^2 = 6$, $D_1^3 = 6$.

Thus,

$$t_N = 0 + (N-1)\,1 + \frac{(N-1)(N-2)}{2!}\,6 + \frac{(N-1)(N-2)(N-3)}{3!}\,6$$

$$t_N = (N-1)^3 .$$

Comparing the sequences s_n and t_N, we have that : $s_2 = t_1$, $s_3 = t_2$, . . . , $s_{N+1} = t_N$, ... Thus if $n = N+1$ (or $n - 1 = N$), $s_n = t_N$ and substituting $N = n - 1$ in t_N above, we get $s_n = ((n-1) - 1)^3$

or $s_n = (n-2)^3$.

The general formula for unit cubes with zero faces painted is

$$s_n = \begin{cases} 0 , & \text{for } n = 1 \\ (n-2)^3 , & \text{for } n \geq 2 \end{cases}$$

where n is the length of the edge in a stacked cube .

One face painted . See table on page 53 .

Length of edge n	Unit cubes with 1 face painted	D^1	D^2	D^3	D^4
1	0				
2	0				
3	___				
4	___				
5	___				
6	96				
.	.				
.	.				
n	s_n				

Two faces painted . See table on page 53 .

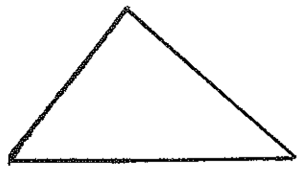
Length of edge n	Unit cubes with 2 faces painted	D^1	D^2	D^3	D^4
1	0				
2	0				
3	___				
4	___				
5	___				
6	48				
.					
n	s_n				

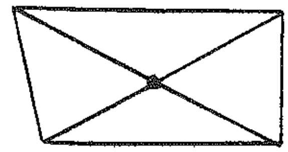
$s_n =$ ______________________

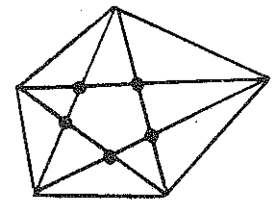
Three faces painted. Answer : ______________________

72. Intersecting points of diagonals in convex polygons .

Take convex polygons of 3, 4, 5 , . . . , n sides . Draw diagonals so that they intersect in the largest number of intersection points . Do not count vertices .







Intersection points of diagonals	0	1	5

Note: In order to get the maximum number of intersections, no more than 2 diagonals can pass through an intersection point . For convex polygons of 6, 8, 10, 12, . . . sides, the regular polygons will not have this property.

For example, in a regular hexagon, Figure 1, the diagonals have 13 points of intersection, while a convex hexagon can have 15 intersection points , Figure 2 . A regular octagon has only 49 points of intersection while octagons with no more than two diagonals through a point have 70 .

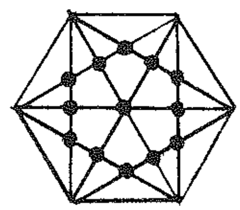


Figure 1

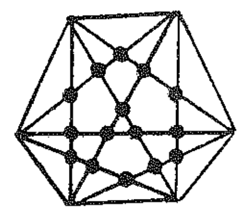


Figure 2

For an odd number of sides, the regular polygon will yield the maximum number of intersection points.

a. Find a general formula for the largest number of intersections of diagonals for a convex polygon of n sides .

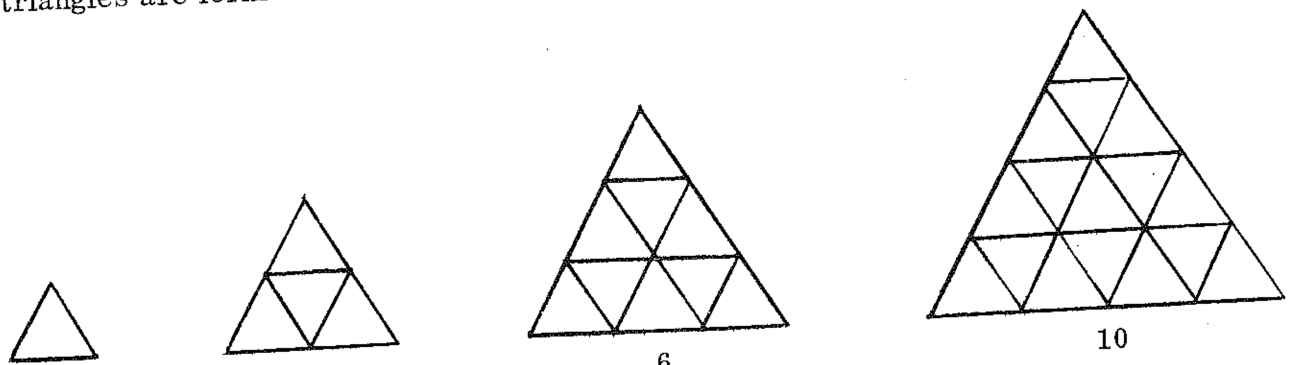
Fill in the table

Polygons Number of sides n	Number of intersections	D^1	D^2	D^3	D^4
3	0				
4	1				
5	5				
6	15				
7	____				
8	____				
9	126				
⋮	⋮				
n	s_n				

s_n = ______________________

73. Counting triangles

A. Unit triangles are formed into triangular patterns as shown below .



Vertex up	1	3	6	10
Vertex down	0	1	3	6
Rows	1	2	3	4

We continue the pattern by adding more rows of unit triangles .

a. Find the number of small triangles (unit triangles) with a vertex pointing up (△) in the larger triangles of 1, 2, 3, 4, 5, 6 rows of unit triangles .
Find a general formula for the number of unit triangles , vertex up, in a large triangle of n rows . Fill in the table

Number of rows n	Number of unit triangles (△)	D^1	D^2	D^3	D^4
1	1				
2	___				
3	___				
4	___				
5	___				
6	___				
.					
n	s_n				

s_n = ______________________

b. Find the number of unit triangles , vertex down (▽) in the larger triangles triangles of 1, 2, 3, 4, 5, 6, rows of unit triangles .

Find a general formula for the number of unit triangles, vertex down , in a large triangle of n rows .

Fill in the table

Number of rows n	Number of unit triangles (▽)	D^1	D^2	D^3	D^4
1	0				
2	___				
3	___				
4	___				
5	___				
6	___				
⋮					
n	s_n				

s_n = ________________________________

c. Find a general formula for the total number of unit triangles, vertex up (△) and vertex down (▽) in a larger triangle of n rows.

Total number S_n = ________________________________

d. Based on the answers to parts (a), (b), (c), what is the sum of two consecutive triangular numbers ?

Answer : ________________________________

B. Look at the triangular patterns on page 56. Take in particular the triangle made up of 3 rows of unit triangles. What is the total number of all sizes triangles with vertex up (△). We have shown the analysis of the problem below.

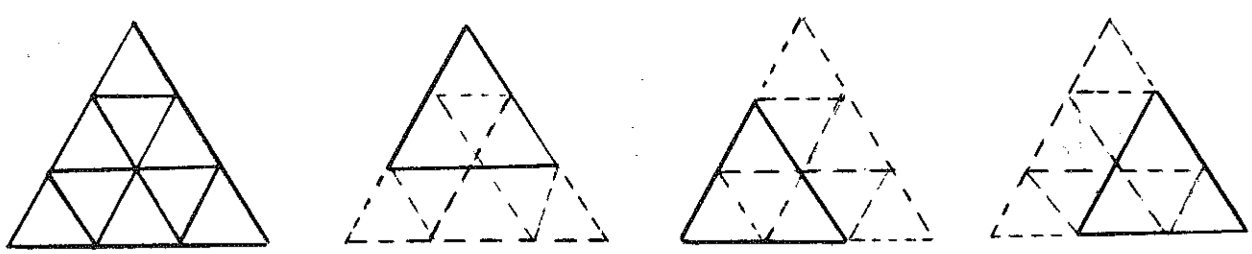


Figure 1 Figure 2 Figure 3 Figure 4

In figure 1 : there are 6 unit triangles with vertex up.

In figure 2 : there is 1 triangle, 2 units on a side, with vertex up.
The same is also true in figures 3, 4.

In figure 1 : there is 1 triangle, 3 units on a side, with vertex up

Thus, the total number of triangles, vertex up, in a triangle of 3 rows of unit triangles is : 6 + 3 + 1 = 10.

a. Find a general formula for the total number of triangles, vertex up, of all sizes in a triangle of n rows of unit triangles.

Fill in the table

Number of rows n	Number of triangles all sizes (△)	D^1	D^2	D^3	D^4
1	____				
2	____				
3	10				
4	____				
5	35				
6	56				
⋮	⋮				
n	s_n				

s_n = ______________________________

C. Look at the triangular patterns on page 56 and page 57. Find the total number of all sizes triangles with vertex down (▽) in a triangle of 1 to 12 rows of unit triangles.

Fill in the table

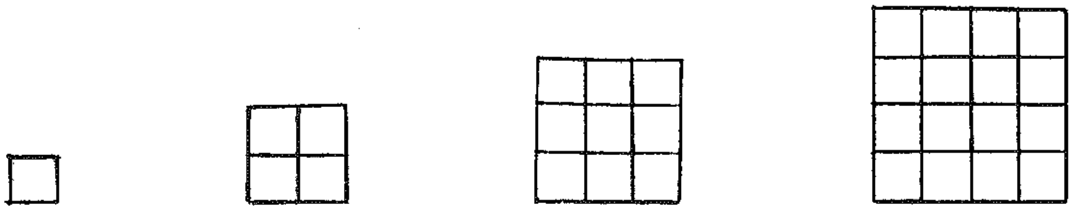
Number of rows n	Number of triangles all sizes (▽)	D^1	D^2	D^3	D^4
1	____				
2	____				
3	____				
4	____				
5	____				
6	22				
7	34				
8	50				
9	70				
10	95				
11	125				
12	161				
⋮					
n					

s_n = ______________________________

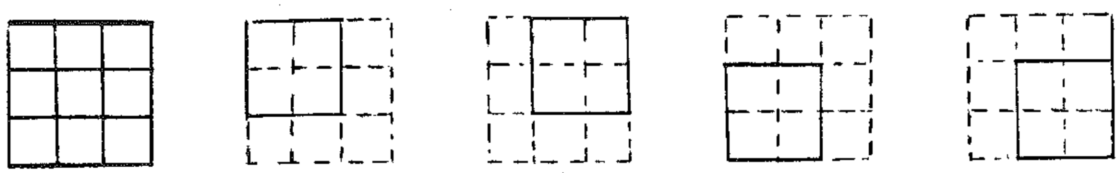
____

74. Counting squares

A. Unit squares are formed into square patterns as shown below .



How many squares of all sizes are there in square pattern . Take in particular the case of 3 unit squares on a side . The analysis is shown below .



1 3 x 3 square
9 1 x 1 squares

4 different 2 x 2 squares

Total number of squares of all sizes = 1 + 9 + 4 = 14

a. How many squares of all sizes are there in a square pattern made up of 1, 2, 3, 4, 5, 6 unit squares on a side .

b. Find a general formula for the total number of squares of all sizes in an n x n pattern of unit squares .

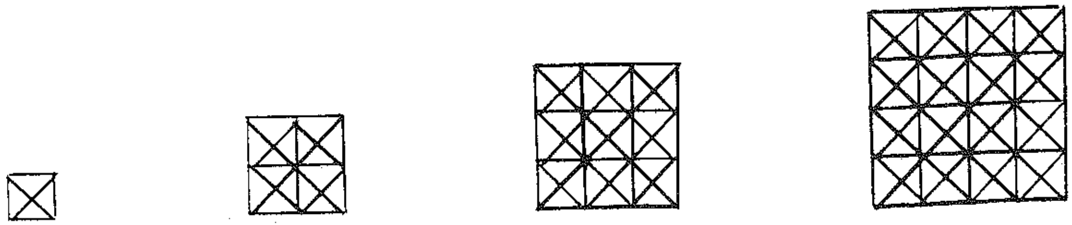
Fill in the table

a.

Number of unit squares on a side n	Total number of squares of all sizes	D^1	D^2	D^3
1	1			
2	___			
3	___			
4	30			
5	55			
6	91			
⋮	⋮			
n	s_n			

b. s_n = ______________________________

B. Draw the diagonals of each of the unit squares in the square arrangement shown on page 59. We get the following .



a. Count the number of squares of all sizes whose sides lie on the diagonals in a square pattern made up of 1 to 10 unit squares on a side .

b. Find a general formula for the number of slanted squares in an n x n square pattern of unit squares .

c. Use the result of (b) above and the result (b) of part (A), page 59, and give the general formula for the total number of squares of type in part (A) and in the present problem .

Fill in the table

a.

Number of unit squares on a side n	Total number of slanted squares	D^1	D^2	D^3	D^4
1	0				
2	___				
3	___				
4	___				
5	82				
6	143				
7	227				
8	340				
9	484				
10	665				
⋮					
n	s_n				

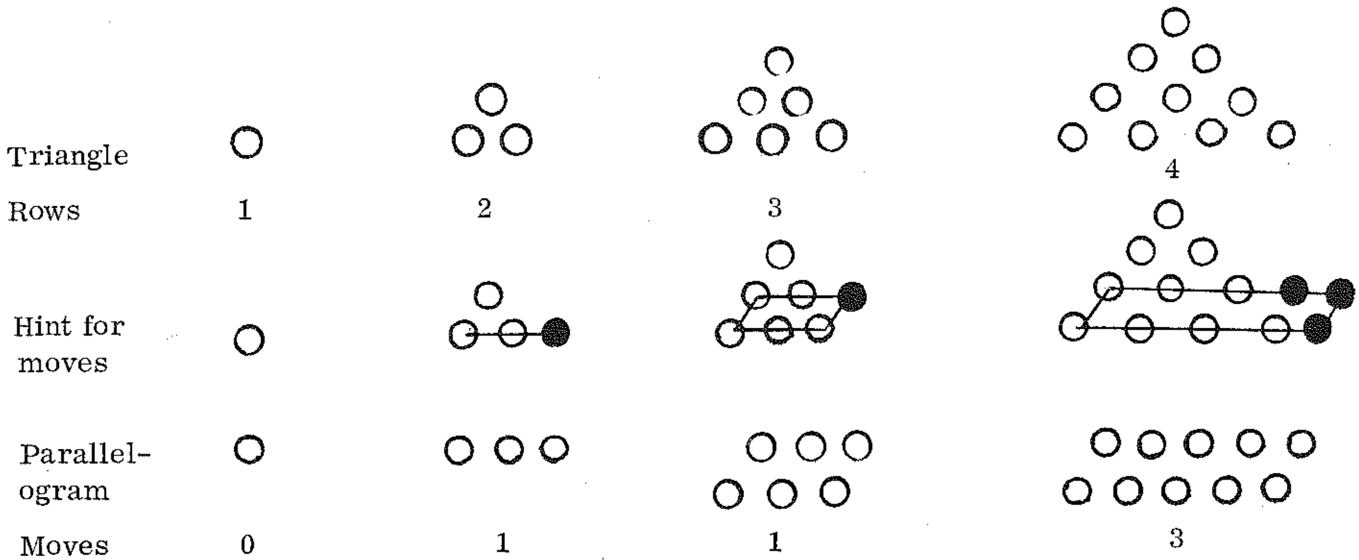
b. s_n = ______________________________

c. Total number of both types : ______________________________

____

75. Pennies : triangles to parallelograms

A. Begin with pennies in a triangular arrangement. Moving 1 penny at a time, what is the least number of moves needed to change the triangle arrangement to a parallelogram array . Below we show the results for the first few cases .



a. Find the least number of moves needed for triangular arrangements of 1 to 10 rows.

b. Find a general formula which gives the least number of moves for a triangular array of n rows .

Fill in the table

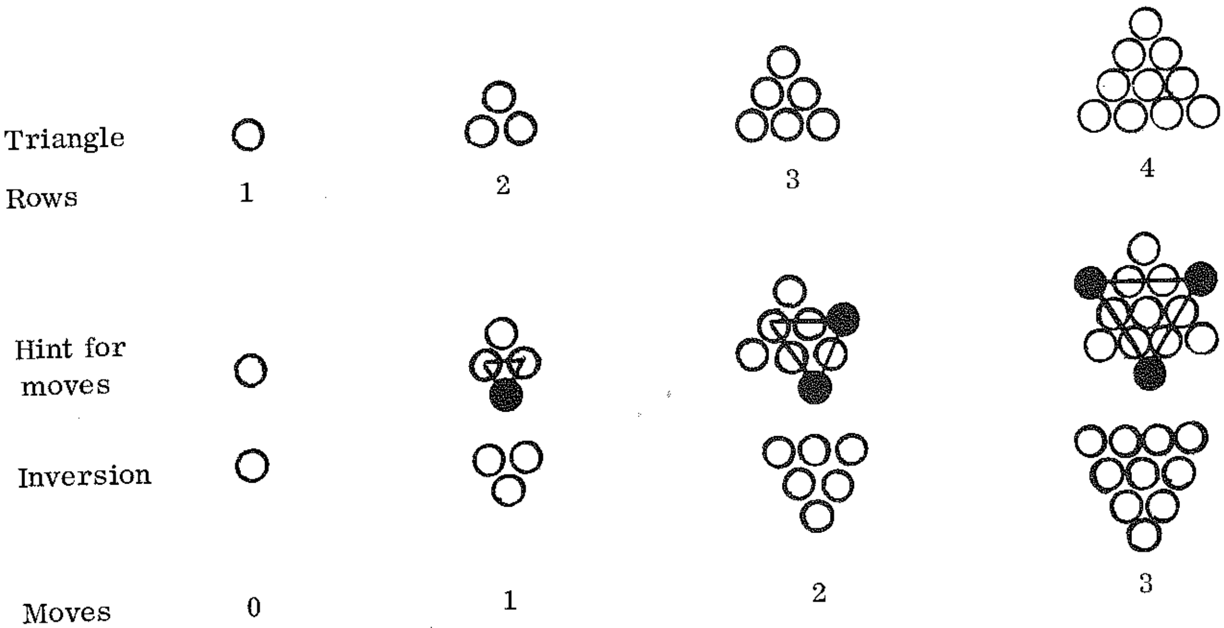
a.

Rows in the triangle n	Least number of moves	D^1	D^2	D^3	D^4
1	0				
2	1				
3	1				
4	3				
5	___				
6	___				
7	___				
8	___				
9	___				
10	15				
n	s_n				

b. s_n = ________________________________

B. Pennies : inverting the triangle

Begin with pennies in a triangular arrangement. Moving 1 penny at a time, find the least number of moves needed to invert the triangular array . Below we show the results for the first few cases .



a. Find the least number of moves needed to invert the triangle for 1 to 12 rows .

b. Find a general formula which gives the least number of moves for a triangular array of n rows .

