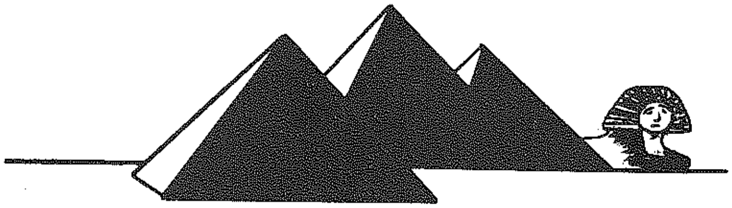
BOSTON COLLEGE MATHEMATICS INSTITUTE

Motivated Math Project Activity

Booklet 5

Fraction Action



> " Complete and thorough study of
> all things, insight into all that
> exists, knowledge of all secrets,
> all mysteries. . . "
>
> Ahmes
> Rhind Papyrus

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PREFACE

The series of MOTIVATED MATH PROJECT ACTIVITY BOOKLETS has been written for students and teachers in elementary and secondary schools. Each Booklet treats a topic generally found in the school curriculum or material that is interesting and motivational which may or may not be included in the usual class room textbook . Some topics are treated in two Booklets: one on an elementary level suitable for the middle grades and the other on an advanced level appropriate for junior and senior high schools.

The Booklets can be used in a variety of ways.

Because each Booklet treats a single topic, it is a handy summary and resource unit which can be expanded by both the student and the teacher.

Many of the Booklets, because they summarize and organize a topic in detail, can be used as mini-course modules to supplement standard class instruction or for individualized study. They also provide an invaluable review of a topic and can serve as a criterion for what has been covered on the topic .

The Booklets, unlike a textbook in which a topic may be treated in several nonconsecutive chapters, provide a convenient and readily accessible reference source. The material on a topic can be quickly and easily found .

Each Booklet contains problems which not only reenforce the class room instruction but also provide motivation, interest and challenge . Many problems are open-ended so that all students can achieve some measure of success. These problems are suitable not only for the routine pencil and paper activity but also may be extended by the use of hand electronic calculators or programmed on a computer .

Each Booklet contains solutions to the problems and in many instances comments, explanations and derivations of the key formulas and algorithms . The Booklets are relatively independent of each other and may be studied in any sequence depending on the background and personal preference of the student .

FRACTION ACTION

The more you know about fractions, the more
you will enjoy Fraction Action .

SOMETHING OLD

1. Common Fraction

a. Here is the set of natural numbers : $N = \{ 0, 1, 2, 3, 4, 5, \ldots \}$.

b. These are examples of common fractions : $\frac{2}{3}, \frac{5}{2}, \frac{0}{7}$ or 2/3 , 5/2 , 0/7 .

c. These numbers and numbers like them can be written as common fractions :

$$\begin{array}{cccc} 0, & 3, & 1\frac{1}{2}, & 7\frac{2}{3} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \frac{0}{1}, & \frac{3}{1}, & \frac{3}{2}, & \frac{23}{3} \end{array} .$$

2. Mixed numbers

$1\frac{1}{2}$, $7\frac{2}{3}$ and so on, are often called 'mixed numbers' .

To write a mixed number as a fraction, you may use the methods shown below .

a. $1\frac{1}{2} = 1 + \frac{1}{2} = \frac{2}{2} + \frac{1}{2} = \frac{2 + 1}{2} = \frac{3}{2}$.

b. $\underset{x}{\overset{+}{1\frac{1}{2}}} = \frac{2(1) + 1}{2} = \frac{3}{2}$.

3. Numerator and denominator

$$\text{Given a common fraction} \quad \begin{array}{c} 2 \\ \hline 3 \end{array} \quad \begin{array}{l} \text{is called the } \underline{\text{numerator}} \\ \text{is called the } \underline{\text{denominator}} \end{array} .$$

The numerator and denominator of a common fraction are natural numbers except that the denominator of a common fraction cannot be zero .

4. Proper, improper common fractions

a. A fraction whose numerator is greater than zero and whose denominator is greater than the numerator is called a proper fraction .

$\frac{1}{2}, \frac{2}{3}, \frac{7}{8}$ are examples of proper fractions .

b. A fraction which is not a proper fraction is called an improper fraction .

$\frac{0}{1}, \frac{2}{2}, \frac{5}{4}$ are examples of improper fractions .

c. We will often refer to 'common fractions' simply as 'fractions' .

PROBLEMS

Represent each of the following as common fractions .

1. $2 =$ ______
2. $3\frac{1}{4} =$ ______
3. $4\frac{1}{3} =$ ______
4. $2\frac{5}{6} =$ ______
5. $6 =$ ______
6. $0 =$ ______
7. Write all the proper fractions which have 5 as a denominator .

____

5. Equal fractions by multiplication and division .

a. Multiplication

If you multiply both the numerator and denominator of a fraction by any number except zero you get a fraction equal to (equivalent to) the original fraction .

$$\frac{2}{3} = \frac{(4)\,2}{(4)\,3} = \frac{8}{12}$$, multiplying by 4

Thus, $\frac{2}{3} = \frac{8}{12}$.

b. Division

A common divisor of two numbers is a number that divides both numbers exactly. (Zero is never a common divisor .)

If you divide both the numerator and the denominator of a fraction by a common divisor, you get a fraction equal to (equivalent to) the original fraction .

$$\frac{12}{18} = \frac{12 \div 2}{18 \div 2} = \frac{6}{9}$$, dividing by 2

Thus, $\frac{12}{18} = \frac{6}{9}$.

PROBLEMS

8. Write 5 fractions equal to $\frac{4}{5}$. ______________________________
9. Write 5 fractions equal to $\frac{8}{3}$. ______________________________
10. Write 5 fractions equal to $\frac{0}{4}$. ______________________________
11. Write the fraction with the smallest denominator which is equal to

a. $\frac{20}{8} =$ ______
b. $\frac{5}{60} =$ ______
c. $\frac{12}{30} =$ ______
d. $\frac{65}{5} =$ ______
e. $\frac{21}{4} =$ ______
f. $\frac{0}{12} =$ ______

12. Write $\frac{2}{3}$ so that it is equal to the fraction with the denominator shown.

a. $\frac{2}{3} = \frac{\quad}{6}$
b. $\frac{2}{3} = \frac{\quad}{12}$
c. $\frac{2}{3} = \frac{\quad}{18}$

13. Write $\frac{4}{5}$ so that it is equal to the fraction with the denominator shown.

a. $\frac{4}{5} = \frac{\qquad}{15}$ b. $\frac{4}{5} = \frac{\qquad}{20}$ c. $\frac{4}{5} = \frac{\qquad}{35}$

14. Circle the fraction(s) which are equal to $\frac{3}{4}$.

$\frac{6}{8}$, $\frac{18}{24}$, $\frac{9}{16}$, $\frac{21}{28}$, $\frac{39}{52}$.

15. Circle the fraction(s) which are equal to $\frac{5}{3}$.

$\frac{25}{9}$, $\frac{20}{12}$, $\frac{30}{18}$, $\frac{30}{50}$, $\frac{70}{42}$

6. Fractions and common denominators

Write the fractions $\frac{2}{3}$ and $\frac{4}{5}$ so that they have common (same) denominators .

Solution: A common denominator of $\frac{2}{3}$ and $\frac{4}{5}$ is any denominator that is exactly divisible by 3 and 5 .

Thus, 15 is exactly divisible by 3 and 5 , and we have

$$\frac{2}{3} = \frac{(5)\ 2}{(5)\ 3} = \frac{10}{15}\text{ , multiplying by 5,}$$

$$\frac{4}{5} = \frac{(3)\ 4}{(3)\ 5} = \frac{12}{15}\text{ , multiplying by 3 .}$$

Also, 30 is exactly divisible by 3 and 5 , and we have

$$\frac{2}{3} = \frac{(10)\ 2}{(10)\ 3} = \frac{20}{30}\text{ , multiplying by 10,}$$

$$\frac{4}{5} = \frac{(6)\ 4}{(6)\ 5} = \frac{24}{30}\text{ , multiplying by 6 .}$$

The smallest natural number (except 0) that is exactly divisible by 3 and 5 is called the 'least common denominator' for the fractions $\frac{2}{3}$ and $\frac{4}{5}$.

15 is the smallest natural number greater than zero that is exactly divisible by 3 and 5. Thus, 15 is the LCD for the denominators of $\frac{2}{3}$ and $\frac{4}{5}$.

PROBLEMS

16. Rewrite $\frac{1}{4}$ and $\frac{5}{6}$ as fractions with common denominator equal to

a. 24 $\frac{1}{4} = \frac{\qquad}{24}$ $\frac{5}{6} = \frac{\qquad}{24}$ c. 48 $\frac{1}{4} = \frac{\qquad}{48}$ $\frac{5}{6} = \frac{\qquad}{48}$

b. 36 $\frac{1}{4} = \frac{\qquad}{36}$ $\frac{5}{6} = \frac{\qquad}{36}$ d. 108 $\frac{1}{4} = \frac{\qquad}{108}$ $\frac{5}{6} = \frac{\qquad}{108}$

17. Is there a number smaller than 24 which can be used as a common denominator for $\frac{1}{4}$ and $\frac{5}{6}$? Yes ____ No ____ .

18. Circle the number(s) which may be used as a common denominator for $\frac{5}{6}$ and $\frac{11}{8}$.

a. 12 b. 48 c. 16 d. 24

19. A common denominator for the 3 fractions $\frac{1}{2}$, $\frac{2}{3}$, $\frac{5}{6}$ is ________.

7. Equality, inequality of fractions. Common denominators

a. If the denominators of two fractions are equal and the numerators are equal, so are the fractions . $\frac{5}{7} = \frac{5}{7}$

b. If the denominators of two fractions are equal and the numerators are NOT equal, then the fraction with the larger numerator is the greater fraction .

$\frac{11}{12} > \frac{7}{12}$

read: $\frac{11}{12}$ is greater than $\frac{7}{12}$.

Or $\frac{7}{12} < \frac{11}{12}$

read: $\frac{7}{12}$ is less than $\frac{11}{12}$.

Summary : Comparison of fractions with same denominators

$\frac{5}{7}$? $\frac{5}{7}$	$\frac{7}{12}$? $\frac{11}{12}$	$\frac{5}{7}$? $\frac{3}{7}$
Since $5 = 5$	Since $7 < 11$	Since $5 > 3$
then $\frac{5}{7} = \frac{5}{7}$	then $\frac{7}{12} < \frac{11}{12}$	then $\frac{5}{7} > \frac{3}{7}$.

PROBLEMS

20. Arrange the following fractions from smallest to largest :

a. $\frac{5}{12}$, $\frac{2}{12}$, $\frac{6}{12}$, $\frac{4}{12}$, $\frac{12}{12}$ ______________________

b. $\frac{4}{7}$, $\frac{1}{7}$, $\frac{2}{7}$, $\frac{5}{7}$, $\frac{10}{7}$ ______________________

21. Find a common denominator for each set of two fractions and write the larger fraction in the last column .

		Larger fraction
a. $\frac{1}{3}$ =	$\frac{2}{5}$ =	____________
b. $\frac{2}{7}$ =	$\frac{1}{3}$ =	____________
c. $\frac{7}{8}$ =	$\frac{3}{4}$ =	____________

d. $\frac{5}{12} =$ $\frac{1}{3} =$ ____________

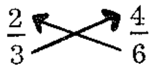
e. $\frac{3}{4} =$ $\frac{25}{32} =$ ____________

f. $\frac{0}{2} =$ $\frac{0}{6} =$ ____________

g. $\frac{4}{2} =$ $\frac{6}{3} =$ ____________

8. Equality, inequality of fractions . Cross product

a. Equality

Given: two common fractions	$\frac{2}{3} \quad ? \quad \frac{4}{6}$
Form the cross product	
The numerators 2, 4 of the fractions are on the same sides as the corresponding fractions .	2(6) 4(3)
Compare the cross products	12 12
Here 12 is equal to 12	$12 = 12$
Since the cross products are equal, the fractions are also equal .	$\frac{2}{3} = \frac{4}{6}$

b. Inequality

Given : two common fractions	$\frac{2}{3} \quad ? \quad \frac{5}{4}$
Form the cross product	
The numerators 2, 5 of the fractions are on the same sides as the corresponding fractions .	2(4) 5(3)
Compare the cross products	8 15
Here 8 is less than 15	$8 < 15$
And we have for the fractions	$\frac{2}{3} < \frac{5}{4}$

Given : two common fractions	$\frac{7}{4} \quad ? \quad \frac{3}{2}$
Form the cross product	
The numerators 7, 3 of the fractions are on the same sides as the corresponding fractions .	7(2) 3(4)
Compare the cross products	14 12
Here 14 is greater than 12	$14 > 12$
And we have for the fractions	$\frac{7}{4} > \frac{3}{2}$

c. Explanation of the cross product

Given : two common fractions (see (a) page 5) $\qquad \frac{2}{3} \qquad \frac{4}{6}$

Multiply the numerator and denominator of $\frac{2}{3}$ by 6, the denominator of $\frac{4}{6}$. $\qquad \frac{2\ (6)}{3\ (6)} \qquad \frac{4\ (3)}{6\ (3)}$

Multiply the numerator and denominator of $\frac{4}{6}$ by 3, the denominator of $\frac{2}{3}$.

