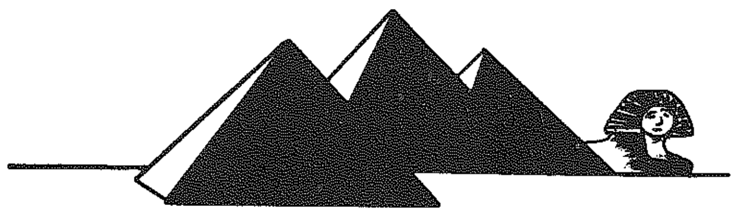
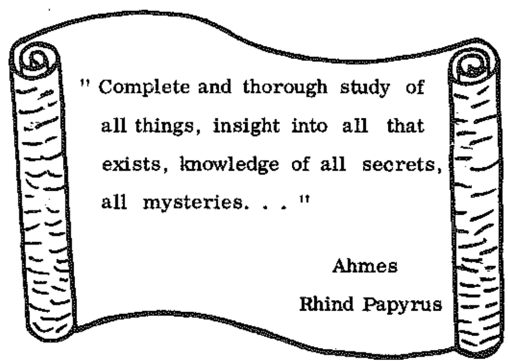
BOSTON COLLEGE MATHEMATICS INSTITUTE

Motivated Math Project Activity

Booklet 6

Fraction Action





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PREFACE

The series of MOTIVATED MATH PROJECT ACTIVITY BOOKLETS has been written for students and teachers in elementary and secondary schools. Each Booklet treats a topic generally found in the school curriculum or material that is interesting and motivational which may or may not be included in the usual class room textbook. Some topics are treated in two Booklets: one on an elementary level suitable for the middle grades and the other on an advanced level appropriate for junior and senior high schools.

The Booklets can be used in a variety of ways.

Because each Booklet treats a single topic, it is a handy summary and resource unit which can be expanded by both the student and the teacher.

Many of the Booklets, because they summarize and organize a topic in detail, can be used as mini-course modules to supplement standard class instruction or for individualized study. They also provide an invaluable review of a topic and can serve as a criterion for what has been covered on the topic.

The Booklets, unlike a textbook in which a topic may be treated in several nonconsecutive chapters, provide a convenient and readily accessible reference source. The material on a topic can be quickly and easily found.

Each Booklet contains problems which not only reenforce the class room instruction but also provide motivation, interest and challenge. Many problems are open-ended so that all students can achieve some measure of success. These problems are suitable not only for the routine pencil and paper activity but also may be extended by the use of hand electronic calculators or programmed on a computer.

Each Booklet contains solutions to the problems and in many instances comments, explanations and derivations of the key formulas and algorithms. The Booklets are relatively independent of each other and may be studied in any sequence depending on the background and personal preference of the student.

FRACTION ACTION

Common fractions provide interesting activities and often produce unexpected and surprising results. The more you know about fractions, the more you will enjoy Fraction Action.

SOMETHING OLD

1. Common fraction

a. Here is the set of natural numbers : $N = \{0, 1, 2, 3, 4, 5, \ldots\}$.

b. These are examples of common fractions : $\frac{2}{3}$ or 2/3 , $\frac{5}{2}$ or 5/2 , $\frac{0}{7}$ or 0/7 .

c. These numbers and numbers like them can be written as common fractions :

0	3	$1\frac{1}{2}$	0.1	2.5	.33333...	.121212...	.466666...
$\frac{0}{1}$	$\frac{3}{1}$	$\frac{3}{2}$	$\frac{1}{10}$	$2\frac{5}{10}$ or $\frac{25}{10}$	$\frac{1}{3}$	$\frac{4}{33}$	$\frac{7}{15}$

d. Definition: a common fraction is a number that can be written in the form $\frac{a}{b}$ where a, b are natural numbers and $b \neq 0$.

In $\frac{a}{b}$ — $a \longleftarrow$ is called the numerator of the fraction; $b \longleftarrow$ is called the denominator of the fraction .

The denominator of a fraction can never be zero.

We will often refer to 'common fractions' simply as 'fractions' .

2. Mixed number

a. $1\frac{1}{2}$, $7\frac{2}{3}$ and so on, are usually called 'mixed numbers' .

b. To write a mixed number as a fraction, you may use the methods shown below .

c. $1\frac{1}{2} = 1 + \frac{1}{2} = \frac{2}{2} + \frac{1}{2} = \frac{2+1}{2} = \frac{3}{2}$

d. $1\frac{1}{2} = \frac{2(1)+1}{2} = \frac{3}{2}$

PROBLEMS

1. Represent each of the following as common fractions.

a. 10 ________ b. $7\frac{8}{9}$ ________ c. 4 ________

2. Which of the following is NOT a common fraction. Please circle .

a. $33\frac{1}{3}$ b. $\sqrt{5}$ c. $\sqrt{9}$

3. Represent each of the following as a natural number or as a mixed number.

a. $\frac{45}{9}$ ________ b. $\frac{43}{5}$ ________ c. $\frac{72}{8}$ ________

4. Represent each of the following as a natural number or as a mixed number.

a. $\frac{25}{6}$ ________ b. $\frac{108}{36}$ ________ c. $\frac{1111}{11}$ ________

3. Decimal numbers (decimals)

a. 0.1 or .1
2.5
.33333...

and numbers like these are called 'decimal numbers' or 'decimals' (some may also call them 'decimal fractions').

b. Decimals like 0.1 , 2.5 , .75 are called terminating decimals .

c. Decimals like

.33333... also written as $.\overline{3}$ where the bar shows the 3 is repeated,

.121212... also written as $.\overline{12}$ where the bar shows the 12 is repeated,

.466666... also written as $.4\overline{6}$ where the bar shows the 6 is repeated,

are called non-terminating repeating decimals .

The number of digits under the bar is called the period of the non-terminating repeating decimal . Thus, $.\overline{3}$ has a period of one,

$.\overline{12}$ has a period of two ,

$.4\overline{6}$ has a period of one .

d. Decimals like .101001000100001... , .12345678910111213... which do not have a regular pattern of repeating digits are called non-terminating non-repeating decimals . Non-terminating non-repeating decimals cannot be expressed as fractions and so we will not discuss such decimals here .

4. Changing decimals to fractions

a. Terminating decimals

Table 1

.1 =	$\frac{1}{10}$	tenths	.0001 =	$\frac{1}{10,000}$	ten-thousandths
.01 =	$\frac{1}{100}$	hundredths	.00001 =	$\frac{1}{100,000}$	hundred-thousandths
.001 =	$\frac{1}{1,000}$	thousandths	.000001 =	$\frac{1}{1,000,000}$	millionths

Example 1 a. $.12 = \frac{12}{100} = \frac{3}{25}$ b. $.255 = \frac{255}{1000} = \frac{51}{200}$

c. $2.5 = 2 + .5 = 2 + \frac{5}{10} = 2\,\frac{5}{10} = \frac{25}{10} = \frac{5}{2}$.

b. Non-terminating repeating decimals

Example 2 Write .33333... as a fraction .

Solution Let $N = .33333...$

Step 1 . The period of the decimal is 1, so multiply both sides above by 10 raised to the power of the period $10\,N = 3.33333...$

Step 2 Subtract $N = .33333...$ from both sides of Step 1

$$\begin{array}{r} 10\,N = 3.33333... \\ -\quad N = \ \ .33333... \\ \hline 9\,N = 3 \\ N = \frac{3}{9} = \frac{1}{3} \end{array}$$

Thus, $N = .33333... = \frac{1}{3}$. To check, perform $1 \div 3$.

<u>Example 3</u> Write $.121212...$ as a fraction .

<u>Solution</u> Let $N = .121212...$

Step 1 The period of the decimal is 2, so multiply both sides above by 10 raised to the power of the period (here 10^2) $\qquad 100N = 12.121212...$

Step 2 Subtract $N = .121212...$ from both sides of Step 1

$$\begin{array}{r} -\quad N = -\,.121212... \\ \hline 99\,N = 12 \\ N = \frac{12}{99} = \frac{4}{33} \end{array}$$

Thus, $N = .121212... = \frac{4}{33}$. To check , perform $4 \div 33$.

<u>Example 4</u> Write $.466666...$ as a fraction .

<u>Solution</u> Let $N = .466666...$

Step 1 Multiply both sides of the above by that power of 10 which is equal to the number of non-repeating digits (here it is 10^1) $\qquad 10\,N = 4.66666...$

Step 2 Separate the non-repeating part from the repeating part $\qquad 10\,N = 4 + .66666...$

Step 3 Use the method in Example 2 (page 2) or Example 3 (above) to get $.66666... = \frac{2}{3}$

and substitute in Step 2 $\qquad 10\,N = 4 + \frac{2}{3} = 4\frac{2}{3} = \frac{14}{3}$

$$N = \frac{14}{30} = \frac{7}{15}$$

Thus, $N = .466666... = \frac{7}{15}$. To check, perform $7 \div 15$.

<u>PROBLEMS</u>

5. Write each of the following non-terminating repeating decimals as a fraction.

a. $.11111...$ ______	b. $.\overline{2}$ ______	c. $.\overline{8}$ ______
d. $.99999...$ ______	e. $.1515...$ ______	f. $.08\overline{3}$ ______
g. $.01\overline{6}$ ______	h. $.\overline{09}$ ______	i. $.\overline{009}$ ______

6. Write $.\overline{142857}$ as a fraction .

7. a. Let a be a digit. Find a fraction representation for $.\overline{a}$.

   b. Let a , b stand for different digits. Find a fraction representation for $.\overline{ab}$.

5. <u>Proper, improper common fractions</u>

a. $\frac{a}{b}$ is a proper fraction if a, b are natural numbers with $a > 0$, $b > a$. The symbol $>$ is read as : is greater than .

Examples of proper fractions : $\frac{1}{2}$, $\frac{2}{3}$, $\frac{7}{8}$.

b. A fraction which is not a proper fraction is called an improper fraction .

Examples of improper fractions: $\frac{0}{1}$, $\frac{2}{2}$, $\frac{5}{4}$.

6. <u>Equality, inequality of fractions : cross product</u>

a. Given : the common fractions $\frac{a}{b}$ and $\frac{c}{d}$

b. form the cross product $\frac{a}{b} \quad \frac{c}{d}$

$a(d)$ $\quad$ $c(b)$ where the numerator of each fraction is on the same side as the fraction ,

c. then (1) $\frac{a}{b} = \frac{c}{d}$ if and only if $a(d) = c(b)$. Thus, $\frac{1}{3} = \frac{2}{6}$ since

$$1(6) = 2(3)$$
$$6 = 6 ,$$

(2) $\frac{a}{b} < \frac{c}{d}$ if and only if $a(d) < c(b)$. Thus $\frac{2}{5} < \frac{3}{4}$ since

$$2(4) < 3(5)$$
$$8 < 15 ,$$

(3) $\frac{a}{b} > \frac{c}{d}$ if and only if $a(d) > c(b)$. Thus, $\frac{2}{3} > \frac{1}{4}$ since

$$2(4) > 1(3)$$
$$8 > 3 .$$

The above is called the 'cross product' test for the equality or inequality of common fractions .

<u>PROBLEMS</u>

8. In each of the following insert one of the relations $=$, $>$, $<$ so that the result is a true statement .

a. $\frac{2}{3}$ _____ $\frac{3}{2}$	b. $\frac{7}{6}$ _____ $\frac{7}{4}$	c. $\frac{3}{8}$ _____ $\frac{5}{8}$
d. 3 _____ $2\frac{3}{5}$	e. $1\frac{1}{3}$ _____ $1\frac{4}{7}$	f. $\frac{0}{5}$ _____ $\frac{0}{10}$

9. Arrange the following fractions from smallest to largest .

a. $\frac{5}{3}$, $\frac{5}{1}$, $\frac{5}{14}$, $\frac{5}{10}$, $\frac{5}{5}$, $\frac{5}{7}$ ______________________________

b. Complete this sentence : If 2 fractions have the same nonzero numerator but different denominators, then the fraction with the larger denominator is the ____________ (smaller, larger) fraction.

10. Arrange the following fractions from smallest to largest .

a. $\frac{4}{9}$, $\frac{13}{9}$, $\frac{9}{9}$, $\frac{8}{9}$, $\frac{5}{9}$, $\frac{1}{9}$ ______________________________

b. Complete this sentence : If 2 fractions have the same denominator but different numerators, then the fraction with the larger numerator is the ____________ (smaller, larger) fraction .

11. Can two fractions with the same nonzero numerator and different denominators

a. be equal ? Yes ______ No ______ Example __________

b. Can two fractions with the same numerator and different denominators be equal ? Yes ________ No ________ Example _________

12. Write all the common fractions which have a denominator 7 and are less

a. than 1 . ______________________________

b. Let $b > 0$ where b is a natural number. How many common fractions have denominator b and are less than 1 ? ________________________

7. Equivalent fractions by multiplication and division

a. Multiplication

If you multiply both the numerator and the denominator of a fraction by any nonzero number you get a fraction equivalent to the original fraction .

The word 'equivalent' is more appropriate here than 'equal' .

The above follows at once from the cross product test (Section 6, page 4), namely,

$\frac{2}{3} = \frac{2(4)}{3(4)} = \frac{8}{12}$, multiplying by 4

Thus, $\frac{2}{3} = \frac{8}{12}$.

If $n \neq 0$, then $\frac{a}{b} = \frac{a(n)}{b(n)}$

since $a(bn) = a(nb)$

that is , $abn = abn$ where the cross products are equal .

b. Division

A common divisor of two or more numbers is a number that divides each of the numbers exactly. (Zero is never a common divisor).

Thus, 3 is a common divisor of the numbers 3, 6, 12.

If you divide both the numerator and the denominator of a fraction by a common divisor, you get a fraction equivalent to the original fraction.

$$\frac{12}{18} = \frac{12 \div 2}{18 \div 2} = \frac{6}{9}, \text{ dividing by } 2$$

Thus, $\frac{12}{18} = \frac{6}{9}$.

The above follows at once from the cross product test (Section 6, page 4), namely,

If $n \neq 0$, then $\frac{n\,a}{n\,b} = \frac{a}{b}$

since $na(b) = a(nb)$

that is, $nab = nab$, where the cross products are equal.

Relatively prime: Given: a common fraction $\frac{a}{b}$ where a, b are natural numbers and $b \neq 0$.

If the only common divisor of a and b is the number 1, then the numbers a, b are said to be relatively prime to each other.

Thus, in $\frac{3}{4}$, 3 and 4 are relatively prime.

Summary

c. $\frac{a}{b} = \frac{n\,a}{n\,b}$ states that given a fraction $\frac{a}{b}$, then if we multiply the numerator and denominator of $\frac{a}{b}$ by $n \neq 0$, the fraction $\frac{a}{b}$ is equivalent to $\frac{n\,a}{n\,b}$.

d. $\frac{n\,a}{n\,b} = \frac{a}{b}$ states that if a fraction can be written so that it has a common divisor n in the numerator and the denominator, then we can divide out this common divisor (factor) and $\frac{n\,a}{n\,b} = \frac{a}{b}$.

Simplifying a fraction: From (d) above: $\frac{12}{24} = \frac{6}{12}$, $\frac{12}{24} = \frac{4}{8}$, $\frac{12}{24} = \frac{3}{6}$, $\frac{12}{24} = \frac{1}{2}$

Now the fractions $\frac{6}{12}$, $\frac{4}{8}$, $\frac{3}{6}$, $\frac{1}{2}$ are simplifications of $\frac{12}{24}$.

The simplest form of $\frac{12}{24}$ is $\frac{1}{2}$.

PROBLEMS

13. Write 5 fractions equivalent to each of the following fractions.

a. $\frac{4}{5}$ ______________ b. $\frac{8}{3}$ ______________

c. $\frac{3}{4}$ ______________ d. $\frac{0}{7}$ ______________

e. $1\frac{2}{3}$ ______________ f. $2\frac{1}{2}$ ______________

14. Simplify each of the following .

a. $\frac{16}{32}$ = ________ b. $\frac{8}{4}$ = ________ c. $1\frac{2}{4}$ = ________

d. $\frac{0}{4}$ = ________ e. $\frac{5}{3}$ = ________ f. $\frac{4}{7}$ = ________

15. Write the fraction with the smallest denominator which is equivalent to

a. $\frac{20}{8}$ ________ b. $\frac{5}{60}$ ________ c. $\frac{12}{30}$ ________

d. $\frac{65}{5}$ ________ e. $\frac{21}{4}$ ________ f. $\frac{0}{12}$ ________

16. What is the only divisor of both the numerator and denominator of $\frac{5}{7}$? _____

Can $\frac{5}{7}$ be written as a fraction with a smaller denominator? Yes___ No___

17. Write $\frac{2}{3}$ so that it is equivalent to the fraction with the denominator shown .

a. $\frac{2}{3} = \frac{\quad}{6}$ b. $\frac{2}{3} = \frac{\quad}{12}$ c. $\frac{2}{3} = \frac{\quad}{18}$

18. Write $\frac{4}{5}$ so that it is equivalent to the fraction with the denominator shown .

a. $\frac{4}{5} = \frac{\quad}{15}$ b. $\frac{4}{5} = \frac{\quad}{20}$ c. $\frac{4}{5} = \frac{\quad}{35}$

19. Circle the fraction(s) which are equivalent to $\frac{3}{4}$.

$\frac{6}{8}$, $\frac{18}{24}$, $\frac{9}{16}$, $\frac{21}{28}$, $\frac{39}{52}$

20. Circle the fraction(s) which are equivalent to $\frac{5}{3}$.

$\frac{25}{9}$, $\frac{20}{12}$, $\frac{30}{18}$, $\frac{30}{50}$, $\frac{70}{42}$.

21. a. How many common fractions are equivalent to $\frac{2}{3}$? ____________

b. Briefly state how you would find the fractions in (a) .

____

8. Fractions and common denominators

Problem: Write the fractions $\frac{2}{3}$ and $\frac{4}{5}$ so that they have common (same) denominators.

Solution: A common denominator of the fractions $\frac{2}{3}$ and $\frac{4}{5}$ is any denominator that is exactly divisible by 3 and 5 .

Thus, 15 is exactly divisible by 3 and 5 , and we have

$$\frac{2}{3} = \frac{2(5)}{3(5)} = \frac{10}{15}, \text{ multiplying by } 5$$

$$\frac{4}{5} = \frac{4(3)}{5(3)} = \frac{12}{15}, \text{ multiplying by } 3 .$$

Also, 30 is exactly divisible by 3 and 5 , and we have

$\frac{2}{3} = \frac{2(10)}{3(10)} = \frac{20}{30}$, multiplying by 10,

$\frac{4}{5} = \frac{4(6)}{5(6)} = \frac{24}{30}$, multiplying by 6 .

Least common denominator

The smallest natural number (except 0) that is exactly divisible by 3 and 5 is called the 'least common denominator' for the fractions $\frac{2}{3}$ and $\frac{4}{5}$.

Thus, 15 is the smallest natural number greater than zero that is exactly divisible by 3 and 5 . 15 is the LCD for the fractions $\frac{2}{3}$ and $\frac{4}{5}$.

Common denominator and least common denominator (general)

A common denominator of the two fractions $\frac{a}{b}$, $\frac{c}{d}$ is any number $m \neq 0$, such that m is divisible by b and d, the denominators of the two fractions .

The smallest such number $m \neq 0$, is called the Least Common Denominator (LCD) for the fractions .

Example 1 Given: the two fractions $\frac{a}{b}$, $\frac{c}{d}$ then one of the simplest choices for the common denominator m is $m = bd$, the product of the denominators (since bd is divisible by both b and d) .

Multiply the numerator and denominator of the fraction $\frac{a}{b}$ by d to get $\frac{a\,d}{b\,d}$,

and multiply the numerator and denominator of the fraction $\frac{c}{d}$ by b to get $\frac{c\,b}{d\,b}$.

The fractions $\frac{a\,d}{b\,d}$ and $\frac{c\,b}{b\,d}$ have a common denominator .

Example 2. Given: the fractions $\frac{2}{3}$ and $\frac{1}{15}$.

Multiply the numerator and denominator of $\frac{2}{3}$ by 15 to get $\frac{2(15)}{3(15)} = \frac{30}{45}$,

and multiply the numerator and denominator of $\frac{1}{15}$ by 3 to get $\frac{1(3)}{15(3)} = \frac{3}{45}$.

The fractions $\frac{2}{3} = \frac{30}{45}$ and $\frac{1}{15} = \frac{3}{45}$ have common denominators .

Prime numbers The above method is a time saver in those cases where the denominators are prime numbers .

A prime natural number is a natural number greater than 1 which has only 1 and itself as divisors .

Example 3 Write the fractions $\frac{2}{3}$ and $\frac{3}{5}$ so that they have a common denominator.

Solution: Write multiples of 3 : 0, 3, 6, 9, 12, 15 18, 21, 24, ...

Write multiples of 5 : 0, 5, 10, 15 20, 25, 30, 35, 40, . . .

The smallest nonzero multiple of 3 which is also a multiple of 5 is 15. Thus, 15 is the LCD of the two fractions and we have

$$\frac{2}{3} = \frac{2(5)}{3(5)} = \frac{10}{15} \quad , \quad \frac{3}{5} = \frac{3(3)}{5(3)} = \frac{9}{15}$$

The method described in this example will always give the LCD.

PROBLEMS

22. Rewrite $\frac{1}{4}$ and $\frac{5}{6}$ as fractions with the common denominator shown.

a. $\frac{1}{4} = \frac{\quad}{24}$ b. $\frac{1}{4} = \frac{\quad}{36}$ c. $\frac{1}{4} = \frac{\quad}{48}$ d. $\frac{1}{4} = \frac{\quad}{108}$

$\frac{5}{6} = \frac{\quad}{24}$ $\frac{5}{6} = \frac{\quad}{36}$ $\frac{5}{6} = \frac{\quad}{48}$ $\frac{5}{6} = \frac{\quad}{108}$

e. Is there a number smaller than 24 which can be used as a common denominator for $\frac{1}{4}$ and $\frac{5}{6}$? Yes _____ No_____ The number _______.

23. Write the following fractions so that they have a common denominator. Use the method in Example 1, page 8.

a. $\frac{2}{3} =$ _______ b. $\frac{2}{5} =$ _______ c. $\frac{1}{2} =$ _______

$\frac{4}{6} =$ _______ $\frac{3}{7} =$ _______ $\frac{5}{12} =$ _______

24. Write each pair of fractions so that they have the least common denominator.

a. $\frac{2}{3} =$ _______ b. $\frac{2}{5} =$ _______ c. $\frac{1}{2} =$ _______

$\frac{4}{6} =$ _______ $\frac{3}{7} =$ _______ $\frac{5}{12} =$ _______

25. Circle the number(s) which may be used as a common denominator for the two fractions $\frac{5}{6}$ and $\frac{7}{8}$.

a. 12 b. 48 c. 16 d. 24

26. A common denominator for the three fractions $\frac{2}{3}$, $\frac{4}{5}$, $\frac{3}{7}$ is _______.

9. Equality, inequality of fractions with common denominator

a. Given: two fractions $\frac{a}{b}$, $\frac{c}{b}$ with a common denominator b,

then (1) the fractions are equal $\frac{a}{b} = \frac{c}{b}$ if and only if $a\,b = c\,b$ (from page 4)

and dividing by b, $a = c$.