Fill in the table

a,

Rows in the triangle n	Least number of moves
1	0
2	1
3	2
4	3
5	____
6	____
7	____
8	____
9	15
10	18
11	22
12	26
⋮	⋮
n	s_n

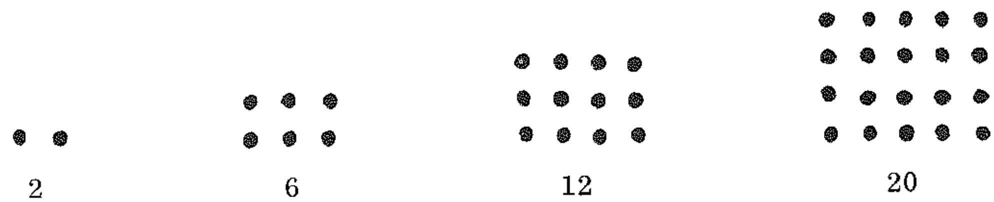
b. s_n = ______________________

76. Rectangular and trapezoidal numbers

The figurate numbers discussed in Section 18, page 38 are not the only sequences of numbers arising from dots arranged in a geometric pattern . Here we discuss 2 other types of figurate numbers .

A. Rectangular numbers (or 'oblong' numbers)

The pattern for rectangular numbers is shown below .



Beginning with a 1 x 2 arrangement , each successive number is determined by adding one more row and one more column to the previous rectangular arrangement.

a. Find the first 6 rectangular numbers.

b. Find a general formula for the n th rectangular number .

Fill in the table

a.

Number of rows n	Total number of dots	D^1	D^2	D^3	D^4
1	2				
2	6				
3	12				
4	20				
5	____				
6	____				
⋮	⋮				
n	s_n				

b. s_n = ______________________________

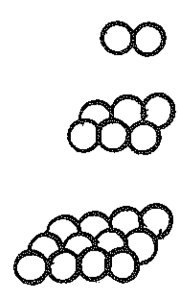
c. What is the relation between :

1. the n th rectangular number and the n th triangular number: ______________

2. the n th rectangular number and the n th square number: ______________

d. We can build a pyramidal arrangement of spheres so that each successive layer consists of a rectangular number of spheres (see section 66, page 46)

The number of spheres in a pyramid of n layers is the n th pyramidal number with rectangular base .



Layer 1

Layer 2

Layer 3

d. 1 Find the first 6 rectangular pyramidal numbers.

d. 2 Find a general formula for the n th rectangular pyramidal number .

Fill in the table

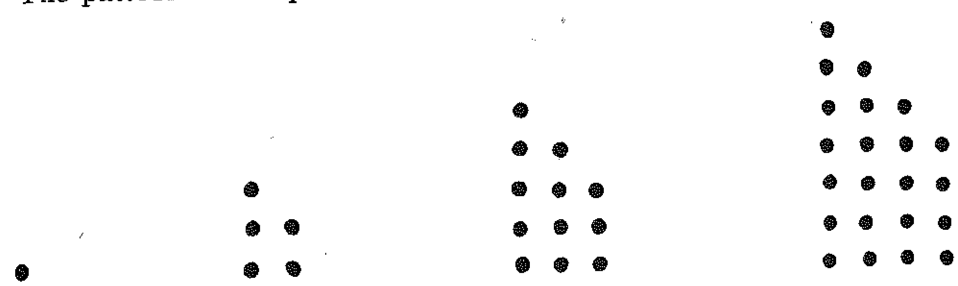
d.1

Number of layers n	Total number of spheres	D^1	D^2	D^3	D^4
1	2				
2	8				
3	___				
4	___				
5	___				
6	112				
⋮					

d.2 S_n = ______________________________

B. Trapezoidal numbers

The pattern for trapezoidal numbers is shown below .



1 5 12 22

a. What are the first 6 trapezoidal numbers .

b. Find a general formula for the n th trapezoidal number . The 1, 2, 3, . . . , n in the table below refer to the lst, 2nd, 3rd, . . . , n th trapezoidal numbers and are not related to the number of rows or columns in the pattern of dots above .

Fill in the table

a.

n	Trapezoidal number	D^1	D^2	D^3	D^4
1	1				
2	5				
3	12				
4	22				
5	___				
6	___				
⋮					
n	s_n				

b. s_n = ______________________________

c. Show that a trapezoidal number can be written as a sum of a triangular number and a square number. ______________________________

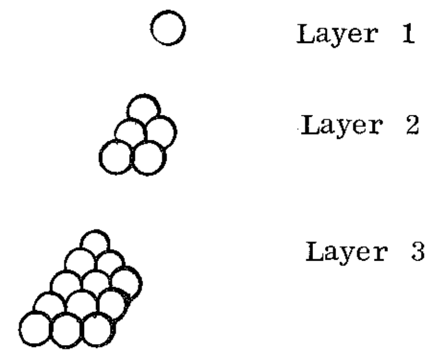
d. Compare the trapezoidal numbers with the pentagonal numbers page 40 .

____

e. We can define pyramidal trapezoidal numbers in the same way that we defined the other pyramidal numbers , pages 46, 63 .

e. 1. What are the first 6 pyramidal trapezoidal numbers .

e. 2. Find a general formula for the n th pyramidal trapezoidal number .



Fill in the table

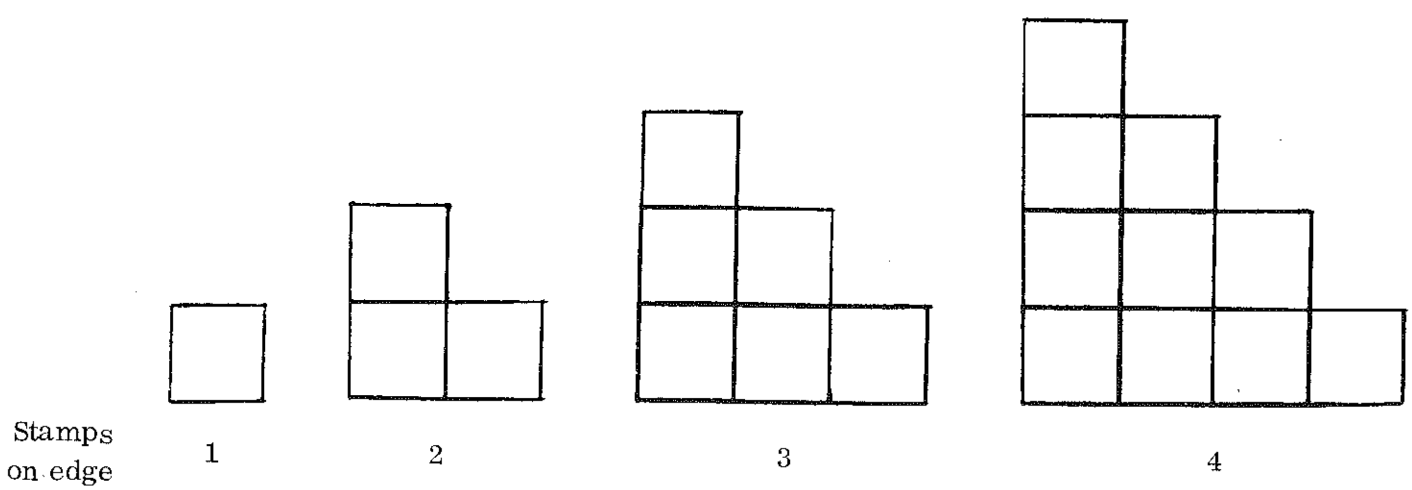
e. 1

Number of layers n	Total number of spheres	D^1	D^2	D^3	D^4
1	1				
2	____				
3	____				
4	____				
5	____				
6	____				

e. 2 S_n = ______________________________

77. Three stamps from a triangular block

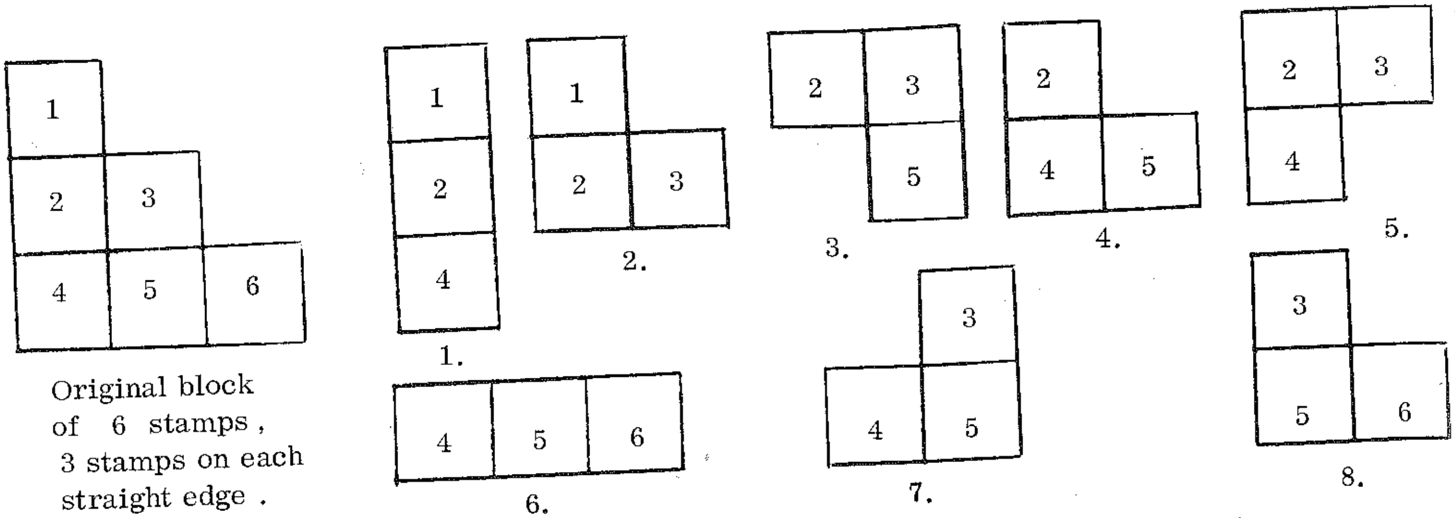
Stamps are arranged in a triangular pattern as shown below .



We have triangular sheets of stamps as shown above . In how many ways can you get connected blocks of three stamps from each triangular array ?

'Connected' means that each stamp in a 3-block must be joined to the rest of the block on at least one side . Separated stamps or stamps joined only at a corner are not counted .

Example 1 Take a triangular sheet of stamps with 3 stamps on each straight edge .



Original block of 6 stamps , 3 stamps on each straight edge .

The triangular sheet contains 8 connected blocks of 3 stamps .

a. How many connected blocks of three stamps are contained in triangular sheets of 1, 2, 3, 4, 5, 6 stamps along the straight edges .

b. Find a general formula for the number of different connected blocks of three stamps that can be taken from a triangular sheet with n stamps along each straight edge . Fill in the table

a,

Stamps along each straight edge n	Total number of blocks of three stamps	D^1	D^2	D^3	D^4
1	0				
2	1				
3	___				
4	___				
5	40				
6	65				
⋮	⋮				
n	s_n				

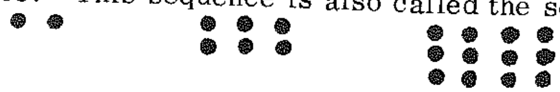
b. s_n = ______________________

Solutions and Comments

1. 0, 2, 4, 6, 8, . . . , $2(n-1)$, ... Substitute $n = 1, 2, 3, \ldots$ in $2(n-1)$ to get the terms of the sequence. This is the sequence of even nonnegative integers. It is also an example of an arithmetic sequence (progression),

$s_n = a + (n-1)d$ where a is the first term (here, $a = 0$),
d is the common difference of the terms (here, $d = 2$),
s_n, see section 4, page 3.

2. 2, 6, 12, 20, 30, . . . , $n(n+1)$, ... Substitute $n = 1, 2, 3, \ldots$ in $n(n+1)$ to get the terms of the sequence. This sequence is also called the sequence of rectangular numbers.



2 6 12 and so on .

3. 2, 5, 8, 11, 14, . . . , $2 + (n-1)3$, ... This sequence is an example of an arithmetic sequence (progression), with first term $a = 2$ and common difference $d = 3$. (See Problem 1)

4. 1, 5, 12, 22, 35, . . . , $\frac{n(3n-1)}{2}$, ... This sequence is usually called the sequence of pentagonal numbers. (See page 40.)

5. 1, 5, 14, 30, 55, . . . , $\frac{n(n+1)(2n+1)}{6}$, ... This sequence is called the sequence of square pyramidal numbers. (See page 40.) A closely related property is that

$$S_n = \frac{n(n+1)(2n+1)}{6} = 1^2 + 2^2 + \ldots + n^2$$

6. 1, 2, 4, 8, 15, . . . , $\frac{n(n^2-3n+8)}{6}$, ... This sequence of numbers may be found by adding the first four numbers in successive rows of Pascal's Triangle. (See page 28.) For the first three rows add in 0's to bring the number of terms to 4. Therefore,

$$1 + 0 + 0 + 0 = 1$$
$$1 + 1 + 0 + 0 = 2$$
$$1 + 2 + 1 + 0 = 4$$
$$1 + 3 + 3 + 1 = 8$$
$$1 + 4 + 6 + 4 = 15$$
$$\vdots$$

Note also that based on the first 4 terms alone, we would probably think that the sequence was 1, 2, 4, 8, 16, . . . , 2^{n-1}, ... It is precisely this sort of ambiguity that is eliminated when we have the generator for the sequence.

Note: For the next 6 problems we will give examples of approaches that might be used to find general terms for the given sequences. The rest of the booklet shows a very special "pattern" to look for in a sequence and develops a general method for finding a generator if we do discover a pattern of that type.

7. 1, 2, 3, 4, 5, 6, <u>7, 8, 9,</u> . . . , <u>n,</u> . . . . It is clear that $s_n = n$ (For s_n, see Section 4, page 3.) fits the information, and in practice we would not require any special method to see that. For the sake of illustration, however, we will show some methods.

7. (Continued)

(1) The given numbers form an arithmetic progression with first term $a = 1$ and common difference $d = 1$. The general term is given by $s_n = 1 + (n - 1)1 = n$.

(See the solution to Problem 1 for the general form of an arithmetic progression.)

(2) We divide each term by its position in the sequence.

$$1 \div 1 = 1 \quad \text{or} \quad s_1 = 1 \cdot 1$$
$$2 \div 2 = 1 \quad \text{or} \quad s_2 = 1 \cdot 2$$
$$3 \div 3 = 1 \quad \text{or} \quad s_3 = 1 \cdot 3$$

Continuing this pattern we will have $s_n \div n = 1$ or $s_n = 1 \cdot n = n$.

8. 1, 3, 5, 7, 9, 11, 13, 15, 17, . . . , $2n - 1$, ...

(1) The given numbers form an arithmetic progression with first term $a = 1$ and common difference $d = 2$. The general term is given by $s_n = 1 + (n - 1) \cdot 2 = 2n - 1$.

(2) We divide each term by its position in the sequence.

$$1 \div 1 = 1 \text{ R } 0 \quad \text{or} \quad 1 = 1 \cdot 1 + 0$$
$$3 \div 2 = 1 \text{ R } 1 \quad \text{or} \quad 3 = 1 \cdot 2 + 1$$
$$5 \div 3 = 1 \text{ R } 2 \quad \text{or} \quad 5 = 1 \cdot 3 + 2$$
$$7 \div 4 = 1 \text{ R } 3 \quad \text{or} \quad 7 = 1 \cdot 4 + 3$$

Continuing this pattern we will have $s_n \div n = 1 \text{ R } (n - 1)$ or $s_n = 1 \cdot n + (n - 1) = 2n - 1$.

9. 3, 5, 7, 9, 11, 13, 15, 17, 19, . . . , $2n + 1$, ...

(1) The given numbers form an arithmetic progression with first term $a = 3$ and common difference $d = 2$. The general term is given by $s_n = 3 + (n - 1) \cdot 2 = 2n + 1$.

(2) We divide each term by its position in the sequence.

$$3 \div 1 = 3 \text{ R } 0 \quad \text{or} \quad 3 = 3 \cdot 1 + 0 \quad (= 2 \cdot 1 + 1)$$
$$5 \div 2 = 2 \text{ R } 1 \quad \text{or} \quad 5 = 2 \cdot 2 + 1$$
$$7 \div 3 = 2 \text{ R } 1 \quad \text{or} \quad 7 = 2 \cdot 3 + 1$$
$$9 \div 4 = 2 \text{ R } 1 \quad \text{or} \quad 9 = 2 \cdot 4 + 1$$

Continuing this pattern we will have $s_n \div n = 2 \text{ R } 1$ or $s_n = 2n + 1$.

10. 1, 3, 6, 10, 15, 21, 28, 36, 45, . . . , $\frac{n(n + 1)}{2}$, ...

These numbers do not form an arithmetic progression, and so $s_n = a + (n - 1)d$ cannot be

(1) We divide each term by its position in the sequence.

$$1 \div 1 = 1 \text{ R } 0 \quad \text{or} \quad 1 = 1 \cdot 1 + 0$$
$$3 \div 2 = 1 \text{ R } 1 \quad \text{or} \quad 3 = 1 \cdot 2 + 1$$
$$6 \div 3 = 2 \text{ R } 0 \quad \text{or} \quad 6 = 2 \cdot 3 + 0$$
$$10 \div 4 = 2 \text{ R } 2 \quad \text{or} \quad 10 = 2 \cdot 4 + 2$$
$$15 \div 5 = 3 \text{ R } 0 \quad \text{or} \quad 15 = 3 \cdot 5 + 0$$
$$21 \div 6 = 3 \text{ R } 3 \quad \text{or} \quad 21 = 3 \cdot 6 + 3$$

Although it is easy to continue this pattern for any finite number of additional terms, it is more difficult to write a general term.

10. (Continued)

If n is odd, that is, $n = 2N - 1$ or $N = \frac{n+1}{2}$, for a positive integer N, we have

$$s_n \div n = N \text{ R } 0 \quad \text{or} \quad s_n = N \cdot n \quad \text{and substituting for } N$$
$$= \left(\frac{n+1}{2}\right) n = \frac{n(n+1)}{2} .$$

If n is even, that is, $n = 2N$ or $N = \frac{n}{2}$, for a positive integer N, we have

$$s_n \div n = N \text{ R } N \quad \text{or} \quad s_n = N \cdot n + N \quad \text{and substituting for } N$$
$$= \left(\frac{n}{2}\right) n + \left(\frac{n}{2}\right) = \frac{n(n+1)}{2} .$$

Combining the results, we have $s_n = \frac{n(n+1)}{2}$ for all positive integers n.