Now the denominators in the two fractions are the same, namely 18 .

To compare the fractions, we need only compare the numerators, namely $\qquad 2\ (6)$ and $4\ (3)$

But, 2(6) and 4(3) are precisely the cross products in $\qquad \frac{2}{3} \times \frac{4}{6}$

$2\ (6) \qquad 4\ (3)$.

The above shows why the cross product method is a short cut method for determining the equality and inequality of fractions.

Summary: Comparison of fractions by cross product

$\frac{2}{3}$? $\frac{4}{6}$	$\frac{2}{3}$? $\frac{3}{4}$	$\frac{5}{6}$? $\frac{1}{2}$
$\frac{2}{3} \times \frac{4}{6}$	$\frac{2}{3} \times \frac{3}{4}$	$\frac{5}{6} \times \frac{1}{2}$
2(6) 4(3)	2(4) 3(3)	5(2) 1(6)
12 12	8 9	10 6
12 = 12	8 < 9	10 > 6
Thus $\frac{2}{3} = \frac{4}{6}$	Thus $\frac{2}{3} < \frac{3}{4}$	Thus $\frac{5}{6} > \frac{1}{2}$

PROBLEMS

22. Insert = , < , > so that the relation is true .

a. $\frac{1}{3} \qquad \frac{1}{4}$ $\qquad$ b. $\frac{2}{8} \qquad \frac{2}{10}$ $\qquad$ c. $\frac{7}{6} \qquad \frac{7}{4}$

d. $\frac{2}{3} \qquad \frac{3}{2}$ $\qquad$ e. $\frac{4}{9} \qquad \frac{7}{9}$ $\qquad$ f. $\frac{0}{5} \qquad \frac{0}{10}$

23. Insert = , < , > so that the relation is true .

a. $\frac{4}{5} \qquad \frac{4}{9}$ $\qquad$ b. $\frac{7}{4} \qquad \frac{7}{2}$ $\qquad$ c. $\frac{9}{5} \qquad \frac{9}{3}$ $\qquad$ d. $\frac{12}{4} \qquad \frac{12}{12}$

What do you notice about the numerators of each pair of fractions above?

24. Complete this sentence: If 2 fractions have the same nonzero numerator but different denominators, then the fraction with the larger denominator is the ____________ fraction .
(smaller, larger)

25. Arrange these fractions from smallest to largest .

a. $\frac{3}{11}, \frac{3}{4}, \frac{3}{10}, \frac{3}{2}, \frac{3}{1}, \frac{3}{8}, \frac{3}{7}$ ____________________

b. $\frac{1}{2}, \frac{2}{3}, 1, \frac{4}{5}$ ____________________

9. Addition of fractions

a. Given : two fractions with a common denominator $\frac{2}{7}, \frac{4}{7}$

The sum of the two fractions is : $\frac{2}{7} + \frac{4}{7} = \frac{2 + 4}{7} = \frac{6}{7}$

Add the numerators of the fractions and write the sum over the common denominator.

b. Given: two fractions with different denominators $\frac{1}{3}, \frac{7}{10}$

First, rewrite the fractions so that they have a common denominator - 30 in this case .

$$\frac{1}{3} = \frac{1\ (10)}{3\ (10)} = \frac{10}{30}$$

$$\frac{7}{10} = \frac{7\ (3)}{10\ (3)} = \frac{21}{30}$$

The sum is : $\frac{1}{3} + \frac{7}{10} = \frac{10}{30} + \frac{21}{30} = \frac{10 + 21}{30} = \frac{31}{30}$

Add the numerators and write the sum over the common denominator .

Some prefer to add fractions in this form :

$$\begin{array}{r} \frac{1}{3} = \frac{10}{30} \\ \frac{7}{10} = \frac{21}{30} \\ \hline \frac{31}{30} \end{array}$$

The sum of $\frac{1}{3}$ and $\frac{7}{10}$ is $\frac{31}{30}$

PROBLEMS

26. Find the following sums :

a. $\frac{5}{14} + \frac{3}{14} =$ ____________

b. $\frac{13}{10} + \frac{19}{10} =$ ____________

c. $\frac{4}{5} + \frac{2}{3} =$ ____________________

d. $4 + 1\frac{2}{3} =$ ____________________

e. $4\frac{1}{3} + 2\frac{2}{3} =$ ____________________

f. $2\frac{1}{3} + 3\frac{1}{6} =$ ____________________

27. Find the following sums .

a. $\frac{1}{3} + \frac{1}{6} =$ ____________ b. $\frac{2}{7} + \frac{5}{6} =$ ____________

c. $\frac{3}{11} + \frac{17}{22} =$ ____________ d. $\frac{8}{3} + \frac{14}{5} =$ ____________

28. Find the following sums .

a. $\frac{1}{3} + \frac{3}{4} + \frac{7}{12} =$ ____________________

b. $\frac{3}{5} + \frac{1}{2} + \frac{6}{7} =$ ____________________

10. Subtraction of fractions

a. Given: two fractions with a common denominator $\frac{5}{7}$, $\frac{2}{7}$

Find the difference $\frac{5}{7} - \frac{2}{7}$.

The difference is $\frac{5}{7} - \frac{2}{7} = \frac{5 - 2}{7} = \frac{3}{7}$.

Subtract the numerators and write the difference over the common denominator .

b. Given : two fractions with different denominators $\frac{3}{4}$, $\frac{2}{3}$

First, rewrite the fractions so that they have a common denominator - 12 in this case.

$$\frac{3}{4} = \frac{3\ (3)}{4\ (3)} = \frac{9}{12}$$

$$\frac{2}{3} = \frac{2\ (4)}{3\ (4)} = \frac{8}{12}$$

Now $\frac{3}{4} - \frac{2}{3} = \frac{9}{12} - \frac{8}{12} = \frac{9 - 8}{12} = \frac{1}{12}$.

Subtract the numerators and write the difference over the common denominator .

Some prefer to subtract fractions in this form $\frac{3}{4} = \frac{9}{12}$

$$\begin{array}{r} \frac{2}{3} = \frac{8}{12} \\ \hline \end{array}$$

The difference $\frac{3}{4} - \frac{2}{3}$ is $\frac{1}{12}$

PROBLEMS

29. Find the following differences .

a. $\frac{7}{10} - \frac{3}{10} =$ ______ b. $\frac{15}{13} - \frac{7}{13} =$ ______

c. $\frac{4}{5} - \frac{3}{7} =$ ______

d. $2\frac{1}{4} - 1\frac{1}{4} =$ ______

e. $3\frac{1}{6} - 2\frac{1}{3} =$ ______

30. Find the following differences .

a. $\frac{1}{2} - \frac{1}{3} =$ ______ b. $\frac{2}{5} - \frac{1}{6} =$ ______

c. $\frac{3}{4} - \frac{3}{8} =$ ______ d. $4\frac{1}{2} - 2\frac{1}{4} =$ ______

11. Multiplication of fractions

Given : two fractions $\frac{2}{3}$, $\frac{4}{5}$

The product of the fractions is $\frac{2}{3}\left(\frac{4}{5}\right) = \frac{2\,(4)}{3\,(5)} = \frac{8}{15}$

Multiply the numerators of the fractions and write the product over the product of the denominators .

The word ' of ' in statements such as : $\frac{1}{2}$ of $\frac{3}{4}$ means multiplication .

Thus, find : $\frac{1}{2}$ of $\frac{3}{4}$. Solution : $\frac{1}{2}$ of $\frac{3}{4} = \frac{1}{2}\left(\frac{3}{4}\right) = \frac{1\,(3)}{2\,(4)} = \frac{3}{8}$.

PROBLEMS

31. Find the following products .

a. $\frac{2}{5}\left(\frac{8}{3}\right) =$ ______ b. $\frac{7}{16} \times \frac{4}{7} =$ ______

c. $4\left(3\frac{1}{2}\right) =$ ______ d. $\frac{5}{3} \times \frac{3}{5} =$ ______

e. $\frac{2}{9} \bullet \frac{7}{4} =$ ______

32. Find

a. $\frac{1}{3}$ of $\frac{2}{7} =$ ______ b. $\frac{1}{5}$ of $\frac{1}{3} =$ ______

c. $\frac{1}{3}$ of $6 =$ ______ d. $\frac{1}{4}$ of $0 =$ ______

33. Find $\frac{2}{3}\left(\frac{3}{5} + \frac{6}{5}\right) =$ ______________________

$\frac{5}{8}\left(2 - 1\frac{1}{4}\right) =$ ______________________

12. Reciprocals

Two numbers are reciprocals of each other if their product is 1 .

a. The reciprocal of 2 is $\frac{1}{2}$ since $2\left(\frac{1}{2}\right) = \frac{2}{1}\left(\frac{1}{2}\right) = \frac{2(1)}{1(2)} = \frac{2}{2} = 1$.

Also, it follows at once, that the reciprocal of $\frac{1}{2}$ is 2 .

b. The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$ since $\frac{2}{3}\left(\frac{3}{2}\right) = \frac{2(3)}{3(2)} = \frac{6}{6} = 1$.

Also, it follows at once, that the reciprocal of $\frac{3}{2}$ is $\frac{2}{3}$.

In practice: 1. to find the reciprocal of a number like 2 , write $\frac{2}{1}$ and then invert the fraction to get $\frac{1}{2}$. $\frac{1}{2}$ is the reciprocal of 2 ,

2. to find the reciprocal of a number like $\frac{2}{3}$, invert the fraction to get $\frac{3}{2}$. $\frac{3}{2}$ is the reciprocal of $\frac{2}{3}$,

3. to find the reciprocal of a number like $1\frac{3}{4}$, first write $1\frac{3}{4}$ as $\frac{7}{4}$, then use (2) above .

PROBLEMS

34. Fill in the blanks .

a. The reciprocal of $\frac{5}{4}$ is ________ since $\frac{5}{4}$ x ______ = ______.

b. The reciprocal of $2\frac{1}{2}$ is ________ since $2\frac{1}{2}$ x ______ = ______.

c. The reciprocal of 1 is ________ since 1 x ______ = ______.

35. Does 0 have a reciprocal ? Yes ______ No ______ .

Give a reason for your answer ______________________

13. Division of fractions : Making the divisor 1 .

Division of fractions is rather simple if you remember : Any number divided by 1 is that number .

Thus, $4 \div 1 = 4$, which can also be written as $\frac{4}{1} = 4$,

$\frac{2}{3} \div 1 = \frac{2}{3}$, which can also be written as $\frac{\frac{2}{3}}{1} = \frac{2}{3}$.

a. Find the quotient of $$\begin{array}{cl} \frac{2}{3} & \longleftarrow \text{dividend} \\ \cline{1-1} \frac{4}{5} & \longleftarrow \text{divisor} \end{array}$$

We will make the divisor 1 in this problem by multiplying the dividend and divisor by $\frac{5}{4}$ which is the reciprocal of the divisor $\frac{4}{5}$.

Thus, $$\frac{\frac{2}{3}}{\frac{4}{5}} = \frac{\frac{2}{3} \text{ x } \frac{5}{4}}{\frac{4}{5} \text{ x } \frac{5}{4}} = \frac{\frac{2}{3} \text{ x } \frac{5}{4}}{1} = \underbrace{\underset{\text{dividend} \qquad \text{reciprocal of divisor}}{\boxed{\frac{2}{3} \text{ x } \frac{5}{4}}}}_{\text{This gives a simple rule for finding the quotient of fractions.}} = \frac{2(5)}{3(4)} = \frac{10}{12} \quad (\text{ or } \frac{5}{6})$$

The quotient $$\frac{\frac{3}{2}}{\frac{5}{7}} = \underset{\text{dividend}}{\underset{\uparrow}{\frac{3}{2}}} \text{ x } \underset{\text{reciprocal of divisor}}{\underset{\uparrow}{\frac{7}{5}}} = \frac{3(7)}{2(5)} = \frac{21}{10} \;.$$

b. $$\underset{\text{dividend}}{\underset{\uparrow}{\frac{3}{5}}} \div \underset{\text{divisor}}{\underset{\uparrow}{\frac{2}{9}}} = \underset{\text{dividend}}{\underset{\uparrow}{\frac{3}{5}}} \text{ x } \underset{\text{reciprocal of divisor}}{\underset{\uparrow}{\frac{9}{2}}} = \frac{3(9)}{5(2)} = \frac{27}{10} \;.$$

PROBLEMS

36. Find the quotients .

a. $\frac{2}{5} \div \frac{3}{4} =$ ________________

b. $1\frac{1}{2} \div 3 =$ ________________

c. $2\frac{1}{3} \div 3\frac{1}{5} =$ ________________

d. $\dfrac{\frac{3}{8}}{\frac{2}{3}} =$

e. $\dfrac{\frac{3}{4}}{\frac{4}{3}} =$

f. $\dfrac{5}{\frac{1}{3}} =$

g. $\dfrac{\frac{2}{5}}{3} =$

h. $\dfrac{1\frac{2}{3}}{2\frac{1}{5}} =$

i. $\dfrac{0}{\frac{1}{6}} =$

14. Division of fractions. Same denominators, divide the numerators.

a. Given : two fractions with a common denominator .

$$\frac{2}{3} \div \frac{5}{3} = \frac{2}{3} \times \frac{3}{5} = \frac{6}{15} = \frac{2}{5},$$

here 2 is the numerator of the first fraction and 5 is the numerator of the second fraction .