Thus, two fractions with a common denominator are equal if and only if the numerators of the two fractions are equal.

<u>Example 1</u> $\frac{4}{7} = \frac{4}{7}$ since $4 = 4$ and the denominators are the same.

b. Given: two fractions $\frac{a}{b}$, $\frac{c}{b}$ with a common denominator b,

then (2) the fractions are unequal $\frac{a}{b} < \frac{c}{b}$ if and only if $a\,b < c\,b$ (from page 4)

and dividing by b $a < c$.

Thus, two fractions with a common denominator are unequal if and only if the numerators are unequal. The fraction with the larger numerator is the larger fraction.

<u>Example 2</u> $\frac{2}{9} < \frac{5}{9}$ since $2 < 5$ and the denominators are the same.

<u>Summary: comparison of fractions with same denominators</u>

$\frac{5}{7}$? $\frac{5}{7}$	$\frac{7}{12}$? $\frac{11}{12}$	$\frac{5}{7}$? $\frac{3}{7}$
Since $5 = 5$	since $7 < 11$	since $5 > 3$
then $\frac{5}{7} = \frac{5}{7}$,	then $\frac{7}{12} < \frac{11}{12}$,	then $\frac{5}{7} > \frac{3}{7}$.

<u>PROBLEMS</u>

27. Arrange the following fractions from smallest to largest.

a. $\frac{5}{12}$, $\frac{2}{12}$, $\frac{6}{12}$, $\frac{4}{12}$, $\frac{12}{12}$ ________________

b. $\frac{4}{7}$, $\frac{1}{7}$, $\frac{2}{7}$, $\frac{5}{7}$, $\frac{10}{7}$ ________________

28. Find a common denominator for each set of two fractions and write the larger fraction in the last column.

		<u>Larger fraction</u>
a. $\frac{1}{3}$ = ______	$\frac{2}{5}$ = ______	______
b. $\frac{2}{7}$ = ______	$\frac{1}{3}$ = ______	______
c. $\frac{7}{8}$ = ______	$\frac{3}{4}$ = ______	______

29. Find a common denominator for the following fractions and then arrange them in increasing order.

$\frac{2}{5}$, $\frac{7}{12}$, $\frac{2}{3}$, $\frac{3}{10}$, $\frac{11}{15}$, $\frac{3}{20}$ ________________

30. If $\frac{a}{b} < \frac{c}{b}$, how many fractions are there with denominator b which are greater than $\frac{a}{b}$ and less than $\frac{c}{b}$? ____________

10. Addition and subtraction of fractions

Common denominator: If two fractions $\frac{a}{b}$, $\frac{c}{b}$ have a common denominator, then

a. the sum of the fractions is given by $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$, add the numerators of the fractions and write the sum over the common denominator,

b. the difference of the fractions (where $\frac{a}{b} > \frac{c}{b}$) is given by $\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$, subtract the numerators of the fractions and write the difference over the common denominator .

Example 1 Sum $\frac{2}{5} + \frac{1}{5} = \frac{2+1}{5} = \frac{3}{5}$

Example 2 Difference $\frac{2}{5} - \frac{1}{5} = \frac{2-1}{5} = \frac{1}{5}$.

Different denominators: If two fractions $\frac{a}{b}$, $\frac{c}{d}$ have different denominators, first rewrite the fractions so that they have a common denominator and then use (a) or (b) above .

c. The sum of the fractions is given by $\frac{a}{b} + \frac{c}{d} = \frac{a(d)}{b(d)} + \frac{c(b)}{d(b)} = \frac{ad + cb}{bd}$,

d. the difference of the fractions (where $\frac{a}{b} > \frac{c}{d}$) $\frac{a}{b} - \frac{c}{d} = \frac{a(d)}{b(d)} - \frac{c(b)}{d(b)} = \frac{ad - cb}{bd}$.

is given by

Example 3 Sum $\frac{1}{3} + \frac{1}{7} = \frac{1(7)}{3(7)} + \frac{1(3)}{7(3)} = \frac{7+3}{21} = \frac{10}{21}$

Example 4 Difference $\frac{3}{4} - \frac{2}{3} = \frac{3(3)}{4(3)} - \frac{2(4)}{3(4)} = \frac{9-8}{12} = \frac{1}{12}$.

PROBLEMS

31. Find the following sums .

a. $\frac{2}{9} + \frac{7}{9} =$ ____________

b. $\frac{3}{8} + \frac{1}{4} =$ ____________

c. $\frac{2}{7} + \frac{3}{5} =$ ____________

d. $4\frac{1}{3} + \frac{5}{6} =$ ____________

e. $\frac{4}{9} + 3\frac{1}{3} =$ ____________

f. $5\frac{1}{2} + 2\frac{2}{5} =$ ____________

32. Find the following differences.

a. $\frac{3}{11} - \frac{2}{11} =$ ____________

b. $\frac{7}{7} - \frac{2}{3} =$ ____________

c. $\frac{10}{4} - 1\frac{5}{6} =$ ________ e. $3\frac{1}{4} - 2\frac{5}{6} =$ ________

d. $2\frac{1}{3} - \frac{3}{5} =$ ________ f. $5\frac{2}{3} - \frac{0}{13} =$ ________

11. Multiplication of fractions

Given: two fractions $\frac{a}{b}$, $\frac{c}{d}$

then the product of the two fractions is given by $\frac{a}{b}\left(\frac{c}{d}\right) = \frac{a\,c}{b\,d}$, multiply the numerators and write the product over the product of the denominators .

Notice that the word 'of' in sentences like "What is $\frac{1}{2}$ of $\frac{3}{4}$? " means multiplication of the fractions .

Solution: $\frac{1}{2}$ of $\frac{3}{4} = \frac{1}{2}\left(\frac{3}{4}\right) = \frac{1\ (3)}{2\ (4)} = \frac{3}{8}$.

PROBLEMS

33. What is

a. $\frac{5}{4}$ of $\frac{2}{3} =$ ________ b. $\frac{5}{4}$ of $\frac{5}{6} =$ ________ c. $\frac{5}{4}$ of $\frac{7}{8} =$ ________

34. a. Is $\frac{5}{4}$ less than or greater than 1 ? Less than _____ Greater than _____

b. Is $\frac{5}{4}$ of $\frac{a}{b}$ (where $a \neq 0$) greater than $\frac{a}{b}$ or less than $\frac{a}{b}$?

Greater than _______ Less than _______

35. What is

a. $\frac{4}{5}$ of $\frac{2}{3} =$ ________ b. $\frac{4}{5}$ of $\frac{5}{6} =$ ________ c. $\frac{4}{5}$ of $\frac{7}{8} =$ ________

36. a. Is $\frac{4}{5}$ less than or greater than 1 ? Less than ______ Greater than ____

b. Is $\frac{4}{5}$ of $\frac{a}{b}$ (where $a \neq 0$) greater than $\frac{a}{b}$ or less than $\frac{a}{b}$?

Greater than _______ Less than ________

37. If $\frac{a}{b}$, $\frac{c}{d}$ are proper fractions, is the product $\frac{a}{b}\left(\frac{c}{d}\right) = \frac{a\,c}{b\,d}$ also a proper fractions ? Yes ______ No ______ Why ____________________

12. Reciprocals

Two numbers are reciprocals of each other if their product is 1 .

a. The reciprocal of 2 is $\frac{1}{2}$ since $2\left(\frac{1}{2}\right) = \frac{2}{1}\left(\frac{1}{2}\right) = \frac{2(1)}{1(2)} = \frac{2}{2} = 1$.

It follows at once that the reciprocal of $\frac{1}{2}$ is 2 .

b. The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$ since $\frac{2}{3}\left(\frac{3}{2}\right) = \frac{2(3)}{3(2)} = \frac{6}{6} = 1$.

It follows at once that the reciprocal of $\frac{3}{2}$ is $\frac{2}{3}$.

Thus, c. if $b \neq 0$, then b and $\frac{1}{b}$ are reciprocals of each other ,

d. if $a \neq 0$, then $\frac{a}{b}$ and $\frac{b}{a}$ are reciprocals of each other .

PROBLEMS

38. Fill in the blanks

a. The reciprocal of $\frac{3}{7}$ is ________ since $\frac{3}{7}$ x _____ = ______ .

b. The reciprocal of $2\frac{1}{3}$ is ________ since $2\frac{1}{3}$ x _____ = ______ .

c. The reciprocal of 1 is ________ since 1 x _____ = ______ .

39. Find the reciprocal of each of the following .

a. 5 ______ c. $3\frac{2}{5}$ ______ e. $\frac{22}{7}$ ______

b. $\frac{1}{5}$ ______ d. $\frac{2}{11}$ ______ f. $1\frac{1}{2}$ ______

40. Does 0 have a reciprocal ? Yes ____ No ______ Why not? ______________

Does $\frac{0}{1}$ have a reciprocal ? Yes ____ No ______ Why not ? ______________

41. Given two fractions $\frac{a}{b}$, $\frac{c}{d}$ where $a \neq 0$, $c \neq 0$ and $\frac{a}{b} < \frac{c}{d}$.

The reciprocal of $\frac{a}{b}$ is ________ , the reciprocal of $\frac{c}{d}$ is ________ .

Which reciprocal is the larger ? __________

Use the fractions $\frac{2}{3}$ and $\frac{4}{5}$ to illustrate your answer . ______________

____

13. Division of fractions

Division of fractions is rather simple if you remember : any number divided by 1 is that number .

Example 1 $4 \div 1 = 4$, $\frac{2}{3} \div 1 = \frac{2}{3}$.

Example 2 $\dfrac{\frac{4}{5}}{1} = \frac{4}{5}$, $\dfrac{1\frac{2}{3}}{1} = 1\frac{2}{3}$.

a. Divisor 1 Find the quotient of $\dfrac{\frac{2}{3}}{\frac{4}{5}}$ $\begin{matrix} \leftarrow \text{dividend} \\ \leftarrow \text{divisor} \end{matrix}$

Solution: The divisor will become 1 if we multiply it by the reciprocal of $\frac{4}{5}$. Thus, multiply the dividend and divisor by $\frac{5}{4}$, the reciprocal of $\frac{4}{5}$.

$$\frac{\frac{2}{3}}{\frac{4}{5}} = \frac{\frac{2}{3} \times \frac{5}{4}}{\frac{4}{5} \times \frac{5}{4}} = \frac{\frac{2}{3} \times \frac{5}{4}}{1} = \underbrace{\boxed{\underset{\text{dividend}}{\frac{2}{3}} \times \underset{\text{reciprocal of divisor}}{\frac{5}{4}}}}_{\text{This gives us a simple rule for finding the quotient of fractions.}} = \frac{2(5)}{3(4)} = \frac{10}{12} \text{ (or } \frac{5}{6}\text{)}$$

Example 1 Find the quotient of $\dfrac{\frac{3}{2}}{\frac{5}{7}}$.

Solution: $$\frac{\frac{3}{2}}{\frac{5}{7}} = \underset{\text{dividend}}{\frac{3}{2}} \times \underset{\text{reciprocal of divisor}}{\frac{7}{5}} = \frac{3(7)}{2(5)} = \frac{21}{10}$$

Example 2 $$\underset{\text{dividend}}{\frac{3}{5}} \div \underset{\text{divisor}}{\frac{2}{9}} = \underset{\text{dividend}}{\frac{3}{5}} \times \underset{\text{reciprocal of divisor}}{\frac{9}{2}} = \frac{3(9)}{5(2)} = \frac{27}{10}$$

In general : $$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b} \times \frac{d}{c}}{\frac{c}{d} \times \frac{d}{c}} = \frac{\frac{a}{b} \times \frac{d}{c}}{1} = \frac{a}{b} \times \frac{d}{c} = \frac{a\,d}{b\,c}$$

b. Division: same denominators Find the quotient $\dfrac{\frac{1}{3}}{\frac{2}{5}}$ $\begin{matrix} \leftarrow \text{dividend} \\ \leftarrow \text{divisor} \end{matrix}$

Solution: Make the denominators of the fractions in the dividend and divisor the same. Then use (a) above.

$$\frac{\frac{1}{3}}{\frac{2}{5}} = \frac{\frac{1\ (5)}{3\ (5)}}{\frac{2\ (3)}{5\ (3)}} = \frac{\frac{5}{15}}{\frac{6}{15}} = \frac{5}{15} \times \frac{15}{6} = \frac{5}{6}$$

If the denominators of the fractions in the dividend and divisor are the same, divide the numerators of the fractions in the dividend and divisor.

Example 3 $\frac{2}{7} \div \frac{5}{7} = \frac{2}{5}$, since the denominators are the same.

PROBLEMS

42. Find each quotient. Use the method in (a), page 14.

a. $\dfrac{\frac{5}{6}}{\frac{1}{2}} =$ __________

c. $\dfrac{2}{2\frac{1}{3}} =$ __________

b. $\dfrac{1\frac{1}{2}}{3} =$ __________

d. $\dfrac{2\frac{2}{5}}{1\frac{3}{4}} =$ __________

e. $2\frac{3}{7} \div 1\frac{1}{4} =$ __________

43. Find each quotient. Use the method in (b), page 14.

a. $\dfrac{\frac{3}{10}}{\frac{7}{10}} =$ __________

c. $\dfrac{\frac{3}{5}}{10} =$ __________

b. $\dfrac{4}{\frac{2}{3}} =$ __________

d. $\dfrac{1\frac{1}{2}}{2\frac{1}{4}} =$ __________

e. $3\frac{1}{3} \div 1\frac{1}{5} =$ __________

44. Circle the larger number in each of the following pairs.

a. $\dfrac{\frac{1}{2}}{\frac{3}{2}}$, $\dfrac{1}{2}$

c. $\dfrac{\frac{5}{2}}{\frac{3}{2}}$, $\dfrac{5}{2}$

b. $\dfrac{\frac{1}{2}}{\frac{1}{3}}$, $\dfrac{1}{2}$

d. $\dfrac{\frac{5}{2}}{\frac{1}{3}}$, $\dfrac{5}{2}$

45. Consider the quotient $\dfrac{\frac{a}{b}}{\frac{c}{d}}$.

a. If $\frac{c}{d} > 1$, then the quotient is __________ (greater , less) than $\frac{a}{b}$.

b. If $\frac{c}{d} < 1$, then the quotient is __________ (greater , less) than $\frac{a}{b}$.

Once you are familiar with fractions and operations on fractions you can enjoy Fraction Action . Perhaps you can find other fraction novelties to add to this collection .

THE NEW

14. The Magic of Lo-Shu . Fraction magic squares

Magic squares, according to historical records, originated in China . Below is the ancient and fascinating Lo-Shu 3 by 3 normal magic square .

Columns

		1	2	3
	1	6	7	2
Rows	2	1	5	9
	3	8	3	4

Diagonal 1 Diagonal 2

PROBLEMS

46. Find the sum of the numbers in :

row 1 ______ column 1 ______ diagonal 1 ______

row 2 ______ column 2 ______ diagonal 2 ______

row 3 ______ column 3 ______

a. Was the sum of each row, column, and diagonal the same ? Yes____ No____

In those cases where your answer is 'yes', that number is called the Magic Number of the square .

47. Find the Magic Number of each magic square .

Magic square 1

4	$\frac{1}{2}$	3
$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$
2	$\frac{9}{2}$	1

Magic number____

Magic square 2

$\frac{1}{3}$	$\frac{3}{2}$	$\frac{2}{3}$
$\frac{7}{6}$	$\frac{5}{6}$	$\frac{1}{2}$
1	$\frac{1}{6}$	$\frac{4}{3}$

Magic number_____

Magic square 3

$\frac{1}{2}$	$\frac{1}{12}$	$\frac{2}{3}$
$\frac{7}{12}$	$\frac{5}{12}$	$\frac{1}{4}$
$\frac{1}{6}$	$\frac{3}{4}$	$\frac{1}{3}$

Magic number ______

48. Write fractions in the blank cells so that each magic square will have the Magic Number shown above the square .

Magic number 4

$2\frac{1}{6}$		
$\frac{7}{6}$	$\frac{5}{6}$	

Magic number $4\frac{1}{2}$

		$\frac{21}{12}$
	$\frac{3}{2}$	
	$\frac{3}{4}$	

Magic number 1

		$\frac{8}{15}$
$\frac{3}{5}$	$\frac{1}{3}$	

15. Making Magic Squares

You can make magic squares. Below is the secret for the 3 by 3 magic square .

x - y	x + y - z	x + z
x + y + z	x	x - y - z
x - z	x + z - y	x + y

Select fractions or natural numbers for x, y, z such that $x > y + z$. Then perform the operations on the selected numbers which are shown in each cell . The result will be a magic square .

PROBLEMS

49. a. $x = 5/4$
$y = 3/4$
$z = 1/4$

Magic square

Magic number ______

b. $x = 7/12$
$y = 1/4$
$z = 1/12$

Magic square

Magic number _____

c. $x = 1\frac{1}{2}$
$y = 2/3$
$z = 1/2$

Magic square

Magic number _______

d. $x = 1$
$y = 7/15$
$z = 1/5$

Magic square

Magic number ______

e. $x = 1$
$y = 1/2$
$z = 1/6$

Magic square

Magic number ______

f. $x = 2/3$
$y = 1/3$
$z = 1/12$

Magic square

Magic number ______

16. Fraction add-a-trails

To do an add-a-trail :

Begin at any of the 9 boxes .
Draw a continuous trail to the end-of-trail box.
A trail can only go vertically or horizontally.
A trail cannot retrace or cross itself .

The sum of all the numbers on the trail must be equal to the number in the end-of-trail box.

Here we began with $\frac{1}{4}$ and the sum of the 6 numbers is 3 .

$\frac{7}{8}$	$\frac{1}{8}$	$\frac{3}{4}$
2	$\frac{3}{8}$	$\frac{1}{2}$
$\frac{5}{8}$	1	$\frac{1}{4}$
	3 End	

1/4 + 1/2 + 3/4 + 1/8 + 3/8 + 1 = 3

PROBLEMS

50. Do the following add-a-trails .

a.

$\frac{5}{8}$	$\frac{2}{3}$	$\frac{2}{5}$
$\frac{3}{4}$	$\frac{1}{6}$	$\frac{1}{3}$
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{8}$
	2 End	

b.

$\frac{2}{3}$	$\frac{1}{6}$	1
$\frac{3}{4}$	$\frac{1}{2}$	$\frac{5}{6}$
$\frac{5}{8}$	$\frac{1}{3}$	$\frac{1}{8}$
	$3\frac{1}{2}$ End	

c.

$\frac{7}{12}$	$\frac{2}{3}$	$\frac{1}{4}$
$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{5}$
$\frac{3}{4}$	$\frac{1}{2}$	$\frac{7}{8}$
	2 End	

51. Do the following add-a-trails . The procedure is the same as in problem 50 above .

a.

$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{1}{2}$
$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{6}{8}$
$\frac{1}{2}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{7}{8}$	$\frac{2}{8}$
$\frac{5}{8}$	$\frac{1}{8}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{2}{4}$
$\frac{7}{8}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{5}{8}$	$\frac{1}{4}$
		6 End		

b.

$\frac{1}{3}$	$\frac{19}{20}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{5}$
$\frac{7}{8}$	$\frac{2}{5}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{1}{4}$
$\frac{1}{6}$	$\frac{3}{10}$	$\frac{3}{5}$	$\frac{5}{8}$	$\frac{5}{6}$
$\frac{2}{3}$	$\frac{3}{8}$	$\frac{7}{10}$	$\frac{4}{5}$	$\frac{3}{8}$
$\frac{4}{5}$	$\frac{1}{3}$	$\frac{4}{5}$	$\frac{2}{3}$	$\frac{1}{6}$
		$4\frac{1}{10}$ End		

17. The Mystery of the extra camel

Omar, rich in camels, left his three sons 17 prize camels. To the

eldest he gave $\frac{4}{9}$ of the camels,

second he gave $\frac{1}{3}$ of the camels,

youngest he gave $\frac{1}{6}$ of the camels .

The three young boys wondered how they could divide the camels according to their father's wish without killing any of the animals .

While they were discussing the problem, a neighbor came by riding on a camel . He stopped and listened to the boys. Getting off the camel, the neighbor generously gave the boys his own animal .

The boys now had 18 camels. Their father's wish could be fulfilled .

The eldest boy took $\frac{4}{9}$ of the 18 camels which was 8 camels,

the second boy took $\frac{1}{3}$ of the 18 camels which was 6 camels,

the youngest boy took $\frac{1}{6}$ of the 18 camels which was 3 camels.

But behold ! The total added up to exactly 17 camels .

The neighbor mounted the remaining camel, which happened to be his own, and rode away leaving everyone happy !

Can you explain what happened ?

PROBLEMS

52. a. Find the sum of the fractions $\frac{4}{9} + \frac{1}{3} + \frac{1}{6} =$ ______________

b. Did the sum of the fractions come out to be 1 ? Yes_____ No _____

c. What is $\frac{17}{18} \times 18 =$ __________

d. Do you know the answer to the mystery of the extra camel ?

Explain briefly ____

In the Omar camel puzzle,

a. there is a starting number , namely 17 ,

b. then 1 is added to the starting number so that $17 + 1 = 18$,

c. and there is a set of fractions (here 3 of them, 4/9, 1/3, 1/6) .

How do we find fractions for a given starting number so that we end up with a puzzle like the Omar camel mystery ? Below we describe one method .

Example 1 a. Let the starting number be 11 ,

b. you add 1 to the starting number for a total of $11 + 1 = 12$.

Procedure: c. Form all the fractions with 1, 2, 3, 4, ... , $(11 - 1 = 10)$ as numerators and 12 as a denominator ,

d. multiply each of the fractions in (c) by 12 .

Table 1

Thus,

$\frac{1}{12} \times 12 = 1$	$\frac{6}{12} \times 12 = 6$
$\frac{2}{12} \times 12 = 2$	$\frac{7}{12} \times 12 = 7$
$\frac{3}{12} \times 12 = 3$	$\frac{8}{12} \times 12 = 8$
$\frac{4}{12} \times 12 = 4$	$\frac{9}{12} \times 12 = 9$
$\frac{5}{12} \times 12 = 5$	$\frac{10}{12} \times 12 = 10$.

f. Now, in how many ways can you select 2, 3 or more different numbers from 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 (that is, 11 - 1), which add up to 11 ? Once you have found these numbers, use the fractions corresponding to your selected numbers from the table above

g. Two numbers that add up to 11 and their corresponding fractions .