(2) Another approach is the following.

$$s_1 = 1$$
$$s_2 = 3 = s_1 + 2 = 1 + 2$$
$$s_3 = 6 = s_2 + 3 = 1 + 2 + 3$$
$$s_4 = 10 = s_3 + 4 = 1 + 2 + 3 + 4$$

Continuing this pattern we will have $s_n = 1 + 2 + 3 + \ldots + n$. The formula for this sum is well-known, and so we have $s_n = \frac{n(n+1)}{2}$.

11. 3, 6, 11, 18, 27, 38, <u>51, 66, 83,</u> . . . , <u>$n^2 + 2$,</u> . . . .

These numbers do not form an arithmetic progression, and so $s_n = a + (n-1)d$ cannot be used.

(1) We divide each term by its position in the sequence.

$$3 \div 1 = 3 \text{ R } 0 \quad \text{or} \quad 3 = 3 \cdot 1 + 0 \quad (= 1 \cdot 1 + 2)$$
$$6 \div 2 = 3 \text{ R } 0 \quad \text{or} \quad 6 = 3 \cdot 2 + 0 \quad (= 2 \cdot 2 + 2)$$
$$11 \div 3 = 3 \text{ R } 2 \quad \text{or} \quad 11 = 3 \cdot 3 + 2$$
$$18 \div 4 = 4 \text{ R } 2 \quad \text{or} \quad 18 = 4 \cdot 4 + 2$$
$$27 \div 5 = 5 \text{ R } 2 \quad \text{or} \quad 27 = 5 \cdot 5 + 2$$

Continuing the pattern from the third term on, we have

$$s_n \div n = n \text{ R } 2 \quad \text{or} \quad s_n = n \cdot n + 2 = n^2 + 2 .$$

Note that the first two terms can be rewritten in this form.

(2) Another approach is the following.

$$s_1 = 3$$
$$s_2 = 6 = s_1 + 3 = 3 + 3$$
$$s_3 = 11 = s_2 + 5 = 3 + 3 + 5$$
$$s_4 = 18 = s_3 + 7 = 3 + 3 + 5 + 7$$
$$s_5 = 27 = s_4 + 11 = 3 + 3 + 5 + 7 + 11$$

Continuing this pattern we have

$$s_n = 3 + 3 + 5 + 7 + \ldots + (2n - 1)$$
$$= 2 + 1 + 3 + 5 + \ldots + (2n - 1) = 2 + n^2 .$$

(Here we assume that we already know the formula for the sum of the first n odd integers.)

12. 3, 10, 29, 66, 127, <u>218, 345, 514,</u> . . . , <u>$n^3 + 2$,</u> . . . .

These numbers do not form an arithmetic progression, and so we cannot use $s_n = a + (n-1)d$.

12. (Continued)

(1) We divide each term by its position in the sequence.

$$
\begin{array}{lll}
3 \div 1 = 3 \text{ R } 0 & \text{or} & 3 = 3\cdot 1 + 0 \quad (= 1\cdot 1 + 2) \\
10 \div 2 = 5 \text{ R } 0 & \text{or} & 10 = 5\cdot 2 + 0 \quad (= 4\cdot 2 + 2) \\
29 \div 3 = 9 \text{ R } 2 & \text{or} & 29 = 9\cdot 3 + 2 \\
66 \div 4 = 16 \text{ R } 2 & \text{or} & 66 = 16\cdot 4 + 2 \\
127 \div 5 = 25 \text{ R } 2 & \text{or} & 127 = 25\cdot 5 + 2
\end{array}
$$

Continuing the pattern from the third term on, we have

$$s_n \div n = n^2 \text{ R } 2 \quad \text{or} \quad s_n = n^2 \cdot n + 2 = n^3 + 2 \, .$$

Note that the first two terms can also be written in this form.

(2) Suppose that we try to develop a formula as in Problems 10 and 11.

$$
\begin{aligned}
s_1 &= 3 \\
s_2 &= s_1 + 7 = 3 + 7 \\
s_3 &= s_2 + 19 = 3 + 7 + 19 \\
s_4 &= s_3 + 37 = 3 + 7 + 19 + 37 \\
s_5 &= s_4 + 61 = 3 + 7 + 19 + 37 + 61
\end{aligned}
$$

It is not clear how to continue. We might take the numbers 3, 7, 19, 37, 61, ... as a new sequence, but even if we found a formula for these terms, we would still need a formula for their sum. In any case this procedure will not always be easy to apply.

Note: The procedure that we will develop is related to method (1) of Problems 7, 8, 9 and method (2) of Problems 10, 11, 12. It is more effective because the "pattern" that we are seeking is clearly defined, and if the pattern does appear, we can always get an explicit formula for the generator.

It is not completely accurate to call the formulas above the generators. What we can say is that the formulas above do produce the given terms and do generalize some pattern or behavior that we detected in the given terms. No finite number of terms uniquely determines a sequence unless we also have some sort of general rule for the terms of the sequence.

13.

n	s_n	D^1
1	0	
		2
2	2	
		2
3	4	
		2
4	6	
		2
5	8	

The first order differences are all equal to 2, and all the higher order differences are equal to 0.

14.

n	s_n	D^1
1	2	
		3
2	5	
		3
3	8	
		3
4	11	
		3
5	14	

15.

n	s_n	D^1	D^2
1	2		
		4	
2	6		2
		6	
3	12		2
		8	
4	20		2
		10	
5	30		

16.

n	s_n	D^1	D^2
1	1		
		4	
2	5		3
		7	
3	12		3
		10	
4	22		3
		13	
5	35		

17.

n	s_n	D^1	D^2	D^3
1	1			
2	5	4		
3	14	9	5	
4	30	16	7	2
5	55	25	9	2

18.

n	s_n	D^1	D^2	D^3
1	1			
2	2	1		
3	4	2	1	
4	8	4	2	1
5	15	7	3	1

19.

n	$2n$	D^1
1	2	
2	4	2
3	6	2
4	8	2
⋮	⋮	
n	$2n$	

20.

n	$4n - 5$	D^1
1	-1	
2	3	4
3	7	4
4	11	4
⋮	⋮	
n	$4n - 5$	

21.

n	$1 - 2n$	D^1
1	-1	
2	-3	-2
3	-5	-2
4	-7	-2
⋮	⋮	
n	$1 - 2n$	

22.

n	$\frac{3}{4}n + 1$	D^1
1	7/4	
2	10/4	3/4
3	13/4	3/4
4	16/4	3/4
⋮	⋮	
n	$\frac{3}{4}n + 1$	

Note to page 9, Summary
The second remark of the summary has not really been demonstrated, but it does follow immediately from the general formula developed in section 14 and given on page 29. For the present we will be content to develop techniques to find the generator of a sequence, assuming the truth of the statements in the summary.

Note to page 11, remark (b)
Remark (b) is crucial to our technique of solving for the general term of a sequence. Our methods rely on patterns of a certain type, namely those where for some order, the differences are all the same. From these methods we can obtain formulas which will fit the numbers given.
The more terms given, the more confident we can be of our results, but no matter how many terms we are given, we must still assume that 'the pattern continues' in precisely the sense defined here.

23.

n	Sequence	D^1
1	9	
2	14	5
3	19	5
4	24	5
5	29	5

1. $a = 5$
2. $s_n = an + b$
3. $s_n = 5n + b$
4. $9 = 5 + b$
5. $b = 4$
6. $s_n = 5n + 4$

24.

n	Sequence	D^1
1	1	
2	4	3
3	7	3
4	10	3
5	13	3

1. $a = 3$
2. $s_n = an + b$
3. $s_n = 3n + b$
4. $1 = 3 + b$
5. $b = -2$
6. $s_n = 3n - 2$

25.

n	Sequence	D^1
1	9	
		-4
2	5	
		-4
3	1	
		-4
4	-3	
		-4
5	-7	

1. $a = -4$
2. $s_n = an + b$
3. $s_n = -4n + b$
4. $b = 13$
5. $s_n = -4n + 13$

26.

n	Sequence	D^1
1	2	
		1/2
2	5/2	
		1/2
3	3	
		1/2
4	7/2	
		1/2
5	4	

1. $a = 1/2$
2. $s_n = an + b$
3. $s_n = \frac{1}{2}n + b$
4. $2 = 1/2 + b$
5. $b = 3/2$
6. $s_n = \frac{1}{2}n + \frac{3}{2}$

Note to page 13, formula (A)
Although we only have to know the first term of the sequence and one difference to use the formula, we must also have some reason to assume that the general term does have the form $s_n = an + b$. In other words we would still want to compute a difference table for the first several terms in order to check that the first order differences all have the same nonzero value. In order to assume that the formula (A) works for all n, we must have that the D_i^1 s all have the same nonzero value.

The formula (A) allows us to illustrate concretely some of the remarks made on pages 10-11. Consider the sequence which begins 5, 8, 11, 14, 17, 25, 47, 100, 60, ...

n	Sequence	D^1
1	5	
		3
2	8	
		3
3	11	
		3
4	14	
		3
5	17	
		8
6	25	
		22
7	47	
		53
8	100	
		-40
9	60	

a. For $n = 1, 2, 3, 4, 5$, the D_i^1, $i = 1, 2, 3, 4$ are all one and the same constant 3. If we had stopped at $n = 5$, then $s_1 = 5$, $D_1^1 = 3$ and using formula (A), page 13, we get

$$s_n = s_1 + (n-1)D_1^1 = 5 + (n-1)3$$
$$s_n = 3n + 2,$$

as the general term of the sequence. Notice that this s_n does give the terms of the sequence for $n = 1, 2, 3, 4, 5$ but not for $n = 6, 7, 8, \ldots$.

b. For s_n to be the general term for all n, the D_i^1 must all be the same value for all i.

c. Quite different from an arbitrary sequence given by someone, is the sequence that one gets from considering a problem such as the number of diagonals in a triangle, a square, a pentagon, and so on. Here apparently, if one does the problem correctly, and one gets an nth term from the particular instances tried, then it is almost certain that the nth term will give the correct answer for those instances not tried. But again, this procedure is not a proof. There is no assurance that someplace along the line, the pattern may not change radically. In many instances, however, the pattern does not change, and from the few particular instances we usually generalize and are correct. Actually, what we are doing in finite differences is a type of induction, but it is not the mathematical type of induction.

27.

n	Sequence	D^1
1	9	
		5
2	14	
		5
3	19	
		5
4	24	
		5
5	29	

$s_1 = 9$

$D_1^1 = 5$

$s_n = 9 + (n-1)5$

$= 5n + 4$

28.

n	Sequence	D^1
1	1	
		3
2	4	
		3
3	7	
		3
4	10	
		3
5	13	

$s_1 = 1$

$D_1^1 = 3$

$s_n = 1 + (n-1)$

$= 3n - 2$

29.

n	Sequence	D^1
1	9	
2	5	-4
3	1	-4
4	-3	-4
5	-7	-4

$s_1 = 9$

$D_1^1 = -4$

$s_n = 9 + (n-1)(-4)$

$= -4n + 13$

30.

n	Sequence	D^1
1	2	
2	5/2	1/2
3	3	1/2
4	7/2	1/2
5	4	1/2

$s_1 = 2$

$D_1^1 = \frac{1}{2}$

$s_n = 2 + (n-1)(1/2)$

$= \frac{1}{2}n + \frac{3}{2}$

31.

n	Sequence	D^1
1	-1	
2	-3	-2
3	-5	-2
4	-7	-2
5	-9	-2

$s_1 = -1$

$D_1^1 = -2$

$s_n = -1 + (n-1)(-2)$

$= -2n + 1$

32.

n	Sequence	D^1	D^2
1	1		
2	4	3	
3	9	5	2
4	16	7	2
5	25	9	2
⋮	⋮		
n	$1 \cdot n^2$		

$1 = \frac{1}{2}(2)$

33.

n	Sequence	D^1	D^2
1	4		
2	16	12	
3	36	20	8
4	64	28	8
5	100	36	8
⋮	⋮		
n	$4n^2$		

$4 = \frac{1}{2}(8)$

34.

n	Sequence	D^1	D^2
1	1/2		
2	2	3/2	
3	9/2	5/2	1
4	8	7/2	1
5	25/2	9/2	1
⋮	⋮		
n	$\frac{1}{2}n^2$		

$\frac{1}{2} = \frac{1}{2}(1)$

35.

n	Sequence	D^1	D^2
1	-1		
2	-4	-3	
3	-9	-5	-2
4	-16	-7	-2
5	-25	-9	-2
⋮	⋮		
n	$-1 \cdot n^2$		

$-1 = \frac{1}{2}(-2)$

36.

n	Sequence	D^1	D^2
1	-3		
2	-4	-1	
3	-3	1	2
4	0	3	2
5	5	5	2
⋮	⋮		
n	$1n^2 - 4n$		

$1 = \frac{1}{2}(2)$

37.

n	Sequence	D^1	D^2
1	2		
2	6	4	
3	12	6	2
4	20	8	2
5	30	10	2
⋮	⋮		
n	$1n^2 + n$		

$1 = \frac{1}{2}(2)$

38.

n	Sequence	D^1	D^2
1	1		
2	0	-1	
3	-3	-3	-2
4	-8	-5	-2
5	-15	-7	-2
⋮	⋮		
n	$-1n^2+2n$		

$-1 = \frac{1}{2}(-2)$

39.

n	Sequence	D^1	D^2
1	1		
2	-2	-3	
3	-9	-7	-4
4	-20	-11	-4
5	-35	-15	-4
⋮	⋮		
n	$-2n^2+3n$		

$-2 = \frac{1}{2}(-4)$

40.

n	Sequence	D^1	D^2
1	-3		
2	-6	-3	
3	-13	-7	-4
4	-24	-11	-4
5	-39	-15	-4
⋮	⋮		
n	$-2n^2+3n-4$		

$-2 = \frac{1}{2}(-4)$

41.

n	Sequence	D^1	D^2
1	1/2		
2	1	1/2	
3	5/2	3/2	1
4	5	5/2	1
5	17/2	7/2	1
⋮	⋮		
n	$\frac{1}{2}n^2-n+1$		

$\frac{1}{2} = \frac{1}{2}(1)$

Note to page 17, Summary
Remark (2) of the summary really follows from the general formula on page 29. As in the linear case, page 9, we will work with the facts of the summary here in order to obtain the general term of the sequences.

42.

n	Sequence	D^1	D^2
1	4		
2	12	8	
3	26	14	6
4	46	20	6
5	72	26	6

$a = \frac{1}{2}(6) = 3$

$3 \cdot 3 + b = 8$ and so $b = -1$

$3 - 1 + c = 4$ and so $c = 2$

$s_n = 3n^2 - n + 2$

43.

n	Sequence	D^1	D^2
1	2		
2	9	7	
3	20	11	4
4	35	15	4
5	54	19	4

$a = \frac{1}{2}(4) = 2$

$3 \cdot 2 + b = 7$ and so $b = 1$

$2 + 1 + c = 2$ and so $c = -1$

$s_n = 2n^2 + n - 1$

44.

n	Sequence	D^1	D^2
1	-2		
2	1	3	
3	6	5	2
4	13	7	2
5	22	9	2

$a = \frac{1}{2} \cdot 2 = 1$

$3 \cdot 1 + b = 3$ and so $b = 0$

$1 + 0 + c = -2$ and so $c = -3$

$s_n = n^2 - 3$

45.

n	Sequence	D^1	D^2
1	-2		
		-1	
2	-3		-2
		-3	
3	-6		-2
		-5	
4	-11		-2
		-7	
5	-18		

$a = \frac{1}{2}(-2) = -1$

$3(-1) + b = -1$ and so $b = 2$

$-1 + 2 + c = -2$ and so $c = -3$

$s_n = -n^2 + 2n - 3$

Note to page 20, formula (B)
See the note in the commentary for page 13, formula (A). The remark there, with appropriate changes, applies here also.

46.

n	Sequence	D^1	D^2
1	2		
		6	
2	8		4
		10	
3	18		4
		14	
4	32		4
		18	
5	50		

$s_1 = 2,\ D_1^1 = 6,\ D_1^2 = 4$

$s_n = 2 + (n-1)6 + \frac{(n-1)(n-2)}{2}(4)$

$= 2n^2$

47.

n	Sequence	D^1	D^2
1	5		
		9	
2	14		4
		13	
3	27		4
		17	
4	44		4
		21	
5	65		

$s_1 = 5,\ D_1^1 = 9,\ D_1^2 = 4$

$s_n = 5 + (n-1)9 + \frac{(n-1)(n-2)}{2}(4)$

$= 2n^2 + 3n$

48.

n	Sequence	D^1	D^2
1	9		
		9	
2	18		4
		13	
3	31		4
		17	
4	48		4
		21	
5	69		

$s_1 = 9,\ D_1^1 = 9,\ D_1^2 = 4$

$s_n = 9 + (n-1)9 + \frac{(n-1)(n-2)}{2}(4)$

$= 2n^2 + 3n + 4$

49.

n	Sequence	D^1	D^2
1	-2		
		-8	
2	-10		-4
		-12	
3	-22		-4
		-16	
4	-38		-4
		-20	
5	-58		

$s_1 = -2,\ D_1^1 = -8,\ D_1^2 = -4$

$s_n = -2 + (n-1)(-8) + \frac{(n-1)(n-2)}{2}(-4)$

$= -2n^2 - 2n + 2$

50.

n	Sequence	D^1	D^2
1	1/2		
		5/2	
2	6/2		2
		9/2	
3	15/2		2
		13/2	
4	28/2		2
		17/2	
5	45/2		

$s_1 = \frac{1}{2},\ D_1^1 = \frac{5}{2},\ D_1^2 = 2$

$s_n = \frac{1}{2} + (n-1)\frac{5}{2} + \frac{(n-1)(n-2)}{2}(2) = n^2 - \frac{1}{2}n = \frac{n(2n-1)}{2}$

51.

n	Sequence	D^1	D^2	D^3
1	1			
		7		
2	8		12	
		19		6
3	27		18	
		37		6
4	64		24	
		61		
5	125			

$a = \frac{1}{6}(6) = 1$

$12 = 12(1) + 2b$ and so $b = 0$

$7 = 7(1) + 3(0) + c$ and so $c = 0$

$1 = 1 + 0 + 0 + d$ and so $d = 0$

$s_n = n^3$

52.

n	Sequence	D^1	D^2	D^3
1	1			
		13		
2	14		24	
		37		12
3	51		36	
		73		12
4	124		48	
		121		
5	245			

$a = \frac{1}{6}(12) = 2$

$24 = 12(2) + 2b$ and so $b = 0$

$13 = 7(2) + 3(0) + c$ and so $c = -1$

$1 = 2 + 0 + -1 + d$ and so $d = 0$

$s_n = 2n^3 - n$

53.

n	Sequence	D^1	D^2	D^3
1	1			
		11		
2	12		22	
		33		12
3	45		34	
		67		12
4	112		46	
		113		
5	225			

$a = \frac{1}{6}(12) = 2$

$22 = 12(2) + 2b$ and so $b = -1$

$11 = 7(2) + 3(-1) + c$ and so $c = 0$

$1 = 2 - 1 + 0 + d$ and so $d = 0$

$s_n = 2n^3 - n^2$

54.

n	Sequence	D^1	D^2	D^3
1	2			
		12		
2	14		22	
		34		12
3	48		34	
		68		12
4	116		46	
		114		
5	230			

$a = \frac{1}{6}(12) = 2$

$22 = 12(2) + 2b$ and so $b = -1$

$12 = 7(2) + 3(-1) + c$ and so $c = 1$

$2 = 2 - 1 + 1 + d$ and so $d = 0$

$s_n = 2n^3 - n^2 + n$

55.

n	Sequence	D^1	D^2	D^3
1	-3			
		12		
2	9		22	
		34		12
3	43		34	
		68		12
4	111		46	
		114		
5	225			

$a = \frac{1}{6}(12) = 2$

$22 = 12(2) + 2b$ and so $b = -1$

$12 = 7(2) + 3(-1) + c$ and so $c = 1$

$-3 = 2 - 1 + 1 + d$ and so $d = -5$

$s_n = 2n^3 - n^2 + n - 5$

56. (1) Using the difference table in the answer to Problem 51, $s_1 = 1$, $D_1^1 = 7$, $D_1^2 = 12$, $D_1^3 = 6$

By formula (C), page 24,

$$s_n = 1 + (n-1)7 + \frac{(n-1)(n-2)}{2!}(12) + \frac{(n-1)(n-2)(n-3)}{3!}(6)$$

$$= 1 + 7n - 7 + 6n^2 - 18n + 12 + n^3 - 6n^2 + 11n - 6 = n^3$$

(2) Using the difference table in the answer to Problem 52, $s_1 = 1$, $D_1^1 = 13$, $D_1^2 = 24$, $D_1^3 = 12$.