Thus, the quotient of the fractions is the quotient of the numerators .

b. Given: two fractions with different denominators .

$$\frac{1}{2} \div \frac{3}{7} = \underbrace{\frac{7}{14} \div \frac{6}{14}}_{\substack{\text{Rewrite the fractions so} \\ \text{that they have a common} \\ \text{denominator .}}} = \frac{7}{6},$$

divide the numerators of the fractions which have a common denominator.

The above can also be written as

$$\frac{\frac{1}{2}}{\frac{3}{7}} = \frac{\frac{7}{14}}{\frac{6}{14}} = \frac{7}{6} .$$

PROBLEMS

37. Use the method above . Find the quotients .

a. $\frac{5}{7} \div \frac{2}{7} =$ ____________

b. $\frac{4}{11} \div \frac{10}{11} =$ ________________

c. $\frac{2}{5} \div \frac{3}{4} =$ ____________

d. $1\frac{1}{2} \div 3 =$ __________________

e. $2\frac{1}{3} \div 3\frac{1}{5} =$ ______________________________

15. Division of fractions . Reduced to division of natural numbers .

a. Find the quotient

$$\begin{array}{cl} 4 & \leftarrow \text{dividend} \\ \cline{1-1} \frac{2}{3} & \leftarrow \text{divisor} \end{array}$$

Solution: Since the numerator is a natural number, we will make the denominator a natural number also . Thus, multiply the dividend and divisor by 3 (which is the smallest natural number that will make the divisor 2/3 a natural number) .

Thus,

$$\frac{4}{\frac{2}{3}} = \frac{4(3)}{\frac{2}{3}(3)} = \frac{12}{2} = 6 .$$

b. Find the quotient

$$\frac{\frac{2}{7}}{5} = \frac{\frac{2}{7}(7)}{5(7)} = \frac{2}{35},$$

where we multiplied the dividend and divisor by 7 .

c. Find the quotient

$$\frac{\frac{1}{3}}{\frac{2}{5}} = \frac{\frac{1}{3}(15)}{\frac{2}{5}(15)} = \frac{5}{6},$$

where we multiplied the dividend and divisor by the common denominator of 3 and 5 .

PROBLEMS

38. Use the method of Section 15 on page 12. Find the quotients .

a. $\frac{3}{8} \div \frac{5}{8} =$ ______________

b. $5 \div \frac{1}{3} =$ ______________

c. $\frac{2}{5} \div 3 =$ ______________

d. $\frac{3}{8} \div \frac{2}{3} =$ ______________

e. $1\frac{2}{3} \div 2\frac{1}{5} =$ ______________

SOMETHING NEW

16. Fraction gems in the multiplication table

The multiplication table is rich in fraction facts . The number of a row is shown to the left of the double vertical lines, and the number of a column is shown above the double horizontal lines .

Multiplication Table

↓ Columns

Rows →

x	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

a. Examples of equal (equivalent) fractions .

Take a number in row 1 as the numerator and the number in row 2 (in the same column) as the denominator . We have the fractions

Row 1 →
Row 2 → $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \frac{6}{12}, \frac{7}{14}, \frac{8}{16}, \frac{9}{18}, \frac{10}{20}$ All these fractions are equal to each other.

Take numbers in row 5 as numerators , in row 3 as denominators.

Row 5 →
Row 3 → $\frac{5}{3}, \frac{10}{6}, \frac{15}{9}, \frac{20}{12}, \frac{25}{15}, \frac{30}{18}, \frac{35}{21}, \frac{40}{24}, \frac{45}{27}, \frac{50}{30}$

All these fractions are equal to each other .

PROBLEMS

39. Write 5 fractions using the numbers

a. in row 2 as numerators and the corresponding numbers in row 3 as denominators.

____

Are these fractions equal to each other? Yes____ No____

b. in row 6 as numerators and the corresponding numbers in row 3 as denominators.

____

Are these fractions equal to each other? Yes ___ No____

40. Find 5 fractions from the multiplication table equal to the fraction $\frac{4}{9}$.

____

What two rows did you use ?

Rows______________________

41. Find 5 fractions from the multiplication table equal to the fraction $\frac{5}{8}$.

____

What two rows did you use ?

Rows ______________________

42. Find 5 fractions from the multiplication table equal to the fraction $\frac{24}{28}$.

____

What two rows did you use ?

Rows ______________________

43. Find 5 fractions from the multiplication table equal to the fraction $\frac{12}{27}$.

____

What two rows did you use ?

Rows______________________

44. Find 5 fractions from the multiplication table equal to the fraction $\frac{9}{5}$.

____

What two rows did you use ?

Rows ______________________

45. Find 5 fractions from the multiplication table equal to $1\frac{1}{4}$.

____

What two rows did you use ?

Rows ______________________

b. Examples of unequal fractions

Row → 4
Row → 5

$\frac{4}{5}$ is a fraction whose numerator is from row 4 and whose denominator is the corresponding number in row 5. The rows 4 and 5 are consecutive rows.

Now $\frac{4}{5}$ and all the fractions equal to $\frac{4}{5}$ are greater than any proper fraction made up from any two consecutive rows above or including row 4. If row 4 is used, then the numbers from this row must be the denominators of the fractions.

Example 1 $\frac{4}{5} > \frac{3}{4}$ where $\frac{3}{4}$ is made up from the consecutive rows 3 and 4.

$\frac{4}{5}$ is also greater than $\frac{6}{8}$, $\frac{9}{12}$, $\frac{12}{16}$, and so on, fractions equal to 3/4.

PROBLEMS

46. Use the multiplication table .

   a. Write 3 different fractions less than $\frac{5}{6}$. ________________

   b. Write 3 different fractions less than $\frac{6}{7}$. ________________

   c. Write 3 different fractions less than $\frac{21}{24}$. ________________

   d. Write 3 different fractions less than $\frac{56}{63}$. ________________

$\begin{array}{l}\text{Row} \rightarrow \\ \text{Row} \rightarrow\end{array} \frac{4}{5}$ is a fraction whose numerator is from row 4 and whose denominator is the corresponding number in row 5 . The rows 4 and 5 are consecutive rows .

Now $\frac{4}{5}$ and all the fractions equal to $\frac{4}{5}$ are <u>less than</u> any <u>proper fraction</u> made up from any two consecutive rows <u>below or including row 5.</u>

<u>Example 1</u> $\frac{4}{5} < \frac{5}{6}$ where $\frac{5}{6}$ is made up from the consecutive rows 5 and 6 .

$\frac{4}{5}$ is also less than $\frac{10}{12}$, $\frac{15}{18}$, $\frac{20}{24}$ and so on, fractions equal to $\frac{5}{6}$.

PROBLEMS

47. Use the multiplication table .

   a. Write 3 different fractions greater than $\frac{3}{4}$. ________________

   b. Write 3 different fractions greater than $\frac{6}{7}$. ________________

   c. Write 3 different fractions greater than $\frac{20}{30}$. ________________

   d. Write 3 different fractions greater than $\frac{24}{27}$. ________________

c. Getting fractions with common denominator.

Use the multiplication table . Write the fractions $\frac{2}{3}$ and $\frac{4}{5}$ so that they have a common denominator .

<u>Solution :</u> The denominator of $\frac{2}{3}$, namely, 3 is in row 3 .

The denominator of $\frac{4}{5}$, namely, 5 is in row 5 .

Study rows 3 and 5 . Is there a number which appears in both these rows? Yes, the numbers 15 and 30 .

Thus, $\frac{2}{3} = \frac{10}{15}$ (from rows 2 and 3) ,

and $\frac{4}{5} = \frac{12}{15}$ (from rows 4 and 5) .

The two fractions $\frac{2}{3}$, $\frac{4}{5}$ now have a common denominator 15 or 30.

Also $\frac{2}{3} = \frac{20}{30}$ (from rows 2 and 3),

and $\frac{4}{5} = \frac{24}{30}$ (from rows 4 and 5) .

However, 15 is the least common denominator .

PROBLEMS

48. Use the multiplication table. Write the two fractions so that they have a common denominator .

a. $\frac{2}{4}$ = _______ b. $\frac{5}{6}$ = _______ c. $\frac{5}{7}$ = _______

$\frac{3}{5}$ = _______ $\frac{1}{3}$ = _______ $\frac{6}{9}$ = _______

d. Multiplication table: addition, subtraction, division activities .

Use the multiplication table. Take the numbers in one row as the numerators and the corresponding numbers in any other row as the denominators of fractions . Make strips like the following .

Strip 1 .

Row 1 Row 2	$\frac{1}{2}$	$\frac{2}{4}$	$\frac{3}{6}$	$\frac{4}{8}$	$\frac{5}{10}$	$\frac{6}{12}$	$\frac{7}{14}$	$\frac{8}{16}$	$\frac{9}{18}$	$\frac{10}{20}$

Strip 2

Row 1 Row 3	$\frac{1}{3}$	$\frac{2}{6}$	$\frac{3}{9}$	$\frac{4}{12}$	$\frac{5}{15}$	$\frac{6}{18}$	$\frac{7}{21}$	$\frac{8}{24}$	$\frac{9}{27}$	$\frac{10}{30}$

Suppose we want : $\frac{1}{2} + \frac{1}{3}$, $\frac{1}{2} - \frac{1}{3}$, $\frac{1}{2} \div \frac{1}{3}$.

Arrange the strips 1 and 2 as shown below .

Strip 1	$\frac{1}{2}$	$\frac{2}{4}$	$\frac{3}{6}$	$\frac{4}{8}$	$\frac{5}{10}$	$\frac{6}{12}$	$\frac{7}{14}$	$\frac{8}{16}$	$\frac{9}{18}$	$\frac{10}{20}$	
Strip 2		$\frac{1}{3}$	$\frac{2}{6}$	$\frac{3}{9}$	$\frac{4}{12}$	$\frac{5}{15}$	$\frac{6}{18}$	$\frac{7}{21}$	$\frac{8}{24}$	$\frac{9}{27}$	$\frac{10}{30}$

Place strip 1 at the top (the $\frac{1}{2}$ strip) . Then move strip 2 (the $\frac{1}{3}$ strip) until you have two fractions, one in each strip , with the same least common denominator .

Now:

addition $\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{3+2}{6} = \frac{5}{6}$ (See Section 9 (a))

subtraction $\frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6} = \frac{3-2}{6} = \frac{1}{6}$ (See Section 10 (a))

division $\frac{1}{2} \div \frac{1}{3} = \frac{3}{6} \div \frac{2}{6} = 3 \div 2 = \frac{3}{2}$. (See Section 14 (a))

PROBLEMS

49. Make strips for $\frac{2}{3}$ and $\frac{3}{5}$.

Find:
a. $\frac{2}{3} + \frac{3}{5}$ = _______ b. $\frac{2}{3} - \frac{3}{5}$ = _______ c. $\frac{2}{3} \div \frac{3}{5}$ = _______

17. The Magic of Lo-Shu. Fraction magic squares .

Magic squares, according to historical records, originated in China . Below is the ancient and fascinating Lo-Shu 3 by 3 normal magic square .

Columns

		1	2	3
	1	6	7	2
Rows	2	1	5	9
	3	8	3	4

Diagonal 1 Diagonal 2

PROBLEMS

50. Find the sum of the numbers :

in row 1 ______ in column 1 ______ in diagonal 1 ______

in row 2 ______ in column 2 ______ in diagonal 2 ______

in row 3 ______ in column 3 ______

Was the sum of each row, column, and diagonal the same ? Yes___ No___.

In any case where your answer is 'yes' , that number is called the Magic Number of the square .

51. We have formed some magic squares with fractions .

Magic square 1

$\frac{7}{2}$	0	$\frac{5}{2}$
1	2	3
$\frac{3}{2}$	4	$\frac{1}{2}$

Magic square 2

$\frac{1}{2}$	4	$\frac{3}{2}$
3	2	1
$\frac{5}{2}$	0	$\frac{7}{2}$

Magic square 3

1	$\frac{1}{6}$	$1\frac{1}{3}$
$1\frac{1}{6}$	$\frac{5}{6}$	$\frac{1}{2}$
$\frac{1}{3}$	$1\frac{1}{2}$	$\frac{2}{3}$

What is the Magic Number of :

Magic square 1 ______ Magic square 2 ______ Magic square 3 ______

52. Write in fractions in the blank cells in the following magic squares so that each magic square will have the Magic Number that is shown above the square.