Numbers	Fractions
1 , 10	$\frac{1}{12}$, $\frac{10}{12} = \frac{5}{6}$
2, 9	$\frac{2}{12} = \frac{1}{6}$, $\frac{9}{12} = \frac{3}{4}$
3, 8	$\frac{3}{12} = \frac{1}{4}$, $\frac{8}{12} = \frac{2}{3}$
4, 7	$\frac{4}{12} = \frac{1}{3}$, $\frac{7}{12}$
5, 6	$\frac{5}{12}$, $\frac{6}{12} = \frac{1}{2}$

h. Three numbers that add up to 11 and their corresponding fractions .

Numbers	Fractions
1, 2, 8	$\frac{1}{12}$, $\frac{1}{6}$, $\frac{2}{3}$
1, 3, 7	$\frac{1}{12}$, $\frac{1}{4}$, $\frac{7}{12}$
1, 4, 6	$\frac{1}{12}$, $\frac{1}{3}$, $\frac{1}{2}$
2, 3, 6	$\frac{1}{6}$, $\frac{1}{4}$, $\frac{1}{6}$
2, 4, 5	$\frac{1}{6}$, $\frac{1}{3}$, $\frac{5}{12}$

i. Four numbers that add up to 11 and their corresponding fractions .

Numbers	Fractions
1, 2, 3, 5	$\frac{1}{12}$, $\frac{1}{6}$, $\frac{1}{4}$, $\frac{5}{12}$.

PROBLEMS

53. In Example 1, (c), page 19 , why do we form fractions from 1 through 10 and do not use 11 as a numerator ?

____

____

PROBLEMS

Follow the procedure in Example 1, pages 19-20. For each starting number fill in the blanks . In each case select 2, 3, or 4 different numbers.

54. Starting number: 9 .

a. Two numbers that add up to 9 and their corresponding fractions .(Use 1, 2, . . . 8)

Numbers	Fractions	Numbers	Fractions
____	____	____	____
____	____	____	____

b. Three numbers that add up to 9 and their corresponding fractions .(Use 1, 2, . . 8)

Numbers	Fractions	Numbers	Fractions
____	____	____	____
____	____		

c. Can you find four numbers that add up to 9 ? Yes_____ No _____
(Use 1, 2, . . . 8)

55. Starting number: 13 .

a. Two numbers that add up to 13 and their corresponding fractions . (Use 1, 2, . . 12)

Numbers	Fractions	Numbers	Fractions
____	____	____	____
____	____	____	____
____	____	____	____

b. Three numbers that add up to 13 and their corresponding fractions .(Use 1, 2, . . 12)

Numbers	Fractions	Numbers	Fractions
____	____	____	____
____	____	____	____
____	____	____	____
____	____	____	____

c. Four numbers that add up to 13 and their corresponding fractions . (Use 1, 2, . . 12)

Numbers	Fractions	Numbers	Fractions
____	____	____	____
____	____	____	____

d. Can you find five numbers that add up to 13 ? Yes _____ No _____

56. Take the set of numbers 1, 2, 3, 4, ... , 14 .

a. List all pairs of numbers from the set that add up to 15 .

____

b. List all triplets of numbers from the set that add up to 15 .

____

____

c. List all quadruplets of numbers from the set that add up to 15 .

____

d. Are there any sextuplets of numbers that add up to 15 ? Yes _____ No _____

18. Sums of unit fractions. Denominators are exact divisors .

Example 1 Find the sum of the unit fractions $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$.

Solution: a. The largest denominator here is 6 .
b. Moreover, all the exact divisors of 6 are 1, 2, 3, 6 and all these numbers are denominators of the unit fractions .
c. When the above is the case, the sum of the unit fractions can be found quite easily and rapidly :

Add up all the denominators of the unit fractions and write this sum over the largest denominator of the unit fractions.

Thus, $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = \frac{1+2+3+6}{6} = \frac{12}{6} = 2$.

Check the above answer using the method in Section 10, page 11 .

PROBLEMS

57. Follow the procedure in Example 1 above . Find each sum .

a. $\frac{1}{1} + \frac{1}{2} + \frac{1}{5} + \frac{1}{10} =$ ______ c. $\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} =$ ______

b. $\frac{1}{1} + \frac{1}{7} + \frac{1}{49} =$ ______ d. $\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} =$ ______

58. Explain why the method of adding fractions in Example 1 gives the correct answer .

____

19. Sums of unit fractions . Denominators are not all the divisors .

Example 1 Find the sum of the unit fractions $\frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{10}$.

Solution : Only 2, 5, and 10 are exact divisors of 10 and 4 is not .

There is an even number of fractions in the problem .

Here one does not have to consider whether or not the denominators are exact divisors of the largest denominator .

In $\frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{10}$

$4 \times 5 = 20$

$2 \times 10 = 20$

The denominators, when paired, give the same product 20 .

Thus, $\frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{10} = \frac{10+5+4+2}{20} = \frac{21}{20}$, sum the denominators of the fractions and put this sum over 20 .

<u>Check</u>: $\frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{10} = \frac{10}{20} + \frac{5}{20} + \frac{4}{20} + \frac{2}{20} = \frac{10+5+4+2}{20} = \frac{21}{20}$.

<u>Example 2</u> Find the sum of the unit fractions $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$.

<u>Solution</u>: The number of fractions here is odd .

In $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$

$4^2 = 16$

$2 \times 8 = 16$

The square of one of the denominators is equal to the product of pairs of other denominators .

Thus, $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{8+4+2}{16} = \frac{14}{16} = \frac{7}{8}$, sum the denominators of the fractions and put this sum over 16 .

<u>Check</u>: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{8}{16} + \frac{4}{16} + \frac{2}{16} = \frac{8+4+2}{16} = \frac{14}{16} = \frac{7}{8}$.

PROBLEMS

59. Follow the procedure in Examples 1, 2 , pages 22, 23 . Find each sum .

a. $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \frac{1}{9} =$ ________

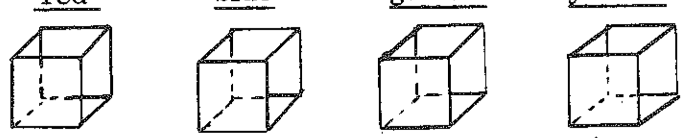
b. $\frac{1}{4} + \frac{1}{5} + \frac{1}{12} + \frac{1}{15} =$ ________

c. $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} =$ ________

d. $\frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12} =$ ________

20. Some Fraction Magic

Take a box of colored cubes (colored beads or colored plastic chips are just as good) . Assume that there are at least 4 distinct colors .

	red	blue	green	yellow
				
Assign this value to each cube	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{4}{7}$

Procedure: Ask someone to pick out 3, 4, 5, ..., colored cubes without showing them to you .

Let him take the values assigned to each colored cube and form the product of the fractions .

Thus, suppose someone picked : 2 red cubes, 1 blue cube and 1 green .

Then the product is $\underbrace{\frac{1}{2} \times \frac{1}{2}}_{\text{red}} \times \underbrace{\frac{2}{3}}_{\text{blue}} \times \underbrace{\frac{3}{5}}_{\text{green}} = \frac{1 \times 1 \times 2 \times 3}{2 \times 2 \times 3 \times 5} = \frac{6}{60}$.

Do NOT simplify the product .

Let him give you just the product $\frac{6}{60}$. You will be able to tell him the color of the cubes he picked and how many of each .

Solution: Take the denominator 60 of the fraction $\frac{6}{60}$.

Write the prime factorization of 60 .

$$60 = (2^2)\,(3)\,(5)$$

The number and colors of the cubes are :

one green (since 5 is the denominator of 3/5, assigned to the green cube)

one blue (since 3 is the denominator of 2/3, assigned to the blue cube)

two reds (since 2 is the denominator of 1/2, assigned to the red cube , the power 2 tells that there are 2 of the red cubes) .

PROBLEMS

60. For each product (of fractions) state the color and number of cubes .

Product	Prime factorization of denominator	Color and number of cubes
a. $\frac{12}{45}$		
b. $\frac{16}{126}$		
c. $\frac{18}{150}$		
d. $\frac{24}{210}$		
e. $\frac{144}{1225}$		

21. Fractions between fractions

a. Mean of two fractions : Given two fractions $\frac{a}{b}$ and $\frac{c}{d}$ where $\frac{a}{b} < \frac{c}{d}$.

The fraction $\dfrac{\frac{a}{b} + \frac{c}{d}}{2}$ called the mean of the two fractions is between the fractions $\frac{a}{b}$ and $\frac{c}{d}$.

Thus, $\frac{a}{b} < \frac{\frac{a}{b} + \frac{c}{d}}{2} < \frac{c}{d}$.

Example 1. Find a fraction between $\frac{1}{3}$ and $\frac{1}{2}$ (where $\frac{1}{3} < \frac{1}{2}$) .

Solution: $\frac{\frac{1}{3} + \frac{1}{2}}{2} = \frac{\frac{2}{6} + \frac{3}{6}}{2} = \frac{\frac{5}{6}}{2} = \frac{5}{12}$ and $\frac{1}{3} < \frac{5}{12} < \frac{1}{2}$.

PROBLEMS

61. Use the method of the mean to find a fraction between each pair of fractions given below .

a. $\frac{1}{5}$, $\frac{2}{3}$ __________   c. $\frac{4}{5}$, $2\frac{1}{6}$ __________

b. $2\frac{1}{8}$, $3\frac{1}{3}$ __________   d. $1\frac{1}{2}$, $\frac{9}{4}$ __________

62. Show that the fraction $\frac{\frac{a}{b} + \frac{c}{d}}{2}$ is exactly midway between the fractions $\frac{a}{b}$, $\frac{c}{d}$ (where $\frac{a}{b} < \frac{c}{d}$) .

b. Add numerators and denominators of the two fractions .

Given: two fractions $\frac{a}{b}$, $\frac{c}{d}$ where $\frac{a}{b} < \frac{c}{d}$.

The fraction $\frac{a+c}{b+d}$, where we added the numerators and denominators of the two fractions is between the fractions $\frac{a}{b}$, $\frac{c}{d}$.

Thus, $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

Example 2 Find a fraction between $\frac{1}{3}$ and $\frac{1}{2}$ (where $\frac{1}{3} < \frac{1}{2}$) .

Solution : Adding the numerators and denominators of the two fractions, we get $\frac{1 + 1}{3 + 2} = \frac{2}{5}$ and $\frac{1}{3} < \frac{2}{5} < \frac{1}{2}$.

PROBLEMS

63. Use the method of adding numerators and denominators to find a fraction between each pair of fractions given below .

a. $\frac{1}{5}$, $\frac{2}{3}$ __________   c. $\frac{4}{5}$, $2\frac{1}{6}$ __________

b. $2\frac{1}{8}$, $3\frac{1}{3}$ __________   d. $1\frac{1}{2}$, $\frac{9}{4}$ __________

64. Find 5 fractions between $\frac{2}{5}$ and $\frac{3}{7}$ by the repeated use of the method in this section . ____________________

65. If $\frac{a}{b} < \frac{c}{d}$, show that $\frac{a}{b} < \frac{a+c}{b+d}$ and $\frac{a+c}{b+d} < \frac{c}{d}$.

c. Common denominator method

Example 1 Find a fraction between $\frac{2}{3}$ and $\frac{5}{6}$.

Solution

Step 1 Write the fractions $\qquad \frac{2}{3} \qquad \frac{5}{6}$

Step 2 Write the fractions so that they have a common denominator (least common denominator if it is easy to find) . $\qquad \frac{4}{6} \qquad \frac{5}{6}$

Step 3 To find 1 fraction between the given fractions 2/3 and 5/6 , multiply the numerator and the denominator of each fraction by 2 (the multiplier is 1 more than the number of fractions sought) . $\qquad \frac{4(2)}{6(2)} \qquad \frac{5(2)}{6(2)}$

or $\frac{8}{12} \qquad \frac{10}{12}$

Step 4 It is clear that $\frac{9}{12}$ is between 8/12 and 10/12 and we have $\qquad \frac{2}{3} = \frac{8}{12} < \frac{9}{12} < \frac{10}{12} = \frac{5}{6}$

or $\frac{2}{3} < \frac{3}{4} < \frac{5}{6}$

Note: Read step 3 carefully. The multiplier must be 1 more than the number of fractions you are looking for .

PROBLEMS

66. Use the method in (c) above .

a. Find 1 fraction between 1/4 and 3/4 . ____________

b. Find 2 fractions between 2/5 and 9/10 . ____________

c. Find 10 fractions between 1/2 and 3/4 . ____________

d. Find 100 fractions between 5/8 and 3/4 . ____________

67. a. Could you find a million fractions between 1/3 and 1/2 ?

Yes___ No___ What would you use as a multiplier ? ____________

b. Can you find any number of fractions you want between 1/3 and 1/2 ?

Yes ____ No ____

68. The number 2 is the immediate successor of the number 1, that is, there are no other natural numbers between 1 and 2 .

Is there an immediate successor of the fraction 1/3 ? Yes ___ No____

Explain briefly____

69. Given the two fractions $\frac{a}{b}$, $\frac{c}{d}$ where $\frac{a}{b} < \frac{c}{d}$.

Find 10 fractions between a/b and c/d .

____

d. Arithmetic progression method

Given : any two natural numbers or fractions a, b where $a < b$.

The formula $a < \frac{1}{n}[(n-m)a + mb] < b$, where n, m are natural numbers and

$1 \leq m \leq n-1$

will give $(n-1)$ numbers between a and b .

Example 1 Take $a = 2$, $b = 10$.

Select $n = 4$, then from $1 \leq m \leq n-1$ we get $m = 1, 2, 3$.

From $a < \frac{1}{n}[(n-m)a + mb] < b$, substituting for a, b, n ,

we have $2 < \frac{1}{4}[(4-m)2 + 10m] < 10$

or $2 < 2 + 2m < 10$,

so that for $m = 1$, $2 < 4 < 10$,

$m = 2$, $2 < 6 < 10$,

$m = 3$, $2 < 8 < 10$.

Thus, 3 numbers between 2 and 10 are : 4, 6, 8 .

Example 2 Take $a = \frac{1}{3}$, $b = \frac{1}{2}$.

Select $n = 3$, then from $1 \leq m \leq n-1$ we get $m = 1, 2$.

From $a < \frac{1}{n}[(n-m)a + mb] < b$, substituting for a, b, n ,

we have $\frac{1}{3} < \frac{1}{3}\left[(3-m)\frac{1}{3} + \frac{m}{2}\right] < \frac{1}{2}$

or $\frac{1}{3} < \frac{1}{3} + \frac{m}{18} < \frac{1}{2}$

so that for $m = 1$, $\frac{1}{3} < \frac{7}{18} < \frac{1}{2}$

$m = 2$, $\frac{1}{3} < \frac{4}{9} < \frac{1}{2}$.

Thus, 2 numbers between 1/3 and 1/2 are : $\frac{7}{18}$, $\frac{4}{9}$.

PROBLEMS

70. Use the formula method above .

a. Find 2 fractions between 1/5 and 2/3 . ____________

b. Find 3 fractions between 1/3 and 3/4 . ____________

71. In $\frac{a}{b} < \frac{1}{n}\left[(n-m)\frac{a}{b} + m\frac{c}{d}\right] < \frac{c}{d}$, substitute $n = 2$ and $m = 1$

and solve . What did you get ? See 21, (a) .

____

(b) Is your answer in (a) the same as that in Section 21, (a) page 24 ?
Yes_____ No _____ .

22. Fractions between fractions with a given denominator .

Given : two fractions $\frac{a}{b}$, $\frac{c}{d}$ where $\frac{a}{b} < \frac{c}{d}$.

Find : fractions $\frac{e}{f}$ with a given denominator f such that $\frac{a}{b} < \frac{e}{f} < \frac{c}{d}$.

Solution: From $\frac{a}{b} < \frac{e}{f}$

we get $af < eb$.

In the above a, b, f will be known and e will have a set of values satisfying $af < eb$.

From $\frac{e}{f} < \frac{c}{d}$

we get $ed < cf$.

In the above, c, d, f will be known and e will have a set of values satisfying $ed < cf$.

Take the intersection of the two sets of values for e.

The values for e in the intersection set will be the numerators of the fractions with denominator f .

Example 1 Find all the fractions between $\frac{1}{3}$ and $\frac{1}{2}$ with denominator 20 .

Solution: From $af < eb$

with $a = 1$, $b = 3$, $f = 20$

we get $1(20) < 3e$

so that $e = 7, 8, 9, \ldots$.

From $ed < cf$

with $c = 1$, $d = 2$, $f = 20$

we get $2e < 1(20)$

so that $e = 0, 1, 2, 3, \ldots 9$.

The intersection of the two sets of values for e gives

$$e = 7, 8, 9 .$$

These values are the numerators of the fractions with denominator 20 .

Thus, $\frac{1}{3} < \frac{7}{20} , \frac{8}{20} , \frac{9}{20} < \frac{1}{2}$.

PROBLEMS

72. Find all the fractions
a. between 1/2 and 2/3 with denominator 16, ____________
b. between 3/7 and 1/2 with denominator 21, ____________
c. between 3/2 and 5/3 with denominator 15 ____________

73. Find all the fractions between 1/3 and 1/2 with denominator 4 . ____________

23. The mystery of the Egyptian unit fractions

A unit fraction is a fraction with 1 in the numerator .

Examples of unit fractions

$\frac{1}{2}$, $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{13}$

Except for the fraction $\frac{2}{3}$, the Egyptians replaced all their proper fractions by a sum of distinct unit fractions .

Examples

$$\frac{2}{15} = \frac{1}{10} + \frac{1}{30}$$

$$\frac{3}{7} = \frac{1}{4} + \frac{1}{7} + \frac{1}{28}$$

What method did the Egyptians use to write proper fractions as sums of unit fractions ? This is still somewhat of a mystery. We present a few methods here . There is fun and challenge in trying to solve the mystery .

a. Greatest unit fraction method

Example 1 Write $\frac{2}{5}$ as a sum of unit fractions by the greatest unit fraction method.

Solution: Write $\frac{2}{5} > \frac{1}{z}$, where $\frac{1}{z}$ is the greatest unit fraction less than 2/5, and we solve for z ,

so that $2z > 5$, and find the smallest value of z that makes this inequality true ,

and $z = 3$.

Thus, $\frac{2}{5} = \frac{1}{3} + R$, where $\frac{1}{3}$ is the greatest unit fraction less than 2/5 and R stands for remainder .

The remainder is given by : $R = \frac{2}{5} - \frac{1}{3} = \frac{1}{15}$.

Since 1/15 is a unit fraction, we have

$\frac{2}{5} = \frac{1}{3} + \frac{1}{15}$. Check: $\frac{2}{5} = \frac{1}{3} + \frac{1}{15} = \frac{5}{15} + \frac{1}{15} = \frac{6}{15} = \frac{2}{5}$.

Example 2 Write $\frac{4}{5}$ as a sum of unit fractions by the greatest unit fraction method .

Solution: Write $\frac{4}{5} > \frac{1}{z}$

so that $4z > 5$

and $z = 2$.

Thus, $\frac{4}{5} = \frac{1}{2} + R$, where $R = \frac{4}{5} - \frac{1}{2} = \frac{8}{10} - \frac{5}{10} = \frac{3}{10}$

so that $\frac{4}{5} = \frac{1}{2} + \frac{3}{10}$.

Now repeat the above procedure on 3/10 .

We have $\frac{3}{10} > \frac{1}{z}$

so that $3z > 10$

and $z = 4$.

We have $\frac{3}{10} = \frac{1}{4} + R$,

where $R = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}$.

Substituting for R we have $\frac{3}{10} = \frac{1}{4} + \frac{1}{20}$.

Finally, substituting for $\frac{3}{10}$ in $\frac{4}{5}$ gives : $\frac{4}{5} = \frac{1}{2} + \frac{1}{4} + \frac{1}{20}$.

PROBLEMS

74. Write each of the following fractions as a sum of unit fractions using the greatest unit fraction method .

a. $\frac{5}{6} =$ ____________ c. $\frac{5}{9} =$ ____________

b. $\frac{4}{7} =$ ____________ d. $\frac{7}{15} =$ ____________

e. $\frac{13}{21} =$ ____________

b. Formulas for unit fractions

There are several formulas which can be used to express certain fractions as sums of unit fractions .

1. $\frac{2}{n} = \frac{1}{\frac{n+1}{2}} + \frac{1}{\frac{n(n+1)}{2}}$, where n must be an odd number

Example 1 Take $\frac{2}{5}$ so that in this case $n = 5$.

Then $\frac{2}{5} = \frac{1}{3} + \frac{1}{15}$.

2. $\frac{2}{p\,q} = \frac{1}{p\left(\frac{p+q}{2}\right)} + \frac{1}{q\left(\frac{p+q}{2}\right)}$, where p, q are both odd or both even .

Example 2 Take $\frac{2}{35}$ and write $35 = 5(7)$ so that $p = 5,\ q = 7$.

Then $\frac{2}{35} = \frac{1}{30} + \frac{1}{42}$.

3. $\frac{2}{3\,p} = \frac{1}{2p} + \frac{1}{6\,p}$

Example 3 Take $\frac{2}{15} = \frac{2}{3(5)}$ so that $p = 5$.

Then $\frac{2}{15} = \frac{1}{10} + \frac{1}{30}$.

4. $\frac{z}{p\,q} = \frac{1}{p\left(\frac{p+q}{z}\right)} + \frac{1}{q\left(\frac{p+q}{z}\right)}$, where z must divide $(p+q)$.

Example 4 Take $\frac{4}{7}$ where $z = 4$, and write $7 = 7(1)$ so that $p = 7$, $q = 1$.

Then $\frac{4}{7} = \frac{1}{2} + \frac{1}{14}$.

PROBLEMS

Below we have the $\frac{2}{n}$ Table (where n is an odd number) from the Rhind Papyrus.