By formula (C), page 24,

$$s_n = 1 + (n-1)13 + \frac{(n-1)(n-2)}{2!}(24) + \frac{(n-1)(n-2)(n-3)}{3!}(12)$$

$$= 1 + 13n - 13 + 12n^2 - 36n + 24 + 2n^3 - 12n^2 + 22n - 12 = 2n^3 - n\ .$$

56. (Continued)

(3) Using the difference table in the answer to Problem 53, $s_1 = 1$, $D_1^1 = 11$, $D_1^2 = 22$, $D_1^3 = 12$.

By formula (C), page 24,

$$s_n = 1 + (n-1)\,11 + \frac{(n-1)(n-2)}{2!}(22) + \frac{(n-1)(n-2)(n-3)}{3!}(12)$$

$$= 1 + 11n - 11 + 11n^2 - 33n + 22 + 2n^3 - 12n^2 + 22n - 12 = 2n^3 - n^2$$

(4) Using the difference table in the answer to Problem 54, $s_1 = 2$, $D_1^1 = 12$, $D_1^2 = 22$, $D_1^3 = 12$.

By formula (C), page 24,

$$s_n = 2 + (n-1)12 + \frac{(n-1)(n-2)}{2!}(22) + \frac{(n-1)(n-2)(n-3)}{3!}(12)$$

$$= 2 + 12n - 12 + 11n^2 - 33n + 22 + 2n^3 - 12n^2 + 22n - 12 = 2n^3 - n^2 + n$$

(5) Using the difference table in the answer to Problem 54, $s_1 = -3$, $D_1^1 = 12$, $D_1^2 = 22$, $D_1^3 = 12$.

By formula (C), page 24,

$$s_n = -3 + (n-1)\,12 + \frac{(n-1)(n-2)}{2!}(22) + \frac{(n-1)(n-2)(n-3)}{3!}(12)$$

$$= -3 + 12n - 12 + 11n^2 - 33n + 22 + 2n^3 - 12n^2 + 22n - 12 = 2n^3 - n^2 + n - 5$$

57. a) $2n$, Degree: 1

b) $2 + (n-1)3 = 3n - 1$, Degree: 1

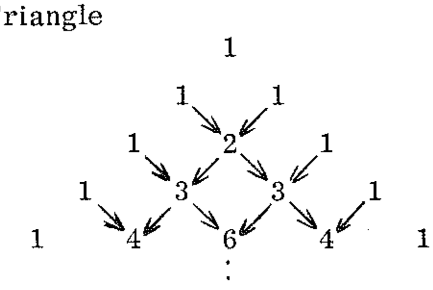
c) $n(n+1) = n^2 + n$, Degree: 2

d) $\frac{n(3n-1)}{2} = \frac{3}{2}n^2 - \frac{1}{2}n$, Degree: 2

e) $\frac{n(n+1)(2n+1)}{6} = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$, Degree: 3

f) $\frac{n(n^2 - 3n + 8)}{6} = \frac{1}{6}n^3 - \frac{1}{2}n^2 + \frac{4}{3}n$, Degree: 3

Note to page 27, Pascal's Triangle



Recall that Pascal's Triangle may be devoloped one row at a time in the following manner. Each row begins and ends with a 1 (except the first row which consists of 1 alone). Each of the other entries in a row is the sum of the two elements in the row above, immediately to its right and left. (See the figure above.)

Note to formula (D), page 29

Example: Consider the sequence which begins 1, 2, 4, 8, ... We form a difference table.

n	Sequence	D^1	D^2	D^3
1	1			
		1		
2	2		1	
		2		1
3	4		2	
		4		
4	8			

Note to formula (D), page 29 (Continued)
Formula (D) says the following:

$$s_1 = s_1 = 1$$
$$s_2 = s_1 + (2-1)D_1^1 = 1 + 1\cdot 1 = 2$$
$$s_3 = s_1 + (3-1)D_1^1 + \frac{(3-1)(3-2)}{2!}(1) = 1 + 2\cdot 1 + \frac{2\cdot 1}{2}(1) = 4$$
$$s_4 = s_1 + (4-1)D_1^1 + \frac{(4-1)(4-2)}{2!}(1) + \frac{(4-1)(4-2)(4-3)}{3!}(1) = 1 + 3\cdot 1 + \frac{3\cdot 2}{2}(1) + \frac{3\cdot 2\cdot 1}{6}(1) = 8$$

We have no information which allows us to apply formula (D) for $n = 5, 6, 7, \ldots$.
If we assumed that $D_i^3 = 1$ for all $i = 1, 2, 3, \ldots$ (a very rash assumption based on the given information), it would follow that $D_1^n = 0$ for $n = 4, 5, 6, \ldots$. Formula (D) could then be applied for all values of n since we already have all the nonzero terms D_1^i which appear in the formula. The generator would be

$$s_n = 1 + (n-1)\,1 + \frac{(n-1)(n-2)}{2!}(1) + \frac{(n-1)(n-2)(n-3)}{3!}(1) = \frac{n(n^2 - 3n + 8)}{6}$$

Is this the correct answer?

It is correct in that it generates a sequence whose first four terms are 1, 2, 4, 8 and whose third order differences are all 1.

It is doubtful in that the key assumption about the third order differences is based on very little evidence. If we were told that the fifth term of the sequence we are seeking is 15, we would have that $D_2^3 = 1$, and our assumption would be better supported.

If the fifth term is anything other than 15, our assumption fails, and the generator found on the basis of that assumption generates a different sequence.

Therefore the assumption that for some k, the D_i^k all have the same nonzero value is not needed to verify formula (D), but it is essential in using formula (D) to find a generator for the sequence being studied.

Note to section 15, page 30
The use of finite differences in the study of series dates from the 17th century. The following scheme for the series generated by n^3 appears in a letter written by Leibniz in 1673.

				0		0		0				
			6		6		6		6			
		6		12		18		24		30		
	1		7		19		37		61		91	
0		1		8		27		64		125		216

Leibniz wrote that according to the English mathematician, John Pell, the original discovery was made by Gabriel Mouton of Lyons.

Derivation of S_3, page 33(d)

Take S_3 . From table 14, page 33, we have

$$S_3 = S_2 + D_3^1 \qquad \ldots (1)$$

Substituting the result of (c), page 33, we get

$$S_3 = (\,1\cdot 0 + 2\,D_1^1 + 1\,D_1^2\,) + D_3^1 \qquad \ldots (2)$$

Derivation of S_3, page 33(d) (Continued)

D_3^1 is not a lead difference. Using

$$D_i^{k+1} = D_{i+1}^k - D_i^k \qquad \text{(table 14, page 33)}$$

with k = 1, i = 2 we get

$$D_2^2 = D_3^1 - D_2^1 \quad \text{or} \quad D_3^1 = D_2^1 + D_2^2 \qquad \ldots (3)$$

From (c), page 33,

$$D_2^1 = D_1^1 + D_1^2 \qquad \ldots (4)$$

Using the formula again with k = 2, i = 1

$$D_1^3 = D_2^2 - D_1^2 \quad \text{or} \quad D_2^2 = D_1^2 + D_1^3 \qquad \ldots (5)$$

Combining (3), (4), and (5)

$$D_3^1 = (D_1^1 + D_1^2) + (D_1^2 + D_1^3) = D_1^1 + 2 D_1^2 + D_1^3 \qquad \ldots (6)$$

Finally,

$$S_3 = 1\cdot 0 + 2 D_1^1 + 1 D_1^2 + D_3^1 \qquad \text{, using (2) above}$$

$$= 1\cdot 0 + 2 D_1^1 + 1 D_1^2 + (D_1^1 + 2 D_1^2 + D_1^3) \qquad \text{, using (6) above}$$

$$= 1\cdot 0 + 3 D_1^1 + 3 D_1^2 + 1 D_1^3 \qquad \ldots (7)$$

Notes to formula (E), page 35

1. In some particular problems it is more natural to assign a nonzero value to S_o ($S_o = s_o$, and $S_n = s_o + \ldots + s_n$.). A glance at the arguments in section 17 show that the only change needed in formula (E) is to add the term $1 \cdot S_o = S_o$ to the rest of the sum.
2. See the note in the commentary for formula (D), page 29. The significance of the D_i^k being constant is the same in this formula.
3. Formula (E) and table 13, page 32, say that if we are given a sequence which satisfies the condition that the kth order differences are constant, then the same table of differences contains the information needed to find the generator of the series as well as the formula for the nth partial sum of the series. See example 2, page 36.

58. (1)

n	Series	D^1	D^2	D^3
1	1			
		3		
2	4		2	
		5		
3	9		2	
		7		
4	16		2	
		9		
5	25			
⋮	⋮			
n	n^2			

Using formula (E), page 35, with $D_1^1 = 1$, $D_1^2 = 3$, $D_1^3 = 2$

$$S_n = n\cdot 1 + \frac{n(n-1)}{2!}(3) + \frac{n(n-1)(n-2)}{3!}(2) = \frac{n(n+1)(2n+1)}{6}$$

(2)

n	Series	D^1	D^2	D^3	D^4
1	1				
		7			
2	8		12		
		19			
3	27		18	6	
		37			
4	64		24	6	
		61			
5	125				
⋮	⋮				
n	n^3				

Using formula (E), page 35, with $D_1^1 = 1$, $D_1^2 = 7$, $D_1^3 = 12$, $D_1^4 = 6$,

$$S_n = n\cdot 1 + \frac{n(n-1)}{2!}(7) + \frac{n(n-1)(n-2)}{3!}(12) + \frac{n(n-1)(n-2)(n-3)}{4!}(6) = \frac{n^2(n+1)^2}{4}$$

58. (Continued)

(3)

n	Series D^1	D^2
1	2	
2	4	2
3	6	2
4	8	2
5	10	2
⋮	⋮	
n	2n	

Using formula (E), page 35, with $D_1^1 = 2$, $D_1^2 = 2$,

$$S_n = n \cdot 2 + \frac{n(n-1)}{2!}(2)$$
$$= n(n+1)$$

(4)

n	Series D^1	D^2
1	1	
2	3	2
3	5	2
4	7	2
5	9	2
⋮	⋮	
n	2n - 1	

Using formula (E), page 35, with $D_1^1 = 1$, $D_1^2 = 2$,

$$S_n = n \cdot 1 + \frac{n(n-1)}{2!}(2)$$
$$= n^2$$

(5)

n	Series D^1	D^2
1	2	
2	7	5
3	12	5
4	17	5
5	22	5
⋮	⋮	
n	5n - 3	

Using formula (E), page 35, with $D_1^1 = 2$, $D_1^2 = 5$,

$$S_n = n \cdot 2 + \frac{n(n-1)}{2!}(5)$$
$$= \frac{n(5n-1)}{2}$$

(6)

n	Series D^1	D^2
1	a	
2	a + d	d
3	a + 2d	d
4	a + 3d	d
5	a + 4d	d
⋮	⋮	
n	a + (n - 1)d	

Using formula (E), page 35, with $D_1^1 = a$, $D_1^2 = d$,

$$S_n = n \cdot a + \frac{n(n-1)}{2!}(d)$$
$$= \frac{n(dn + (2a - d))}{2}$$

(7)

n	Series D^1	D^2	D^3
1	3		
2	8	5	
3	15	7	2
4	24	9	2
5	35	11	2
⋮	⋮		
n	n(n + 2)		

Using formula (E), page 35, with $D_1^1 = 3$, $D_1^2 = 5$, $D_1^3 = 2$,

$$S_n = n \cdot 3 + \frac{n(n-1)}{2!}(5) + \frac{n(n-1)(n-2)}{3!}(2)$$
$$= \frac{n(n+1)(2n+7)}{6}$$

(8)

n	Series D^1	D^2	D^3
1	2		
2	6	4	
3	12	6	2
4	20	8	2
5	30	10	2
⋮	⋮		
n	n(n + 1)		

Using formula (E), page 35, with $D_1^1 = 2$, $D_1^2 = 4$, $D_1^3 = 2$,

$$S_n = n \cdot 2 + \frac{n(n-1)}{2!}(4) + \frac{n(n-1)(n-2)}{3!}(2)$$
$$= \frac{n(n+1)(n+2)}{3}$$

(9)

n	Series D^1	D^2	D^3
1	1		
2	9	8	
3	25	16	8
4	49	24	8
5	81	32	8
⋮	⋮		
n	$(2n-1)^2$		

Using formula (E), page 35, with $D_1^1 = 1$, $D_1^2 = 8$, $D_1^3 = 8$,

$$S_n = n \cdot 1 + \frac{n(n-1)}{2!}(8) + \frac{n(n-1)(n-2)}{3!}(8)$$
$$= \frac{n(4n^2 - 1)}{3}$$

58. (Continued)

(10)

n	Series D^1	D^2	D^3
1	4		
2	16	12	
3	36	20	8
4	64	28	8
5	100	36	8
⋮	⋮		
n	$(2n)^2$		

Using formula (E), page 35, with $D_1^1 = 4$, $D_1^2 = 12$, $D_1^3 = 8$,

$$S_n = n \cdot 4 + \frac{n(n-1)}{2!}(12) + \frac{n(n-1)(n-2)}{3!}(8) = \frac{2n(n+1)(2n+1)}{3}$$

(11)

n	Series D^1	D^2	D^3
1	1		
2	16	15	
3	49	33	18
4	100	51	18
5	169	69	18
⋮	⋮		
n	$(3n-2)^2$		

Using formula (E), page 35, with $D_1^1 = 1$, $D_1^2 = 15$, $D_1^3 = 18$,

$$S_n = n \cdot 1 + \frac{n(n-1)}{2!}(15) + \frac{n(n-1)(n-2)}{3!}(18) = \frac{n(6n^2-3n-1)}{2}$$

(12)

n	Series D^1	D^2	D^3
1	2		
2	5	3	
3	10	5	2
4	17	7	2
5	26	9	2
⋮	⋮		
n	n^2+1		

Using formula (E), page 35, with $D_1^1 = 2$, $D_1^2 = 3$, $D_1^3 = 2$,

$$S_n = n \cdot 2 + \frac{n(n-1)}{2!}(3) + \frac{n(n-1)(n-2)}{3!}(2) = \frac{n(2n^2+3n+7)}{6}$$

(13)

n	Series D^1	D^2	D^3	D^4
1	1			
2	27	26		
3	125	98	72	
4	343	218	120	48
5	729	386	168	48
⋮	⋮			
n	$(2n-1)^3$			

Using formula (E), page 35, with $D_1^1 = 1$, $D_1^2 = 26$, $D_1^3 = 72$, $D_1^4 = 48$,

$$S_n = n \cdot 1 + \frac{n(n-1)}{2!}(26) + \frac{n(n-1)(n-2)}{3!}(72) + \frac{n(n-1)(n-2)(n-3)}{4!}(48) = n^2(2n^2-1)$$

(14)

n	Series D^1	D^2	D^3	D^4
1	1			
2	4	3		
3	10	6	3	
4	20	10	4	1
5	35	15	5	1
⋮	⋮			
n	$\frac{n(n+1)(n+2)}{6}$			

Using formula (E), page 35, with $D_1^1 = 1$, $D_1^2 = 3$, $D_1^3 = 3$, $D_1^4 = 1$,

$$S_n = n \cdot 1 + \frac{n(n-1)}{2!}(3) + \frac{n(n-1)(n-2)}{3!}(3) + \frac{n(n-1)(n-2)(n-3)}{4!}(1) = \frac{n(n+1)(n+2)(n+3)}{24}$$

Note to Problem 58(14)
Each term S_n is the sum of the first n triangular pyramidal numbers (See section 18(a), page 39.). The terms $S_1, S_2, S_3, \ldots$ form a sequence of 'fourth dimensional' figurate numbers. The name is used to indicate that this is the next generalization of the plane figurate numbers and pyramidal numbers discussed in section 18.

Note to section 17: Method for finding S_n, the nth partial sum of a series

The following procedure for finding S_n is based on the principle of mathematical induction and is suitable for those series where the generator of the series is a polynomial. (We will not go into the details of mathematical induction. The step outline of the procedure is sufficient for our purposes here.)

Example 1. Given: the series $S_n = 1 + 2 + 3 + \ldots + n$ ... (1)

Find a formula for S_n.

Solution

Step 1. If the nth term (the generator) of a series is a polynomial of degree k, assume that the formula for S_n will be a polynomial of degree $(k+1)$. In the series above the generator of the series, namely $s_n = n$, is of degree 1. Assume

$$S_n = an^2 + bn + c \qquad \ldots (2)$$

The task now is to find the coefficients a, b, c.