Magic Number 3

		$1\frac{1}{6}$
	1	
	$\frac{1}{2}$	

Magic Number 3

$1\frac{5}{8}$		
$\frac{7}{8}$	$\frac{5}{8}$	

Magic Number $2\frac{1}{2}$

		$1\frac{1}{3}$
$1\frac{1}{2}$	$\frac{5}{6}$	

53. Here are two 4 by 4 magic squares with fractions .

Magic square 1

$\frac{1}{2}$	$1\frac{1}{8}$	$\frac{5}{8}$	2
$1\frac{7}{8}$	$\frac{3}{4}$	$1\frac{1}{4}$	$\frac{3}{8}$
$1\frac{3}{4}$	$\frac{7}{8}$	$1\frac{3}{8}$	$\frac{1}{4}$
$\frac{1}{8}$	$1\frac{1}{2}$	1	$1\frac{5}{8}$

Magic square 2

$1\frac{5}{6}$	$1\frac{1}{3}$	$\frac{5}{6}$	$1\frac{2}{3}$
$\frac{1}{3}$	$2\frac{1}{6}$	$2\frac{2}{3}$	$\frac{1}{2}$
$2\frac{1}{3}$	$\frac{1}{6}$	$\frac{2}{3}$	$2\frac{1}{2}$
$1\frac{1}{6}$	2	$1\frac{1}{2}$	1

What is the Magic Number of : Magic square 1 _____

Magic square 2 _____

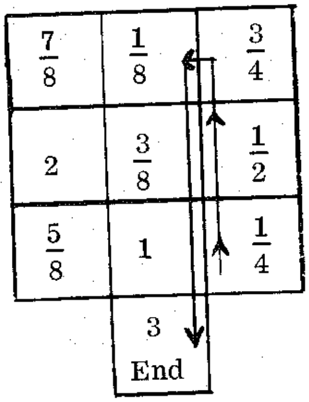
18. Fraction add-a-trails

To do an add-a-trail :

Begin at any of the 9 boxes .
Draw a continuous trail to the end-of-trail box.
A trail can only go vertically or horizontally.
A trail cannot retrace or cross itself .

The sum of all the numbers on the trail must be equal to the number in the end-of-trail box .

Here we began with $\frac{1}{4}$ and the sum of the 6 numbers is 3 .



$\frac{7}{8}$	$\frac{1}{8}$	$\frac{3}{4}$
2	$\frac{3}{8}$	$\frac{1}{2}$
$\frac{5}{8}$	1	$\frac{1}{4}$
	3 End	

PROBLEMS

54. Do the following add-a-trails .

a.

$\frac{1}{2}$	$\frac{7}{16}$	$\frac{3}{4}$
$\frac{3}{16}$	$\frac{5}{8}$	$\frac{5}{16}$
$\frac{7}{8}$	$\frac{3}{8}$	$\frac{1}{4}$
	3 End	

b.

$\frac{1}{5}$	$\frac{3}{5}$	$\frac{3}{10}$
$\frac{3}{4}$	$\frac{7}{10}$	$\frac{1}{20}$
$\frac{5}{6}$	$\frac{2}{5}$	$\frac{4}{5}$
	$2\frac{1}{4}$ End	

c.

$\frac{1}{16}$	$\frac{1}{3}$	$\frac{1}{2}$
$\frac{1}{8}$	$\frac{15}{16}$	$\frac{7}{8}$
$\frac{5}{6}$	$\frac{1}{4}$	$\frac{3}{4}$
	$3\frac{1}{3}$ End	

19. The mystery of the Egyptian unit fractions

A unit fraction is a fraction with 1 in the numerator .

For example, $\frac{1}{2}$, $\frac{1}{5}$, $\frac{1}{7}$ are unit fractions .

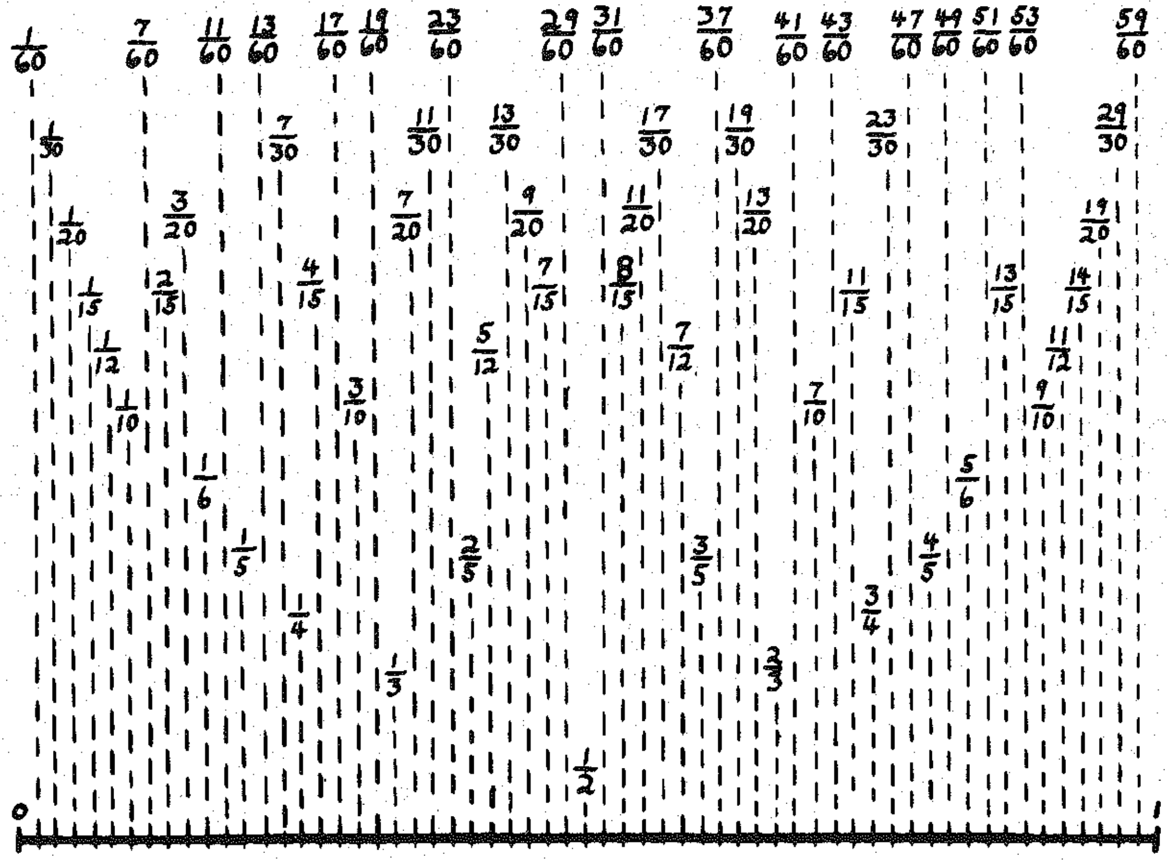
Except for the fraction $\frac{2}{3}$, the Egyptians replaced all their proper fractions by a sum of distinct unit fractions . For example, we can write $\frac{2}{15}$ and $\frac{3}{7}$ as a sum of unit fractions .

$\frac{2}{15} = \frac{1}{10} + \frac{1}{30}$ $\qquad$ $\frac{3}{7} = \frac{1}{4} + \frac{1}{7} + \frac{1}{28}$

What method did the Egyptians use to write proper fractions as sums of unit fractions ? This is still somewhat of a mystery. We present a few methods here . There is fun and challenge in trying to solve the mystery .

a. The ruler for unit fractions

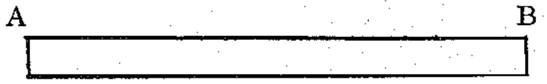
Below we have a ruler where the distance between 0 and 1 has been divided into 60 parts. Each part is marked by a fraction .



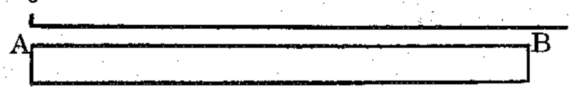
How to find some proper fractions as the sum of unit fractions using the ruler .

Find $\frac{2}{5}$ as a sum of unit fractions.

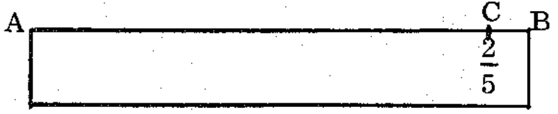
1. Take a strip of paper about 8 inches long .



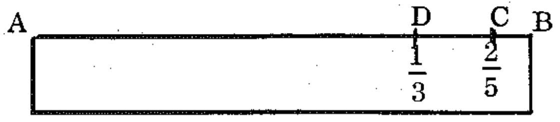
2. Place the point marked A carefully at the point 0 on the ruler with the edge of the paper from A to B along the ruler .



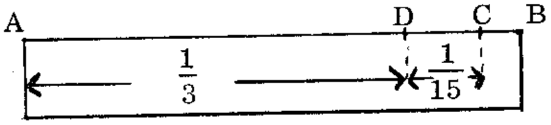
3. Find the point for $\frac{2}{5}$ on the ruler. Put a mark on your strip of paper that corresponds to $\frac{2}{5}$. Call this point C .



4. With the point A at the point 0 on the ruler, find the largest unit fraction that is less than the distance from A to C . It is $\frac{1}{3}$. Mark This point D . Your strip of paper should look like the strip at the right .



5. Place the point marked D at the point 0 on the ruler. Find the largest unit fraction that is less than or equal to the distance from D to C. If you have done this carefully, you will find that the distance from D to C is exactly $\frac{1}{15}$.



Thus, $\frac{2}{5} = \frac{1}{3} + \frac{1}{15}$.

PROBLEMS

55. Follow the steps described for the ruler on page 19 and write each of the following fractions as a sum of unit fractions.

a. $\frac{5}{12}$ = ____________

b. $\frac{8}{15}$ = ____________

c. $\frac{7}{20}$ = ____________

d. $\frac{5}{6}$ = ____________

e. $\frac{2}{3}$ = ____________

f. $\frac{3}{5}$ = ____________

g. $\frac{4}{15}$ = ____________

h. $\frac{13}{60}$ = ____________

i. $\frac{41}{60}$ = ____________

b. Unit fractions by trial

This method is based on the idea of equal fractions and the addition of fractions with a common denominator. We illustrate the method by examples.

Example 1 Can we write $\frac{2}{5}$ as a sum of unit fractions with denominator 5 ?

a. It is clear that we can only have $\frac{2}{5} = \frac{1}{5} + \frac{1}{5}$.

But the above violates the requirement that the unit fractions must be distinct (different).

b. Therefore, try the next smallest multiple of 5, namely 10 as a denominator.

We have $\frac{2}{5} = \frac{4}{10}$.

Can we write the fraction $\frac{4}{10}$ as a sum of unit fractions by considering fractions with a denominator 10 ?

Write $\frac{2}{5} = \frac{4}{10} = \frac{?}{10} + \frac{?}{10} + \frac{?}{10} + \frac{?}{10}$

We write several fractions with denominator 10 since we do not know at this stage how many fractions we will need.

c. The task now is to find the numerators of the above fractions with denominators 10. Remember that the sum of these numerators must add up to 4.

What is the largest number less than the numerator 4 which divides 10 ?

It is 2. We now write

$\frac{2}{5} = \frac{4}{10} = \frac{2}{10} + \frac{4-2}{10} = \frac{2}{10} + \frac{2}{10}$, where the numerators add up to 4.

But simplifying the above, we get $\frac{2}{5} = \frac{4}{10} = \frac{1}{5} + \frac{1}{5}$

which violates the requirement that the unit fractions be distinct.

d. Try the next multiple of 5, namely 15 as the denominator .

We have $\frac{2}{5} = \frac{6}{15}$.

Can we write $\frac{6}{15}$ as a sum of unit fractions by considering fractions with a denominator 15 ?

We write as in (b) above $\frac{2}{5} = \frac{6}{15} = \frac{?}{15} + \frac{?}{15} + \frac{?}{15} + \ldots + \frac{?}{15}$

Now , what is the largest number less than the numerator 6 which exactly divides 15 ? It is 5 .

We have $\frac{2}{5} = \frac{6}{15} = \frac{5}{15} + \frac{6-5}{15} = \frac{5}{15} + \frac{1}{15}$ where the sum of the numerators adds up to 6.

The above after simplifying becomes $\frac{2}{5} = \frac{1}{3} + \frac{1}{15}$,

which agrees with the answer from the ruler method (pages 19, 20) .

Example 2 Can we write $\frac{41}{60}$ as a sum of unit fractions by considering fractions with denominators 60 ?

a. As in example 1 , part (b), we write

$$\frac{41}{60} = \frac{?}{60} + \frac{?}{60} + \frac{?}{60} + \ldots + \frac{?}{60}$$

We must now find the numerators of these fractions (which add up to 41) .

b. What is the largest number less than 41 which exactly divides 60 ?

It is 30. Write 30 as the numerator of the first fraction with denominator 60.

$$\frac{41}{60} = \frac{30}{60} + \frac{?}{60} + \frac{?}{60} + \ldots + \frac{?}{60}$$

Now subtract 30 from 41 . We have 41 - 30 = 11 .

What is the largest number less than 11 which exactly divides 60 ? It is 10 .

Write 10 as the numerator of the second fraction with denominator 60 .

$$\frac{41}{60} = \frac{30}{60} + \frac{10}{60} + \frac{?}{60} + \ldots + \frac{?}{60}$$

Now subtract 10 from 11 . We have 11 - 10 = 1 .

The 1 is the numerator of the third fraction with denominator 60 .

$$\frac{41}{60} = \frac{30}{60} + \frac{10}{60} + \frac{1}{60}$$, where the numerators add up to 41 .

The above after simplifying becomes $\frac{41}{60} = \frac{1}{2} + \frac{1}{6} + \frac{1}{60}$.

See Problem 55 (i) page 20 .

PROBLEMS

56. Project Below is a list of fractions from the Rhind Papyrus. Write the fractions as sums of unit fractions by the trial method (pages 20-21). To save space , we write fractions with a slash .