Table 1

1. $\frac{2}{5} = \frac{1}{3} + \frac{1}{15}$ ____________

2. $\frac{2}{7} = \frac{1}{4} + \frac{1}{28}$ ____________

3. $\frac{2}{9} = \frac{1}{6} + \frac{1}{18}$ ____________

4. $\frac{2}{11} = \frac{1}{6} + \frac{1}{66}$ ____________

5. $\frac{2}{13} = \frac{1}{8} + \frac{1}{52} + \frac{1}{104}$ ____________

6. $\frac{2}{15} = \frac{1}{10} + \frac{1}{30}$ ____________

7. $\frac{2}{17} = \frac{1}{12} + \frac{1}{51} + \frac{1}{68}$ ____________

8. $\frac{2}{19} = \frac{1}{12} + \frac{1}{76} + \frac{1}{114}$ ____________

9. $\frac{2}{21} = \frac{1}{14} + \frac{1}{42}$ ____________

10. $\frac{2}{23} = \frac{1}{12} + \frac{1}{276}$ ____________

11. $\frac{2}{25} = \frac{1}{15} + \frac{1}{75}$ ____________

12. $\frac{2}{27} = \frac{1}{18} + \frac{1}{54}$ ____________

13. $\frac{2}{29} = \frac{1}{24} + \frac{1}{58} + \frac{1}{174} + \frac{1}{232}$ ____________

14. $\frac{2}{31} = \frac{1}{20} + \frac{1}{124} + \frac{1}{155}$ ____________

15. $\frac{2}{33} = \frac{1}{22} + \frac{1}{66}$ ____________

16. $\frac{2}{35} = \frac{1}{30} + \frac{1}{42}$ ____________

17. $\frac{2}{37} = \frac{1}{24} + \frac{1}{111} + \frac{1}{296}$ ____________

18. $\frac{2}{39} = \frac{1}{26} + \frac{1}{78}$ ____________

19. $\frac{2}{41} = \frac{1}{24} + \frac{1}{246} + \frac{1}{328}$ ____________

20. $\frac{2}{43} = \frac{1}{42} + \frac{1}{86} + \frac{1}{129} + \frac{1}{301}$ ____________

21. $\frac{2}{45} = \frac{1}{30} + \frac{1}{90}$ ____________

22. $\frac{2}{47} = \frac{1}{30} + \frac{1}{141} + \frac{1}{470}$ ____________

23. $\frac{2}{49} = \frac{1}{28} + \frac{1}{196}$ ____________

24. $\frac{2}{51} = \frac{1}{34} + \frac{1}{102}$ ____________

25. $\frac{2}{53} = \frac{1}{30} + \frac{1}{318} + \frac{1}{795}$ ____________

26. $\frac{2}{55} = \frac{1}{30} + \frac{1}{330}$ ____________

27. $\frac{2}{57} = \frac{1}{38} + \frac{1}{114} =$ ____________

28. $\frac{2}{59} = \frac{1}{36} + \frac{1}{236} + \frac{1}{531}$ ____________

29. $\frac{2}{61} = \frac{1}{40} + \frac{1}{244} + \frac{1}{488} + \frac{1}{610}$ ____________

30. $\frac{2}{63} = \frac{1}{42} + \frac{1}{126}$ __________

31. $\frac{2}{65} = \frac{1}{39} + \frac{1}{195}$ __________

32. $\frac{2}{67} = \frac{1}{40} + \frac{1}{335} + \frac{1}{536}$ __________

33. $\frac{2}{69} = \frac{1}{46} + \frac{1}{138}$ __________

34. $\frac{2}{71} = \frac{1}{40} + \frac{1}{568} + \frac{1}{710}$ __________

35. $\frac{2}{73} = \frac{1}{60} + \frac{1}{219} + \frac{1}{292} + \frac{1}{365}$ __________

36. $\frac{2}{75} = \frac{1}{50} + \frac{1}{150}$ __________

37. $\frac{2}{77} = \frac{1}{44} + \frac{1}{308}$ __________

38. $\frac{2}{79} = \frac{1}{60} + \frac{1}{237} + \frac{1}{316} + \frac{1}{790}$ __________

39. $\frac{2}{81} = \frac{1}{54} + \frac{1}{162}$ __________

40. $\frac{2}{83} = \frac{1}{60} + \frac{1}{332} + \frac{1}{415} + \frac{1}{498}$ __________

41. $\frac{2}{85} = \frac{1}{51} + \frac{1}{255}$ __________

42. $\frac{2}{87} = \frac{1}{58} + \frac{1}{174}$ __________

43. $\frac{2}{89} = \frac{1}{60} + \frac{1}{356} + \frac{1}{534} + \frac{1}{890}$ __________

44. $\frac{2}{91} = \frac{1}{70} + \frac{1}{130}$ __________

45. $\frac{2}{93} = \frac{1}{62} + \frac{1}{186}$ __________

46. $\frac{2}{95} = \frac{1}{60} + \frac{1}{380} + \frac{1}{570}$ __________

47. $\frac{2}{97} = \frac{1}{56} + \frac{1}{679} + \frac{1}{776}$ __________

48. $\frac{2}{99} = \frac{1}{66} + \frac{1}{198}$ __________

49. $\frac{2}{101} = \frac{1}{101} + \frac{1}{202} + \frac{1}{303} + \frac{1}{606}$ __________

What method(s) did the Egyptians use to get their results. This is the challenge !

75. <u>Project</u>

a. Try the greatest unit fraction method (Section 23, page 29) on each of the fractions in Table 1 (page 31) . If your answer agrees with the one in Table 1 , write a G beside the fraction.

b. Try each of the formulas 1, 2, 3 on page 30 on each of the fractions in Table 1 (page 31) . If the result from a formula agrees with the answer in Table 1, write the number of the formula beside the fraction .

c. If neither (a) nor (b) give the answer in Table 1 , write N beside the fraction .

24. Fractions as sums of 2 unit fractions . Diophantine

Given: a fraction of the form $\frac{2}{n}$, where n is an odd number .

Find: unit fractions $\frac{1}{x}$, $\frac{1}{y}$ such that : $\frac{2}{n} = \frac{1}{x} + \frac{1}{y}$.

Example 1 Take $\frac{2}{15} = \frac{1}{x} + \frac{1}{y}$, where $n = 15$ and find $\frac{1}{x}$, $\frac{1}{y}$.

Solution: Write all the divisors of 15 . $D_{15} = \{1, 3, 5, 15\}$.

Step 1 Write pairs (P, Q) of relatively prime distinct divisors from D_{15}.

Thus, { (1,3), (1,5), (1, 15), (3, 5) } .

Also, define : P' to be such that $P P' = 15$ (that is, n here is 15)

Example 2 If $P = 1$, then $P' = 15$,

$P = 3$, then $P' = 5$.

define: Q' to be such that $Q Q' = 15$ (that is, n here is 15) .

Example 3 If $Q = 3$, then $Q' = 5$

$Q = 1$, then $Q' = 15$.

define : $k = \frac{P + Q}{2}$

Step 2 Solutions to the problem are given by

$$\frac{2}{15} = \frac{1}{k P'} + \frac{1}{k Q'} \quad \ldots\ldots\ldots\ldots\ldots \text{(A)}$$

for each pair (P, Q) of relatively prime divisors in Step 1 .

Step 3 a. for (1, 3) , $P = 1$, $Q = 3$. Thus, $P' = 15$, $Q' = 5$, $k = \frac{1+3}{2} = 2$.

From (A) , $\frac{2}{15} = \frac{1}{2(15)} + \frac{1}{2(5)} = \frac{1}{30} + \frac{1}{10}$.

b. for (1, 5) , $P = 1$, $Q = 5$. Thus, $P' = 15$, $Q' = 3$, $k = \frac{1+5}{2} = 3$.

From (A) , $\frac{2}{15} = \frac{1}{3(15)} + \frac{1}{3\ (3)} = \frac{1}{45} + \frac{1}{9}$.

c. for (1, 15) , $P = 1$, $Q = 15$. Thus, $P' = 15$, $Q' = 1$, $k = \frac{1+15}{2} = 8$.

From (A), $\frac{2}{15} = \frac{1}{8\ (15)} + \frac{1}{8\ (1)} = \frac{1}{120} + \frac{1}{8}$.

d. for (3, 5) , $P = 3$, $Q = 5$. Thus, $P' = 5$, $Q' = 3$, $k = \frac{3+5}{2} = 4$.

From (A) , $\frac{2}{15} = \frac{1}{4\ (5)} + \frac{1}{4\ (3)} = \frac{1}{20} + \frac{1}{12}$.

PROBLEMS

76. Use the procedure above . Express each fraction as a sum of 2 unit fractions .

a. 2/21 ____________________

b. $\frac{2}{33}$ ____

c. $\frac{2}{35}$ ____

25. <u>How many fractions are there ?</u>

In Section 21, page 26, Problem 67, we saw that we could find as many fractions as we wish between the fractions 1/3 and 1/2 . As a matter of fact, there are infinitely many fractions between <u>any two</u> fractions .

The above raises an interesting question :

Are there more natural numbers than fractions
or are there more fractions than natural numbers ?

We examine this question now .

<u>One-to - one correspondence</u>

The members of two sets A, B are in one-to-one correspondence if all the members of set A <u>can</u> be matched exactly (one-to-one) with all the members of set B and vice versa .

<u>Example 1</u> Is it possible to put the members of set $A = \{a, b, c\}$ into one-to-one correspondence with the members of set $B = \{x, y, z\}$?

<u>Solution:</u> One of many solutions is given below .

Perform the following matching:

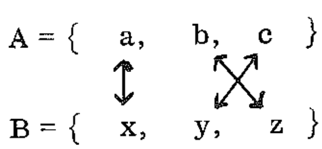
$$\begin{array}{cccc} A = \{ & a, & b, & c \ \} \\ & \updownarrow & \updownarrow & \updownarrow \\ B = \{ & x, & y, & z \ \} \end{array}$$

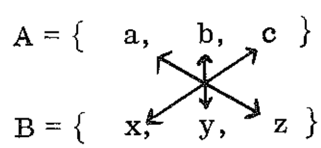
where a is matched only with x, and x is matched only with a,
b is matched only with y, and y is matched only with b ,
c is matched only with z, and z is matched only with c .

All the members of set A are matched exactly (one-to-one) with all the members of set B , and all the members of set B are matched exactly (one-to-one) with all the members of set A .

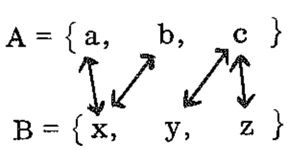
Thus, the elements of set A can be put into one-to-one correspondence with the elements of set B .

Other solutions are shown below.





The matching



is not a one-to-one correspondence since x is matched with 2 members of set A and c is matched with 2 members of set B .

Example 1 Is it possible to put the members of set $C = \{r, s, t, u\}$ into one-to-one correspondence with the members of set $D = \{x, y\}$?

Solution: Take

$$\begin{array}{l} C = \{\, r,\ s,\ t,\ u \,\} \\ \quad\ \ \ \updownarrow\ \ \updownarrow \\ D = \{\, x\ ,\ y \,\}\ . \end{array}$$

Here, r is matched only with x , and x is matched only with r , s is matched only with y, and y is matched only with s.

All the elements of set D are matched with only some of the elements of set C . There are some elements of set C, namely, t, u which have no partners in set D .

Moreover, no matter what other matching procedure we use, we will still not be able to match exactly (one-to-one) the members of set C with the members of set D .

Thus, the members of set C cannot be put into one-to-one correspondence with the members of set D .

Equivalent sets : Two sets whose members can be put into one-to-one correspondence are called equivalent sets .

Same number of members : Two sets have the same number of members if the sets are equivalent , that is, if the members of the two sets can be put into one-to-one correspondence .

Example 2 The set $A = \{a, b, c\}$ and the set $B = \{x, y, z\}$ have the same number of members since the sets are equivalent .

Example 3. The set $C = \{r, s, t, u\}$ and the set $D = \{x, y\}$ do not have the same number of members since the sets are not equivalent .

PROBLEMS

77. Which of the following matchings are a one-to-one correspondence?

a. $A = \{a, b, c, d\}$

$B = \{1, 2, 3\}$

Is ___ Is not ___

b. $C = \{a, b, c, d\}$

$D = \{1, 2, 3\}$

Is ___ Is not ___

c. $E = \{a, b, c\}$

$F = \{1, 2, 3\}$

Is ___ Is not ___

d. $G = \{a, b, c\}$

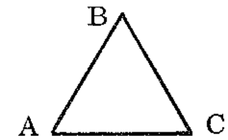
$H = \{1, 2, 3\}$

Is ___ Is not ___

78. If two sets have the same number of members , can the members of the two sets be put into one-to-one correspondence ?

Yes _____ No _____

79. Given: the triangle ABC .



Can the vertices of the triangle ABC be put into one-to-one correspondence with the sides of the triangle ABC ? Yes ___ No ___

If yes, how ? ______________________________

If no , why not? ______________________________

80. Can the members of the set of United States Senators be put into one-to-one correspondence with the members of the set of all States of the United States ? Yes _____ No _____

If yes , how ? ______________________________

If no , why not? ______________________________

81. Can the members of set $A = \{0, 1, 2, 3, 4, 5, \ldots, n, \ldots\}$ be put into one-to-one correspondence with the members of the set $B = \{0, 1, 4, 9, 16, 25, \ldots, n^2, \ldots\}$?

If yes, how ? ______________________________

If no, why not? ______________________________

Fill in the blanks.

The set B of the squares of all the natural numbers __________ equivalent to the set A of all natural numbers. (is) (is not)

The set B __________ have the same number of members as set A. (does)(does not)

82. In how many ways can the members of set A be put into one-to-one correspondence with the members of set B ?

	Number of ways
a. $A = \{a\}$, $B = \{1\}$	________
b. $A = \{a, b\}$, $B = \{1, 2\}$	________
c. $A = \{a, b, c\}$, $B = \{1, 2, 3\}$	________
d. $A = \{a, b, c, d\}$, $B = \{1, 2, 3, 4\}$	________
e. $A = \{a_1, a_2, \ldots, a_n\}$, $B = \{1, 2, \ldots, n\}$	________

To find out whether there are just as many fractions as there are natural numbers, we must (on the basis of Example 1, page 34 and Example 2, page 35),

1. list all the natural numbers, and
2. list all the fractions, and
3. see if we can put the natural numbers into one-to-one correspondence with the fractions .

Now, it is simple to list all the natural numbers. Thus,

$$0, 1, 2, 3, 4, 5, \ldots, n, (n+1), \ldots$$

To list all the natural numbers means:

1. no natural number has been left out of the list ,
2. we can tell exactly where each natural number is in the list .

The listing (bottom, page 36) if continued contains all the natural numbers . We add 1 to the preceding number to get the next number .

Moreover, for a particular number, say 100, we know exactly where it is in the list. 100 is the 101 st element in the list .

Can we list all the fractions ?

Because there are infinitely many fractions between any two fractions, we cannot list the fractions according to increasing or decreasing magnitude (there is, for example, no one unique fraction that follows 1/3 in the same way that there is only one number, namely 2, which follows 1) .

However, there are methods for listing all the fractions .

a. Method 1. Row and column method for listing all the fractions.

Study Chart 1 .

CHART 1

$\frac{0}{1}$	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$	$\frac{6}{1}$	. . . .
	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{2}$	$\frac{6}{2}$	. . . .
	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{6}{3}$	. . . .
	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$	$\frac{6}{4}$	. . . .
	.	.	.	.	.	.	. . . .
	.	.	.	.	.	.	. . . .

In Chart 1,

1. each row has as numerators the numbers 1, 2, 3, 4, 5, . . . , (the first row also has 0 as a numerator) ,
2. the first row has denominators 1,
   the second row has denominators 2 ,
   the third row has denominators 3 , and so on .

Do all the fractions appear in this Chart 1 ? The answer will be 'yes' if given any fraction , we can tell exactly where it is .

Example 1 Where is the fraction 5/3 ?

Solution : Given any fraction, the numerator will indicate the column and the denominator will indicate the row where the fraction is to be found. Thus, 5/3 is in the 5 th column and 3 rd row .

Example 2 Where is the fraction a/b ?

Solution: The fraction a/b is in the a th column and in the b th row .

Thus, all the fractions are in the Chart 1, page 37. We now devise a method for listing the fractions horizontally . Study Chart 2 . It is a copy of Chart 1, p. 37 .

<u>CHART 2</u>

$$
\begin{array}{ccccccccccccc}
\frac{0}{1} & \rightarrow & \frac{1}{1} & \rightarrow & \frac{2}{1} & & \frac{3}{1} & \rightarrow & \frac{4}{1} & & \frac{5}{1} & \frac{6}{1} & \cdots\cdots \\
 & & \frac{1}{2} & & \frac{2}{2} & & \frac{3}{2} & & \frac{4}{2} & & \frac{5}{2} & \frac{6}{2} & \cdots\cdots \\
 & & \downarrow & & & & & & & & & & \\
 & & \frac{1}{3} & & \frac{2}{3} & & \frac{3}{3} & & \frac{4}{3} & & \frac{5}{3} & \frac{6}{3} & \cdots\cdots \\
 & & \frac{1}{4} & & \frac{2}{4} & & \frac{3}{4} & & \frac{4}{4} & & \frac{5}{4} & \frac{6}{4} & \cdots\cdots \\
 & & \downarrow & & \cdot & & \cdot & & \cdot & & \cdot & \cdot & \cdots\cdots \\
 & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & \cdot & \cdots\cdots
\end{array}
$$

To list the fractions horizontally, start with 0/1 and follow the pattern of the arrows in Chart 2 to get :

$$\frac{0}{1},\quad \frac{1}{1},\quad \frac{2}{1},\quad \frac{1}{2},\quad \frac{1}{3},\quad \frac{2}{2},\quad \frac{3}{1},\quad \frac{4}{1},\quad \frac{3}{2},\quad \frac{2}{3},\quad \frac{1}{4},\quad \ldots \qquad (A).$$

Since all the fractions are in Chart 2, then all the fractions are also in the listing (A) directly above . Now take the elements of the set of natural numbers and the above list of fractions and match 0 with the fraction 0/1 ,
match 1 with the fraction 1/1 ,
match 2 with the fraction 2/1 , and so on as shown below:

$$
\begin{array}{cccccccccccc}
\frac{0}{1}, & \frac{1}{1}, & \frac{2}{1}, & \frac{1}{2}, & \frac{1}{3}, & \frac{2}{2}, & \frac{3}{1}, & \frac{4}{1}, & \frac{3}{2}, & \frac{2}{3}, & \frac{1}{4}, & \ldots\ . \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \ldots\ .
\end{array}
$$

The top list contains all the fractions. The bottom list contains all the natural numbers. Notice that you can match exactly (one-to-one) all the fractions with all the natural numbers .

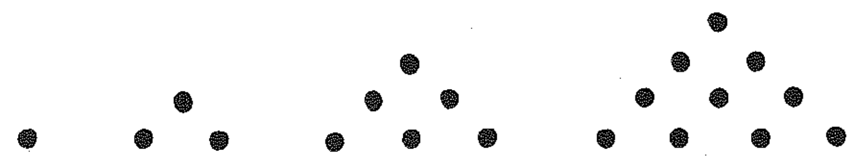
There are only as many fractions as
there are natural numbers !

Infinity has many surprises !

Return again to the horizontal listing (A) of all the fractions .

There is a formal way of finding the natural number to associate with each of the fractions in the list (A) . But first we will define and illustrate a few figurate numbers called ' triangular numbers ' .

Triangular numbers: a triangular number is a figurate number. The first few triangular numbers are given below :



$T_1 = 1$ $\quad T_2 = 3$ $\quad T_3 = 6$ $\quad T_4 = 10$

$T_5 = 15$ $\quad T_7 = 28$ $\quad T_9 = 45$ $\quad T_{11} = 66$

$T_6 = 21$ $\quad T_8 = 36$ $\quad T_{10} = 55$ $\quad T_{12} = 78$.

The n th triangular number is given by the formula : $T_n = \dfrac{n(n+1)}{2}$.

We now give a formula for finding the position of a fraction in the list (A), page 38.

1. Let $\frac{a}{b}$ (where $\frac{a}{b} \neq \frac{0}{1}$, since we know that $\frac{0}{1}$ is the first fraction) be a fraction in unreduced form that appears in the horizontal listing (A), of the fractions .

2. Write $k = a + b - 2$, then substitute for a, b and perform the calculation.

   a. If k turns out to be an EVEN number, then the position of $\frac{a}{b}$ in the list (A), page 38, is given by

   Position of $\frac{a}{b} = T_k + a + 1$, where T_k is the k th triangular number, (B)

   b. if k turns out to be an ODD number, then the position of $\frac{a}{b}$ in the list (A), page 38, is given by

   Position of $\frac{a}{b} = T_k + b + 1$, where T_k is the k th triangular number . (C)

Example 1. What is the position of the fraction $\frac{1}{3}$ in the list (A), page 38 ?

Solution: Here $a = 1$, $b = 3$.

From $k = a + b - 2$, we get $k = 1 + 3 - 2 = 2$.

Since $k = 2$ is an even number, use (B) above to get

$$\text{Position of } \frac{1}{3} = T_2 + 1 + 1 = 3 + 1 + 1 = 5 \ .$$

The fraction $\frac{1}{3}$ is the 5 th fraction in the list (A).

Example 1 What is the position of the fraction $\frac{2}{3}$ in the list (A), page 38 ?

Solution: Here $a = 2$, $b = 3$. From $k = a + b - 2$, we get $k = 2 + 3 - 2 = 3$. Since $k = 3$ is an odd number, use (C) on page 39 to get

$$\text{Position of } \frac{2}{3} = T_3 + 3 + 1 = 6 + 3 + 1 = 10 .$$

The fraction 2/3 is the 10 th fraction in the list (A), page 38 .

PROBLEMS

83. Find the position of each of the following fractions in the list (A), page 38.

a. $\frac{2}{2}$ ___ b. $\frac{3}{2}$ ___ c. $\frac{1}{4}$ ___ d. $\frac{19}{20}$ ___ e. $\frac{99}{100}$ ___

84. Can you find which fraction is the 100 th in the list (A), page 38, without writing out the first 100 fractions ? How ? ________________

____

85. What is the n th (n a natural number not zero) fraction in the list (A) , page 38 ? ________________________

b. Method 2 . Sum of numerator and denominator method for listing fractions .

In this method, we arrange the fractions according to the sums of the numerators and denominators in the following way :

CHART 3

Sum of numerator and denominator.	Fractions
1	$\frac{0}{1}$
2	$\frac{1}{1}$
3	$\frac{1}{2}$, $\frac{2}{1}$
4	$\frac{1}{3}$, $\frac{2}{2}$, $\frac{3}{1}$
5	$\frac{1}{4}$, $\frac{2}{3}$, $\frac{3}{2}$, $\frac{4}{1}$
6	$\frac{1}{5}$, $\frac{2}{4}$, $\frac{3}{3}$, $\frac{4}{2}$, $\frac{5}{1}$
.	. . . . .
.	. . . . .
and so on .	

Do all the fractions appear in Chart 3 ? It is quite clear that they do .

Example 1 Where is the fraction $\frac{124}{275}$ in Chart 3 ?

Solution : The sum of the numerator and denominator is $124 + 275 = 399$. Thus, the fraction 124/275 is in the 399 th row and is the

124 th entry in this row .

In general, the fraction $\frac{a}{b}$ (where $\frac{a}{b} \neq \frac{0}{1}$) will be in the $(a + b)$ th row and will be the a th entry in the row in Chart 3, page 40 .