Step 2. Set (1) equal to (2).

$$1 + 2 + 3 + \ldots + n = an^2 + bn + c \qquad \ldots (3)$$

Write S_{n+1}, that is, from (1) and (2) we have

$$(1 + 2 + 3 + \ldots + n) + (n + 1) = a(n+1)^2 + b(n+1) + c \qquad \ldots (4)$$

Step 3. Replace $(1 + 2 + 3 + \ldots + n)$ on the left side by (2).

$$an^2 + bn + c + (n+1) = a(n+1)^2 + b(n+1) + c$$

or

$$an^2 + bn + c + (n+1) = a(n^2 + 2n + 1) + b(n + 1) + c$$

which simplifies to

$$(1 - 2a)n - (a + b - 1) = 0 \qquad \ldots (5)$$

Step 4. For equation (5) to be valid for all values of n, we must have coefficients of the powers of n and constant term equal to 0.

$$1 - 2a = 0 \qquad \ldots (6)$$

$$a + b - 1 = 0 \qquad \ldots (7)$$

Solve (6) for a and substitute in (7) to get b. Thus, $a = \frac{1}{2}$, $b = \frac{1}{2}$.

Step 5. Substitute the values of a, b in (2) to get

$$S_n = \frac{1}{2}n^2 + \frac{1}{2}n + c \qquad \ldots (8)$$

Step 6. Use (1) and (8). Since they represent the same nth partial sum, set $n = 1$ to get

$$1 = \frac{1}{2} + \frac{1}{2} + c \quad \text{or} \quad c = 0$$

Substitute $c = 0$ in (8) and

$$S_n = \frac{n(n+1)}{2}$$

is the nth partial sum for the series (1), a result we have seen on pages 31, 32, 36.

Example 2. Given: the series $S_n = 1^2 + 2^2 + 3^2 + \ldots n^2$ ... (1)

Find a formula for S_n

Solution: Use the step procedure in example 1.

Step 1. The generator, $s_n = n^2$ is of degree 2. Assume

$$S_n = an^3 + bn^2 + cn + d \qquad \ldots (2)$$

Step 2.

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = an^3 + bn^2 + cn + d \qquad \ldots (3)$$

Write S_{n+1}.

$$1^2 + 2^2 + 3^2 + \ldots + n^2 + (n + 1)^2 = a(n+1)^3 + b(n+1)^2 + c(n+1) + d$$

Note to section 17: Method for finding S_n, the nth partial sum of a series (Continued)

Step 3. $an^3 + bn^2 + cn + d + (n+1)^2 = a(n+1)^3 + b(n+1)^2 + c(n+1) + d$

which simplifies to

$$(1 - 3a)n^2 + (2 - 3a - 2b)n - (a + b + c - 1) = 0 \qquad \ldots (4)$$

Step 4. Set coefficients of powers of n and the constant term equal to 0.

$$1 - 3a = 0$$
$$2 - 3a - 2b = 0$$
$$a + b + c - 1 = 0$$

Solve for a, b, c so that $a = \frac{1}{3}$, $b = \frac{1}{2}$, $c = \frac{1}{6}$.

Step 5. Substitute for a, b, c in (2).

$$S_n = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n + d \qquad \ldots (5)$$

Step 6. $1^2 = \frac{1}{3} + \frac{1}{2} + \frac{1}{6} + d$ and so $d = 0$.

Substitute d = 0 in (5) to get

$$S_n = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

or $S_n = \frac{n(n+1)(2n+1)}{6}$, the nth partial sum for the series (1) .

59. Since the same table of differences is used for both s_n and S_n , we will write out the table only once, without headings. See example 2, page 36, for the appropriate designations.

(1)

n			
1	6		
		12	
2	18		6
		18	
3	36		6
		24	
4	60		6
		30	
5	90		

Using formula (D), page 29,

with $s_1 = 6$, $D_1^1 = 12$, $D_1^2 = 6$,

$$s_n = 6 + (n-1)12 + \frac{(n-1)(n-2)}{2!}(6)$$
$$= 3n(n+1)$$

Using formula (E), page 35,

with $D_1^1 = 6$, $D_1^2 = 12$, $D_1^3 = 6$,

$$S_n = n \cdot 6 + \frac{n(n-1)}{2!}(12) + \frac{n(n-1)(n-2)}{3!}(6)$$
$$= n(n+1)(n+2)$$

(2)

n				
1	6			
		18		
2	24		18	
		36		6
3	60		24	
		60		6
4	120		30	
		90		
5	210			

Using formula (D), page 29,

with $s_1 = 6$, $D_1^1 = 18$, $D_1^2 = 18$, $D_1^3 = 6$,

$$s_n = 6 + (n-1)18 + \frac{(n-1)(n-2)}{2!}(18) + \frac{(n-1)(n-2)(n-3)}{3!}(6)$$
$$= n(n+1)(n+2)$$

Using formula (E), page 35,

with $D_1^1 = 6$, $D_1^2 = 18$, $D_1^3 = 18$, $D_1^4 = 6$,

$$S_n = n \cdot 6 + \frac{n(n-1)}{2!}(18) + \frac{n(n-1)(n-2)}{3!}(18) + \frac{n(n-1)(n-2)(n-3)}{4!} = \frac{n(n+1)(n+2)(n+3)}{4}$$

(3)

n		
1	3	
		3
2	6	
		3
3	9	
		3
4	12	
		3
5	15	

Using formula (D), page 29, with $s_1 = 3$, $D_1^1 = 3$,

$$s_n = 3 + (n-1)3 = 3n$$

Using formula (E), page 35, with $D_1^1 = 3$, $D_1^2 = 3$,

$$S_n = n \cdot 3 + \frac{n(n-1)}{2!}(3) = \frac{3n(n+1)}{2} .$$

59. (Continued)

(4)

n		D^1
1	4	
		4
2	8	
		4
3	12	
		4
4	16	
		4
5	20	

Using formula (D), page 29, with $s_1 = 4$, $D_1^1 = 4$,

$$s_n = 4 + (n-1)4 = 4n$$

Using formula (E), page 35, with $D_1^1 = 4$, $D_1^2 = 4$,

$$S_n = n \cdot 4 + \frac{n(n-1)}{2!}(4) = 2n(n+1)$$

(5)

n		D^1
1	5	
		5
2	10	
		5
3	15	
		5
4	20	
		5
5	25	

Using formula (D), page 29, with $s_1 = 5$, $D_1^1 = 5$,

$$s_n = 5 + (n-1)5 = 5n$$

Using formula (E), page 35, with $D_1^1 = 5$, $D_1^2 = 5$,

$$S_n = n \cdot 5 + \frac{n(n-1)}{2!}(5) = \frac{5n(n+1)}{2}$$

60.

n	Series	D^1	D^2	D^3
1	1			
		5		
2	6		4	
		9		
3	15		4	
		13		
4	28			

Using formula (E), page 35, with $D_1^1 = 1$, $D_1^2 = 5$, $D_1^3 = 4$,

$$S_n = n \cdot 1 + \frac{n(n-1)}{2!}(5) + \frac{n(n-1)(n-2)}{3!}(4)$$
$$= \frac{n(n+1)(4n-1)}{6}$$

Note to Problem 60

S_n is also the formula for hexagonal pyramidal numbers. The remark on page 41 about geometric models applies to these numbers as well.

61. (a)

n	Heptagonal Number	D^1	D^2
1	1		
		6	
2	7		5
		11	
3	18		5
		16	
4	34		

Using formula (D), page 29, with $s_1 = 1$, $D_1^1 = 6$, $D_1^2 = 5$,

$$s_n = 1 + (n-1)6 + \frac{(n-1)(n-2)}{2!}(5)$$
$$= \frac{n(5n-3)}{2}$$

(b)

n	Octagonal Number	D^1	D^2
1	1		
		7	
2	8		6
		13	
3	21		6
		19	
4	40		

Using formula (D), page 29, with $s_1 = 1$, $D_1^1 = 7$, $D_1^2 = 6$,

$$s_n = 1 + (n-1)7 + \frac{(n-1)(n-2)}{2!}(6)$$
$$= n(3n-2)$$

(c)

n	Nonagonal Numbers	D^1	D^2
1	1		
		8	
2	9		7
		15	
3	24		7
		22	
4	46		

Using formula (D), page 29, with $s_1 = 1$, $D_1^1 = 8$, $D_1^2 = 7$,

$$s_n = 1 + (n-1)8 + \frac{(n-1)(n-2)}{2!}(7) = \frac{n(7n-5)}{2}$$

61. (Continued)

(d)

n	Decagonal Number	D^1	D^2
1	1		
		9	
2	10		8
		17	
3	27		8
		25	
4	52		

Using formula (D), page 29, with $s_1 = 1$, $D_1^1 = 9$, $D_1^2 = 8$,

$$s_n = 1 + (n-1)9 + \frac{(n-1)(n-2)}{2!}(8) = n(4n-3)$$

62. (a)

n	Sequence	D^1	D^2
1	1		
		k - 1	
2	k		k - 2
		2k - 3	
3	3k - 3		k - 2
		3k - 5	
4	6k - 8		k - 2
		4k - 7	
5	10k - 15		k - 2
		5k - 9	
6	15k - 24		

We show how to obtain s_3. The other terms are found in a similar manner.

From table 10, page 27,

$$D_1^2 = D_2^1 - D_1^1 \quad \text{or} \quad D_2^1 = D_1^1 + D_1^2 \qquad \ldots (1)$$

By assumption $D_1^1 = k - 1$ and $D_1^2 = k - 2$. Substituting in (1),

$$D_2^1 = (k-1) + (k-2) = 2k - 3 \qquad \ldots (2)$$

From table 10, page 27

$$D_2^1 = s_3 - s_2 \quad \text{or} \quad s_3 = s_2 + D_2^1 \qquad \ldots (3)$$

By assumption $s_2 = k$ and by (2) $D_2^1 = 2k - 3$. Substituting in (3),

$$s_3 = k + (2k-3) = 3k - 3 \qquad \ldots (4)$$

(b) By the assumption of example 3, part C, page 43, $D_i^2 = k - 2$ for all $i = 1, 2, 3, \ldots$. We can use finite differences to find the generator of the sequence. Using formula (D), page 29, with $s_1 = 1$, $D_1^1 = k - 1$, $D_1^2 = k - 2$,

$$(s_n)_k = 1 + (n-1)(k-1) + \frac{(n-1)(n-2)}{2!}(k-2) = \frac{n[(k-2)(n-1) + 2]}{2}$$

See formula (F), page 43.

63. (a) Here $s_n = 1 + (n-1)1 = n$

n	Series	D^1	D^2
1	1		
		1	
2	2		
		1	
3	3		
		1	
4	4		

Using formula (E), page 35, with $D_1^1 = 1$, $D_1^2 = 1$,

$$S_n = n \cdot 1 + \frac{n(n-1)}{2}(1) = \frac{n(n+1)}{2}$$

Compare with section 18(a), page 38

(b) Here $s_n = 1 + (n-1)2 = 2n - 1$

n	Series	D^1	D^2
1	1		
		2	
2	3		
		2	
3	5		
		2	
4	7		

Using formula (E), with $D_1^1 = 1$, $D_1^2 = 2$,

$$S_n = n \cdot 1 + \frac{n(n-1)}{2} = n^2$$

Compare with section 18(b), page 39.

63. (Continued)

(c) Here $s_n = 1 + (n - 1)3 = 3n - 2$

n	Series D^1	D^2
1	1	
		3
2	4	
		3
3	7	
		3
4	10	

Using formula (E), with $D_1^1 = 1$, $D_1^2 = 3$,

$$S_n = n \cdot 1 + \frac{n(n-1)}{2!}(3) = \frac{n(3n-1)}{2}$$

Compare with section 18(c), pages 40-41.

(d) Here $s_n = 1 + (n-1)(k-2)$

n	Series D^1	D^2
1	1	
		k - 2
2	k - 1	
		k - 2
3	2k - 3	
		k - 2
4	3k - 5	

Using formula (E), with $D_1^1 = 1$, $D_1^2 = k - 2$,

$$S_n = n \cdot 1 + \frac{n(n-1)}{2!}(k-2) = \frac{n\ [(k-2)(n-1) + 2]}{2}$$

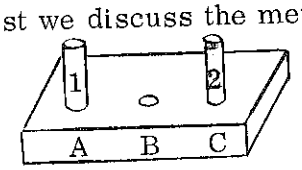
Compare with formula (F), page 43.

Notes to Problem 63(d)

1. This development of the plane figurate numbers is essentially the same idea as the procedure of example 3, page 43.
2. This purely arithmetic approach to the figurate numbers shows how the plane figurate numbers can be defined as sums of series whose terms form an arithmetic sequence just as the pyramidal numbers are defined as sums of series whose terms are the plane figurate numbers. See section 18.

64. First we discuss the method for interchanging the pegs in the smallest number of moves.

1.



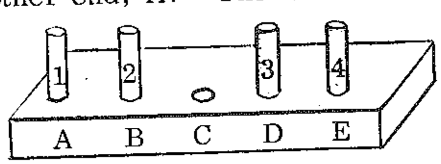
We mark the pegs 1, 2 (left to right) and the holes A, B, C (left to right).

MOVES

1. 2 to B   2. 1 to C   3. 2 to A

Note that after move 1 the pegs alternate in color white-black, and the empty hole is at C. After move 2 the pegs alternate in color black-white, and the empty hole is at the other end, A. The last move separates the different color pegs again.

2.

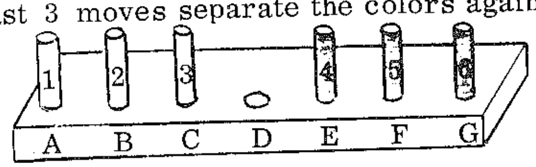


MOVES

1. 3 to C
2. 2 to D
3. 1 to B
4. 3 to A
5. 4 to C
6. 2 to E
7. 1 to D
8. 4 to B

After move 3 the pegs alternate in color white-black, and the empty hole is at A. The next 2 moves interchange the colors and move the empty hole to the other end, E. The last 3 moves separate the colors again and so are a kind of inverse to the first 3 moves.

3.



MOVES

1. 4 to D
2. 3 to E
3. 2 to C
4. 4 to B
5. 5 to D
6. 6 to F
7. 3 to G
8. 2 to E
9. 1 to C
10. 4 to A
11. 5 to B
12. 6 to D
13. 2 to F
14. 1 to E
15. 6 to C

After move 6 the pegs alternate in color white-black, and the empty hole is at G. The next 3 moves interchange the colors and move the empty hole to the other end, A. The last 6 moves separate the colors again.

In other words the first part of the strategy is to mix the colors so that the pegs alternate in color white-black, with the empty hole at one end of the board. If there are n pairs, the next n moves change the color alternation to black-white and move the empty hole to the other end of the board. The remaining moves reverse the effect of the first set

64. (Continued)
and separate the colors.

The moves above indicate the details for bringing this about. Minor variations are possible. For example, the moves have a mirror symmetry in that every move of a white peg to the right can be replaced by a move of the corresponding black peg to the left, and similarly moves by the black pegs must be replaced by moves of the corresponding white pegs. The final result would be unchanged.

Since the rules do not allow any backtracking or jumping over more than one peg of the opposite color, sequences of moves much different from those listed above will often lead to a 'deadend' in which pegs will be stranded with no possible moves before the interchange is completed.

With the pegs marked 1 to 8 and the holes marked A to I, we list moves for four pairs of pegs. The pegs are mixed at move 10.

1. 5 to E	5. 6 to E	9. 2 to D	13. 7 to E	17. 2 to F	21. 8 to E
2. 4 to F	6. 7 to G	10. 1 to B	14. 8 to G	18. 1 to D	22. 2 to G
3. 3 to D	7. 4 to H	11. 5 to A	15. 4 to I	19. 6 to B	23. 1 to F
4. 5 to C	8. 3 to F	12. 6 to C	16. 3 to H	20. 7 to C	24. 4 to D

Pairs of Pegs	Number of Moves	D^1	D^2
1	3		
		5	
2	8		2
		7	
3	15		2
		9	
4	24		2
		11	
5	35		

Using formula (D), page 29, with

$s_1 = 3, \ D_1^1 = 5, \ D_1^2 = 2,$

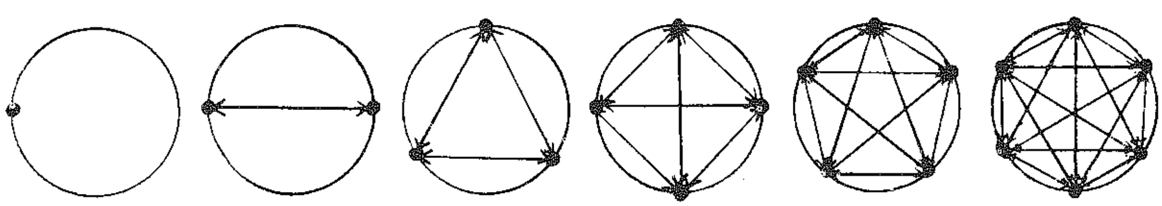
$$s_n = 3 + (n-1)5 + \frac{(n-1)(n-2)}{2!}(2)$$

$$= n(n+2)$$

Note to Problem 64
We might note that the number of moves needed to mix the pegs, that is, to place them in alternate colors with the hole at one end , (which is the same as the number of moves to separate them again with the hole at the center) is 1 for 1 pair, 3 for 2 pairs, 6 for 3 pairs, 10 for 4 pairs, and in general will be $n(n+1)/2$ for n pairs. This is the sequence of triangular numbers. (See section 18(a), page 38.) The minimum number number of moves to interchange the pegs becomes

$$\underbrace{\frac{n(n+1)}{2}}_{\text{Mix the colors}} + \underbrace{n}_{\text{Interchange the colors}} + \underbrace{\frac{n(n+1)}{2}}_{\text{Separate the colors}} = n(n+2)$$

65. (A)



Knights	Handshakes	D^1	D^2
1	0		
		1	
2	1		1
		2	
3	3		1
		3	
4	6		1
		4	
5	10		1
		5	
6	15		

Using formula (D), page 29, with

$s_1 = 0, \ D_1^1 = 1. \ D_1^2 = 1,$

$$s_n = 0 + (n-1)1 + \frac{(n-1)(n-2)}{2!}(1)$$

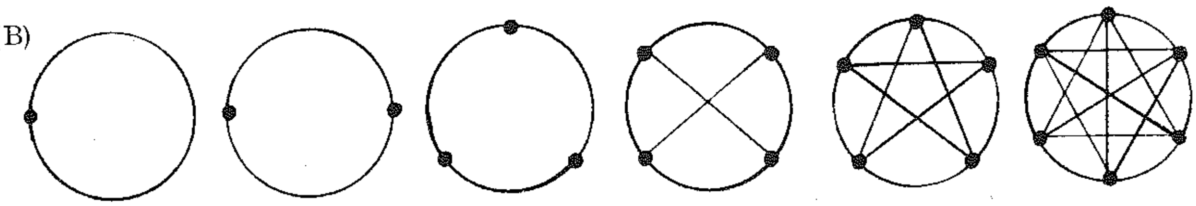
$$= \frac{n(n-1)}{2}$$

Notes to Problem 65(A)

1. This problem is equivalent to asking for the number of combinations of n things (here, the knights) taken 2 at a time. The number of combinations of n things r at a time is $C_r^n = \frac{n!}{r!\,(n-r)!}$. In our problem, r = 2. Thus, $C_2^n = \frac{n!}{2!\,(n-2)!} = \frac{n(n-1)}{2} = s_n$, the same result as before.