1. 2/5 = __________ 2. 2/7 = __________ 3. 2/9 = __________

4. 2/11= __________
5. 2/13 = __________
6. 2/15 = __________
7. 2/17 = __________
8. 2/19 = __________
9. 2/21 = __________
10. 2/23= __________
11. 2/25 = __________
12. 2/27= __________
13. 2/29= __________
14. 2/31= __________
15. 2/33= __________
16. 2/35= __________
17. 2/37= __________
18. 2/39= __________
19. 2/41= __________
20. 2/43= __________
21. 2/45= __________
22. 2/47= __________
23. 2/49= __________
24. 2/51= __________
25. 2/53= __________
26. 2/55= __________
27. 2/57= __________
28. 2/59= __________
29. 2/61= __________
30. 2/63= __________
31. 2/65= __________
32. 2/67= __________
33. 2/ 69= __________
34. 2/71= __________
35. 2/73= __________
36. 2/75= __________
37. 2/77= __________
38. 2/79= __________
39. 2/81= __________
40. 2/83= __________
41. 2/85= __________
42. 2/87= __________
43. 2/89= __________
44. 2/91= __________
45. 2/93= __________
46. 2/95= __________
47. 2/97= __________
48. 2/99= __________
49. 2/101 = __________

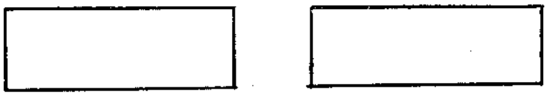
c. Unit fractions : Ahmes and the loaves of bread

This method of writing fractions as a sum of unit fractions is based on a problem given by the scribe Ahmes in the Rhind Papyrus.

Problem 1 If you divide 2 loaves of bread among 5 men, how much will each man get ?

Solution : This problem is solved by writing $\frac{2}{5}$ as a sum of unit fractions , that is, 2 loaves ÷ 5 men .

a. Draw the 2 loaves of bread .

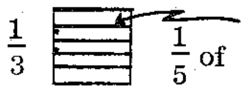


b. Now divide the loaves of bread into the smallest number of equal parts just larger than 5, the denominator of $\frac{2}{5}$. If you divide each loaf into 3 equal parts, the condition will be satisfied .

Now since there are six $\frac{1}{3}$'s , and only 5 men , each man will get at least $\frac{1}{3}$ of a loaf of bread .

So far we have : $\frac{2}{5} = \frac{1}{3}$ + more , (since $\frac{1}{3}$ of a loaf is left over).

c. Take that part of the bread that is left over, namely $\frac{1}{3}$ loaf, and divide it into 5 equal parts (since there are 5 men) .



$\frac{1}{5}$ of $\frac{1}{3} = \frac{1}{15}$

Each man will also get $\frac{1}{5}$ of $\frac{1}{3}$ of a loaf.

Thus, $\frac{2}{5} = \frac{1}{3} + \frac{1}{15}$, which is the same answer that we saw on page 19-20.

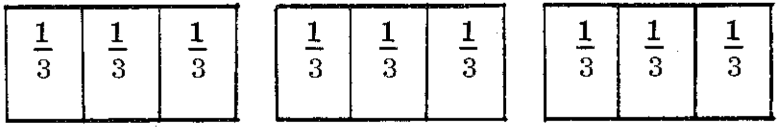
Problem 2 If you divide 3 loaves of bread among 7 men, how much will each man get ?

Solution This problem is solved by writing $\frac{3}{7}$ as a sum of unit fractions .

a. Draw the 3 loaves of bread .

Now divide the loaves of bread into the smallest number of equal parts just larger than 7.

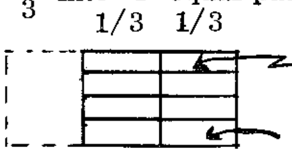
Thus, divide each loaf into 3 equal parts.



Now since there are nine $\frac{1}{3}$'s and only 7 men , each man will get at least $\frac{1}{3}$ of a loaf .

So far we have : $\frac{3}{7} = \frac{1}{3}$ + more , (since there are two $\frac{1}{3}$'s left over) .

b. Take that part of the bread that is left over, namely two $\frac{1}{3}$'s and divide it into the smallest number of equal parts just larger than 7, the denominator of $\frac{3}{7}$. Thus, divide each $\frac{1}{3}$ into 4 equal parts .



$\frac{1}{4}$ of $\frac{1}{3} = \frac{1}{12}$

$\frac{1}{12}$ left over

We now have $\frac{3}{7} = \frac{1}{3} + \frac{1}{12}$ + more , since $\frac{1}{12}$ is still left over .

c. Divide the $\frac{1}{12}$ into 7 equal pieces: $\frac{1}{12}$

$\frac{1}{7}$ of $\frac{1}{12} = \frac{1}{84}$

Each man will also get $\frac{1}{7}$ of $\frac{1}{12}$ of a loaf .

Thus, $\frac{3}{7} = \frac{1}{3} + \frac{1}{12} + \frac{1}{84}$ and the problem is solved .

Check: Since there are 7 men , if we multiply the individual share by 7 $7\left(\frac{1}{3} + \frac{1}{12} + \frac{1}{84}\right) = 3$.

PROBLEMS

57. Use the method shown in the two problems on pages 22-23 . Find the amount of bread that each man gets if

a. 2 loaves are divided among 7 men ______________

b. 3 loaves are divided among 5 men ______________

c. 2 loaves are divided among 9 men ______________

d. 5 loaves are divided among 11 men ______________

20. The puzzle of the extra horse

One day, King Arthur called Sir Lancelot, Sir Galahad and Sir Gawain and said to them:

" For your faithful service, I am giving you 17 of my best horses .

Sir Lancelot, you may have $\frac{1}{2}$ of the horses ,

Sir Galahad, you may have $\frac{1}{3}$ of the horses,

and Sir Gawain, you may have $\frac{1}{9}$ of the horses . "

The Knights thanked King Arthur and went to the King's stables to divide the horses. But when they got there, they ran into a problem .

How could they divide the 17 horses according to the King's bequest without killing any of them ?

While they were discussing how the division could possibly be carried out, Merlin the Court Magician came riding by on his horse. He listened to the Knights for awhile and then generously gave the Knights his own wonderful horse .

The Knights now had 18 horses . It was quite simple to carry out the division .

Sir Lancelot took 1/2 of the 18 horses which was 9 horses,

Sir Galahad took 1/3 of the 18 horses which was 6 horses,

and Sir Gawain took 1/9 of the 18 horses which was 2 horses .

But behold ! The total added up to exactly 17 horses !

Merlin mounted the remaining horse, which happened to be his own and with a friendly wave rode away leaving everyone happy but a trifle puzzled .

PROBLEMS

58. Can you explain what happened ?

a. Find the sum of the fractions $\frac{1}{2} + \frac{1}{3} + \frac{1}{9} =$ __________

b. Did the sum of the fractions come out to be 1 ? Yes _____ No _____

c. What is $\frac{17}{18} \times 18 =$ __________

d. Do you know the answer to the mystery ? What is it ?

____

In the King Arthur story,

a. there is a starting number 17,
b. then 1 is added to the starting number,
c. and there is a set of fractions, namely $\frac{1}{2}, \frac{1}{3}, \frac{1}{9}$.

How do we find fractions for a given starting number so that we end up with a puzzle like the King Arthur story? We describe two methods.

Method 1

Example 1 a. Let the starting number be 17,
b. you add 1 to the starting number for a total of $17 + 1 = 18$.
c. Next, try to find 3 fractions that add up to $\frac{17}{18}$.

Solution: Write : $\frac{17}{18} = \frac{?}{18} + \frac{?}{18} + \frac{?}{18}$.

Now, find 3 numbers : each less than 18,
whose sum is 17,
such that each of the numbers divides 18 exactly.

After a little trial and error, you will find that 2, 6, 9 are 3 such numbers since : each is less than 18
their sum $2 + 6 + 9 = 17$
each of the numbers divides 18 exactly.

Thus, $\frac{17}{18} = \frac{2}{18} + \frac{6}{18} + \frac{9}{18} = \frac{1}{9} + \frac{1}{3} + \frac{1}{2}$.

The fractions for a starting number 17 are $\frac{1}{2}, \frac{1}{3}, \frac{1}{9}$.

Example 2 a. Let the starting number be 11,
b. you add 1 to the starting number for a total of $11 + 1 = 12$.
Next, try to find 3 fractions that add up to $\frac{11}{12}$.

Solution: Write : $\frac{11}{12} = \frac{?}{12} + \frac{?}{12} + \frac{?}{12}$.

Now, find 3 numbers : each less than 12,
whose sum is 11,
such that each of the numbers divides 12 exactly.

After a little trial and error, you will find that there are two sets of such numbers, namely 1, 4, 6 and 2, 3, 6.

Thus, $\frac{11}{12} = \frac{1}{12} + \frac{4}{12} + \frac{6}{12} = \frac{1}{12} + \frac{1}{3} + \frac{1}{2}$

and $\frac{11}{12} = \frac{2}{12} + \frac{3}{12} + \frac{6}{12} = \frac{1}{6} + \frac{1}{4} + \frac{1}{2}$

The fractions for a starting number 11 are $\frac{1}{2}, \frac{1}{3}, \frac{1}{12}$

or $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}$.

PROBLEMS

59. Use Method 1 (page 25) to find 3 fractions for each of the starting numbers below. There may be more than 1 solution in some cases.

	Starting number	Fractions		Starting number	Fractions
a.	5	________________	b.	19	________________
c.	15	________________	d.	23	________________

e. Starting number 9 . Can you find 3 fractions here ? Yes___ No___

Method 2 In Method 1, Example 2 (page 25) , we had to find 3 numbers each less than 12, whose sum was 11 and such that each number divided 12 exactly. This provides a clue for the present method .

Example 1
a. Let the starting number be 11,
b. you add 1 to the starting number for a total of $11 + 1 = 12$.
c. Next, try to find 3 fractions that add up to $\frac{11}{12}$.

Solution: 1. Find all the exact divisors of 12, namely $D_{12} = \{1, 2, 3, 4, 6, 12\}$.

2. Disregard the divisor 1, and form unit fractions with the other divisors of 12 as denominators .
Thus, $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{12}$.

3. Multiply each of the unit fractions in (2) above by 12 as shown below.

$$\begin{array}{lcccc}
 & & \text{Number} & \text{selection} & \\
\frac{1}{2} \text{ x } 12 = 6 & \longrightarrow & 6 & \longrightarrow & 6 \\
\frac{1}{3} \text{ x } 12 = 4 & \longrightarrow & 4 & & \\
\frac{1}{4} \text{ x } 12 = 3 & \longrightarrow & & \longrightarrow & 3 \\
\frac{1}{6} \text{ x } 12 = 2 & \longrightarrow & & \longrightarrow & 2 \\
\frac{1}{12} \text{ x } 12 = \underset{\uparrow}{1} & \longrightarrow & \underline{\;1\;} & & \underline{\quad\;} \\
 & & 11 & & 11
\end{array}$$

From this column pick 3 numbers that add up to 11.

These numbers add up to 11 .

These numbers add up to 11

There are 2 solutions here .

4. Since the numbers 6, 4, 1 add up to 11 , use the corresponding fractions $\frac{1}{2}, \frac{1}{3}, \frac{1}{12}$.

Since the numbers 6, 3, 2 add up to 11, use the corresponding fractions $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}$.

PROBLEMS

60. Use Method 2 (page 26) to find 3 fractions for each of the starting numbers below. There may be more than 1 solution in some cases.

Starting number	Fractions	Starting number	Fractions
a. 5	________________	b. 19	____________________
c. 15	________________	d. 23	____________________

e. Starting number 9 . Can you find 3 fractions here ? Yes___ No___

21. Sums of unit fractions. Denominators are exact divisors .

Example 1. Find the sum of the unit fractions $\frac{1}{1} + \frac{1}{2} + \frac{1}{4}$.

Solution:
a. The largest denominator here is 4.
b. Moreover, all the exact divisors of 4 are 1, 2, 4 and all these numbers are denominators of the unit fractions .
c. When the above is the case, the sum of the unit fractions can be found quite easily and rapidly :

Add up all the denominators of the unit fractions and write this sum over the largest denominator of the unit fractions.

Thus, $\frac{1}{1} + \frac{1}{2} + \frac{1}{4} = \frac{1+2+4}{4} = \frac{7}{4}$.

Check the above answer using the method in Section 9 (page 7).

PROBLEMS

61. Use the method above and find the sums of the unit fractions. Check your answers using the method in Section 9 (page 7) .

a. $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} =$ ________ b. $\frac{1}{1} + \frac{1}{3} + \frac{1}{9} =$ ____________

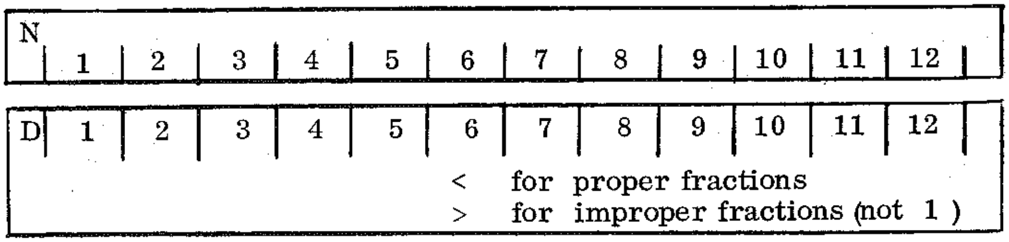
c. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12} =$ ____________________

d. $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} =$ __________

22. Comparison of fractions . Fraction slide rule .

We will make a simple device that will help you in some cases to compare fractions quickly .

a. Take 2 strips of paper, each about 1 inch wide and about 10 inches long .
b. Mark each strip of paper as shown below .