In Chart 3, page 40, if we take the fractions row by row, we have the following horizontal listing :

$$\frac{0}{1}, \quad \frac{1}{1}, \quad \frac{1}{2}, \quad \frac{2}{1}, \quad \frac{1}{3}, \quad \frac{2}{2}, \quad \frac{3}{1}, \quad \frac{1}{4}, \quad \frac{2}{3}, \quad \frac{3}{2}, \quad \frac{4}{1}, \quad \frac{1}{5}, \ldots \quad (D)$$

The listing (D), if continued, contains all the fractions. Moreover, it is quite obvious that we can now put the elements of (D) into one-to-one correspondence with the elements of the set of natural numbers as follows :

$$\begin{array}{cccccccccc} \frac{0}{1}, & \frac{1}{1}, & \frac{1}{2}, & \frac{2}{1}, & \frac{1}{3}, & \frac{2}{2}, & \frac{3}{1}, & \frac{1}{4}, & \frac{2}{3}, & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8, & \ldots \end{array}$$

Since the elements of the set of fractions can be matched exactly with the elements of the set of natural numbers, the two sets have the same number of elements. There are only as many fractions as there are natural numbers.

Can we tell the position of a given fraction in the listing (D) ?

1. Let $\frac{a}{b}$ (where $\frac{a}{b} \neq \frac{0}{1}$, since we know that $\frac{0}{1}$ is the first fraction) be a fraction in <u>unreduced form</u> that appears in the horizontal listing (D) above.

2. Write $k = a + b - 2$, then substitute for a, b and perform the calculation .

Position of $\frac{a}{b}$ in list (D) $= T_k + a + 1$, where T_k is the k th triangular number (page 39). .. (E)

<u>Example 1</u> What is the position of the fraction $\frac{1}{3}$ in the list (D) above ?

<u>Solution:</u> Here $a = 1$, $b = 3$.

From $k = a + b - 2$, we get $k = 1 + 3 - 2 = 2$.

Then T_k becomes $T_2 = 3$. Then from (E), we have

Position of $\frac{1}{3}$ in list (D) $= T_k + a + 1 = 3 + 1 + 1 = 5$.

The fraction 1/3 is the 5 th fraction in the list (D) above .

PROBLEMS

86. Find the position of each of the following fractions in the list (D) above.

a. $\frac{3}{2}$ ___ b. $\frac{1}{4}$ ___ c. $\frac{7}{9}$ ___ d. $\frac{19}{20}$ ___

e. $\frac{99}{100}$ ___

87. Can you find which fraction is the 100 th in the list (D), page 41 without writing out the first 100 fractions ? How ? ______________

____

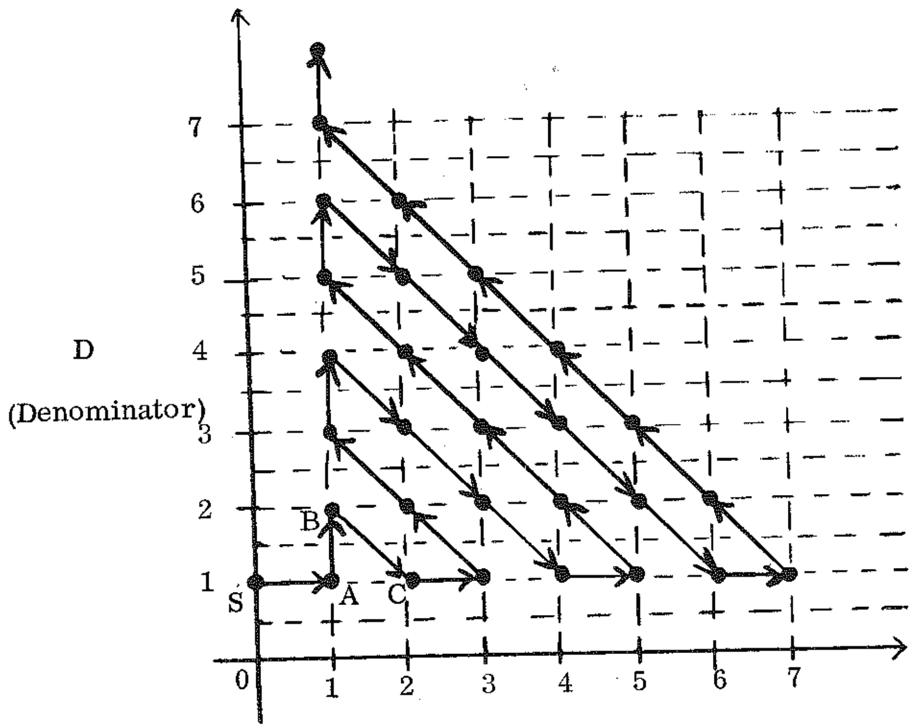
88. What is the n th (n a natural number not zero) fraction in the list (D), page 41 ? ______________

89. Which fractions (if any) are in the same position in both the listing (A), page 38 and (D) page 41 . ______________

c. Method 3 . Graph method for listing fractions .

Study the following graph .

GRAPH 1



Each point on the graph is associated with two numbers (N, D) (just like the pair (x, y) in the Cartesian coordinate system) which we will write as the fraction $\frac{N}{D}$.

Begin at S .	Here N = 0, D = 1 , and we have the fraction 0/1 .
Follow the arrow to A.	Here N = 1, D = 1, and we have the fraction 1/1.
Follow the arrow to B.	Here N = 1, D = 2, and we have the fraction 1/2 .

If after the fractions at S and A , we list in one row all the fractions on a given diagonal, we will have the results shown in Chart 4 .

CHART 4

Fractions	Sum of numerator and denominator
$\frac{0}{1}$	1
$\frac{1}{1}$	2

Fractions	Sum of numerator and denominator
$\frac{1}{2}, \frac{2}{1}$	3
$\frac{3}{1}, \frac{2}{2}, \frac{1}{3}$	4
$\frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1}$	5
$\frac{5}{1}, \frac{4}{2}, \frac{3}{3}, \frac{2}{4}, \frac{1}{5}$	6
. . . . .	.
and so on .	

It is clear that all the fractions are in Graph 1 (page 42) and in Chart 4 (page 42) if they were continued .

If we take the fractions row by row from Chart 4, we have the horizontal listing (F) shown below .

$$\frac{0}{1},\ \frac{1}{1},\ \frac{1}{2},\ \frac{2}{1},\ \frac{3}{1},\ \frac{2}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{2}{3},\ \frac{3}{2},\ \frac{4}{1}, \ldots \qquad (F)$$

As on page 41 , we can now match exactly all the fractions with all the natural numbers . There are just as many fractions as there are natural numbers .

Can we tell the position of a given fraction in the listing (F) ?

1. Let $\frac{a}{b}$ (where $\frac{a}{b} \neq \frac{0}{1}$, since we know that $\frac{0}{1}$ is the first fraction) be a fraction in unreduced form that appears in the listing (F) of the fractions.

2. Write $k = a + b - 2$, then substitute for a, b and perform the calculation ,

   a. if k turns out to be an EVEN number, then the position of $\frac{a}{b}$ in the list (F) is given by

   $$\text{Position of } \frac{a}{b} = T_k + b + 1 , \quad \text{where } T_k \text{ is the } k \text{ th triangular number (page 39)} \qquad (G)$$

   b. if k turns out to be an ODD number, then the position of $\frac{a}{b}$ in the list (F) is given by

   $$\text{Position of } \frac{a}{b} = T_k + a + 1 , \quad \text{where } T_k \text{ is the } k \text{ th triangular number (page 39)} \qquad (H)$$

<u>Example 1</u> What is the position of $\frac{1}{3}$ in the list (F) ?

<u>Solution</u> : Here $a = 1$, $b = 3$.

From $k = a + b - 2$, we get $k = 1 + 3 - 2 = 2$.

Since $k = 2$ is an even number, then from (G), we have that

$$\text{Position of } \frac{1}{3} = T_2 + 3 + 1 = 3 + 3 + 1 = 7 .$$

<u>Example 1</u> What is the position of $\frac{3}{2}$ in the list (F), page 43 ?

<u>Solution:</u> Here $a = 3$, $b = 2$.

From $k = a + b - 2$, we get $k = 3 + 2 - 2 = 3$.

Since $k = 3$ is an odd number, then from (H), page 43 , we have

$$\text{Position of } \frac{3}{2} = T_3 + 3 + 1 = 6 + 3 + 1 = 10 .$$

The fraction 3/2 is the 10 th fraction in the list (F), page 43 .

PROBLEMS

90. Find the position of each of the following fractions in the list (F), page 43 .

a. $\frac{2}{3}$ _____ b. $\frac{1}{4}$ _____ c. $\frac{7}{9}$ _____

d. $\frac{19}{20}$ _____ e. $\frac{99}{100}$ _____

91. Can you find which fraction is the 100 th fraction in the list (F) page 43, without writing out the first 100 fractions ?

How ? ____________________

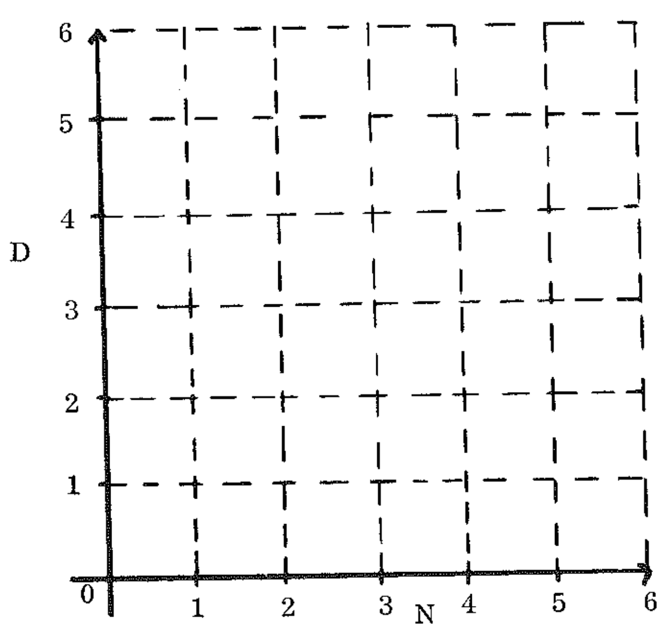
92. What is the n th (where n is a natural number) fraction in the list (F), p. 43 ? ____________________

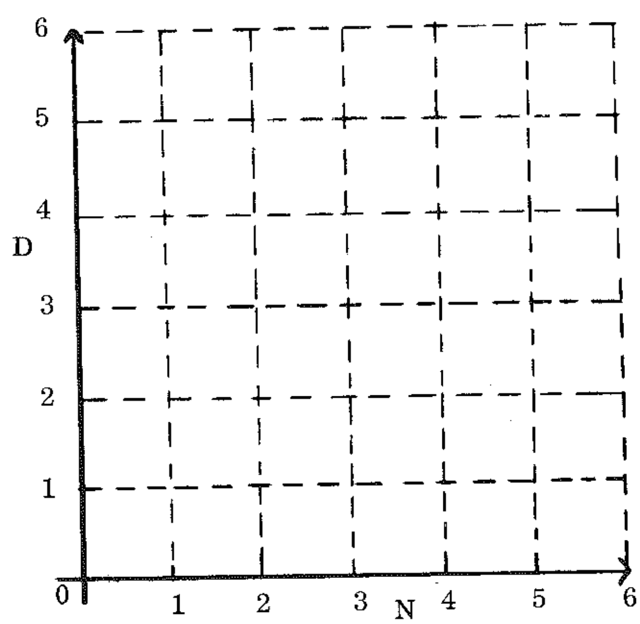
93. Which fractions, if any, are in the same position in

a. List (A), page 38 and (F),page 43 ____________________

b. List (D), page 41 and (F), page 43 ____________________

94. Draw a graph similar to the one for list (F), page 42 , for the listings (A), page 38 and (D), page 41 .





Solutions and Comments

page

1 PROB: 1

a) $10 = 10/1$; b) $7\frac{8}{9} = 7 + \frac{8}{9} = \frac{63}{9} + \frac{8}{9} = \frac{71}{9}$ or $7\frac{8}{9} = \frac{9(7)+8}{9} = \frac{71}{9}$; c) $4 = 4/1$

1 PROB: 2

a) $33\frac{1}{3} = \frac{3(33)+1}{3} = \frac{100}{3}$, which is a common fraction.

b) Since 5 is not a perfect square, $\sqrt{5}$ cannot be represented by a common fraction.

c) $\sqrt{9} = 3 = 3/1$, which is a common fraction.

Note: Problems 2(b) and 2(c) can be generalized. $\sqrt{n}$ can be represented as a rational number if and only if there is another integer p such that $p^2 = n$. In that case $\sqrt{n} = p/1$. $\sqrt{0}$, $\sqrt{1}$, $\sqrt{4}$, $\sqrt{9}$, . . . , $\sqrt{p^2}$, . . . can be represented by common fractions while $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, . . . cannot.

1 PROB: 3

a) $45/9 = 5(9)/9 = 5$; b) $43/5 = \frac{5(8)+3}{5} = 5(8)/5 + 3/5 = 8 + 3/5 = 8\frac{3}{5}$;
c) $72/8 = 9(8)/8 = 9$.

Note: You could have used simple division to get the answers to these problems.

2 PROB: 4

Use the procedure in Problem 3 . a) $25/6 = 4\frac{1}{6}$; b) $108/36 = 3$; c) $1111/11 = 101$.

3 PROB: 5

a) Let
$$\begin{aligned} N &= .1111\ldots \\ 10N &= 1.1111\ldots \\ -N &= -.1111\ldots \\ \hline 9N &= 1 \text{ or } N = 1/9 \end{aligned}$$

b) Let
$$\begin{aligned} N &= .\overline{2} \\ 10N &= 2.\overline{2} \\ -N &= -.\overline{2} \\ \hline 9N &= 2 \text{ or } N = 2/9 \end{aligned}$$

c) Let
$$\begin{aligned} N &= .\overline{8} \\ 10N &= 8.\overline{8} \\ -N &= -.\overline{8} \\ \hline 9N &= 8 \text{ or } N = 8/9 \end{aligned}$$

d) Let
$$\begin{aligned} N &= .9999\ldots \\ 10N &= 9.9999\ldots \\ -N &= -.9999\ldots \\ \hline 9N &= 9 \text{ or } N = 1 \end{aligned}$$

e) Let
$$\begin{aligned} N &= .1515\ldots \\ 100N &= 15.1515\ldots \\ -N &= -.1515\ldots \\ \hline 99N &= 15 \text{ or } N = 15/99 = 5/33 \end{aligned}$$

f) Let
$$N = .08\overline{3}$$
$$100N = 8.\overline{3} = 8 + .\overline{3}$$
By Problem 5(c), $.\overline{3} = 1/3$.
$$100N = 8\frac{1}{3} = 25/3$$
or $N = 25/300 = 1/12$

g) Let
$$N = .01\overline{6}$$
$$100N = 1.\overline{6} = 1 + .\overline{6}$$
By Example 4, p. 3, $.\overline{6} = 2/3$.
$$100N = 1\frac{2}{3} = 5/3$$
or $N = 5/300 = 1/60$

h) Let
$$N = .0\overline{9}$$
$$10N = .\overline{9}$$
By Problem 5(d), $.\overline{9} = 1$.
$$10N = 1$$
or $N = 1/10 = .1$

page

3 PROB: 5 (Continued)

i) Let $N = .00\overline{9}$ $\qquad 100N = .\overline{9}$

By Problem 5(d), $.\overline{9} = 1$. $\qquad 100N = 1$ or $N = 1/100 = .01$

4 PROB: 6

Let $N = .\overline{142857}$

$$\begin{array}{rcl} 1000000N & = & 142857.\overline{142857} \\ -\quad N & = - & .\overline{142857} \\ \hline 999999N & = & 142857 \\ \text{or} \quad N & = & 142857/999999 = 1/7 \\ & & (\text{ since } 999999 = 7(142857)\ .) \end{array}$$

4 PROB: 7

a) Let

$$\begin{array}{rcl} N & = & .\overline{a} \\ 10N & = & a.\overline{a} \\ -\ N & = - & .a \\ \hline 9N & = & a \quad \text{or} \quad N = a/9 \end{array}$$

Note: Using the result of Problem 7(a), we could immediately conclude that for $N = .6666\ldots$, we have $N = 6/9 = 2/3$.

b) Let

$$\begin{array}{rcl} N & = & .\overline{ab} \\ 100N & = & ab.\overline{ab} \\ -\ N & = & -\ .ab \\ \hline 99N & = & ab \quad \text{or} \quad N = ab/99 \end{array}$$

Note: Using the result of Problem 7(b), we could immediately conclude that for $N = .1515\ldots$, we have $N = 15/99 = 5/33$.

4 PROB: 8

a) $2/3 < 3/2$ since $2(2) < 3(3)$

b) $7/6 < 7/4$ since $7(4) < 7(6)$

c) $3/8 < 5/8$ since $3(8) < 5(8)$

d) Rewrite: $3 = 3/1$; $2\frac{3}{5} = 13/5$. $\quad 3/1 > 13/5$ since $3(5) > 13(1)$

e) Rewrite: $1\frac{1}{3} = 4/3$; $1\frac{4}{7} = 11/7$. $\quad 4/3 < 11/7$ since $4(7) < 11(3)$

f) $0/5 = 0/10$ since $0(10) = 0(5)$

Note: There is another approach for mixed numbers such as appear in Problems 8(d) and 8(e). If the numbers are written as the sum of a whole number and a proper fraction (if necessary), the number with the larger whole number part is the larger number. If the whole number parts are the same, we need to compare only the proper fraction parts.

Examples 1) In 8(d) to compare 3 and $2\frac{3}{5}$ we compare 3 and 2.

Since $3 > 2$, $3 > 2\frac{3}{5}$.

2) In 8(e) to compare $1\frac{1}{3}$ and $1\frac{4}{7}$ we compare $1/3$ and $4/7$ since $1 = 1$. $\quad 1/3 < 4/7$ since $1(7) < 4(3)$.

Therefore $1\frac{1}{3} < 1\frac{4}{7}$.

In fact for any whole number n, $n\frac{1}{3} < n\frac{4}{7}$.

5 PROB: 9

a) $5/14 < 5/10 < 5/7 < 5/5 < 5/3 < 5/1$

b) If 2 fractions have the same nonzero numerator but different denominators, then the fraction with the larger denominator is the smaller fraction.

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5 PROB: 9(b) (Continued)

To see this result let n be a whole number different from 0, and let a, b be any whole numbers satisfying $0 < a < b$.

$n/a > n/b$ since $n(b) > n(a)$ which is equivalent to $b > a$.

5 PROB:10

a) $1/9 < 4/9 < 5/9 < 8/9 < 9/9 < 13/9$

b) If 2 fractions have the same denominator but different numerators, then the fraction with the larger numerator is the larger fraction.

To see this result let n be a whole number different from 0, and let a, b be any whole numbers satisfying $a < b$.

$a/n < b/n$ since $a(n) < b(n)$ which is equivalent to $a < b$.

5 PROB:11

a) No. If a, b, n are nonzero whole numbers and $a \neq b$, then

$n/a = n/b$ implies $n(b) = n(a)$ which implies $b = a$.

Since this contradicts our assumption that $a \neq b$, we cannot have $n/a = n/b$.

b) Yes. $0/1 = 0/2 = 0/3 = \ldots$. The difference is that we have not excluded 0 from the numerator. We now have for $a \neq 0$, $b \neq 0$

$0/a = 0/b$, that is, $0(b) = 0(a)$ and a, b do not have to be the same.

5 PROB:12

a) $0/7$, $1/7$, $2/7$, $3/7$, $4/7$, $5/7$, $6/7$.

b) There are exactly b such fractions, namely, $0/b$, $1/b$, $\ldots$, $(b-2)/b$, $(b-1)/b$.

Note: We may now characterize proper fractions as those which are strictly greater than 0 and strictly less than 1. (See Section 5, page 4 .)

6 PROB:13

a) $4/5 = 8/10 = 12/15 = 16/20 = 20/25 = 24/30 = \ldots = 4n/5n = \ldots$.

b) $8/3 = 16/6 = 24/9 = 32/12 = 40/15 = 48/18 = \ldots = 8n/3n = \ldots$.

c) $3/4 = 6/8 = 9/12 = 12/16 = 15/20 = 18/24 = \ldots = 3n/4n = \ldots$.

d) $0/7$ is equivalent to any fraction $0/n$, where $n \neq 0$.

e) We could first write $1\frac{2}{3} = 5/3$, and then

$5/3 = 10/6 = 15/9 = 20/12 = 25/15 = 30/18 = \ldots = 5n/3n = \ldots$.

Another approach is to work with the proper fraction portion.

$$1\frac{2}{3} = 1\frac{4}{6} = 1\frac{6}{9} = 1\frac{8}{12} = 1\frac{10}{15} = \ldots = 1\frac{2n}{3n} = \ldots .$$

Check: $1\frac{2n}{3n} = \frac{3n(1) + 2n}{3n} = \frac{5n}{3n} = \frac{5}{3} = 1\frac{2}{3}$.

f) Method 1: $2\frac{1}{2} = 5/2 = 10/4 = 15/6 = 20/8 = 25/10 = 30/12 = \ldots = 5n/2n = \ldots$.

Method 2: $2\frac{1}{2} = 2\frac{2}{4} = 2\frac{3}{6} = 2\frac{4}{8} = 2\frac{5}{10} = 2\frac{6}{12} = \ldots = 2\frac{n}{2n} = \ldots$.

page

7 PROB:14

a) $16/32 = 16(1)/16(2) = 1/2$

b) $8/4 = 4(2)/4(1) = 2/1 = 2$

c) Method 1: $1\frac{2}{4} = 6/4 = 2(3)/2(2) = 3/2$; Method 2: $2/4 = 2(1)/2(2) = 1/2$ so $1\frac{2}{4} = 1\frac{1}{2}$.

d) $0/4$ is equal to any fraction $0/n$, where $n \neq 0$. In particular $0/4 = 4(0)/4(1) = 0/1$.

e) Since 3 and 5 are relatively prime, $5/3$ is already in simplest form.

f) Since 4 and 7 are relatively prime, $4/7$ is already in simplest form.

7 PROB:15

In each case we divide out the greatest common divisor of the numerator and denominator. In the resulting fraction the numerator and denominator are relatively prime, and so the denominator is the smallest of any equivalent fraction.

a) $20/8 = 4(5)/4(2) = 5/2$

b) $5/60 = 5(1)/5(12) = 1/12$

c) $12/30 = 6(2)/6(5) = 2/5$

d) $65/5 = 5(13)/5(1) = 13/1$

e) $21/4$ has the smallest possible denominator since 21 and 4 are relatively prime.

f) $0/12 = 12(0)/12(1) = 0/1$.