2. From the schematic representation we see that the problem is also the same as the geometric problem of finding the number of chords determined by n points on a circle or the total number of sides and diagonals in an n-gon.

65. (B)



Number of knights n	Number of handshakes	D^1	D^2	D^3	D^4
1	0				
		0			
2	0		0		
		0		2	
3	0		2		-3
		2		-1	
4	2		1		1
		3		0	
5	5		1		0
		4		0	
6	9		1		
		5			
7	14				
⋮	⋮				
n	s_n				

It appears that we never have the D_i^k all the same for any k ; however, if we look at D^2, we see that all the terms from D_3^2 on are equal to 1.

We begin again, keeping only the terms 0, 2, 5, 9, 14, ... from the original sequence. Call the terms of this new sequence t_1, t_2, t_3, and so on.

N	Sequence		
1	0		
		2	
2	2		1
		3	
3	5		1
		4	
4	9		1
		5	
5	14		
⋮	⋮		
N	t_N		

Using formula (D), page 29

$$t_N = 0 + (N-1)2 + \frac{(N-1)(N-2)}{2!}(1)$$
$$= \frac{N^2 + N - 2}{2} = \frac{N(N+1)}{2} - 1$$

Comparing this sequence to our original sequence we have $t_1 = s_3$, $t_2 = s_4$, $t_3 = s_5$, and in general we have $t_N = s_{N+2}$. Therefore if $n \geq 3$, $n = N+2$ for some positive integer n. Then $N = n - 2$, and we have $s_n = t_N = t_{n-2} = \frac{(n-2)((n-2)+1)}{2} - 1 = \frac{n(n-3)}{2}$.

The general rule for the sequence is $s_n = \begin{cases} 0, & n = 1,\ 2 \\ \frac{n(n-3)}{2}, & n = 3,\ 4,\ 5,\ \ldots \end{cases}$

Note to Problem 65(B)
Since the differences for the original sequence were never constant at any order, no single polynomial will generate the terms of the sequence for all values of n.

Again, from the diagram we can interpret the terms from s_3 on as giving the number of diagonals in an n-gon.

66. 1.

Number of layers n	Total number of grapefruit	D^1	D^2	D^3
1	1			
		3		
2	4		3	
		6		1
3	10		4	
		10		1
4	20		5	
		15		1
5	35		6	
		21		
6	56			

Using formula (D), page 29, with $s_1 = 1$, $D_1^1 = 3$, $D_1^2 = 3$, $D_1^3 = 1$,

$$s_n = 1 + 3(n-1) + \frac{(n-1)(n-2)}{2!}(3) + \frac{(n-1)(n-2)(n-3)}{3!}(1) = \frac{n(n+1)(n+2)}{6}$$

The sequence s_n in this problem is the sequence of triangular pyramidal numbers. (See section 18(a), page 39.)

2.

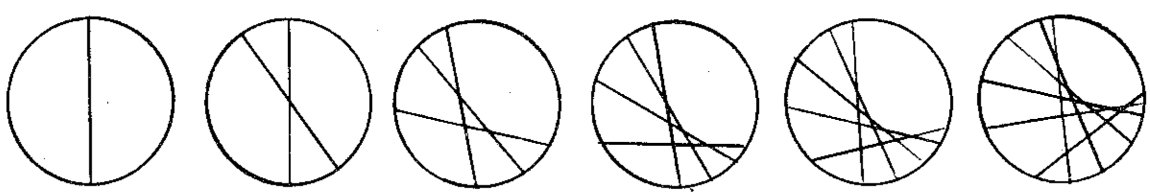
Number of layers n	Total number of peaches	D^1	D^2	D^3
1	1			
		4		
2	5		5	
		9		2
3	14		7	
		16		2
4	30		9	
		25		2
5	55		11	
		36		
6	91			

Using formula (D), page 29, with $s_1 = 1$, $D_1^1 = 4$, $D_1^2 = 5$, $D_1^3 = 2$,

$$s_n = 1 + (n-1)4 + \frac{(n-1)(n-2)}{2!}(5) + \frac{(n-1)(n-2)(n-3)}{3!}(2) = \frac{n(n+1)(2n+1)}{6}$$

The sequence s_n of this problem is the sequence of square pyramidal numbers. (See section 18(b), page 40.)

67. 1.



Number of cuts n	Number of pieces	D^1	D^2
1	2		
		2	
2	4		1
		3	
3	7		1
		4	
4	11		1
		5	
5	16		1
		6	
6	22		

Using formula (D), page 29, with $s_1 = 2$, $D_1^1 = 2$, $D_1^2 = 1$,

$$s_n = 2 + (n-1)2 + \frac{(n-1)(n-2)}{2}(1) = \frac{n^2+n+2}{2} = \frac{n(n+1)}{2} + 1$$

Note to Problem 67.1
Remembering that the top row of Pascal's triangle corresponds to n = 0, the number of pieces from n slices is the same as the sum of the first three numbers in row n of

Note to Problem 67.1
Pascal's triangle. (For the first two rows add in 0s to bring the number of terms to 3.) This correspondence even works for 0 cuts which gives 1 piece. The first few sums are

$$\begin{aligned} 1 &= 1 + 0 + 0 \\ 2 &= 1 + 1 + 0 \\ 4 &= 1 + 2 + 1 \\ 7 &= 1 + 3 + 3 \\ 11 &= 1 + 4 + 6 \end{aligned}$$

2. The numbers in the table can be verified by looking at the illustrations for part 1 of the problem.

Number of cuts n	Number of edges	D^1	D^2
1	3		
		5	
2	8		2
		7	
3	15		2
		9	
4	24		2
		11	
5	35		

Using formula (D), page 29, with $s_1 = 3$, $D_1^1 = 5$, $D_1^2 = 2$,

$$s_n = 3 + (n-1)5 + \frac{(n-1)(n-2)}{2!}(2)$$
$$= n(n+2)$$

3. The numbers in the table can be verified by looking at the illustrations for part 1 of the problem.

Number of cuts n	Number of edges	D^1	D^2
1	4		
		8	
2	12		4
		12	
3	24		4
		16	
4	40		4
		20	
5	60		

Using formula (D), page 29, with $s_1 = 4$, $D_1^1 = 8$, $D_1^2 = 4$,

$$s_n = 4 + (n-1)8 + \frac{(n-1)(n-2)}{2!}(4)$$
$$= 2n(n+1)$$

Note to Problem 67.3
We could also calculate this number directly by using the result of the previous problem. First we note that 1 cut gives 2 curved edges. Each additional cut divides each of 2 curved edges into 2 new curved edges, and so for n cuts we have 2n curved edges. These edges do not belong to more than one piece and so are counted only once. The straight edges are each common to 2 pieces and so must be counted twice. From 67.2 the number of such edges is given by

$$n(n+2) - 2n = (n^2 + 2n) - 2n = n^2$$

Therefore the total number of edges is

$$2n^2 + 2n = 2n(n+1),$$ the same answer as above.

4. The numbers in the table can be verified by looking at the illustrations for part 1 of the problem.

Number of cuts n	Number of points	D^1	D^2
1	2		
		3	
2	5		1
		4	
3	9		1
		5	
4	14		1
		6	
5	20		

Using formula (D), page 29, with $s_1 = 2$, $D_1^1 = 3$, $D_1^2 = 1$,

$$s_n = 2 + (n-1)3 + \frac{(n-1)(n-2)}{2!}(1)$$
$$= \frac{n(n+3)}{2}$$

67. 4. (Continued)

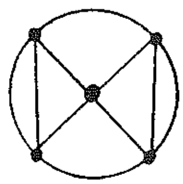
In answering part (c) the number of edges means the number found in 67.2 . In that case

$$(\text{number of pieces}) + (\text{number of points}) - (\text{number of edges})$$
$$= \left(\frac{n(n+1)}{2} + 1\right) + \left(\frac{n(n+3)}{2}\right) - n(n+2) = 1$$

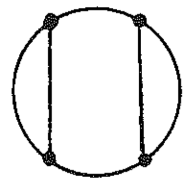
This is a special case of Euler's Theorem, which is true for a large class of figures that includes the diagrams used in our pie cutting problem. For example, to stay with this problem, if we cut the pie in any way whatever,

$$(\text{number of pieces}) + (\text{number of points}) - (\text{number of edges}) = 1 .$$

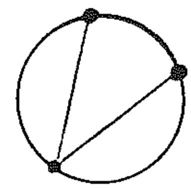
We will not prove this, but we will give some examples.



6 + 5 - 10 = 1



3 + 4 - 6 = 1



3 + 3 - 5 = 1

5. The numbers in the table can be verified by looking at the illustrations for part 1 of the problem.

Number of cuts n	Number of points	D^1	D^2
1	4		
		8	
2	12		4
		12	
3	24		4
		16	
4	40		4
		20	
5	60		

Using formula (D), page 29,
with $s_1 = 4$, $D_1^1 = 8$, $D_1^2 = 4$,

$$s_n = 4 + (n-1)8 + \frac{(n-1)(n-2)}{2!}(4)$$
$$= 2n(n+1)$$

Notes to Problem 67.5

1. The number of points found in 5 is the same as the number of edges found in 3. The number of pieces is 1 more than 1/4 of this number.

2. We can calculate this number directly by using the result of part 4.
The number of points on the edge of the pie is twice the number of cuts, and so for n cuts there will be 2n such points. These points are common to 2 pieces each and so must be counted twice.
Using the result of 4 , the number of points in the interior of the pie is

$$\frac{n(n+3)}{2} - 2n = \frac{n(n-1)}{2}$$

Each of these points is common to 4 pieces and so must be counted four times.
The total number of points is therefore

$$4\left(\frac{n(n-1)}{2}\right) + 2(2n) = 2n^2 + 2n = 2n(n+1)$$, the same answer as above.

68.A

Number of blocks n	Number of destinations	D^1
1	0	
		1
2	1	
		1
3	2	
		1
4	3	
		1
5	4	

Using formula (D), page 29,
with $s_1 = 0$, $D_1^1 = 1$,

$$s_n = 0 + (n-1)1 = n - 1$$

68. B Note to Problem 68. B

By analogy with the hint for Problem 68.A, the number of destinations which are n blocks from A is the number of solutions x, y counting order, where x and y are integers, to the equation

$$|x| + |y| = n$$

To see this we use a rectangular coordinate system, origin at A, with the length of a block as the unit of length. The intersections are the lattice points (points with integral coordinates) in this system.

Number of blocks n	Number of destinations	D^1
1	4	
		4
2	8	
		4
3	12	
		4
4	16	
		4
5	20	

Using formula (D), page 29, with $s_1 = 4$, $D^1_1 = 4$,

$$s_n = 4 + (n - 1)4 = 4n$$

Since the total number of destinations n or fewer blocks from A is

$$S_n = s_1 + s_2 + \dots + s_n ,$$

we use formula (E), page 35, with $D^1_1 = 4$, $D^2_1 = 4$.

$$S_n = n \cdot 4 + \frac{n(n-1)}{4}(4) = 2n(n + 1)$$

Notes to Problem 68. B

1. If you count A as the single destination which is 0 blocks from A, S_n becomes

$$S_n = 2n(n + 1) + 1 .$$

This number is the same as the number of lattice points which satisfy

$$|x| + |y| \leq n .$$

2. We can also compute s_n by using the results of Problem 68.A. By symmetry the number of destinations in each quadrant is the same; moreover, this number is the same as the number found in Problem 68, A. In addition to the points in the interior of the quadrants each of the two streets which intersect at A has 2 points which are n blocks from A (one in each direction along the street). Using this we have

$$s_n = 2 \cdot 2 + 4(n - 1) = 4n ,$$ the same answer as above.

69.

Number of blocks n	Number of destinations	D^1
1	6	
		6
2	12	
		6
3	18	
		6
4	24	
		6
5	30	

Using formula (D), page 29, with $s_1 = 6$, $D^1_1 = 6$,

$$s_n = 6 + (n - 1)6 = 6n$$

As in the square problem, Problem 68. B, the number of destinations n or fewer blocks from A is

$$S_n = s_1 + s_2 + \dots + s_n .$$

Using formula (E), page 35, with $D^1_1 = 6$, $D^2_1 = 6$,

$$S_n = n \cdot 6 + \frac{n(n-1)}{2}(6) = 3n(n + 1)$$

Note to Problem 69

If you count A as the single destination which is 0 blocks from A , then S_n becomes

$$S_n = 3n(n + 1) + 1 \quad .$$

70. A

Number of tube lengths n	Number of rooms	D^1	D^2
1	0		
		0	
2	0		1
		1	
3	1		1
		2	
4	3		1
		3	
5	6		1
		4	
6	10		

Using formula (D), page 29, with $s_1 = 0,\ D_1^1 = 0,\ D_1^2 = 1,$

$$s_n = 0 + (n-1)0 + \frac{(n-1)(n-2)}{2!}(1) = \frac{(n-1)(n-2)}{2}$$

B. Note to Problem 70. B.

By analogy with the hint for Problem 70. A, the number of rooms which are n tubelengths from A is the number of integer solutions x, y, z, counting order, to the equation

$$|x| + |y| + |z| = n \quad .$$

To see this we use a 3-dimensional Cartesian coordinate system with origin at A and unit length equal to the tubelength. The rooms correspond to the lattice points in this system.

Number of tube lengths n	Number of rooms	D^1	D^2
1	6		
		12	
2	18		8
		20	
3	38		8
		28	
4	66		8
		36	
5	102		

Using formula (D), page 29, with $s_1 = 6,\ D_1^1 = 12,\ D_1^2 = 8,$

$$s_n = 6 + (n-1)12 + \frac{(n-1)(n-2)}{2!}(8) = 4n^2 + 2$$

Since the number of rooms n or fewer tubelengths from A is

$$S_n = s_1 + s_2 + \ldots + s_n \ ,$$

we use formula (E), page 35, with $D_1^1 = 6,\ D_1^2 = 12,\ D_1^3 = 8$.

$$S_n = n \cdot 6 + \frac{n(n-1)}{2!}(12) + \frac{n(n-1)(n-2)}{3!}(8) = \frac{2n(2n^2 + 3n + 4)}{3} \quad .$$

Notes to Problem 70. B.

1. If you count A as the single room which is 0 tubelengths from A,

$$S_n = \frac{2n(2n^2 + 3n + 4)}{3} + 1 \quad .$$

   This number is the number of lattice points (x, y, z) which satisfy

$$|x| + |y| + |z| \leq n \quad .$$

2. It is also possible to compute s_n using the results of previous problems, but the calculation is more difficult than the one for Problem 68. B . By symmetry the interior of each of the 8 octants contains the same number of points which are n tubelengths from A; moreover, this number is the same as the number found in Problem 70. A . The 3 mutually perpendicular tubes which pass through A determine 3 mutually perpendicular planes which pass through A. The number of rooms in each of these planes which is n tubelengths from A is the same as s_n of Problem 68. B . There is one remaining correction. Each tube passing

Notes to Problem 70. B (Continued)
through A contains 2 points which are n tubelengths from A. These points will be in 2 planes and so have been counted twice. Therefore we must subtract 6 from our calculation.

$$s_n = 8\left(\frac{(n-1)(n-2)}{2}\right) + 3(4n) - 6 = 4n^2 + 2 .$$

3. We can describe the results of the last three problems in geometric terms.
In Problem 68. B the points which were a distance n from A were all the lattice points on the square whose vertices were the 4 points which were n blocks from A along a straight line.
In Problem 69 the points which were a distance n from A were all the lattice points on a regular hexagon whose vertices were the 6 points which were n blocks from A along a straight line.
In Problem 70. B the points which were a distance n from A were all the lattice points on the surface of a regular octahedron whose vertices were the 6 points whose distance from A is n tubelengths along a straight line.

71. a)

Length of edge n	Unit cubes with 0 faces painted	1 face painted	2 faces painted	3 faces painted
1	0	0	0	0
2	0	0	0	8
3	1	6	12	8
4	8	24	24	8
5	27	54	36	8
6	64	96	48	8

b) One face painted

Length of edge n	Unit cubes with 1 face painted	D^1	D^2
1	0		
		0	
2	0		6
		6	
3	6		12
		18	
4	24		12
		30	
5	54		12
		42	
6	96		

As in the example on pages 53-54, there is one term which breaks the pattern. $D_1^2 = 6$ but $D_i^2 = 12$ for $i = 2, 3, 4, \ldots$. We form a new sequence by dropping s_1. Call the new sequence t_N, where $t_1 = 0$, $t_2 = 6$, $t_3 = 24, \ldots$.

N	Sequence t_N	D^1	D^2
1	0		
		6	
2	6		12
		18	
3	24		12
		30	
4	54		12
		42	
5	96		

Using formula (D), page 29, with $t_1 = 0$, $D_1^1 = 6$, $D_1^2 = 12$,

$$t_N = 0 + (N-1)6 + \frac{(N-1)(N-2)}{2!}(12)$$

$$= 6(N-1)^2$$

Comparing the sequences s_n and t_N, we have that: $s_2 = t_1$, $s_3 = t_2$, $\ldots$, $s_{N+1} = t_N$, ...

71. b) (Continued)

Thus if $n = N + 1$, for a positive integer N (or $n - 1 = N$), $s_n = t_N$. Substituting for N in the expression for t_N we have $s_n = 6((n-1) - 1)^2 = 6(n-2)^2$.

The general rule for unit cubes with 1 painted face is given by $s_n = \begin{cases} 0, & \text{for } n = 1 \\ (n-2)^2, & \text{for } n \geq 2 \end{cases}$, where n is the length of the edge of the stacked cube .

Two faces painted

Length of edge n	Unit cubes with 2 faces painted	D^1
1	0	
2	0	0
3	12	12
4	24	12
5	36	12
6	48	12

Here $D^1_1 = 0$ breaks the pattern since $D^1_i = 12$ for $i = 2, 3, 4, \ldots$. We form a new sequence by dropping s_1. Call the new sequence t_N , where $t_1 = 0$, $t_2 = 12$, $t_3 = 24, \ldots$.