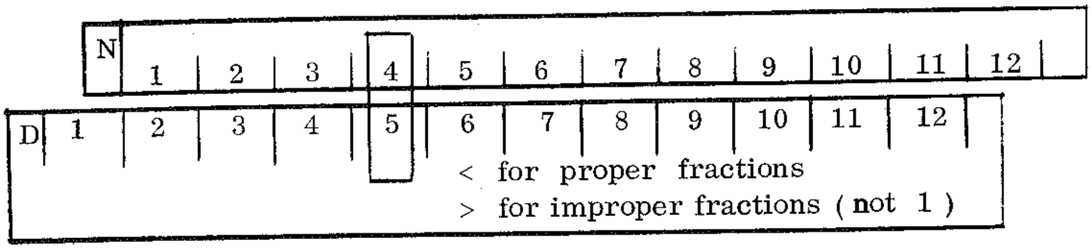


See bottom page 27 : the N on the top strip of paper stands for 'numerator' and the D on the lower strip of paper stands for 'denominator' .

Also on the D strip of paper we have indicated the inequality relation for proper and improper fractions (see explanation below) .

c. <u>How to use the slide rule .</u>

1. Take the proper fraction $\frac{4}{5}$. Place the strip N on the desk. Move the strip of paper marked D until the 5 is under the 4 on the strip marked N. You should have the set up shown below .



The proper fraction $\frac{4}{5}$ has been boxed . Now according to the inequality relation for proper fractions (shown on the D strip) , we have that

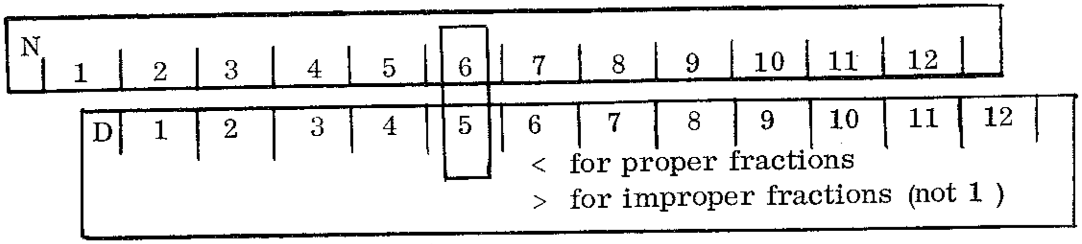
$\frac{4}{5}$ is greater than any fraction to the <u>left</u> of $\frac{4}{5}$, that is

$\frac{4}{5}$ is <u>greater</u> than $\frac{3}{4}$, $\frac{2}{3}$, $\frac{1}{2}$, and

$\frac{4}{5}$ is less than any fraction to the <u>right</u> of $\frac{4}{5}$, that is ,

$\frac{4}{5}$ is <u>less</u> than $\frac{5}{6}$, $\frac{6}{7}$, $\frac{7}{8}$ and so on .

2. Take the improper fraction $\frac{6}{5}$. Place the strip N on the desk . Move the strip of paper marked D until the 5 is under the 6 on the strip marked N . You should have the set up shown below .



The improper fraction $\frac{6}{5}$ has been boxed . Now according to the inequality relation for improper fractions (shown on the D strip), we have that

6/5 is less than the fractions to the <u>left</u> of 6/5, that is

6/5 is <u>less</u> than 5/4, 4/3, 3/2 , 2/1 , and

6/5 is greater than the fractions to the <u>right</u> of 6/5 , that is

6/5 is <u>greater</u> than 7/6, 8/7, 9/8, and so on .

PROBLEMS

62. Make a slide rule like the one on page 27. Use the slide rule for the following problems.

Name two fractions

a. 4/7 is greater than the fractions ____________

4/7 is less than the fractions ____________

b. 7/4 is greater than the fractions ____________

7/4 is less than the fractions ____________

c. 2/7 is greater than the fractions ____________

2/7 is less than the fractions ____________

d. 5/9 is greater than the fractions ____________

5/9 is less than the fractions ____________

23. Fractions between fractions

There is no natural number between 2 and 3, or between 4 and 5, or between any two consecutive natural numbers.

What about fractions ?
Is there a fraction between any two nonequal fractions ?

This is an interesting question and the answer to the question will reveal some startling results.

We indicate several methods for getting fractions between two fractions.

a. Mean of two fractions

Example 1 Find a fraction between the two fractions $\frac{2}{3}$ and $\frac{5}{6}$, that is, find a fraction greater than $\frac{2}{3}$ and less than $\frac{5}{6}$.

Solution: To find a fraction between $\frac{2}{3}$ and $\frac{5}{6}$, add the two fractions and then take $\frac{1}{2}$ of the sum. Thus,

1. $\frac{2}{3} + \frac{5}{6} = \frac{4}{6} + \frac{5}{6} = \frac{9}{6}$ and $\frac{1}{2}$ of $\frac{9}{6} = \frac{9}{12} = \frac{3}{4}$.

Therefore, $\frac{3}{4}$ is between $\frac{2}{3}$ and $\frac{5}{6}$, that is

$$\frac{2}{3} < \frac{3}{4} < \frac{5}{6} .$$

Use your slide rule to check the above result. The fraction $\frac{3}{4}$ is called the mean of the two fractions $\frac{2}{3}$ and $\frac{5}{6}$.

2. We could have solved the above problem in one step as follows:

$$\frac{\frac{2}{3} + \frac{5}{6}}{2} = \frac{\frac{4}{6} + \frac{5}{6}}{2} = \frac{\frac{9}{6}}{2} = \frac{9}{12} = \frac{3}{4} .$$

PROBLEMS

63. Find a fraction between each pair of fractions using the mean of two fractions. Check your answer by the slide rule or by the cross product.

a. 1/2 and 3/5 ____________   b. 3/2 and 5/3 ____________

c. 2/7 and 1/3 ________ d. 1/4 and 1/3 ________

e. How would you show that the fraction 3/4 is exactly half-way between 2/3 and 5/6 .

b. Add numerators and denominators

Example 1 Find a fraction between the two fractions $\frac{2}{3}$ and $\frac{5}{6}$.

Solution: To find a fraction between $\frac{2}{3}$ and $\frac{5}{6}$, add the numerators of the fractions and put this sum over the sum of the denominators of the fractions . Thus,

$$\frac{2}{3} < \frac{2+5}{3+6} < \frac{5}{6}$$

and we have $\frac{2}{3} < \frac{7}{9} < \frac{5}{6}$.

Use your slide rule or the cross product to check the above result.

PROBLEMS

64. Use the method of addition of numerators and denominators . Find a fraction between each pair of fractions . Check your answer by the slide rule or by the cross product page 6 .

a. 1/4 and 1/3 ________ b. 2/3 and 3/4 ________

c. 3/8 and 5/13 ________ d. 3/7 and 4/9 ________

e. Show that the fraction 7/9 is not exactly half-way between 2/3 and 5/6 .

c. Common denominators

Example 2 Find a fraction between 2/3 and 5/6 .

Solution Step 1 Write the fractions $\frac{2}{3}$ $\frac{5}{6}$

Step 2 Write the fractions so that they have a common denominator (least common denominator if it is easy to find) . $\frac{4}{6}$ $\frac{5}{6}$

Step 3 To find 1 fraction between the given fractions 2/3 and 5/6 , multiply the numerator and denominator of 4/6 and 5/6 by 2 (the multiplier is 1 more than the number of fractions sought). $\frac{4(2)}{6(2)}$ $\frac{5(2)}{6(2)}$ $\frac{8}{12}$ $\frac{10}{12}$

Step 4 It is clear that $\frac{9}{12}$ is between 8/12 and 10/12 and we have $\frac{2}{3} = \frac{8}{12} < \frac{9}{12} < \frac{10}{12} = \frac{5}{6}$

or $\frac{2}{3} < \frac{3}{4} < \frac{5}{6}$.

Note: read step 3 carefully. The multiplier must be 1 more than the number of fractions you are looking for .

PROBLEMS

65. Use the method in (c) on page 30 .

  a. Find 1 fraction between 1/3 and 2/5 ______________

  b. Find 2 fractions between 5/6 and 11/12 ______________

  c. Find 2 fractions between 23/15 and 8/5 ______________

  d. Find 4 fractions between 1/2 and 5/6 ______________

66. a. Could you find a million fractions between 1/3 and 1/2 ?
  Yes____ No____ What would you use as a multiplier ?______________

  b. Could you find a billion fractions between 1/3 and 1/2 ?
  Yes___ No ___ What would you use as a multiplier ?_____________

  c. Can you find any number of fractions you want between 1/3 and 1/2 ? Yes ____ No _____

67. The number 3 is the immediate successor of the number 2 , that is, there are no natural numbers between 2 and 3 .

  Is there an immediate successor of the fraction 1/2 ? Yes ___ No ___ .

  Explain briefly ____

d. Fractions between fractions with a given denominator

Example 1 Find all the fractions between 1/3 and 1/2 that have a denominator 20.

Solution: Step 1 Write the fractions $\frac{1}{3}$ $\frac{1}{2}$

Step 2 We want all the numerators of the fractions with denominator 20 such that $\frac{1}{3} < \frac{?}{20} < \frac{1}{2}$

Step 3 Find the least common denominator for 1/3 , 1/2 and fractions with denominators 20 . It is 60 . Write $\frac{20}{60} < \frac{?}{60} < \frac{30}{60}$

Step 4 Now all the fractions $\frac{21}{60}, \frac{22}{60}, \ldots \frac{29}{60}$ are between the fractions $\frac{1}{3}$ and $\frac{1}{2}$. But not all these fractions , when reduced will have a denominator 20 . By trial, we find that we can reduce the following fractions to a fraction with denominator 20 :

$$\frac{21}{60} = \frac{7}{20}, \quad \frac{24}{60} = \frac{8}{20}, \quad \frac{27}{60} = \frac{9}{20} .$$

We have : $\frac{1}{3} < \frac{7}{20}, \frac{8}{20}, \frac{9}{20} < \frac{1}{2}$ as the solution to the problem .

PROBLEMS

68. a. Find all the fractions between 1/2 and 2/3 that have a denominator 16 . ________________

  b. Find all the fractions between 3/7 and 1/2 that have a denominator 21 . ________________

c. Find all the fractions between 3/2 and 5/3 that have a denominator 15 . ____________

d. Find all the fractions between 2/5 and 1/2 that have a denominator 25 . ____________

24. Equality, inequality of fractions : denominator - numerator

Example 1 Which of the proper fractions 5/17 , 6/19 is the larger ?

Solution: This method though a trifle longer than the cross product (page 6) requires simpler computations .

Step 1 Write the two fractions $\frac{5}{17}$ $\frac{6}{19}$

Step 2 Rewrite the numerator of each fraction, and then subtract the numerator from the denominator. $\frac{5}{17-5} = \frac{5}{12}$ $\frac{6}{19-6} = \frac{6}{13}$

Step 3 Compare the results, namely, $\frac{5}{12}$ and $\frac{6}{13}$. It is not obvious which of the two fractions is the larger. Repeat Step 2 on 5/12 and 6/13 $\frac{5}{12-5} = \frac{5}{7}$ $\frac{6}{13-6} = \frac{6}{7}$

Step 4 Compare the results, namely $\frac{5}{7}$ and $\frac{6}{7}$. We have $\frac{5}{7} < \frac{6}{7}$ and so $\frac{5}{17} < \frac{6}{19}$.

Check the solution using the cross product (page 6) .

Just enough steps should be used until it becomes clear which of the two fractions is the larger .

PROBLEMS

69. Use the procedure shown above . For each pair of fractions determine which fraction is the larger. Use one of the symbols < , > .

a. $\frac{10}{21}$ ______ $\frac{13}{27}$

b. $\frac{5}{31}$ ______ $\frac{2}{13}$

c. $\frac{11}{23}$ ______ $\frac{23}{48}$

d. $\frac{15}{68}$ ______ $\frac{5}{27}$

70. Use the procedure above in Section 24 . Are the following pairs of fractions equal or unequal .

a. $\frac{4}{5}$ ______ $\frac{44}{55}$

b. $\frac{44}{55}$ ______ $\frac{444}{555}$

c. $\frac{4}{5}$ ______ $\frac{444}{555}$

d. $\frac{444}{555}$ ______ $\frac{4444}{5555}$

71. Project Take two improper fractions, say 7/4 and 11/5 . Follow the procedure above only now recopy the denominators and then subtract the denominators from the numerators. Do you get correct results ? Yes ______ No ______

FRACTION ACTION - ANSWERS
Level 1

We will use the notation 2/3, 1/2 in many answers instead of $\frac{2}{3}$, $\frac{1}{2}$ in order to save space.