7 PROB:16

The only common divisor of 5 and 7 is 1. Therefore $5/7$ is not equivalent to any fraction with a smaller denominator.

7 PROB:17

a) Since $6 = 3(2)$, $2/3 = 2(2)/3(2) = 4/6$

b) Since $12 = 3(4)$, $2/3 = 2(4)/3(4) = 8/12$

c) Since $18 = 3(6)$, $2/3 = 2(6)/3(6) = 12/18$.

7 PROB:18

a) Since $15 = 5(3)$, $4/5 = 4(3)/5(3) = 12/15$

b) Since $20 = 5(4)$, $4/5 = 4(4)/5(4) = 16/20$.

c) Since $35 = 5(7)$, $4/5 = 4(7)/5(7) = 28/35$.

7 PROB:19

$6/8 = 3(2)/4(2) = 3/4$

$18/24 = 3(6)/4(6) = 3/4$

$21/28 = 3(7)/4(7) = 3/4$

$39/52 = 3(13)/4(13) = 3/4$

9 and 16 are relatively prime. Therefore $9/16$ is not equivalent to any fraction with denominator less than 16.

7 PROB:20

$20/12 = 5(4)/3(4) = 5/3$

$30/18 = 5(6)/3(6) = 5/3$

$70/42 = 5(14)/3(14) = 5/3$

$25/9$ is not equivalent to $5/3$ since 25 and 9 are relatively prime.
$30/50 = 3(10)/5(10) = 3/5$ which is not equal to $5/3$.

7 PROB:21

a) There are infinitely many common fractions equivalent to $2/3$.

page

7 PROB:21 (Continued)

b) To find all such fractions we list all fractions 2n/3n, for each whole number $n \neq 0$.

$\{ 2/3, 4/6, 6/9, \ldots, 2n/3n, \ldots \}$

9 PROB:22

a) 1/4 = 1(6)/4(6) = 6/24 5/6 = 5(4)/6(4) = 20/24

b) 1/4 = 1(9)/4(9) = 9/36 5/6 = 5(6)/6(6) = 30/36

c) 1/4 = 1(12)/4(12) = 12/48 5/6 = 5(8)/6(8) = 40/48

d) 1/4 = 1(27)/4(27) = 27/108 5/6 = 5(18)/6(18) = 90/108

e) Yes. 12 is the least common denominator for 1/4 and 5/6, and we have 1/4 = 1(3)/4(3) = 3/12 and 5/6 = 5(2)/6(2) = 10/12 .

9 PROB:23

a) 2/3 = 2(6)/3(6) = 12/18 4/6 = 4(3)/6(3) = 12/18

b) 2/5 = 2(7)/5(7) = 14/35 3/7 = 3(5)/7(5) = 15/35

c) 1/2 = 1(12)/2(12) = 12/24 5/12 = 5(2)/12(2) = 10/24

9 PROB:24

a) 2/3 = 2(2)/3(2) = 4/6 4/6

b) 2/5 = 2(7)/5(7) = 14/35 3/7 = 3(5)/7(5) = 15/35

c) 1/2 = 1(6)/2(6) = 6/12 5/12

9 PROB:25

a) 12 is not a common denominator for 5/6 and 7/8 since 12 is not a multiple of 8 .

b) 48 is a multiple of both 6 and 8 and so will serve as a common denominator.

c) 16 is not a common denominator for 5/6 and 7/8 since 16 is not a multiple of 6.

d) 24 is a multiple of both 6 and 8 and so will serve as a common denominator. In fact 24 is the least common denominator.

9 PROB:26

A common denominator is $3 \times 5 \times 7 = 105$. Since 3, 5, 7 are distinct primes, 105 is the least common denominator for 2/3, 4/5, and 3/7 .

10 PROB:27

a) 2/12 < 4/12 < 5/12 < 6/12 < 12/12 since 2 < 4 < 5 < 6 < 12 and the denominators are the same.

b) 1/7 < 2/7 < 4/7 < 5/7 < 10/7 since 1 < 2 < 4 < 5 < 10 and the denominators are the same.

10 PROB:28

a) 1/3 = 1(5)/3(5) = 5/15 2/5 = 2(3)/5(3) = 6/15 Therefore $\frac{1}{3} < \frac{2}{5}$.

b) 2/7 = 2(3)/7(3) = 6/21 1/3 = 1(7)/3(7) = 7/21 Therefore $\frac{2}{7} < \frac{1}{3}$.

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10 PROB:28 (Continued)

c) 7/8 $\quad$ 3/4 = 3(2)/4(2) = 6/8 $\quad$ Therefore $\frac{3}{4} < \frac{7}{8}$.

10 PROB:29

60 will serve as a common denominator for the given fractions. In fact it is the least common denominator.

2/5 = 2(12)/5(12) = 24/60				7/12 = 7(5)/12(5) = 35/60
2/3 = 2(20)/3(20) = 40/60				3/10 = 3(6)/10(6) = 18/60
11/15 = 11(4)/15(4) = 44/60				3/20 = 3(3)/20(3) = 9/60

Therefore $\quad 3/20 < 3/10 < 2/5 < 7/12 < 2/3 < 11/15$.

10 PROB:30

If a/b < c/b , then a < c.

$$a < \underbrace{a+1 < a+2 < \ldots < a+(c-a)-1}_{c\,-\,a\,-\,1 \text{ numbers}} < a+(c-a) = c$$

Therefore there are c - a - 1 fractions with denominator b between a/b and c/b .
(Note that if c = a + 1, then c - a - 1 = 0 .)

Example: 9/17 < 13/17 $\quad$ Here a = 9, b = 17, c = 13.
c - a - 1 = 13 - 9 - 1 = 3. The fractions are 10/17, 11/17, 12/17 .

11 PROB:31

a) 2/9 + 7/9 = (2+7)/9 = 9/9 = 1

b) 3/8 + 1/4 = 3(4)/8(4) + 1(8)/4(8) = (12 + 8)/32 = 20/32 (= 5/8)

(8 could have served as a common denominator.)

c) 2/7 + 3/5 = 2(5)/7(5) + 3(7)/5(7) = (10 + 21)/35 = 31/35

d) Method 1: $4\frac{1}{3} + \frac{5}{6}$ = 4 + (1/3 + 5/6)
= 4 + (1(6)/3(6) + 5(3)/6(3)) = 4 + (6 + 15)/18 = $4 + \frac{21}{18} = 4 + 1\frac{3}{18}$
$= 5\frac{3}{18} = 5\frac{1}{6}$

Method 2: $4\frac{1}{3} + \frac{5}{6}$ = 13/3 + 5/6
= 13(6)/3(6) + 5(3)/6(3) = (78 + 15)/18 = 93/18 = $5\frac{3}{18} = 5\frac{1}{6}$

In the remaining problems we will use only Method 2.

e) $\frac{4}{9} + 3\frac{1}{3}$ = 4/9 + 10/3
= 4(3)/9(3) + 10(9)/3(9) = (12 + 90)/27 = 102/27 = $3\frac{21}{27} = 3\frac{7}{9}$.

(9 could have served as a common denominator.)

f) $5\frac{1}{2} + 2\frac{2}{5}$ = 11/2 + 12/5
= 11(5)/2(5) + 12(2)/5(2) = (55 + 24)/10 = 79/10 = $7\frac{9}{10}$.

11 PROB:32

a) 3/11 - 2/11 = (3 - 2)/11 = 1/11

b) 7/7 - 2/3 = 7(3)/7(3) - 2(7)/3(7) = (21 - 14)/21 = 7/21 = 1/3

12 c) $\frac{10}{4} - 1\frac{5}{6}$ = 10/4 - 11/6 = 10(6)/4(6) - 11(4)/6(4) = (60 - 44)/24 = 16/24 = 2/3

12 PROB:32 (Continued)

d) $2\frac{1}{3} - \frac{3}{5} = 7/3 - 3/5 = 7(5)/3(5) - 3(3)/5(3) = (35 - 9)/15 = 26/15 = 1\frac{11}{15}$

e) $3\frac{1}{4} - 2\frac{5}{6} = 13/4 - 17/6 = 13(6)/4(6) - 17(4)/6(4) = (78 - 68)/24 = 10/24 = 5/12$

(12 could be used as a common denominator.)

f) $5\frac{2}{3} - \frac{0}{13} = 17/3 - 0/13 = 17(13)/3(13) - 0(3)/13(3) = (221 - 0)/39 = 221/39 = 17/3 = 5\frac{2}{3}$

(More simply $0/13 = 0$, so $5\frac{2}{3} - 0 = 5\frac{2}{3}$. In general $\frac{a}{b} - \frac{0}{c} = \frac{a}{b}$, $b, c \neq 0$.)

12 PROB:33

a) $\frac{5}{4}(\frac{2}{3}) = \frac{10}{12}$ ($= 5/6$)

b) $\frac{5}{4}(\frac{5}{6}) = \frac{25}{24}$

c) $\frac{5}{4}(\frac{7}{8}) = \frac{35}{32}$

12 PROB:34

a) $5/4 > 1 = 1/1$ since $5(1) > 1(4)$

b) $5/4$ of $a/b = 5a/4b > a/b$ since $5a(b) > a(4b)$

12 PROB:35

a) $\frac{4}{5}(\frac{2}{3}) = \frac{8}{15}$

b) $\frac{4}{5}(\frac{5}{6}) = \frac{20}{30}$ ($= 2/3$)

c) $\frac{4}{5}(\frac{7}{8}) = \frac{28}{40}$ ($= 7/10$)

12 PROB:36

a) $4/5 < 1 = 1/1$ since $4(1) < 1(5)$

b) $4/5$ of $a/b = 4a/5b < a/b$ since $4a(b) < a(5b)$.

12 PROB:37

Since a/b and c/d are proper fractions, neither equals 0 , and so their product $ac/bd \neq 0$. Furthermore since $a/b < 1$ and $c/d < 1$, $ac/bd < c/d < 1$ using the result of Problem 36(b). (It also follows from Problem 36(b) that $ac/bd < a/b$.) Thus ac/bd is greater than 0 and less than 1, that is, ac/bd is a proper fraction.

13 PROB:38

a) $7/3$ since $(3/7) \times (7/3) = 21/21 = 1$

b) $(2\frac{1}{3} = \frac{7}{3})$ $3/7$ since $(7/3) \times (3/7) = 1$

(It also follows directly from part (a).)

c) The reciprocal of 1 is 1 since $1 \times 1 = 1$.

13 PROB:39

a) The reciprocal of 5 is $1/5$

b) The reciprocal of $1/5$ is 5

c) The reciprocal of $3\frac{2}{5}$ is $5/17$

d) The reciprocal of $2/11$ is $11/2$

e) The reciprocal of $22/7$ is $7/22$

f) The reciprocal of $1\frac{1}{2}$ is $2/3$

13 PROB: 40

0 or $0/1$ does not have a reciprocal since the product of 0 with any number is 0, not 1 .

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13 PROB:41

The reciprocal of a/b is b/a. The reciprocal of c/d is d/c.
Now a/b < c/d means ad < bc.
Therefore for the reciprocals we have b/a > d/c since bc > ad .
Example: 2/3 < 4/5 but 3/2 > 5/4. The reciprocal of the smaller 2/3 is greater than the reciprocal of the larger 4/5 .

15 PROB:42

a) $\frac{5/6}{1/2} = \frac{5}{6} \times \frac{2}{1} = 10/6$ (= 5/3)

b) $\frac{1\frac{1}{2}}{3} = \frac{3}{2} \times \frac{1}{3} = 3/6$ (= 1/2)

c) $\frac{2}{2\frac{1}{3}} = \frac{2}{1} \times \frac{3}{7} = 6/7$

d) $\frac{2\frac{2}{5}}{1\frac{3}{4}} = \frac{12}{5} \times \frac{4}{7} = \frac{48}{35}$

e) $2\frac{3}{7} \div 1\frac{1}{4} = \frac{17}{7} \times \frac{4}{5} = 68/35$

15 PROB:43

a) $\frac{3/10}{7/10} = 3/7$

b) $\frac{4}{2/3} = \frac{4(3)/1(3)}{2/3} = \frac{12/3}{2/3} = 12/2$ (= 6)

c) $\frac{3/5}{10} = \frac{3/5}{10(5)/1(5)} = \frac{3/5}{50/5} = 3/50$

d) $\frac{1\frac{1}{2}}{2\frac{1}{4}} = \frac{3(2)/2(2)}{9/4} = \frac{6/4}{9/4} = \frac{6}{9}$ ($= \frac{2}{3}$)

e) $3\frac{1}{3} \div 1\frac{1}{5} = 10(5)/3(5) \div 6(3)/5(3) = 50/15 \div 18/15 = 50/18 = 25/9$

15 PROB:44

a) $\frac{1/2}{3/2} = \frac{1}{2} \times \frac{2}{3} = \frac{2}{6} = \frac{1}{3}$ and $\frac{1}{3} < \frac{1}{2}$

b) $\frac{1/2}{1/3} = \frac{1}{2} \times \frac{3}{1} = \frac{3}{2}$ and $\frac{3}{2} > \frac{1}{2}$

c) $\frac{5/2}{3/2} = \frac{5}{2} \times \frac{2}{3} = \frac{10}{6} = \frac{5}{3}$ and $\frac{5}{3} < \frac{5}{2}$

d) $\frac{5/2}{1/3} = \frac{5}{2} \times \frac{3}{1} = \frac{15}{2}$ and $\frac{15}{2} > \frac{5}{2}$

15 PROB:45

a) If c/d > 1, then the quotient is less than a/b .

Check: $\frac{a/b}{c/d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$. $\frac{ad}{bc} < \frac{a}{b}$ means ad(b) < a(bc)
so d < c
and 1 < c/d .

b) If c/d < 1, then the quotient is greater than a/b .

Check: $\frac{a/b}{c/d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$. $\frac{ad}{bc} > \frac{a}{b}$ means ad(b) > a(bc)
so d > c
and 1 > c/d .

16 PROB:46

The sum for each row, column, and diagonal is 15.

16 PROB:47

Magic Square 1: Magic Number = $7\frac{1}{2}$

Magic Square 2: Magic Number = $2\frac{1}{2}$

Magic Square 3: Magic Number = $1\frac{1}{4}$

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16 PROB:48

$\frac{2}{3}$	$1\frac{5}{6}$	$1\frac{1}{2}$
$2\frac{1}{6}$	$1\frac{1}{3}$	$\frac{1}{2}$
$1\frac{1}{6}$	$\frac{5}{6}$	2

Magic Number: 4

$\frac{1}{2}$	$\frac{9}{4}$	$\frac{21}{12}$
$\frac{11}{4}$	$\frac{3}{2}$	$\frac{1}{4}$
$\frac{5}{4}$	$\frac{3}{4}$	$\frac{5}{2}$

Magic Number: $4\frac{1}{2}$

$\frac{4}{15}$	$\frac{1}{5}$	$\frac{8}{15}$
$\frac{3}{5}$	$\frac{1}{3}$	$\frac{1}{15}$
$\frac{2}{15}$	$\frac{7}{15}$	$\frac{2}{5}$

Magic Number: 1

17 PROB:49

a)

$\frac{1}{2}$	$\frac{7}{4}$	$\frac{3}{2}$
$\frac{9}{4}$	$\frac{5}{4}$	$\frac{1}{4}$
1	$\frac{3}{4}$	2

Magic Number: $\frac{15}{4}$

b)

$\frac{1}{3}$	$\frac{3}{4}$	$\frac{2}{3}$
$\frac{11}{12}$	$\frac{7}{12}$	$\frac{1}{4}$
$\frac{1}{2}$	$\frac{5}{12}$	$\frac{5}{6}$

Magic Number: $1\frac{3}{4}$

c)

$\frac{5}{6}$	$1\frac{2}{3}$	2
$2\frac{2}{3}$	$1\frac{1}{2}$	$\frac{1}{3}$
1	$1\frac{1}{3}$	$2\frac{1}{6}$

Magic Number: $4\frac{1}{2}$

d)

$\frac{8}{15}$	$1\frac{4}{15}$	$1\frac{1}{5}$
$1\frac{2}{3}$	1	$\frac{1}{3}$
$\frac{4}{5}$	$\frac{11}{15}$	$1\frac{7}{15}$

Magic Number: 3

e)

$\frac{1}{2}$	$1\frac{1}{3}$	$1\frac{1}{6}$
$1\frac{2}{3}$	1	$\frac{1}{3}$
$\frac{5}{6}$	$\frac{2}{3}$	$1\frac{1}{2}$

Magic Number: 3

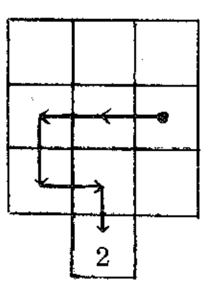
f)

$\frac{1}{3}$	$\frac{11}{12}$	$\frac{3}{4}$
$1\frac{1}{12}$	$\frac{2}{3}$	$\frac{1}{4}$
$\frac{7}{12}$	$\frac{5}{12}$	1

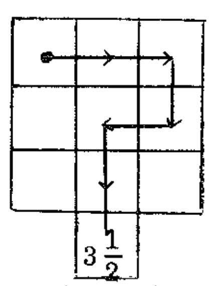
Magic Number: 2

18 PROB:50

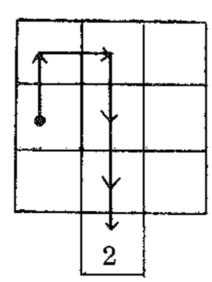
a)



b)



c)



a) Check: $1/3 + 1/6 + 3/4 + 1/2 + 1/4 = 4/12 + 2/12 + 9/12 + 6/12 + 3/12 = 24/12 = 2$
One procedure to solve an add-a-trail: add up all the numbers in the 9 squares.

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18 PROB:50 (a) (Continued)

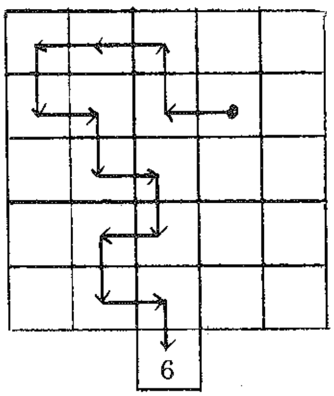
Thus, in Problem 50(a) the sum is $4\frac{1}{15}$. Subtract the 2 from the sum: $4\frac{1}{15} - 2 = 2\frac{1}{15}$. Now cross out those fractions that add up to $2\frac{1}{15}$. The path through the remaining fractions will solve the problem.

b) Use the procedure in (a).

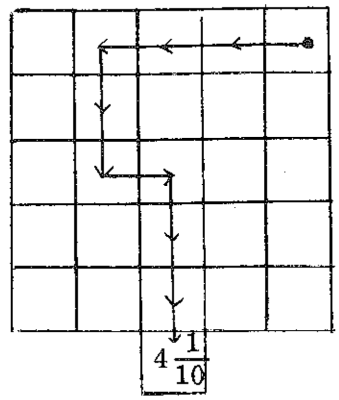
c) Use the procedure in (a).

18 PROB:51

a)



b)



a) Check: $5/8 + 1/2 + 3/8 + 2/4 + 1/4 + 3/4 + 1/4 + 3/8 + 1/4 + 3/4 + 1/8 + 1/2 + 3/4 = \frac{48}{8} = 6$

b) Check: $1/5 + 1/20 + 1/10 + 19/20 + 2/5 + 3/10 + 3/5 + 7/10 + 4/5 = 82/20 = 4\frac{2}{20} = 4\frac{1}{10}$

19 PROB:52

a) $4/9 + 1/3 + 1/6 = 8/18 + 6/18 + 3/18 = 17/18$

b) No

c) $(17/18) \times 18 = 17$

d) The sum of the fractions represents that portion of the new total, 18, which consists exactly of the original 17 camels.

20 PROB:53

Since we are looking for 2 or more positive integers whose sum is 11, 11 itself cannot be one of the numbers. Therefore 11/12 cannot be used since $(11/12) \times 12 = 11$.

21 PROB:54

a)

Numbers	Fractions
1, 8	1/10, 8/10 = 4/5
2, 7	2/10 = 1/5, 7/10
3, 6	3/10, 6/10 = 3/5
4, 5	4/10 = 2/5, 5/10 = 1/2

b)

Numbers	Fractions
1, 2, 6	1/10, 2/10 = 1/5, 6/10 = 3/5
1, 3, 5	1/10, 3/10, 5/10 = 1/2
2, 3, 4	2/10 = 1/5, 3/10, 4/10 = 2/5

c) There are no solutions involving 4 or more numbers since the smallest such sum is $1 + 2 + 3 + 4 = 10 > 9$.

21 PROB:55

a)

Numbers	Fractions	Numbers	Fractions	Numbers	Fractions
1, 12	1/14, 12/14 = 6/7	3, 10	3/14, 10/14 = 5/7	5, 8	5/14, 8/14 = 4/7
2, 11	2/14 = 1/7, 11/14	4, 9	4/14 = 2/7, 9/14	6, 7	6/14 = 3/7, 7/14 = 1/2

page

21 PROB: 55 (Continued)

b)

Numbers	Fractions	Numbers	Fractions
1, 2, 10	$\frac{1}{14}, \frac{2}{14} = \frac{1}{7}, \frac{10}{14} = \frac{5}{7}$	2, 3, 8	$\frac{2}{14} = \frac{1}{7}, \frac{3}{14}, \frac{8}{14} = \frac{4}{7}$
1, 3, 9	$\frac{1}{14}, \frac{3}{14}, \frac{9}{14}$	2, 4, 7	$\frac{2}{14} = \frac{1}{7}, \frac{4}{14} = \frac{2}{7}, \frac{7}{14} = \frac{1}{2}$
1, 4, 8	$\frac{1}{14}, \frac{4}{14} = \frac{2}{7}, \frac{8}{14} = \frac{4}{7}$	2, 5, 6	$\frac{2}{14} = \frac{1}{7}, \frac{5}{14}, \frac{6}{14} = \frac{3}{7}$
1, 5, 7	$\frac{1}{14}, \frac{5}{14}, \frac{7}{14} = \frac{1}{2}$	3, 4, 6	$\frac{3}{14}, \frac{4}{14} = \frac{2}{7}, \frac{6}{14} = \frac{3}{7}$

c)

Numbers	Fractions
1, 2, 3, 7	1/14, 2/14 = 1/7, 3/14, 7/14 = 1/2
1, 2, 4, 6	1/14, 2/14 = 1/7, 4/14 = 2/7, 6/14 = 3/7
1, 3, 4, 5	1/14, 3/14, 4/14 = 2/7, 5/14

d) There are no solutions involving 5 or more numbers since the smallest such sum is $1 + 2 + 3 + 4 + 5 = 15 > 13$.