N	Sequence t_N	D^1
1	0	
2	12	12
3	24	12
4	36	12
5	48	12

Using formula (D), page 29, with $t_1 = 0$, $D^1_1 = 12$,

$t_N = 0 + (N-1)12 = 12(N-1)$.

Comparing the sequences s_n and t_N, we have that: $s_2 = t_1$, $s_3 = t_2$, $s_4 = t_3$, $\ldots, s_{N+1} = t_N$, and so on. Therefore if $n = N + 1$, for a positive integer N (or $n - 1 = N$), $s_n = t_N$. Substituting for N in the expression for t_N we have $s_n = 12((n-1) - 1) = 12(n-2)$.

The general rule for unit cubes with 2 painted faces is given by $s_n = \begin{cases} 0, & \text{for } n = 1 \\ 12(n-2), & \text{for } n \geq 2 \end{cases}$, where n is the length of the edge of the stacked cube.

Three faces painted

After the case $n = 1$, there are always 8 unit cubes with 3 painted faces corresponding to the 8 corners of the stacked cube.

Note: The procedure used in this problem and in Problem 65(B) can be generalized. Suppose that we are looking for a general rule for the sequence which begins s_1, s_2, s_3, s_4, s_5, ... According to the techniques that we have developed, we form a difference table and hope to find that for some k the D^k_i all have the same nonzero value. From Problems 65(B) and 70 we see that it will sometimes happen that what we do find is that for some k D^k_i all have the same value for $i = n_o+1, n_o+2, n_o+3, \ldots$, where $n_o \neq 0$ and that $D^k_1, \ldots, D^k_{n_o}$ do not follow this pattern.

We form a new sequence by dropping the terms $s_1, s_2, \ldots, s_{n_o}$. Call the new sequence t_N, where $t_1 = s_{n_o+1}$, $t_2 = s_{n_o+2}$, $t_3 = s_{n_o+3}$, ...

The difference table for t_N will have D_i^k all with the same nonzero value. We use formula (D), page 29, taking $t_1, D_1^1, \ldots, D_k^1$ from the difference table for the t_N. Note carefully that these lead differences are not the same as the lead differences for the s_n. We then have

$$t_N = t_1 + (N-1)D_1^1 + \frac{(N-1)(N-2)}{2!} D_1^2 + \ldots + \frac{(N-1)(N-2)\ldots(N-k)}{k!} D_1^k \quad \ldots (G)$$

We compare the sequences s_n and t_N. If $n = N + n_o$ for a positive integer N (or $N = n - n_o$), then $s_n = t_N$. When we compare the differences from the two sequences we find

Differences from s_n		Differences from t_N
$D_{n_o+1}^1$	=	D_1^1
$D_{n_o+1}^2$	=	D_1^2
$\vdots$		$\vdots$
$D_{n_o+1}^k$	=	D_1^k

Table A

Substituting for N in the expression (G) for t_N and for the D_1^j, j = 1, ..., k in (G) with the differences in table A, we get for $n > n_o$

$$s_n = s_{n_o+1} + (n-n_o-1)D_{n_o+1}^1 + \ldots + \frac{(n-n_o-1)\ldots(n-n_o-k)}{k!} D_{n_o+1}^k \quad \ldots (G')$$

This formula says nothing about the first n_o terms of the sequence, but under our assumption about the pattern that appeared in the kth order differences it will generate all the terms from s_{n_o+1} on.

Having done this derivation once we can use formula (G') directly without introducing the new sequence t_N.

Example: Find a formula for the generator of the sequence beginning 1, 2, 3, 6, 12, 20, 30, 42, ...

Solution:

n	Sequence	D^1	D^2
1	1		
		1	
2	2		0
		1	
3	3		2
		3	
4	6		3
		6	
5	12		2
		8	
6	20		2
		10	
7	30		2
		12	
8	42		

Based on this table we assume that $D_i^2 = 2$ for i = 4, 5, 6, ... In formula (G') above, $n_o = 3$, $k = 2$, $s_{3+1} = 6$, $D_{3+1}^1 = 6$, $D_{3+1}^2 = 2$.

For $n \geq 4$,

$$s_n = 6 + (n-3-1)6 + \frac{(n-3-1)(n-3-2)}{2!}(6)$$

$$= 6 + (n-4)6 + \frac{(n-4)(n-5)}{2!}(6)$$

$$= n^2 - 3n + 2$$

Example: (Continued)
The formula derived in this fashion will generate only the terms from s_{n_o+1} on.

72.

Number of sides	Number of intersections	D^1	D^2	D^3	D^4
3	0				
4	1	1			
5	5	4	3		
6	15	10	6	3	
7	35	20	10	4	1
8	70	35	15	5	1
9	126	56	21	6	1

If we let n designate the number of sides in the figure, the first case corresponds to n = 3.
We use formula (G'), page 96, with $n_o = 2$, $k = 4$, $s_{2+1} = 0$, $D^1_{2+1} = 1$, $D^2_{2+1} = 3$, $D^3_{2+1} = 3$,

$$D^4_{2+1} = 1,$$

For $n \geq 3$,

$$s_n = 0 + (n-2-1)1 + \frac{(n-2-1)(n-2-2)}{2!}(3) + \frac{(n-2-1)(n-2-2)(n-2-3)}{3!}(3) + \frac{(n-2-1)(n-2-2)(n-2-3)(n-2-4)}{4!}(1)$$

$$= 0 + (n-3) + \frac{(n-3)(n-4)}{2!}(3) + \frac{(n-3)(n-4)(n-5)}{3!}(3) + \frac{(n-3)(n-4)(n-5)(n-6)}{4!}(1)$$

$$= \frac{n^4 - 6n^3 + 11n^2 - 6n}{4!} = \frac{n(n-1)(n-2)(n-3)}{4!}$$

Notes to Problem 72

1. If, instead of formula (G'), we used formula (D), page 29, with $s_1 = 0$, $D^1_1 = 1$, $D^2_1 = 3$, $D^3_1 = 3$, $D^4_1 = 1$, we would get a formula for the general term in which n represented a number which was 2 less than the number of sides in the figure. As long as it is clear that n = (Number of sides) - 2, the formula obtained in that way is perfectly correct; however, it is more natural for the n of the formula to represent the number of sides of the figure, and it is for that reason that we made use of formula (G').
2. The answer, $\frac{n(n-1)(n-2)(n-3)}{4!} = C^n_4$, the number of combinations of n things taken 4 at a time. The geometric interpretation of this is simple. Each point of intersection picks out exactly 2 diagonals which in turn pick out 4 distinct vertices of the n vertices in the figure. On the other hand, for each choice of 4 distinct vertices we get a unique intersection point by taking the diagonals joining every other point. If we had used formula (D) rather than formula (G'), this simple interpretation of the answer would be obscured.
3. If we want to count the number of vertices as well as the number of intersection points we simply add n to the formula.

73.A.a

Rows	Triangles	D^1	D^2
1	1		
2	3	2	
3	6	3	1
4	10	4	1
5	15	5	1
6	21	6	1

Using formula (D), page 29, with $s_1 = 1$, $D^1_1 = 2$, $D^2_1 = 1$,

$$s_n = 1 + (n-1)2 + \frac{(n-1)(n-2)}{2!}(1) = \frac{n(n+1)}{2}$$

This is the sequence of triangular numbers. (See section 18.a, page 38.)

73.A.b

Number of rows n	Number of unit triangles (∇)	D^1	D^2
1	0		
		1	
2	1		1
		2	
3	3		1
		3	
4	6		1
		4	
5	10		1
		5	
6	15		

Using formula(D), page 29, with $s_1 = 0$, $D_1^1 = 1$, $D_1^2 = 1$,

$$s_n = 0 + (n-1)1 + \frac{(n-1)(n-2)}{2!}(1)$$

$$= \frac{n(n-1)}{2}$$

After the initial term, $s_1 = 0$, the terms of the sequence are successive triangular numbers.

c. We find S_n by adding the number of vertex up triangles and the number of vertex down triangles.

$$S_n = \frac{n(n+1)}{2} + \frac{n(n-1)}{2} = n^2$$

We could also obtain the same result using finite differences.

d. If we examine the sequence of triangular numbers (See section 18.a, page 38.), we see that the sum computed in part (c) above is the sum of the (n-1) st triangular number and the n th triangular number. The result is the n th square number. (See section 18.b, page 39.) In symbols if $t_n = n(n+1)/2$ and $s_n = n^2$,

$$t_{n-1} + t_n = s_n$$

73.B.a

Number of rows n	Number of triangles all sizes ($\triangle$)	D^1	D^2	D^3
1	1			
		3		
2	4		3	
		6		1
3	10		4	
		10		1
4	20		5	
		15		1
5	35		6	
		21		
6	56			

Using formula (D), page 29, with $s_1 = 1$, $D_1^1 = 3$, $D_1^2 = 3$, $D_1^3 = 1$,

$$s_n = 1 + (n-1)3 + \frac{(n-1)(n-2)}{2!}(3) + \frac{(n-1)(n-2)(n-3)}{3!}(1) = \frac{n(n+1)(n+2)}{6}$$

This sequence is the same as the sequence of triangular pyramidal numbers. (See 18.a, page 39.)

73.C In order to count the number of triangles efficiently it is easiest to count all the triangles of a given size (number of unit triangles on each edge) and then go to the next size which appears. We summarize the first several cases below.

Number of rows n	Unit triangles per edge 1	2	3	4	5	6	Total number of triangles
1	0	0	0	0	0	0	0
2	1	0	0	0	0	0	1
3	3	0	0	0	0	0	3
4	6	1	0	0	0	0	7
5	10	3	0	0	0	0	13
6	15	6	1	0	0	0	22
7	21	10	3	0	0	0	34
8	28	15	6	1	0	0	50

73. C (Continued)

The pattern in the table is clear. The nonzero entries in each column are successive triangular numbers. The first nonzero entry for triangles of n unit triangles on an edge occurs in the triangle of 2n rows. Although this table does not provide us with a method for calculating the general formula, it does allow us to calculate the total number of triangles for as many cases as we need.

Number of rows n	Number of triangles all sizes (▽)	D^1	D^2	D^3
1	0			
		1		
2	1		1	
		2		1
3	3		2	
		4		0
4	7		2	
		6		1
5	13		3	
		9		0
6	22		3	
		12		1
7	34		4	
		16		0
8	50		4	
		20		1
9	70		5	
		25		0
10	95		5	
		30		1
11	125		6	
		36		
12	161			

No matter how far we carry out the differences we will not find a k such that D_i^k all have the same value; however, if we examine the behavior of the second and third order differences we see that it might be possible to use our techniques if we break the original sequence into two new sequences, one consisting of the even numbered terms and the other consisting of the odd numbered terms.

1) $t_1 = 0 = s_1,\ t_2 = 3 = s_3,\ t_3 = 13 = s_5,\ t_4 = 34 = s_7,\ \ldots$.

2) $u_1 = 1 = s_2,\ u_2 = 7 = s_4,\ u_3 = 22 = s_6,\ u_4 = 50 = s_8,\ \ldots$.

1)

N	Sequence	D^1	D^2	D^3
1	0			
		3		
2	3		7	
		10		4
3	13		11	
		21		4
4	34		15	
		36		4
5	70		19	
		55		
6	125			
⋮	⋮			
N	t_N			

Using formula (D), page 29, with $t_1 = 0$, $D_1^1 = 3$, $D_1^2 = 7$, $D_1^3 = 4$,

$$t_N = 0 + (N-1)\,3 + \frac{(N-1)(N-2)}{2!}(7) + \frac{(N-1)(N-2)(N-3)}{3!}(4) = \frac{N(N-1)(4N+1)}{6}$$

Comparing the sequences s_n and t_N, $s_1 = t_1$, $s_3 = t_2$, $s_5 = t_3, \ldots, s_{2N-1} = t_N, \ldots$. Therefore if $n = 2N-1$ for some positive integer N (or $(n+1)/2 = N$), $s_n = t_N$. Substituting for N in the expression for t_N we have for odd values of n

$$s_n = \frac{1}{6}\left(\frac{n+1}{2}\right)\left(4\left(\frac{n+1}{2}\right)+1\right)\left(\left(\frac{n+1}{2}\right)-1\right) = \frac{(n-1)(n+1)(2n+3)}{24}$$

73.C (Continued)

2)

N	Sequence	D^1	D^2	D^3
1	1			
		6		
2	7		9	
		15		4
3	22		13	
		28		4
4	50		17	
		45		4
5	95		21	
		66		
6	161			
⋮				
N				

Using formula (D), page 29, with $u_1 = 1$, $D_1^1 = 6$, $D_1^2 = 9$, $D_1^3 = 4$,

$$u_n = 1 + (N-1)\,6 + \frac{(N-1)(N-2)}{2!}(9) + \frac{(N-1)(N-2)(N-3)}{3!}(4)$$

$$= \frac{N(N+1)(4N-1)}{6}$$

Comparing the sequences s_n and u_N, $s_2 = u_1$, $s_4 = u_2$, $s_6 = u_3, \ldots, s_{2N} = u_N, \ldots$ Therefore if $n = 2N$ for some positive integer N (or $n/2 = N$), $s_n = u_N$. Substituting for N in the expression for u_N we have for even values of n,

$$s_n = \frac{1}{6}\left(\frac{n}{2}\right)\left(\left(\frac{n}{2}\right)+1\right)\left(4\left(\frac{n}{2}\right)-1\right) = \frac{n(n+2)(2n-1)}{24}$$

The general rule for s_n requires two formulas.

$$s_n = \begin{cases} \dfrac{(n-1)(n+1)(2n+3)}{24}, & \text{if } n \text{ is odd} \\[2ex] \dfrac{n(n+2)(2n-1)}{24}, & \text{if } n \text{ is even} \end{cases}$$

Note to Problem 73.C
It is also possible to treat the special cases of 73.C as follows.
From the table at the top of page 99 we have

Number of rows	Term of the sequence n	Sequence	D^1	D^2	D^3
1	1	0			
			3		
3	2	3		7	
			10		4
5	3	13		11	
			21		4
7	4	34		15	
			36		
9	5	70			
	⋮	⋮			
	n	t_n			

From here we can use our previous methods, but we must be careful of the interpretation of n .

74.A

Number of unit squares on a side n	Total number of squares of all sizes	D^1	D^2	D^3
1	1			
		4		
2	5		5	
		9		2
3	14		7	
		16		2
4	30		9	
		25		2
5	55		11	
		36		
6	91			

74.A (Continued)
Using formula (D), page 29, with $s_1 = 1$, $D_1^1 = 4$, $D_1^2 = 5$, $D_1^3 = 2$,

$$s_n = 1 + (n-1)\,4 + \frac{(n-1)(n-2)}{2!}(5) + \frac{(n-1)(n-2)(n-3)}{3!}(2) = \frac{n(n+1)(2n+1)}{6}$$

This sequence is the same as the sequence of square pyramidal numbers. (See section 18.a, page 40.)

74.B In order to count the number of squares most efficiently it is helpful to count all the squares of a given size in each figure (where size means the number of unit squares on a side) and then go to the next size which appears in that figure. The numbers obtained show regularities which greatly aid the counting process. We summarize the results for the first ten figures.

Number of unit squares on a side n	Number of slanted squares of the given size										Total number of slanted squares
	1	2	3	4	5	6	7	8	9	10	
1	0	0	0	0	0	0	0	0	0	0	0
2	4	1	0	0	0	0	0	0	0	0	5
3	12	5	0	0	0	0	0	0	0	0	17
4	24	13	4	1	0	0	0	0	0	0	42
5	40	25	12	5	0	0	0	0	0	0	82
6	60	41	24	13	4	1	0	0	0	0	143
7	84	61	40	25	12	5	0	0	0	0	227
8	112	85	60	41	24	13	4	1	0	0	340
9	144	113	84	61	40	25	12	5	0	0	484
10	180	145	112	85	60	41	24	13	4	1	665

As with Problem 73.C it would be simple to continue this table for as many rows as we would want. The nonzero entries in each column belong to two sequences; in the odd numbered columns we have the sequence 4, 12, 24, 40, ... whose terms are just 4 times the corresponding triangular number, and in the even numbered columns we have the sequence which begins with 1 and then each following term is 1 more than the terms of the first sequence. (5 = 4 + 1, 13 = 12 + 1, 25 = 24 + 1, and so on.) The first nonzero entry in an odd numbered column, 2n - 1, occurs at row 2n; the first nonzero entry in an even numbered column, 2n, occurs at row 2n. We now use finite differences to look for a general rule.

Number of unit squares on a side n	Total number of slanted squares	D^1	D^2	D^3	D^4
1	0				
		5			
2	5		7		
		12		6	
3	17		13		-4
		25		2	
4	42		15		4
		40		6	
5	82		21		-4
		61		2	
6	143		23		4
		84		6	
7	227		29		-4
		113		2	
8	340		31		4
		144		6	
9	484		37		
		181			
10	665				

No matter how far we carry out the differences we will not find a k such that D_i^k all have the same value; however, the D^3 and D^4 columns indicate that we might be able to split the original sequence into two new sequences which we can treat by our methods.

74. B (Continued)
One sequence will consist of the odd numbered terms, and the other will consist of the even numbered terms.

1) $t_1 = 0 = s_1,\ t_2 = 17 = s_3,\ t_3 = 82 = s_5,\ \ldots$.

2) $u_1 = 5 = s_2,\ u_2 = 42 = s_4,\ u_3 = 143 = s_6,\ \ldots$.