1. $2 = 2/1$

2. $3\frac{1}{4} = \frac{4(3) + 1}{4} = 13/4$

3. $4\frac{1}{3} = \frac{3(4) + 1}{3} = 13/3$

4. $2\frac{5}{6} = \frac{6(2) + 5}{6} = 17/6$

5. $6 = 6/1$

6. $0 = 0/1$

7. The proper fractions with denominator 5 are 1/5, 2/5, 3/5, 4/5

8. 4/5 = 8/10, 12/15, 16/20, 20/25, 24/30, ..., 4n/5n, ...

9. 8/3 = 16/6, 24/9, 32/12, 40/15, 48/18, ..., 8n/3n, ...

10. 0/4 = 0/8, 0/12, 0/16, ..., 0n/4n = 0/4n, ...
Note: It is true that $0/4 = 0n/4n = 0/4n$ for all $n \neq 0$, but because of the 0 in the numerator it is also true that $0/4 = 0/n$ for any $n \neq 0$.

11. We must divide the numerator and denominator by their greatest common divisor (GCD) in order to find the equivalent fraction with the smallest denominator.

a. GCD (8, 20) = 4 $\frac{20}{8} = \frac{20 \div 4}{8 \div 4} = 5/2$

b. GCD(5, 60) = 5 $\frac{5}{60} = \frac{5 \div 5}{60 \div 5} = 1/12$

c. GCD (12, 30) = 6 $\frac{12}{30} = \frac{12 \div 6}{30 \div 6} = 2/5$

d. GCD(65, 5) = 5 $\frac{65}{5} = \frac{65 \div 5}{5 \div 5} = 13/1$

e. GCD(21, 4) = 1 $\frac{21}{4} = \frac{21 \div 1}{4 \div 1} = 21/4$

f. GCD(0,12) = 12 $\frac{0}{12} = \frac{0 \div 12}{12 \div 12} = 0/1$

Note: If the GCD of the numerator and denominator is 1, the denominator is the smallest of any equivalent fraction. See 11 (e).

12. a. 2/3 = 2(2)/3(2) = 4/6
b. 2/3 = 2(4)/3(4) = 8/12
c. 2/3 = 2(6)/3(6) = 12/18

13. a. 4/5 = 4(3)/5(3) = 12/15
b. 4/5 = 4(4)/5(4) = 16/20
c. 4/5 = 4(7)/5(7) = 28/35

14. $6/8 = \frac{6 \div 2}{8 \div 2} = 3/4$; $18/24 = \frac{18 \div 6}{24 \div 6} = 3/4$; $21/28 = \frac{21 \div 7}{28 \div 7} = 3/4$;
$39/52 = \frac{39 \div 13}{52 \div 13} = 3/4$

Since GCD(9, 16) = 1, 9/16 does not equal any fraction with a denominator smaller than 16.

15. $20/12 = \frac{20 \div 4}{12 \div 4} = 5/3$; $30/18 = \frac{30 \div 6}{18 \div 6} = 5/3$; $70/42 = \frac{70 \div 14}{42 \div 14} = 5/3$

Since GCD(25, 9) = 1, 25/9 is not equal to any fraction with a smaller denominator than 16. 30/50 is a proper fraction and so cannot equal the improper fraction of 5/3. A second argument is that GCD(30, 50) = 10. $30/50 = \frac{30 \div 10}{50 \div 10} = 3/5$ and so cannot equal any fraction with denominator smaller than 5.

16. Rewrite 1/4 and 5/6 as fractions with common denominator equal to

a. 24 1/4 = 1(6)/4(6) = 6/24; 5/6 = 5(4)/6(4) = 20/24
b. 36 1/4 = 1(9)/4(9) = 9/36; 5/6 = 5(6)/6(6) = 30/36

c. 48 — 1/4 = 1(12)/4(12) = 12/48; 5/6 = 5(8)/6(8) = 40/48
d. 108 — 1/4 = 1(27)/4(27) = 27/108; 5/6 = 5(18)/6(18) = 90/108

17. Any common denominator for 1/4 and 5/6 must be a common multiple of 4 and 6.
The multiples of 4 are 4, 8, 12, 16, 20, 24, 28, ...
6 are 6, 12, 18, 24, 30, 36, 42, ...
The smallest common multiple is 12.
1/4 = 1(3)/4(3) = 3/12 and 5/6 = 5(2)/6(2) = 10/12

18. a. 12 is not a multiple of 8 and so cannot be a common denominator for 5/6 and 11/8.
b. 48 is a common denominator for 5/6 and 11/8. 5/6 = 5(8)/6(8) = 40/48; 11/8 = 11(6)/8(6) = 66/48.
c. 16 is not a multiple of 6 and so cannot be a common denominator for 5/6 and 11/8.
d. 24 is a common denominator for 5/6 and 11/8. 5/6 = 5(4)/6(4) = 20/24; 11/8 = 11(3)/8(3) = 33/24

19. A common denominator for 1/2, 2/3, 5/6 is (2)(3)(6) = 36. However, the smallest number which would serve as a common denominator is 6.

20. a. 2/12, 4/12, 5/12, 6/12, 12/12
b. 1/7, 2/7, 4/7, 5/7, 10/7

21.

	Fractions	Fractions with common denominator	Larger fraction
a.	1/3, 2/5	1/3 = 5/15; 2/5 = 6/15	2/5 = 6/15
b.	2/7, 1/3	2/7 = 6/21; 1/3 = 7/21	1/3 = 7/21
c.	7/8, 3/4	7/8; 3/4 = 6/8	7/8
d.	5/12, 1/3	5/12; 1/3 = 4/12	5/12
e.	3/4, 25/32	3/4 = 24/32; 25/32	25/32
f.	0/2, 0/6	0/2 = 0/6	Fractions equal
g.	4/2, 6/3	4/2 = 12/6; 6/3 = 12/6	Fractions equal

22. a. 1/3 > 1/4 since 4(1) > 3(1)
b. 2/8 > 2/10 since 10(2) > 8(2)
c. 7/6 < 7/4 since 4(7) < 6(7)
d. 2/3 < 3/2 since 2(2) < 3(3)
e. 4/9 < 7/9 since 9(4) < 9(7)
f. 0/5 = 0/10 since 10(0) = 5(0)

23. a. 4/5 > 4/9 since 9(4) > 5(4)
b. 7/4 < 7/2 since 2(7) < 4(7)
c. 9/5 < 9/3 since 3(9) < 5(9)
d. 12/4 > 12/12 since 12(12) > 4(12)
The fractions in each pair have the same numerator.

24. If 2 fractions have the same nonzero numerator but different denominators then the fraction with the larger denominator is the smaller fraction.

The fractions in Problem 23 illustrate this fact. The truth of the general statement may be demonstrated in the following way.

Suppose b is a nonzero natural number and m, n are nonzero natural numbers. To compare b/m and b/n, we form the cross product

$$\frac{b}{m} \nwarrow\!\!\!\!\!\nearrow \frac{b}{n}$$

and compare nb and mb. If n < m, nb < mb, and b/m < b/n
n > m, nb > mb, and b/m > b/n

then the fraction with the larger denominator is the smaller fraction. Note, however, that if the common numerator were 0, the two fractions would be equal no matter what the denominators were.

25. a. 3/11, 3/10, 3/8, 3/7, 3/4, 3/2, 3/1
    b. 1/2, 2/3, 4/5, 1. The cross product methodis the simplest here.

26. a. 5/14 + 3/14 = $\frac{5 + 3}{14}$ = 8/14 (= 4/7)
    b. 13/10 + 19/10 = $\frac{13 + 19}{10}$ = 32/10 (= 16/5 = 3 1/5)
    c. 4/5 + 2/3 = 12/15 + 10/15 = $\frac{12 + 10}{15}$ = 22/15 (=1 7/15)
    d. 4 + 1 2/3 = 12/3 + 5/3 = $\frac{12 + 5}{3}$ = 17/3 (= 5 2/3)
    e. 4 1/3 + 2 2/3 = (4 + 2) + (1/3 + 2/3) = 4 + 2 + 1 = 7
    f. 2 1/3 + 3 1/6 = 7/3 + 19/6 = 14/6 + 19/6 = 33/6 (= 11/2 = 5 1/2)

27. Find the following sums.
    a. 1/3 + 1/6 = 2/6 + 1/6 = 3/6 (=1/2)
    b. 2/7 + 5/6 = 12/42 + 35/42 = 47/42 (= 1 5/42)
    c. 3/11 + 17/22 = 6/22 + 17/22 = 23/22 (= 1 1/22)
    d. 8/3 + 14/5 = 40/15 + 42/15 = 82/15 (=5 7/15)

28. a. 1/3 + 3/4 + 7/12 = 4/12 + 9/12 + 7/12 = 20/12 (= 5/3 = 1 2/3)
    b. 3/5 + 1/2 + 6/7 = 42/70 + 35/70 + 60/70 = 137/70 (= 1 67/70)

29. a. 7/10 - 3/10 = $\frac{7 - 3}{10}$ = 4/10 = 2/5
    b. 15/13 - 7/13 = $\frac{15 - 7}{13}$ = 8/13
    c. 4/5 - 3/7 = 4(7)/5(7) - 3(5)/7(5) = $\frac{28 - 15}{35}$ = 13/35
    d. 2 1/4 - 1 1/4 = 9/4 - 5/4 = 4/4 = 1
    e. 3 1/6 - 2 1/3 = 19/6 - 7/3 = 19/6 - 14/6 = 5/6

30. a. 1/2 - 1/3 = 3/6 - 2/6 = 1/6
    b. 2/5 - 1/6 = 12/30 - 5/30 = 7/30
    c. 3/4 - 3/8 = 6/8 - 3/8 = 3/8
    d. 4 1/2 - 2 1/4 = 9/2 - 9/4 = 18/4 - 9/4 = 9/4 (= 2 1/4)
       or
       4 1/2 - 2 1/4 = (4 - 2) + (1/2 - 1/4) = 2 + (2/4 - 1/4) = 2 + 1/4 = 2 1/4

31. a. 2/5 x 8/3 = 16/15 (= 1 1/15)
    b. 7/16 x 4/7 = 28/112 (= 1/4)
    c. 4 x 3 1/2 = 4/1 x 7/2 = 28/2 (= 14)
    d. 5/3 x 3/5 = 15/15 (= 1)
    e. 2/9 x 7/4 = 14/36 (= 7/18)

32. a. 1/3 of 2/7 = 1/3 x 2/7 = 2/21
    b. 1/5 of 1/3 = 1/5 x 1/3 = 1/15
    c. 1/3 of 6 = 1/3 x 6/1 = 6/3 (= 2)
    d. 1/4 of 0 = 1/4 x 0/1 = 0/4 (= 0)

33. a. $2/3\ (3/5 + 6/5) = 2/3\ (\frac{3+6}{5}) = 2/3 \times 9/5 = 18/15\ \ (= 6/5)$

or

$2/3\ (3/5 + 6/5) = (2/3 \times 3/5) + (2/3 \times 6/5) = 6/15 + 12/15 = 18/15\ \ (= 6/5)$

b. $5/8\ (2 - 1\ 1/4) = 5/8\ (8/4 - 5/4) = 5/8 \times 3/4 = 15/32$

or

$5/8\ (2 - 1\ 1/4) = (5/8 \times 8/4) - (5/8 \times 5/4) = 40/32 - 25/32 = 15/32$

34. a. The reciprocal of 5/4 is 4/5 since 5/4 x 4/5 = 20/20 = 1
b. The reciprocal of 2 1/2 is 2/5 since 5/2 x 2/5 = 10/10 = 1
c. The reciprocal of 1 is 1 since 1 x 1 = 1

35. 0 does <u>not</u> have a reciprocal since the product of any number with 0 is 0 and not 1.

36. a. $2/5 \div 3/4 = 2/5 \times 4/3 = 8/15$

b. $1\ 1/2 \div 3 = 3/2 \times 1/3 = 3/6\ \ (= 1/2)$

c. $2\ 1/3 \div 3\ 1/5 = 7/3 \times 5/16 = 35/48$

d. $\dfrac{\frac{3}{8}}{\frac{2}{3}} = 3/8 \times 3/2 = 9/16$

g. $\dfrac{\frac{2}{5}}{3} = 2/5 \times 1/3 = 2/15$

e. $\dfrac{\frac{3}{4}}{\frac{4}{3}} = 3/4 \times 3/4 = 9/16$

h. $\dfrac{1\frac{2}{3}}{2\frac{1}{5}} = 5/3 \times 5/11 = 25/33$

f. $\dfrac{5}{\frac{1}{3}} = 5/1 \times 3/1 = 15/1\ \ (= 15)$

i. $\dfrac{0}{\frac{1}{6}} = 0 \times 6/1 = 0$

37. a. $5/7 \div 2/7 = 5/2$
b. $4/11 \div 10/11 = 4/10\ \ (= 2/5)$
c. $2/5 \div 3/4 = 8/20 \div 15/20 = 8/15$
d. $1\ 1/2 \div 3 = 3/2 \div 3/1 = 3/2 \div 6/2 = 3/6\ \ (= 1/2)$
e. $2\ 1/3 \div 3\ 1/5 = 7/3 \div 16/5 = 35/15 \div 48/15 = 35/48$

38. a. $\dfrac{\frac{3}{8}}{\frac{5}{8}} = \dfrac{\frac{3}{8}(8)}{\frac{5}{8}(8)} = 3/\ 5$

d. $\dfrac{\frac{3}{8}}{\frac{2}{3}} = \dfrac{\frac{3}{8}\ (24)}{\frac{2}{3}\ (24)} = 9/16$

b. $\dfrac{5}{\frac{1}{3}} = \dfrac{5\ (3)}{\frac{1}{3}\ (3)} = 15$

e. $\dfrac{1\frac{2}{3}}{2\frac{1}{5}} = \dfrac{\frac{5}{3}}{\frac{11}{5}} = \dfrac{\frac{5}{3}\ (15)}{\frac{11}{5}\ (15)} = 25/33$

c. $\dfrac{\frac{2}{5}}{3} = \dfrac{\frac{2}{5}\ (5)}{3\ (5)} = 2/15$

39. a. $\begin{array}{l}\text{Row 2} \rightarrow \\ \text{Row 3} \rightarrow\end{array} \frac{2}{3},\ \frac{4}{6},\ \frac{6}{9},\ \frac{8}{12},\ \frac{10}{15},\ \ldots,\ \frac{2n}{3n},\ \ldots$

The fractions are all equal.

b. $\begin{array}{l}\text{Row 6} \rightarrow \\ \text{Row 3} \rightarrow\end{array} \frac{6}{3},\ \frac{12}{6},\ \frac{18}{9},\ \frac{24}{12},\ \frac{30}{15},\ \ldots,\ \frac{6n}{3n},\ \ldots$

The fractions are all equal.