22 PROB:56

a) There are 7 pairs whose sum is 15

1, 14	4, 11	
2, 13	5, 10	7, 8
3, 12	6, 9	

b) There are 12 triplets whose sum is 15 .

1, 2, 12	1, 6, 8	2, 6, 7
1, 3, 11	2, 3, 10	3, 4, 8
1, 4, 10	2, 4, 9	3, 5, 7
1, 5, 9	2, 5, 8	4, 5, 6

c) There are 6 quadruplets whose sum is 15 .

1, 2, 3, 9	1, 2, 5, 7	1, 3, 5, 6
1, 2, 4, 8	1, 3, 4, 7	2, 3, 4, 6

d) There is 1 quintuplet whose sum is 15 . 1, 2, 3, 4, 5

e) There are no solutions involving 6 or more numbers since the smallest such sum is $1 + 2 + 3 + 4 + 5 + 6 = 21 > 15$.

22 PROB:57

a) All the divisors of 10 are 1, 2, 5, 10.
Therefore, $1/1 + 1/2 + 1/5 + 1/10 = (1 + 2 + 5 + 10)/10 = 18/10 = 9/5$
Check: $1/1 + 1/2 + 1/5 + 1/10 = 10/10 + 5/10 + 2/10 + 1/10 = 18/10 = 9/5$

b) All the divisors of 49 are 1, 7, 49.
Therefore, $1/1 + 1/7 + 1/49 = (1 + 7 + 49)/49 = 57/49$
Check: $1/1 + 1/7 + 1/49 = 49/49 + 7/49 + 1/49 = 57/49$

c) All the divisors of 8 are 1, 2, 4, 8 .
Therefore, $1/1 + 1/2 + 1/4 + 1/8 = (1 + 2 + 4 + 8)/8 = 15/8$
Check: $1/1 + 1/2 + 1/4 + 1/8 = 8/8 + 4/8 + 2/8 + 1/8 = 15/8$

d) All the divisors of 16 are 1, 2, 4, 8, 16.
Therefore, $1/1 + 1/2 + 1/4 + 1/8 + 1/16 = (1 + 2 + 4 + 8 + 16)/16 = 31/16$
Check: $1/1 + 1/2 + 1/4 + 1/8 + 1/16 = 16/16 + 8/16 + 4/16 + 2/16 + 1/16 = 31/16$

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22 PROB:58

The number n whose divisors d_1, d_2, ... appear as denominators will serve as a common denominator for all the fractions. Now when the fraction $1/d_1$ is rewritten with denominator n, there is a number d_1' among the divisors of n such that $d_1 \times d_1' = n$ and so $1/d_1$ becomes d_1'/n. In this case each divisor appears as the numerator of exactly one fraction, and so the numerator of the sum is the sum of all the divisors of n.

Example: (Problem 57(a))

For 1/1 we have 1(10)/10 = 10/10 where $d_1 = 1$, $d_1' = 10$.

For 1/2 we have 1(5)/10 = 5/10 where $d_1 = 2$, $d_1' = 5$.

For 1/5 we have 1(2)/10 = 2/10 where $d_1 = 5$, $d_1' = 2$.

For 1/10 we have 1(1)/10 = 1/10 where $d_1 = 10$, $d_1' = 1$.

23 PROB:59

a) 1/2 + 1/3 + 1/6 + 1/9 = (2 + 3 + 6 + 9)/18 = 20/18 = 10/9

3 x 6 = 18

2 x 9 = 18

Check: 1/2 + 1/3 + 1/6 + 1/9
= 9/18 + 6/18 + 3/18 + 2/18
= (9 + 6 + 3 + 2)/18
= 20/18 = 10/9

b) 1/4 + 1/5 + 1/12 + 1/15 = (4 + 5 + 12 + 15)/60 = 36/60 = 3/5

5 x 12 = 60

4 x 15 = 60

Check: Use the procedure in (a).

c) 1/4 + 1/8 + 1/16 = (4 + 8 + 16)/64 = 28/64 = 7/16

$8^2 = 64$

4 x 16 = 64

Check: Use the procedure in (a).

d) 1/3 + 1/4 + 1/6 + 1/9 + 1/12 = (3 + 4 + 6 + 9 + 12)/36 = 34/36 = 17/18

$6^2 = 36$

4 x 9 = 36

3 x 12 = 36

Check: Use the procedure in (a).

24 PROB:60

a) $45 = 3^2 \times 5$ Therefore there are 2 blue cubes and 1 green cube.

Check: (2/3) x (2/3) x (3/5) = 12/45

b) $126 = 2 \times 3^2 \times 7$ Therefore there are 1 red cube, 2 blue cubes, and 1 yellow cube.

Check: (1/2) x (2/3) x (2/3) x (4/7) = 16/126

c) $150 = 2 \times 3 \times 5^2$ Therefore there are 1 red cube, 1 blue cube, and 2 green cubes.

Check: (1/2) x (2/3) x (3/5) x (3/5) = 18/150

d) $210 = 2 \times 3 \times 5 \times 7$ There are 1 red cube, 1 blue cube, 1 green cube, and 1 yellow cube.

Check: (1/2) x (2/3) x (3/5) x (4/7) = 24/210

e) $1225 = 5^2 \times 7^2$ Therefore there are 2 green cubes and 2 yellow cubes.

Check: (3/5) x (3/5) x (4/7) x (4/7) = 144/1225

25 PROB:61

a) (1/5 + 2/3)/2 = (3/15 + 10/15)/2 = (13/15)/2 = 13/30

b) $2\frac{1}{8}$ = 17/8, $3\frac{1}{3}$ = 10/3 ; (17/8 + 10/3)/2 = (51/24 + 80/24)/2 = (131/24)/2 = 131/48

page
25 PROB:61 (continued)

c) $2\frac{1}{6} = 13/6$; $(4/5 + 13/6)/2 = (24/30 + 65/30)/2 = (89/30)/2 = 89/60$

d) $1\frac{1}{2} = 3/2$; $(3/2 + 9/4)/2 = (6/4 + 9/4)/2 = (15/4)/2 = 15/8$

25 PROB:62

First we compute $(a/b + c/d)/2 = ((ad + bc)/bd)/2$ (Section 10(c), page 11)

$= (ad + bc)/2bd$

If $a/b < c/d$, then $ad < bc$.

$(ad + bc)/2bd - a/b = (ad + bc)/2bd - 2ad/2bd = (bc - ad)/2bd$

Note that $bc - ad > 0$ since $bc > ad$.

$c/d - (ad + bc)/2bd = 2bc/2bd - (ad + bc)/2bd = (bc - ad)/2bd$.

Since the 2 differences are the same, $(ad+bc)/2bd$ is midway between a/b and c/d .

25 PROB:63

a) $(1 + 2)/(5 + 3) = 3/8$
and $1/5 < 3/8 < 2/3$

b) $2\frac{1}{8} = 17/8$; $3\frac{1}{3} = 10/3$
$(17 + 10)/(8 + 3) = 27/11$
and $17/8 < 27/11 < 10/3$

c) $2\frac{1}{6} = 13/6$; $(4 + 13)/(5 + 6) = 17/11$
and $4/5 < 17/11 < 13/6$

d) $1\frac{1}{2} = 3/2$; $(3 + 9)/(2 + 4) = 12/6 = 2$
and $1\frac{1}{2} < 2 < 9/4$.

25 PROB:64

The answer to this problem is not unique; one possible solution is shown below.

$2/5 < 5/12 < 3/7$
$2/5 < 7/17 < 5/12 < 8/19 < 3/7$
$2/5 < 9/22 < 7/17 < 5/12 < 8/19 < 11/26 < 3/7$

25 PROB:65

From $a/b < c/d$ it follows that $ad < bc$.

$a/b < (a + c)/(b + d)$ if $a(b + d) < b(a + c)$ or $ab + ad < ab + bc$

Subtracting ab from both sides, $ad < bc$.
Since $ad < bc$ by assumption, it follows that the steps may be reversed, and so we conclude that $a/b < (a + c)/(b + d)$.

$(a + c)/(b + d) < c/d$ if $(a + c)d < (b + d)c$ or $ad + cd < bc + cd$

Subtracting cd from both sides, $ad < bc$.
Since $ad < bc$ by assumption, it follows that the steps may be reversed, and so we conclude that $(a + c)/(b + d) < c/d$.

26 PROB:66

a) $1/4 < 2/4 < 3/4$

In this case "Step 3" is unnecessary.

b) $2/5 < ?/10 < 9/10$

$4/10 < ?/10 < 9/10$

The fractions $5/10$ ($= 1/2$), $6/10$ ($= 3/5$), $7/10$, $8/10$ ($= 4/5$) are between $2/5$ and $9/10$. "Step 3" was unnecessary.

page

26 PROB: 66 (continued)

c) $1/2 < ?/4 < 3/4$
$2/4 < ?/4 < 3/4$
Multiply by 11
$22/44 < ?/44 < 33/44$
The 10 fractions are :
$23/44, \ldots, 32/44$

d) $5/8 < ?/8 < 3/4$
$5/8 < ?/8 < 6/8$
Multiply by 101
$505/808 < ?/808 < 606/808$
The 100 fractions are :
$506/808, 507/808, \ldots, 605/808$

26 PROB: 67

YES for both parts of this problem. Follow the procedure shown in Problem 66, (c) and (d).

a) The common denominator is 6 and the multiplier is 1,000,001. The 1,000,000 fractions are 2,000,003/6,000,006 , 2,000,004/6,000,006 , . . . , 3,000,002/6,000,006 .

b) The common denominator is 6 . To find N fractions, multiply by N + 1. The fractions would be: $(2N+3)/(6N+6)$, $(2N+4)/(6N+6)$, . . . , $(3N+1)/(6N+6)$, $(3N+2)/(6N+6)$.

26 PROB: 68

There is no fraction which is the immediate successor of $1/3$, for if a/b is any fraction satisfying $1/3 < a/b$, the methods discussed in Problem 67 show how to find any number of distinct fractions between $1/3$ and a/b .

26 PROB: 69

Start	a/b	c/d
Common denominator	ad/bd	cb/bd
Multiplier is 11	$11\,ad/11\,bd$	$11\,cb/11\,bd$

Since $a/b < c/d$, it follows that $ad < bc$ and so $11\,ad < 11\,bc$. In fact, since all these numbers are whole numbers, $bc - ad \geq 1$, and so $11\,bc - 11\,ad \geq 11$ or $11\,bc \geq 11\,ad + 11$. The fractions $(11\,ad + 1)/11bd$, $(11ad + 2)/11bd$, . . . , $(11ad+10)/11bd$ are between a/b and c/d . In fact there are $11bc - 11ad - 1$ fractions between a/b and c/d with denominator $11bd$. This number could be much larger than 10; it is at least 10 .

27 PROB: 70

Derivation of the formula :

Divide $b - a$ into n equal parts where $d = \dfrac{b-a}{n}$. Thus, $a + m\,d$, where $1 \leq m \leq n - 1$ is some place between a and b . Then we have $a < a + md < b$. Substitute for d and simplify the result .

$$a < a + m\,d < b \qquad \text{so that} \qquad a < a + m\left(\frac{b-a}{n}\right) < b$$

$$a < \frac{1}{n}\left((n-m)\,a + m\,b \right) < b \ .$$

a) $a = 1/5,\ b = 2/3,\ n = 3$ Since $1 \leq m \leq n - 1$, $m = 1, 2$.

From $a < 1/n\,((n - m)\,a + mb) < b$

we have $1/5 < 1/3\,((3 - m)(1/5) + m(2/3)) < 2/3$

or $1/5 < 1/5 + 7m/45 < 2/3$

$m = 1,\ 1/5 < 16/45 < 2/3$ $m = 2,\ 1/5 < 23/45 < 2/3$.

page

27 PROB:70 (Continued)

b) $a = 1/3,\ b = 3/4,\ n = 4$ Since $1 \le m \le 3,\ m = 1, 2, 3$

From $a < 1/n[(n - m)\ a + m\bar{b}] < b$

we have $1/3 < 1/4[(4 - m)(1/3) + m(3/4)] < 3/4$ or $1/3 < 1/3 + 5m/48 < 3/4$

$m = 1,\ 1/3 < 21/48\ (= 7/16) < 3/4$ $m = 2,\ 1/3 < 26/48\ (=13/24) < 3/4$

$m = 3,\ 1/3 < 31/48 < 3/4$.

27 PROB: 71

Using $n = 2,\ m = 1$, the formula becomes

$a/b < 1/2[(2 - 1)(a/b) + 1(c/d)] < c/d$ or $a/b < 1/2[a/b + c/d] < c/d$

which is the mean of the fractions $a/b,\ c/d$. (See Section 21, page 24.)

28 PROB:72

a) From $af < eb$, with $a = 1,\ b = 2,\ f = 16$ we get

$$1(16) < 2e$$

$e = 9, 10, 11, \ldots$

From $ed < cf$, with $c = 2,\ d = 3,\ f = 16$ we get

$$3e < 2(16)$$

$e = 0, 1, 2, \ldots, 9, 10$

The common values for e are 9, 10, and so the fractions are

$$1/2 < 9/16,\ 10/16 < 2/3\ .$$

b) From $af < eb$, with $a = 3,\ b = 7,\ f = 21$ we get

$$3(21) < 7e$$

$e = 10, 11, 12, \ldots$

From $ed < cf$, with $c = 1,\ d = 2,\ f = 21$ we get

$$2e < 1(21)$$

$e = 0, 1, 2, 3, \ldots, 9, 10$

The common value for e is 10. The fraction is $3/7 < 10/21 < 1/2$.

c) From $af < eb$, with $a = 3,\ b = 2,\ f = 15$ we get

$$3(15) < 2e$$

$e = 23, 24, 25, \ldots$

From $ed < cf$, with $c = 5,\ d = 3,\ f = 15$ we get

$$3e < 5(15)$$

$e = 0, 1, 2, \ldots, 23, 24$

The common values for e are 23, 24, and so the fractions are

$$3/2 < 23/15,\ 24/15 < 5/3$$

28 PROB:73

From $af < eb$, with $a = 1,\ b = 3,\ f = 4$ we get

$$1(4) < 3e$$

$e = 2, 3, 4, \ldots$

From $ed < cf$, with $c = 1,\ d = 2,\ f = 4$ we get

$$2e < 1(4)$$

$e = 0, 1$

There are no common values for e , and so there are no fractions with denominator 4 between $1/3$ and $1/2$. This result should be expected since $1/4 < 1/3$ and $2/4 = 1/2$.

30 PROB:74

a) $5/6 > 1/z$ so $5z > 6$. Therefore $z = 2$.

$5/6 = 1/2 + R$ where $R = 5/6 - 1/2 = (10 - 6)/12 = 4/12 = 1/3$

Thus $5/6 = 1/2 + 1/3$

page
30 PROB:74 (Continued)

b) $4/7 > 1/z$ so $4z > 7$. Therefore $z = 2$.

$4/7 = 1/2 + R$ where $R = 4/7 - 1/2 = (8 - 7)/14 = 1/14$. Thus $4/7 = 1/2 + 1/14$.

c) $5/9 > 1/z$ so $5z > 9$. Therefore $z = 2$.

$5/9 = 1/2 + R$ where $R = 5/9 - 1/2 = (10 - 9)/18 = 1/18$. Thus $5/9 = 1/2 + 1/18$.

d) $7/15 > 1/z$ so $7z > 15$. Therefore $z = 3$.

$7/15 = 1/3 + R$ where $R = 7/15 - 1/3 = (21 - 15)/45 = 6/45 = 2/15$

$2/15 > 1/z$ so $2z > 15$. Therefore $z = 8$.

$2/15 = 1/8 + R$ where $R = 2/15 - 1/8 = (16 - 15)/120 = 1/120$.

Thus $2/15 = 1/8 + 1/120$ and so $7/15 = 1/3 + 1/8 + 1/120$.

e) $13/21 > 1/z$ so $13z > 21$. Therefore $z = 2$.

$13/21 = 1/2 + R$ where $R = 13/21 - 1/2 = (26 - 21)/42 = 5/42$

$5/42 > 1/z$ so $5z > 42$. Therefore $z = 9$.

$5/42 = 1/9 + R$ where $R = 5/42 - 1/9 = (45 - 42)/378 = 3/378 = 1/126$

Thus $5/42 = 1/9 + 1/126$ and so $13/21 = 1/2 + 1/9 + 1/126$.

Note 1. For the special case $2/(2n + 1)$ there is a simple formula for the greatest unit fraction decomposition.
From $2/(2n + 1) > 1/z$ we get $2z > 2n + 1$. The smallest value of z for which this true is $z = n + 1$. Then

$$2/(2n + 1) = 1/(n + 1) + R \quad \text{where } R = 2/(2n + 1) - 1/(n + 1)$$
$$= (2n + 2 - 2n - 1)/(2n + 1)(n + 1) = 1/(2n+1)(n+1)$$

Thus $2/(2n + 1) = 1/(n + 1) + 1/(2n + 1)(n + 1)$

Example: $2/5$ Here $5 = 2 \times 2 + 1$ so $n = 2$.
$2/5 = 1/3 + 1/5(3) = 1/3 + 1/15$

This formula is the same as formula 1 on page 30.

Note 2. There is no one simple formula for the greatest unit fraction decomposition of a general p/q, but the process can be simplified.
Suppose $1 < p < q$, and q is not a multiple of p. Then by the division algorithm there are unique integers m, r such that

$$q = mp + r \ , \quad 1 \le r \le p - 1$$

(Since q is not a multiple of p, r cannot be 0.)
Then $(m + 1)$ is the first z in the greatest unit fraction decomposition, and

$$p/q = 1/(m + 1) + (p - r)/(m + 1)q \ .$$

Note that $1 \le p - r \le p - 1$. If $(p - r)/(m + 1)q$ cannot be rewritten as a unit fraction, repeat the process. Since the numerators are decreasing, we must get a unit fraction after at most p steps.

Example: $13/21$ $\quad 21 = 1(13) + 8$ so $m = 1$, $r = 8$ and
$13/21 = 1/2 + (13 - 8)/42 = 1/2 + 5/42$.
Repeat with $5/42$. $\quad 42 = 8(5) + 2$ so $m = 8$, $r = 2$ and
$5/42 = 1/9 + (5 - 2)/9(42) = 1/9 + 3/378 = 1/9 + 1/126$.

Therefore $13/21 = 1/2 + 1/9 + 1/126$. (See Problem 74(e).)

32 PROB:75

a) We illustrate the greatest unit fraction method for $2/5$. The other fractions may be done by the same procedure.

$2/5 > 1/z$ so $2z > 5$. Therefore $z = 3$.

32 PROB:75 (Continued)

$2/5 = 1/3 + R$ where $R = 2/5 - 1/3 = (6-5)/15 = 1/15$

Thus $2/5 = 1/3 + 1/15$. This answer agrees with the decomposition in the table.

The greatest unit fraction method gives an answer which agrees with the table for the following problems: 1, 2, 4, 10 .

b) Formula 1 gives the greatest unit fraction decomposition for the case $2/n$, n odd. (See Note 1 following Problem 74 in the answers.) By part (a) above Formula 1 agrees with the table for problems 1, 2, 4, 10 .

We illustrate Formula 2 for 2/5.

$5 = 1 \times 5$ Let $p = 1$, $q = 5$.

$2/5 = 1/1((1+5)/2) + 1/5((1+5)/2) = 1/3 + 1/15$
This answer agrees with the decomposition in the table.

Formula 2 agrees with the table in the following problems for the given values of p and q.

1	($p = 1$, $q = 5$)	2	($p = 1$, $q = 7$)	4	($p = 1$, $q = 11$)
10	($p = 1$, $q = 23$)	16	($p = 5$, $q = 7$)	44	($p = 7$, $q = 13$)

Note that Formula 2 includes Formula 1 since for any odd number n we have $n = 1 \times n$, and so we can take $p = 1$ and $q = n$. Then

$2/n = 1/1((1+n)/2) + 1/n((n+1)/2)$, which is just Formula 1.

The reason that Formula 2 agrees with the table in some additional problems is that there may be other choices for p, q which do yield the desired fractions. For example in (16), 2/35, the choice $p = 1$, $q = 35$ does not agree with the table, but $p = 5$, $q = 7$ does.

We illustrate Formula 3 for 2/9.

Since $9 = 3 \times 3$, $p = 3$ and so

$2/9 = 1/2(3) + 1/6(3) = 1/6 + 1/18$.

Formula 3 gives the decomposition in the table for all fractions of the form $2/3p$. The numbers of the problems and the values for p are

3	($p = 3$)	6	($p = 5$)	9	($p = 7$)	12	($p = 9$)
15	($p = 11$)	18	($p = 13$)	21	($p = 15$)	24	($p = 17$)
27	($p = 19$)	30	($p = 21$)	33	($p = 23$)	36	($p = 25$)
39	($p = 27$)	42	($p = 29$)	45	($p = 31$)	48	($p = 33$)

c) The decompositions in the table for the following problems do not agree with the results obtained by any of the formulas above.

5,	7,	8,	11,	13,	14,	17,	19,	20,	22,	23,
25,	26,	28,	29,	31,	32,	34,	35,	37,	38,	40,
41,	43,	46,	47,	49 .						

33 PROB:76

a) The set of divisors of 21 is $D_{21} = \{ 1, 3, 7, 21 \}$.

The set of pairs of relatively prime divisors is $\{ (1,3), (1,7), (1,21), (3,7) \}$.
(1) for (1,3), $P = 1$, $Q = 3$. Thus, $P' = 21$, $Q' = 7$, $k = (1+3)/2 = 2$.
From (A), page 33, $2/21 = 1/2(21) + 1/2(7) = 1/42 + 1/14$.

(2) for (1,7), $P = 1$, $Q = 7$. Thus, $P' = 21$, $Q' = 3$, $k = (1+7)/2 = 4$;
From (A), page 33, $2/21 = 1/4(21) + 1/4(3) = 1/84 + 1/12$.

(3) for (1,21), $P = 1$, $Q = 21$. Thus, $P' = 21$, $Q' = 1$, $k = (1+21)/2 = 11$.
From (A), page 33, $2/21 = 1/11(21) + 1/11(1) = 1/231 + 1/11$.

page

33 PROB:76 (Continued)

(4) for (3,7), P = 3, Q = 7. Thus, P' = 7, Q' = 3, k = (3 + 7)/2 = 5 .
From (A), page 33, 2/21 = 1/5(7) + 1/5(3) = 1/35 + 1/15 .

34 b) The set of divisors of 33 is $D_{33} = \{ 1, 3, 11, 33 \}$.

The set of pairs of relatively prime divisors is { (1,3), (1,11), (1,33), (3, 11) }.
(1) for (1,3), P = 1, Q = 3. Thus, P' = 33, Q' = 11, k = (1 + 3)/2 = 2 .
From (A), page 33, 2/33 = 1/2(33) + 1/2(11) = 1/66 + 1/22 .