1)

N	Sequence	D^1	D^2	D^3
1	0			
		17		
2	17		48	
		65		32
3	82		80	
		145		32
4	227		112	
		257		
5	484			

Using formula (D), page 29, with $t_1 = 0$, $D_1^1 = 17$, $D_1^2 = 48$, $D_1^3 = 32$,

$$t_N = 0 + (N-1)17 + \frac{(N-1)(N-2)}{2!}(48) + \frac{(N-1)(N-2)(N-3)}{3!}(32) = \frac{16N^3 - 24N^2 + 11N - 3}{3}$$

Comparing the sequences s_n and t_N, $s_1 = t_1$, $s_3 = t_2$, $s_5 = t_3$, $\ldots$, $s_{2N-1} = t_N$, $\ldots$.
Therefore if $n = 2N-1$ for some positive integer N (or $(n+1)/2 = N$), $s_n = t_N$.
Substituting for N in the expression for t_N we have for odd values of n,

$$s_n = \frac{16\left(\frac{n+1}{2}\right)^3 - 24\left(\frac{n+1}{2}\right)^2 + 11\left(\frac{n+1}{2}\right) - 3}{3} = \frac{(n-1)(4n^2 + 4n + 3)}{6}$$

2)

N	Sequence	D^1	D^2	D^3
1	5			
		37		
2	42		64	
		101		32
3	143		96	
		197		32
4	340		128	
		325		
5	665			

Using formula (D), page 29, with $u_1 = 5$, $D_1^1 = 37$, $D_1^2 = 64$, $D_1^3 = 32$,

$$u_N = 5 + (N-1)37 + \frac{(N-1)(N-2)}{2!}(64) + \frac{(N-1)(N-2)(N-3)}{3!}(32) = \frac{16N^3 - N}{3}$$

Comparing the sequences s_n and u_N, $s_2 = u_1$, $s_4 = u_2$, $s_6 = u_3$, $\ldots$, $s_{2N} = u_N$, $\ldots$.
Therefore if $n = 2N$ for a positive integer N (or $n/2 = N$), $s_n = u_N$.
Substituting for N in the expression for u_N we have for even values of n,

$$s_n = \frac{16\left(\frac{n}{2}\right)^3 - \left(\frac{n}{2}\right)}{3} = \frac{4n^3 - n}{6}$$

The general rule for s_n requires two formulas.

$$s_n = \begin{cases} \dfrac{(n-1)(4n^2 + 4n + 3)}{6}, & \text{if } n \text{ is odd} \\ \dfrac{n(4n^2 - 1)}{6}, & \text{if } n \text{ is even} \end{cases}$$

From 74. A the $n \times n$ pattern contains $\frac{n(n+1)(2n+1)}{6}$ squares with sides parallel to the sides of the pattern. Therefore the total number of squares of all kinds in the $n \times n$ pattern, v_n is given by

$$v_n = \begin{cases} \dfrac{n(n+1)(2n+1)}{6} + \dfrac{(n-1)(4n^2 + 4n + 3)}{6} = \dfrac{2n^3 + n^2 - 1}{2}, & \text{if } n \text{ is odd} \\ \dfrac{n(n+1)(2n+1)}{6} + \dfrac{n(4n^2 - 1)}{6} = \dfrac{n^2(2n + 1)}{2}, & \text{if } n \text{ is even} \end{cases}$$

75. A

Rows in the triangle n	Least number of moves	D^1	D^2	D^3
1	0			
		1		
2	1		-1	
		0		3
3	1		2	
		2		-4
4	3		-2	
		0		5
5	3		3	
		3		-6
6	6		-3	
		0		7
7	6		4	
		4		-8
8	10		-4	
		0		9
9	10		5	
		5		
10	15			

No matter how far we carry out the differences we will never have k such that the D_i^k all have the same value; however, we can see from the sequence itself that if we split it into two sequences, we should be able to use our techniques to find general formulas. (In fact, by now we should recognize the different numbers which appear as the triangular numbers. For the sake of illustration, however, we will go through the details of the derivation.) One sequence will consist of the odd numbered terms, and the other will consist of the even numbered terms.

1) $t_1 = 0 = s_1,\ t_2 = 1 = s_3,\ t_3 = 3 = s_5,\ \dots$.

2) $u_1 = 1 = s_2,\ u_2 = 3 = s_4,\ u_3 = 6 = s_6,\ \dots$.

1)

N	Sequence	D^1	D^2
1	0		
		1	
2	1		1
		2	
3	3		1
		3	
4	6		1
		4	
5	10		

Using formula (D), page 29, with $t_1 = 0,\ D_1^1 = 1,\ D_1^2 = 1$,

$$t_N = 0 + (N-1)\,1 + \frac{(N-1)(N-2)}{2!}(1) = \frac{N(N-1)}{2}$$

Comparing the sequences s_n and t_N, $s_1 = t_1,\ s_3 = t_2,\ s_5 = t_3, \dots, s_{2N-1} = t_N, \dots$.
Therefore if $n = 2N-1$ for a positive integer N (or $(n+1)/2 = N$), $s_n = t_N$.
Substituting for N in the expression for t_N we have for odd values of n

$$s_n = \frac{\left(\frac{n+1}{2}\right)\left(\left(\frac{n+1}{2}\right) - 1\right)}{2} = \frac{(n-1)(n+1)}{8} .$$

2)

N	Sequence	D^1	D^2
1	1		
		2	
2	3		1
		3	
3	6		1
		4	
4	10		1
		5	
5	15		

Using formula (D), page 29, with $u_1 = 1,\ D_1^1 = 2,\ D_1^2 = 1$,

$$u_N = 1 + (N-1)2 + \frac{(N-1)(N-2)}{2!}(1) = \frac{N(N+1)}{2}$$

Comparing the sequences s_n and u_N , $s_2 = u_1,\ s_4 = u_2,\ s_6 = u_3, \dots, s_{2N} = u_N, \dots$.
Therefore if $n = 2N$ for a positive integer N (or $n/2 = N$), $s_n = u_N$.
Substituting for N in the expression for u_N we have for even values of n

75. A (Continued)

$$s_n = \frac{(\frac{n}{2})((\frac{n}{2})+1)}{2} = \frac{n(n+2)}{8}$$

The general rule for s_n requires two formulas.

$$s_n = \begin{cases} \frac{(n+1)(n-1)}{8}, & \text{if } n \text{ is odd} \\ \frac{n(n+2)}{8}, & \text{if } n \text{ is even} \end{cases}$$

75. B

Rows in the triangle n	Least number of moves	D^1	D^2	D^3
1	0			
		1		
2	1		0	
		1		0
3	2		0	
		1		1
4	3		1	
		2		-1
5	5		0	
		2		0
6	7		0	
		2		1
7	9		1	
		3		-1
8	12		0	
		3		0
9	15		0	
		3		1
10	18		1	
		4		-1
11	22		0	
		4		
12	26			

Although there are many "patterns" in the difference table, none is the right one. No matter how far we carry the differences, we will not find a k such that D_i^k all have the same value. It does appear that if we break the sequence into three new sequences, we will be able to use our techniques on those sequences. One will consist of s_1 and every third term from there on, the second will start with s_2 and then every third term from there, and the third will start with s_3 and include every third term from there on.

1) $t_1 = 0 = s_1,\ t_2 = 3 = s_4,\ t_3 = 9 = s_7,\ \ldots$.

2) $u_1 = 1 = s_2,\ u_2 = 5 = s_5,\ u_3 = 12 = s_8,\ \ldots$.

3) $v_1 = 2 = s_3,\ v_2 = 7 = s_6,\ v_3 = 15 = s_9,\ \ldots$.

1)

N	Sequence	D^1	D^2
1	0		
		3	
2	3		3
		6	
3	9		3
		9	
4	18		

Using formula (D), page 29, with $t_1 = 0$, $D_1^1 = 3$, $D_1^2 = 3$,

$$t_N = 0 + (N-1)3 + \frac{(N-1)(N-2)}{2!}(3)$$

$$= \frac{3N(N-1)}{2}$$

Comparing the sequences s_n and t_N, $s_1 = t_1$, $s_4 = t_2$, $s_7 = t_3, \ldots, s_{3N-2} = t_N, \ldots$.
Therefore if $n = 3N-2$ for a positive integer N (or $(n+2)/3 = N$), $s_n = t_N$.
Substituting for N in the expression for t_N we have for $n = 1, 4, 7, 10, \ldots$,

$$s_n = \frac{3}{2}\left(\frac{n+2}{3}\right)\left(\left(\frac{n+2}{3}\right) - 1\right) = \frac{(n-1)(n+2)}{6}$$

75. B (Continued)

2)

N	Sequence	D^1	D^2
1	1		
2	5	4	
3	12	7	3
4	22	10	3

Using formula (D), page 29, with $u_1 = 1$, $D_1^1 = 4$, $D_1^2 = 3$,

$$u_N = 1 + (N-1)4 + \frac{(N-1)(N-2)}{2!}(3) = \frac{N(3N-1)}{2}$$

Comparing the sequences s_n and u_N, $s_2 = u_1$, $s_5 = u_2$, $s_8 = u_3, \ldots, s_{3N-1} = u_N, \ldots$.
Therefore if $n = 3N-1$ for a positive integer N (or $(n+1)/3 = N$), $s_n = u_N$.
Substituting for N in the expression for u_N, we have for $n = 2, 5, 8, 11, \ldots$,

$$s_n = \frac{1}{2}\left(\frac{n+1}{3}\right)\left(3\left(\frac{n+1}{3}\right) - 1\right) = \frac{n(n+1)}{6} .$$

3)

N	Sequence	D^1	D^2
1	2		
2	7	5	
3	15	8	3
4	26	11	3

Using formula (D), page 29, with $v_1 = 2$, $D_1^1 = 5$, $D_1^2 = 3$,

$$v_N = 2 + (N-1)5 + \frac{(N-1)(N-2)}{2!}(3) = \frac{N(3N+1)}{2}$$

Comparing the sequences s_n and v_N, $s_3 = v_1$, $s_6 = v_2$, $s_9 = v_3, \ldots, s_{3N} = v_N, \ldots$.
Therefore if $n = 3N$ for a positive integer N (or $n/3 = N$), $s_n = v_N$.
Substituting for N in the expression for v_N, we have for $n = 3, 6, 9, 12, \ldots$,

$$s_n = \frac{1}{2}\left(\frac{n}{3}\right)\left(3\left(\frac{n}{3}\right) + 1\right) = \frac{n(n+1)}{6} .$$

Note that from the three different sequences there are only two different formulas in terms of n.

$$s_n = \begin{cases} \frac{(n+2)(n-1)}{6}, & n = 1, 4, 7, 10, \ldots \\ \frac{n(n+1)}{6}, & n = 2, 3, 5, 6, 8, 9, \ldots \end{cases}$$

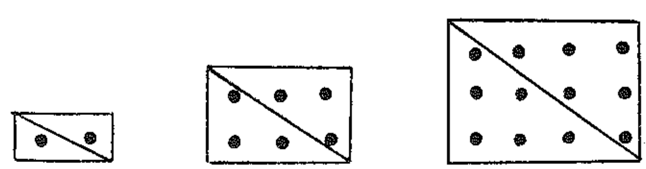
76. A

Number of rows n	Total number of dots	D^1	D^2
1	2		
2	6	4	
3	12	6	2
4	20	8	2
5	30	10	2
6	42	12	2

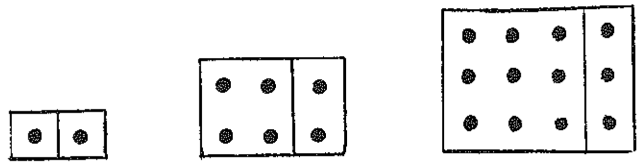
Using formula (D), page 29, with $s_1 = 2$, $D_1^1 = 4$, $D_1^2 = 2$,

$$s_n = 2 + (n-1)4 + \frac{(n-1)(n-2)}{2!}(2)$$
$$= n(n+1)$$

c. 1 The nth triangular number is given by $\frac{n(n+1)}{2}$. (See page 38.) Therefore if we denote the nth rectangular number by s_n and the nth triangular number by t_n, we have $s_n = 2(t_n)$. We can illustrate this geometrically. In the figures above we have divided the rectangular pattern into 2 triangular patterns. Each contains a triangular number of



76.A (Continued)
dots.



c.2 The n th square number is given by n^2. (See page 39.) Therefore if we denote the n th rectangular number by s_n and the n th square number by u_n, we have $s_n = u_n + n$. We can illustrate this geometrically. Each of the rectangular patterns of dots above has been divided into a square arrangement plus a column containing the same number of dots as the edge of the square.

d.

Number of layers n	Total number of spheres	D^1	D^2	D^3
1	2			
		6		
2	8		6	
		12		2
3	20		8	
		20		2
4	40		10	
		30		2
5	70		12	
		42		
6	112			

Using formula (D), page 29, with $S_1 = 2$, $D_1^1 = 6$, $D_1^2 = 6$, $D_1^3 = 2$,

$$S_n = 2 + (n-1)6 + \frac{(n-1)(n-2)}{2!}(6) + \frac{(n-1)(n-2)(n-3)}{3!}(2) = \frac{n(n+1)(n+2)}{3}$$

Since it is also true that $S_n = s_1 + s_2 + \ldots + s_n$, where the s_i are the rectangular numbers. we could use formula (E), page 35, with $D_1^1 = 2$, $D_1^2 = 4$, $D_1^3 = 2$.

$$S_n = n \cdot 2 + \frac{n(n-1)}{2!}(4) + \frac{n(n-1)(n-2)}{3!}(2) = \frac{n(n+1)(n+2)}{3}$$

Note that S_n is twice the n th triangular pyramidal number. (See page 39.)

76.B

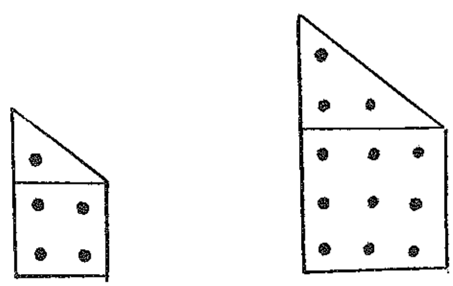
n	Trapezoidal number	D^1	D^2
1	1		
		4	
2	5		3
		7	
3	12		3
		10	
4	22		3
		13	
5	35		3
		16	
6	51		

Using formula (D), page 29, with $s_1 = 1$, $D_1^1 = 4$, $D_1^2 = 3$,

$$s_n = 1 + (n-1)4 + \frac{(n-1)(n-2)}{2!}3 = \frac{n(3n-1)}{2}$$

c. Let s_n be the n th rectangular number, t_n be the n th triangular number, and u_n be the n th square number. Then for $n \geq 2$,

$$t_{n-1} + u_n = \frac{(n-1)n}{2} + n^2 = \frac{n(3n-1)}{2} = s_n.$$



We can illustrate this geometrically. Each of the trapezoidal arrangements above has been divided into a square arrangement and a triangular arrangement. Each edge of the triangular arrangement contains one fewer dot than the edge of the square.

76. B (Continued)

d. From the formula on page 41 the nth pentagonal number is given by $\frac{n(3n-1)}{2}$, and so the nth pentagonal number is the same as the nth trapezoidal number.

e.

Number of layers n	Total number of spheres	D^1	D^2	D^3
1	1			
2	6	5		
3	18	12	7	
4	40	22	10	3
5	75	35	13	3
6	126	51	16	3

Using formula (D) page 29, with $S_1 = 1$, $D_1^1 = 5$, $D_1^2 = 7$, $D_1^3 = 3$,

$$S_n = 1 + (n-1)5 + \frac{(n-1)(n-2)}{2!}(7) + \frac{(n-1)(n-2)(n-3)}{3!}(3) = \frac{n^2(n+1)}{2}$$

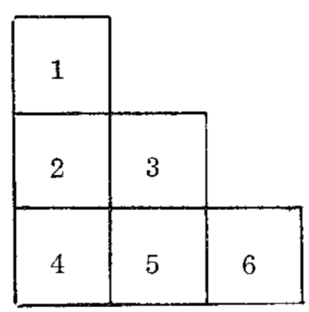
Since it is also true that $S_n = s_1 + s_2 + \ldots + s_n$, where the s_i are the trapezoidal numbers, we can use formula (E), page 35, with $D_1^1 = 1$, $D_1^2 = 4$, $D_1^2 = 3$.

$$S_n = n1 + \frac{n(n-1)}{2!}(4) + \frac{n(n-1)(n-2)}{3!}(3) = \frac{n^2(n+1)}{2}$$

Note to Problem 76. B. (e)
Since the sequence of pentagonal numbers is the same as the sequence of trapezoidal numbers, the sequence of pentagonal pyramidal numbers is also the same as the trapezoidal pyramidal numbers. Even though we cannot construct a pyramidal model for the pentagonal pyramidal numbers by using a pentagonal base, we can construct a pyramidal model with the correct number of spheres if we use a trapezoidal pattern for each level.

77. Before solving the problem we will indicate a method for determining the distinct connected blocks of three stamps in a given sheet.

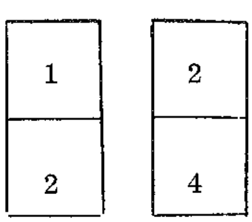
We use the example of the triangular sheet with 3 stamps along each edge.



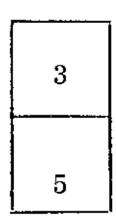
The first step is to list all the connected pairs of stamps. This listing is easily done since the two stamps of the pair must lie in either the same row or the same column. Therefore we need only go through the columns and rows of the sheet to find all the pairs.

The pairs are shown below.

1 st column



2 nd column



3 rd column

None

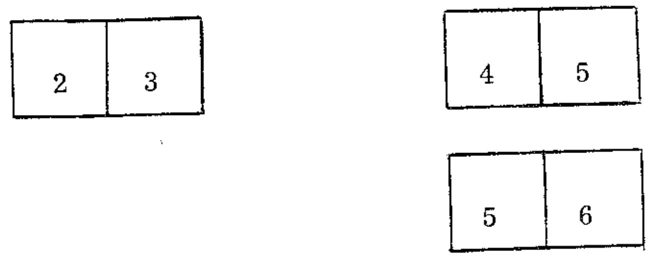
77. (Continued)

1 st row

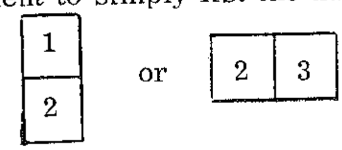
None

2 nd row

3 rd row

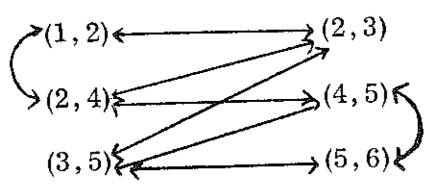


For working with larger sheets it would be more convenient to simply list the numbers of the stamps in each pair. In other words the pairs



would be listed as the pairs (1,2) or (2,3) . We will use this notation for the rest of this example.

Each connected block of three stamps consists of two distinct connected pairs with one stamp in common. Therefore to find all the blocks of three we list all the pairs and draw connecting arrows between any two pairs with a stamp in common. When we we have connected all such pairs, the number of arrows is the number of blocks of three stamps that we want.



The arrow joining (1,2) and (2,4) corresponds to the block consisting of stamps 1, 2, 4 (the first column). There are eight connecting arrows, and so there are eight connected blocks of three stamps. The same process can be used on any size sheet

Stamps along each straight edge n	Total number of blocks of three stamps	D^1	D^2
1	0		
		1	
2	1		6
		7	
3	8		6
		13	
4	21		6
		19	
5	40		6
		25	
6	65		

Using formula (D), page 29, with $s_1 = 0$, $D_1^1 = 1$, $D_1^2 = 6$,

$$s_n = 0 + (n-1)1 + \frac{(n-1)(n-2)}{2!}(6) = (n-1)(3n-5)$$

The nonzero terms of this sequence are the same as the octagonal numbers. (See page 42.)

APPLICATIONS OF FINITE DIFFERENCES

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