(2) for (1,11), P = 1, Q = 11. Thus, P' = 33, Q' = 3, k = (1 + 11)/2 = 6 .
From (A), page 33, 2/33 = 1/6(33) + 1/6(3) = 1/198 + 1/18 .

(3) for (1,33), P = 1, Q = 33. Thus, P' = 33, Q' = 1, k = (1 + 33)/2 = 17 .
From (A), page 33, 2/33 = 1/17(33) + 1/17(1) = 1/561 + 1/17 .

(4) for (3,11), P = 3, Q = 11. Thus, P' = 11, Q' = 3, k = (3 + 11)/2 = 7 .
From (A), page 33, 2/33 = 1/7(11) + 1/7(3) = 1/77 + 1/21 .

c) The set of divisors of 35 is $D_{35} = \{ 1, 5, 7, 35 \}$.

The set of pairs of relatively prime divisors is { (1,5), (1,7), (1,35), (5,7) } .
(1) for (1,5), P = 1, Q = 5. Thus, P' = 35, Q' = 7, k = (1 + 5)/2 = 3 .
From (A), page 33, 2/35 = 1/3(35) + 1/3(7) = 1/105 + 1/21 .

(2) for (1,7), P = 1, Q = 7. Thus, P' = 35, Q' = 5, k = (1 + 7)/2 = 4 .
From (A), page 33, 2/35 = 1/4(35) + 1/4(5) = 1/140 + 1/20 .

(3) for (1,35), P = 1, Q = 35. Thus, P' = 35, Q' = 1, k = (1 + 35)/2 = 18 .
From (A), page 33, 2/35 = 1/18(35) + 1/18(1) = 1/630 + 1/18 .

(4) for (5,7), P = 5, Q = 7. Thus, P' = 7, Q' = 5, k = (5 + 7)/2 = 6 .
From (A), page 33, 2/35 = 1/6(7) + 1/6(5) = 1/42 + 1/30 .

35 PROB:77

a) Is not. 1 in B is matched with 2 elements, a, c, in A.

b) Is not. c in C is not matched with any element in D.

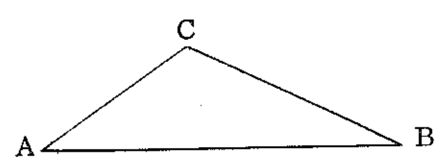
c) Is . d) Is not. 2 in H is not matched with any element in G .

35 PROB:78

Yes. The important word here is <u>can</u>. For finite sets there is no problem if the sets have the same number of elements. However, infinite sets may have the same number of elements , but it takes a little thinking to put the elements of the two sets into one-to-one correspondence (see pages 37 and following).

36 PROB:79

Yes. A is matched with side $\overline{BC}$.
B is matched with side $\overline{AC}$.
C is matched with side $\overline{AB}$.



This matching pairs each vertex with the side opposite the vertex.

36 PROB:80

No. Each state has two senators, and so there is no matching which can pair each state with exactly one senator and each senator with exactly one state. In any matching there must be states matched to more than one senator or senators not matched with any state (or both). (In the matchings discussed, starting on page 37, we shall see that when the sets are infinite the argument given here would not be correct.)

page

36 PROB:81

Yes.

$$\begin{array}{cccccccc} 0 & 1 & 2 & 3 & \ldots & n & \ldots & \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & & \\ 0 & 1 & 4 & 9 & \ldots & n^2 & \ldots & . \end{array}$$

Each natural number n is matched to its own square, and each perfect square is matched to its (non-negative) square root.

The set B of the squares of all the natural numbers <u>is</u> equivalent to the set A of all the natural numbers.

The set B <u>does</u> have the same number of members as set A .

36 PROB:82

a) 1 way $\quad a \longleftrightarrow 1 \quad$ (Since $n! = 1 \times 2 \times 3 \times \ldots \times n$, we can also say $1!$ ways. Read: $n!$ as "n factorial")

b) $2 = 2!$ ways

$$\begin{array}{l} a \longleftrightarrow 1 \\ b \longleftrightarrow 2 \end{array} \qquad \begin{array}{l} a \longleftrightarrow 2 \\ b \longleftrightarrow 1 \end{array}$$

c) $6 = 3!$ ways $\quad$ Procedure: keep a, b, c in their order and permute 1, 2, 3. Since there are 6 permutations of 3 different numbers taken 3 at a time, there are 6 ways to perform the matching.

$$\begin{array}{l} a \longleftrightarrow 1 \\ b \longleftrightarrow 2 \\ c \longleftrightarrow 3 \end{array} \quad \begin{array}{l} a \longleftrightarrow 1 \\ b \longleftrightarrow 3 \\ c \longleftrightarrow 2 \end{array} \quad \begin{array}{l} a \longleftrightarrow 2 \\ b \longleftrightarrow 3 \\ c \longleftrightarrow 1 \end{array} \quad \begin{array}{l} a \longleftrightarrow 2 \\ b \longleftrightarrow 1 \\ c \longleftrightarrow 3 \end{array} \quad \begin{array}{l} a \longleftrightarrow 3 \\ b \longleftrightarrow 1 \\ c \longleftrightarrow 2 \end{array} \quad \begin{array}{l} a \longleftrightarrow 3 \\ b \longleftrightarrow 2 \\ c \longleftrightarrow 1 \end{array}$$

d) $4! = 24$ ways. See part (c) above. $\qquad$ e) $n!$ ways. See part (c) above.

40 PROB:83

a) 2/2 $\quad$ Here $a = 2$, $b = 2$. $\quad$ From $k = a + b - 2$, we have $k = 2 + 2 - 2 = 2$. Since $k = 2$ is even, we have from (B), page 39, Position of $2/2 = T_2 + a + 1 = 3 + 2 + 1 = 6$.

b) 3/2 $\quad$ Here $a = 3$, $b = 2$. $\quad$ From $k = a + b - 2$, we have $k = 3 + 2 - 2 = 3$. Since $k = 3$ is odd, we have from (C), page 39, Position of $3/2 = T_3 + b + 1 = 6 + 2 + 1 = 9$.

c) 1/4 $\quad$ Here $a = 1$, $b = 4$. $\quad$ From $k = a + b - 2$, we have $k = 1 + 4 - 2 = 3$. Since $k = 3$ is odd, we have from (C), page 39, Position of $1/4 = T_3 + b + 1 = 6 + 4 + 1 = 11$.

d) 19/20 $\quad$ Here $a = 19$, $b = 20$. $\quad$ From $k = a + b - 2$, we have $k = 19 + 20 - 2 = 37$. From page 39, $T_k = k(k+1)/2$, so that $T_{37} = 37(38)/2 = 703$.
Since $k = 37$ is odd, we have from (C), page 39, Position of $19/20 = T_{37} + b + 1 = 703 + 20 + 1 = 724$.

e) 99/100 $\quad$ Here $a = 99$, $b = 100$. $\quad$ From $k = a + b - 2$, we have $k = 99 + 100 - 2 = 197$. From page 39, $T_k = k(k+1)/2$, so that $T_{197} = 197(198)/2 = 19604$.
Since $k = 197$ is odd, we have from (C), page 39, Position of $99/100 = T_{197} + b + 1 = 19503 + 100 + 1 = 19604$.

40 PROB:84

To illustrate the procedure we first find which fraction is the 10th in the list (A), page 38.

Let a/b be the 10th fraction in (A), page 38. Make T_k strictly less than $(10 - 1) = 9$.

The (-1) comes from the fact that $0/1$ is not given by the formulas (B), (C), page 39. (See Note, page 64.)

page
40 PROB:84 (Continued)

Now $T_3 = 6 < 9$ and is the largest T_k for which this is true. Therefore, $k = 3$.

Since $k = 3$ is odd, we have from (C), page 39, Position of $a/b = T_k + b + 1$

$$10 = 6 + b + 1 \quad \text{or} \quad 3 = b.$$

Now, from $k = a + b - 2$, page 39, we get $3 = a + 3 - 2$ or $a = 2$.
The 10 th fraction in the list (A), page 38, is $2/3$.

We now solve the problem 84.

Let a/b be the 100 th fraction in (A), page 38. Make $T_k < (100 - 1) = 99$ so that $T_{13} = 91$ and $k = 13$. Since $k = 13$ is odd, we have from (C), page 39,
Position of $a/b = T_k + b + 1$ or $100 = 91 + b + 1$ or $b = 8$.

From $k = a + b - 2$, page 39, we get $13 = a + 8 - 2$ or $a = 7$.
The 100 th fraction in list (A), page 38, is $7/8$.

Note: The reason for taking $T_k < (100 - 1) = 99$ rather than $T_k < 100$ is due to the special position of $0/1$ in the list (A), page 38.
In Chart 2, page 38, after $0/1$, the fractions are listed by diagonals. Thus, after $0/1$ we have $1/1$ in 1 st diagonal
$2/1$, $1/2$ in 2 nd diagonal
and so on.
The number of fractions in the successive diagonals is thus 1, 2, 3, ... and so on. Thus the first k diagonals contain $1 + 2 + 3 + \ldots + k = T_k = k(k+1)/2$ fractions.

To find the position of a/b we need the number of complete diagonals already counted (T_k), the position of a/b within its own diagonal (given by either a or b, depending on whether $a + b$ is even or odd), and an additional 1 to account for $0/1$. When reversing the procedure, we must subtract the 1 at the beginning and then find T_k.

40 PROB:85

To find the nth ($n \neq 1$) fraction, (see Problem 84)

a) Find $T_k < n - 1$

b) If k is even, the position of a/b is given by $n = T_k + a + 1$

so that $\boxed{a = (n - 1) - T_k}$

and from $k = a + b - 2$

we have $b = (k - a) + 2$.

Substituting $b = k - (n - 1) + T_k + 2$

or $\boxed{b = T_k + k + 3 - n}$

Example 1 Find the 5 th fraction in (A) page 38.

Solution: Here $n = 5$. From $T_k < (n - 1)$, we have $T_k < 5 - 1 = 4$ so that $T_k = 3$ and $k = 2$. From above,

$$a = (n - 1) - T_k = 5 - 1 - 3 = 1$$

and

$$b = T_k + k + 3 - n = 3 + 2 + 3 - 5 = 3$$

Therefore, the 5 th fraction in (A), page 38, is $1/3$.

c) If k is odd, the position of a/b is given by $n = T_k + b + 1$

so that $\boxed{b = (n - 1) - T_k}$

page

40 PROB:85 (Continued)

and from $k = a + b - 2$

we have $a = (k - b) + 2$.

Substituting, $a = k - (n - 1) + T_k + 2$

or $\boxed{a = T_k + k + 3 - n}$

Example 2 Find the 9th fraction in (A), page 38 .

Solution: Here $n = 9$. From $T_k < (n - 1)$, we have $T_k < (9 - 1) = 8$ so that $T_k = 6$ and $k = 3$. From above,

$$b = (n - 1) - T_k = 9 - 1 - 6 = 2$$

and

$$a = T_k + k + 3 - n = 6 + 3 + 3 - 9 = 3 .$$

Therefore, the 9th fraction in (A), page 38, is 3/2 .

41 PROB:86

a) 3/2 Here $a = 3$, $b = 2$ From $k = a + b - 2$, we have $k = 3 + 2 - 2 = 3$.
Thus, $T_k = T_3 = 6$, and
Position of $3/2 = T_3 + a + 1 = 6 + 3 + 1 = 10$.

b) 1/4 Here $a = 1$, $b = 4$ From $k = a + b - 2$, we have $k = 1 + 4 - 2 = 3$.
Thus, $T_k = T_3 = 6$, and
Position of $1/4 = T_3 + a + 1 = 6 + 1 + 1 = 8$.

c) 7/9 Here $a = 7$, $b = 9$ From $k = a + b - 2$, we have $k = 7 + 9 - 2 = 14$.
$T_k = k(k+1)/2$, so that $T_{14} = 14(15)/2 = 105$.
Position of $7/9 = T_{14} + a + 1 = 105 + 7 + 1 = 113$.

d) 19/20 Here $a = 19$, $b = 20$ From $k = a + b - 2$, we have $k = 19 + 20 - 2 = 37$.
$T_k = k(k+1)/2$, so that $T_{37} = 37(38)/2 = 703$.
Position of $19/20 = T_{37} + a + 1 = 703 + 19 + 1 = 723$.

e) 99/100 Here $a = 99$, $b = 100$ From $k = a + b - 2$, we have $k = 99 + 100 - 2 = 197$.
$T_k = k(k+1)/2$, so that $T_{197} = 197(198)/2 = 19\,503$.
Position of $99/100 = T_{197} + a + 1 = 19\,503 + 99 + 1 = 19\,603$.

41 PROB:87

To illustrate the procedure, we first find which fraction is the 10th in the list (D), page 41. (See the answer to Problem 84, pages 63-64.)

Let a/b be the 10th fraction in (D), page 41 .

a) Make T_k strictly less than $(10 - 1) = 9$.

b) Here $T_k = T_3 = 6 < 9$, and so $k = 3$.

c) From Position of $a/b = T_k + a + 1$, page 41, we have $10 = T_3 + a + 1$
or $a = 10 - T_3 - 1 = 10 - 6 - 1 = 3$.

From $k = a + b - 2$, page 41, we get $3 = 3 + b - 2$
or $b = 2$.

Thus, 3/2 is the 10th fraction in the list (D), page 41 .

Now find the 100th fraction in the list (D), page 41.

a) $T_k < (100 - 1) = 99$, so that $T_{13} = 91$ and $k = 13$.

page

41 PROB:87 (Continued)

b) Position of $a/b = T_k + a + 1$, page 41

$$100 = 91 + a + 1 \quad \text{or} \quad a = 8 .$$

c) From $k = a + b - 2$, page 41

$$13 = 8 + b - 2 \quad \text{or} \quad b = 7 .$$

Thus, 8/7 is the 100th fraction in list (D), page 41 .

42 PROB:88

For the general case of n ($n \neq 1$) follow the procedure in Problem 87.

a) Let a/b ($a/b \neq 0/1$) be the nth fraction in list (D), page 41.

b) Make $T_k < (n - 1)$, and get k .

c) From Position of $a/b = T_k + a + 1$, page 41

$$n = T_k + a + 1$$

or $$\boxed{a = (n - 1) - T_k}$$

d) From $k = a + b - 2$, page 41

we have $b = k - a + 2$.

Substituting for a, $$\boxed{b = T_k + k + 3 - n}$$

Example 1 Find the 10th fraction in the list (D), page 41.

Solution: $T_k < (10 - 1) = 9$ so that $T_3 = 6 < 9$, and so $k = 3$

From above, $a = (n - 1) - T_3 = 10 - 1 - 6 = 3$

$b = T_k + k + 3 - n = 6 + 3 + 3 - 10 = 2$.

Thus, 3/2 is the 10th fraction in the list (D), page 41 .

42 PROB:89

a) 0/1 is in the same position in both the listings (A), page 38, and (D), page 41 .

b) For $a/b \neq 0/1$, if $k = a + b - 2$ is EVEN, then

Position of $a/b = T_k + a + 1$, from (B), page 39, and from (E), page 41 .

means that the fraction a/b is in the same position in list (A), page 38, and list (D), page 41.

c) For $a/b \neq 0/1$, if $k = a + b - 2$ is ODD, we must have $a \neq b$ (otherwise we have $a + b - 2 = 2a - 2$, which is even), and

Position of $a/b = T_k + b + 1$, from (C), page 39

means a/b will be in a different position in the listings (A), page 38, and (D), page 41.

44 PROB:90

a) 2/3 Here $a = 2$, $b = 3$ From $k = a + b - 2$, we get $k = 2 + 3 - 2 = 3$. Since $k = 3$ is odd, then from (H), page 43, and from $T_k = T_3 = 6$, we have

Position of $2/3 = T_3 + a + 1 = 6 + 2 + 1 = 9$

b) 1/4 Here $a = 1$, $b = 4$ From $k = a + b - 2$, we get $k = 1 + 4 - 2 = 3$. Since $k = 3$ is odd, then from (H), page 43, and

page

44 PROB:90 (Continued)

from $T_k = T_3 = 6$, we have

Position of $1/4 = T_3 + a + 1 = 6 + 1 + 1 = 8$

c) 7/9 Here $a = 7$, $b = 9$

From $k = a + b - 2$, we get $k = 7 + 9 - 2 = 14$.
Since $k = 14$ is even, then from (G), page 43, and from $T_k = k(k+1)/2$, we have $T_{14} = 14(15)/2 = 105$.

Position of $7/9 = T_{14} + b + 1 = 105 + 9 + 1 = 115$.

d) 19/20 Here $a = 19$, $b = 20$

From $k = a + b - 2$, we get $k = 19 + 20 - 2 = 37$.

Since $k = 37$ is odd, then from (H), page 43, and from $T_k = k(k+1)/2$, we have $T_{37} = 37(38)/2 = 703$.

Position of $19/20 = T_{37} + a + 1 = 703 + 19 + 1 = 723$.

e) 99/100 Here $a = 99$, $b = 100$

From $k = a + b - 2$, we get $k = 99 + 100 - 2 = 197$.
Since $k = 197$ is odd, then from (H), page 43, and from $T_k = k(k+1)/2$, we get $T_{197} = 197(198)/2 = 19\,503$.

Position of $99/100 = T_{197} + a + 1 = 19\,503 + 99 + 1 = 19\,603$.

44 PROB:91

To illustrate the procedure, we first find which fraction is the 10th in the list (F), page 43. (See the answers to Problems 84, pages 63-64, and 87, pages 65-66.)

a) Make T_k strictly less than $(10 - 1) = 9$.

b) Here, $T_k = T_3 = 6 < 9$, and $k = 3$.

c) Since $k = 3$ is odd, we use (H), page 43. If a/b is the 10th fraction then from

$$\text{Position of } a/b = T_k + a + 1$$

we have

$$10 = T_3 + a + 1 \quad \text{or} \quad a = 10 - T_3 - 1 = 10 - 6 - 1 = 3.$$

d) From

$$k = a + b - 2$$

we have

$$3 = 3 + b - 2 \quad \text{or} \quad b = 2.$$

Thus, the 10th fraction in list (F), page 43, is 3/2.

Now to find the 100th fraction in list (F), page 43, follow the procedure above.

a) $T_k = T_{13} = 91 < (100 - 1) = 99$, and so $k = 13$.

b) Since $k = 13$ is odd, we use (H), page 43. From

$$\text{Position of } a/b = T_k + a + 1$$

we have

$$100 = 91 + a + 1 \quad \text{or} \quad a = 8.$$

c) From

$$k = a + b - 2$$

we have

$$13 = 8 + b - 2 \quad \text{or} \quad b = 7.$$

The 100th fraction in list (F), page 43, is 8/7.

44 PROB:92

For the general case of n ($n \neq 1$), follow the procedure in Problem 91.

a) Let a/b ($a/b \neq 0/1$) be the nth fraction in list (F), page 43.

b) Make $T_k < (n - 1)$, and also get k.

c) If k is EVEN, then from (G), page 43,

$$\text{Position of } a/b = T_k + b + 1 \quad \text{and so} \quad n = T_k + b + 1$$

$$\text{or} \quad \boxed{b = (n - 1) - T_k}$$

page
44 PROB:92 (Continued)

From $k = a + b - 2$ we have $k = a + (n-1) - T_k - 2$

or $\boxed{a = T_k + k + 3 - n}$

Example 1 Find the 5th fraction in the list (F), page 43.

Solution: a) $T_k = T_2 = 3 < (5-1) = 4$, and so $k = 2$.

b) From above $b = (n-1) - T_k = (5-1) - 3 = 1$.

c) From above $a = T_k + k + 3 - n = 3 + 2 + 3 - 5 = 3$.

Thus, the 5th fraction in list (F), page 43, is 3/1.

d) If k is ODD, then from (H), page 43,

Position of $a/b = T_k + a + 1$ and so $n = T_k + a + 1$

or $\boxed{a = (n-1) - T_k}$

From $k = a + b - 2$ we have $k = (n-1) - T_k + b - 2$

or $\boxed{b = T_k + k + 3 - n}$

Example 2 Find the 11th fraction in the list (F), page 43.

Solution: a) $T_k = T_3 = 6 < (11-1) = 10$, and so $k = 3$. Note: $T_4 = 10$ and so would not be strictly less than 10.

b) From above $a = (n-1) - T_k$
$= (11-1) - 6 = 4$.

c) From above $b = T_k + k + 3 - n = 6 + 3 + 3 - 11 = 1$

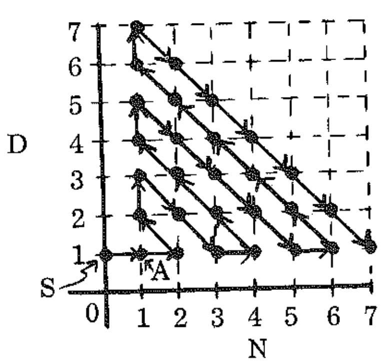
Thus, the 11th fraction in list (F), page 43, is 4/1.

44 PROB:93

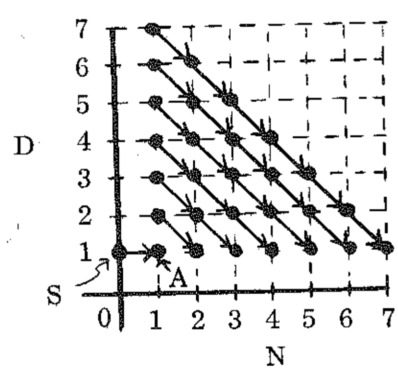
a) 0/1 is in the same position in both the listings (A), page 38, and (F), page 43. For $a/b \neq 0/1$, we have that (B), page 39, will be the same as (G), page 43, if $a = b$. The same is true for (C), page 39 and (H), page 43. But $a = b$ means that the only fractions that are in the same positions are 1/1, 2/2, 3/3, ..., q/q, ...

b) 0/1 is in the same position in both the listings (D), page 41, and (F), page 43. For $a/b \neq 0/1$, the fractions will be in the same positions if $k = a + b - 2$ is odd. Also, if $a = b$ the fractions will be in the same positions in list (D), page 41, and list (F), page 43. (See part (a), above.)

44 PROB:94



This is the listing (A), page 38, shown graphically.
Start at S = 0/1.
Go to A = 1/1.
Follow the arrows to get
0/1, 1/1, 2/1, 1/2, 1/3, 2/2, 3/1, ...



This is the listing (D), page 41, shown graphically. At the end of each diagonal, go to the "top" of the next higher diagonal and follow the arrows. Start at S = 0/1.
Go to A = 1/1. 1 st diagonal 1/2, 2/1
2 nd diagonal 1/3, 2/2, 3/1
Thus, 0/1, 1/1, 1/2, 2/1, 1/3, 2/2, ...

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