40.
$$\begin{array}{l}\text{Row } 4 \rightarrow \\ \text{Row } 9 \rightarrow\end{array} \frac{4}{9},\quad \frac{8}{18},\quad \frac{12}{27},\quad \frac{16}{36},\quad \frac{20}{45},\quad \ldots,\quad \frac{4n}{9n},\quad \ldots$$

Note that since 4 and 9 have 1 as their GCD, column 1 is the only column containing both 4 and 9. Rows 4 and 9 are the only ones which give fractions equal to 4/9, if we want to have 4/9 itself included.

41.
$$\begin{array}{l}\text{Row } 5 \rightarrow \\ \text{Row } 8 \rightarrow\end{array} \frac{5}{8},\quad \frac{10}{16},\quad \frac{15}{24},\quad \frac{20}{32},\quad \frac{25}{40},\quad \ldots,\quad \frac{5n}{8n},\quad \ldots$$

Since GCD (5, 8) = 1, the comment about the choice of rows in Problem 40 applies here also.

42. 24 and 28 are not relatively prime. In fact, 24/28 = 12/14 = 6/7. There are three pairs of rows which yield 24/28 and fractions equal to it.

$$\begin{array}{l}\text{Row } 6 \rightarrow \\ \text{Row } 7 \rightarrow\end{array} \frac{6}{7},\quad \frac{12}{14},\quad \frac{18}{21},\quad \frac{24}{28},\quad \frac{30}{35},\quad \ldots,\quad \frac{6n}{7n},\quad \ldots$$

$$\begin{array}{l}\text{Row } 12 \rightarrow \\ \text{Row } 14 \rightarrow\end{array} \frac{12}{14},\quad \frac{24}{28},\quad \frac{36}{42},\quad \frac{48}{56},\quad \frac{60}{70},\quad \ldots,\quad \frac{12n}{14n},\quad \ldots$$

$$\begin{array}{l}\text{Row } 24 \rightarrow \\ \text{Row } 28 \rightarrow\end{array} \frac{24}{28},\quad \frac{48}{56},\quad \frac{72}{84},\quad \frac{96}{112},\quad \frac{120}{140},\quad \ldots,\quad \frac{24n}{28n},\quad \ldots$$

43. 12/27 = 4/9

$$\begin{array}{l}\text{Row } 4 \longrightarrow \\ \text{Row } 9 \longrightarrow\end{array} \frac{4}{9},\quad \frac{8}{18},\quad \frac{12}{27},\quad \frac{16}{36},\quad \frac{20}{45},\quad \ldots,\quad \frac{4n}{9n},\quad \ldots$$

$$\begin{array}{l}\text{Row } 12 \longrightarrow \\ \text{Row } 27 \longrightarrow\end{array} \frac{12}{27},\quad \frac{24}{54},\quad \frac{36}{81},\quad \frac{48}{108},\quad \frac{60}{135},\quad \ldots,\quad \frac{12n}{27n},\quad \ldots$$

44.
$$\begin{array}{l}\text{Row } 9 \rightarrow \\ \text{Row } 5 \rightarrow\end{array} \frac{9}{5},\quad \frac{18}{10},\quad \frac{27}{15},\quad \frac{36}{20},\quad \frac{45}{25},\quad \ldots,\quad \frac{9n}{5n},\quad \ldots$$

GCD (9, 5) = 1. See comment for Problem 40.

45. Method 1
Write 1 1/4 = 5/4. Proceed as in Problem 44.

$$\begin{array}{l}\text{Row } 5 \longrightarrow \\ \text{Row } 4 \longrightarrow\end{array} \frac{5}{4},\quad \frac{10}{8},\quad \frac{15}{12},\quad \frac{20}{16},\quad \frac{25}{20},\quad \ldots,\quad \frac{5n}{4n},\quad \ldots$$

Method 2
We separate 1 1/4 into a natural number 1 and a proper fraction 1/4. Using rows 1 and 4 we obtain

$$\begin{array}{l}\text{Row } 1 \rightarrow \\ \text{Row } 4 \rightarrow\end{array} \frac{1}{4},\quad \frac{2}{8},\quad \frac{3}{12},\quad \frac{4}{16},\quad \frac{5}{20},\quad \ldots,\quad \frac{n}{4n},\quad \ldots$$

Therefore "fractions" equal to 1 1/4 are 1 2/8, 1 3/12, 1 4/16, 1 5/20, ..., 1 n/4n. If these mixed numbers are rewritten as common fractions, the resulting fractions are precisely the ones obtained in method 1, namely, 10/8, 15/12, 20/16, 25/20, ..., 5n/4n...

46. a.

4/5, 8/10, 12/15, ..., 4n/5n, ...	< 5/6	Row 4/Row 5
3/4, 6/8, 9/12, ..., 3n/4n, ...	< 5/6	Row 3/Row 4
2/3, 4/6, 6/9, ..., 2n/3n, ...	< 5/6	Row 2/Row 3
1/2, 2/4, 3/6, ..., n/2n, ...	< 5/6	Row 1/Row 2

b.

5/6, 10/12, 15/18, ..., 5n/6n, ...	< 6/7	Row 5/Row 6
4/5, 8/10, 12/15, ..., 4n/5n, ...	< 6/7	Row 4/Row 5
3/4, 6/8, 9/12, ..., 3n/4n, ...	< 6/7	Row 3/Row 4
2/3, 4/6, 6/9, ..., 2n/3n, ...	< 6/7	Row 2/Row 3
1/2, 2/4, 3/6, ..., n/2n, ...	< 6/7	Row 1/Row 2

c. First note that 21/24 = 7/8. Then
6/7, 12/14, 18/21, ..., 6n/7n,... < 7/8 Row 6/Row 7
5/6, 10/12, 15/18, ..., 5n/6n, ... < 7/8 Row 5/Row 6
4/5, 8/10, 12/15, ..., 4n/5n, ... < 7/8 Row 4/Row 5
3/4, 6/8, 9/12, ..., 3n/4n, ... < 7/8 Row 3/Row 4
2/3, 4/6, 6/9, ..., 2n/3n, ... < 7/8 Row 2/Row 3
1/2, 2/4, 3/6, ..., n/2n, ... < 7/8 Row 1/Row 2

d. First write 56/63 = 8/9. Then
7/8, 14/16, 21/24, ..., 7n/8n, ... < 8/9 Row 7/Row 8
6/7, 12/14, 18/21, ..., 6n/7n, ... < 8/9 Row 6/Row 7
5/6, 10/12, 15/18, ..., 5n/6n, ... < 8/9 Row 5/Row 6
4/5, 8/10, 12/15, ..., 4n/5n, ... < 8/9 Row 4/Row 5
3/4, 6/8, 9/12, ..., 3n/4n, ... < 8/9 Row 3/Row 4
2/3, 4/6, 6/9, ..., 2n/3n, ... < 8/9 Row 2/Row 3
1/2, 2/4, 3/6, ..., n/2n, ... < 8/9 Row 1/Row 2

47. a. 4/5, 8/10, 12/15, ..., 4n/5n, ... > 3/4 Row 4/Row 5
5/6, 10/12, 15/18, ..., 5n/6n, ...> 3/4 Row 5/Row 6
6/7, 12/14, 18/21, ..., 6n/7n, ...> 3/4 Row 3/Row 4
.
.
.

b. 7/8, 14/16, 21/24, ..., 7n/8n, ... > 6/7 Row 7/Row 8
8/9, 16/18, 24/27, ..., 8n/9n, ... > 6/7 Row 8/Row 9
9/10, 18/20, 27/30, ..., 9n/10n, ... > 6/7 Row 9/Row 10
.
.

c. First rewrite 20/30 = 2/3
3/4, 6/8, 9/12, ..., 3n/4n, ... > 2/3 Row 3/Row 4
4/5, 8/10, 12/15, ..., 4n/5n, ...> 2/3 Row 4/Row 5
5/6, 10/12, 15/18, ..., 5n/6n, ... > 2/3 Row 5/Row 6
.
.
.

d. First rewrite 24/27 = 8/9
9/10, 18/20, 27/30, ..., 9n/10n, ... > 8/9 Row 9/Row 10
10/11, 20/22, 30/33, ..., 10n/11n, ...> 8/9 Row 10/Row 11
11/12, 22/24, 33/36, ..., 11n/12n, ...> 8/9 Row 11/Row 12
.
.
.

Note that the table, as printed, only has the first ten rows and so only contains 9/10. We can extend the table to as many rows as needed. Here at least two more rows are necessary.

48. a. 2/4 and 3/5 The numbers 20, 40, ... appear in both rows 4 and 5
2/4 = 10/20 = 20/40 ... from rows 2 and 4
3/5 = 12/20 = 24/40 ... from rows 3 and 5

b. 5/6 and 1/3 The numbers 6, 12, 18, ... appear in both rows 3 and 6
1/3 = 2/6 = 4/12 ... from rows 1 and 3
5/6 = 5/6 = 10/12 ... from rows 5 and 6

c. 5/7 and 6/9 The numbers 63, 126, ...appear in both rows 7 and 9
5/7 = 45/63 = 90/126 ... from rows 5 and 7
6/9 = 42/63 = 84/126 ... from rows 6 and 9

49. Strip 1

$\frac{2}{3}$	$\frac{4}{6}$	$\frac{6}{9}$	$\frac{8}{12}$	$\frac{10}{15}$	$\frac{12}{18}$	$\frac{14}{21}$	$\frac{16}{24}$

Row 2
Row 3

Strip 2

$\frac{3}{5}$	$\frac{6}{10}$	$\frac{9}{15}$	$\frac{12}{20}$	$\frac{15}{25}$	$\frac{18}{30}$

Row 3
Row 5

a. 2/3 + 3/5 = 10/15 + 9/15 = $\frac{10 + 9}{15}$ = 19/15 (= 1 4/15)

b. 2/3 - 3/5 = 10/15 - 9/15 = $\frac{10 - 9}{15}$ = 1/15

c. 2/3 ÷ 3/5 = 10/15 ÷ 9/15 = 10/9 (using Section 14, page 12)

50. The sum of the numbers in each row, each column, and each diagonal is 15. Thus, 15 is the magic number.

51. Magic Number of magic square 1 : 6
Magic Number of magic square 2 : 6
Magic Number of magic square 3 : 2 1/2

52. a. Magic Number 3

$\frac{1}{3}$	$1\frac{1}{2}$	$1\frac{1}{6}$
$1\frac{5}{6}$	1	$\frac{1}{6}$
$\frac{5}{6}$	$\frac{1}{2}$	$1\frac{2}{3}$

b. Magic Number 3

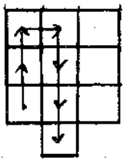
$\frac{1}{2}$	$1\frac{3}{8}$	$1\frac{1}{8}$
$1\frac{5}{8}$	1	$\frac{3}{8}$
$\frac{7}{8}$	$\frac{5}{8}$	$1\frac{1}{2}$

c. Magic Number 2 1/2

$\frac{2}{3}$	$\frac{1}{2}$	$1\frac{1}{3}$
$1\frac{1}{2}$	$\frac{5}{6}$	$\frac{1}{6}$
$\frac{1}{3}$	$1\frac{1}{6}$	1

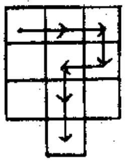
53. Magic square 1 : 4 1/4
Magic square 2 : 5 2/3

54. a.



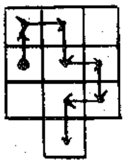
Check: 7/8 + 3/16 + 1/2 + 7/16 + 5/8 + 3/8
= 14/16 + 3/16 + 8/16 + 7/16 + 10/16 + 6/16
= 48/16 = 3

b.



Check: 1/5 + 3/5 + 3/10 + 1/20 + 7/10 + 2/5
= 4/20 + 12/20 + 6/20 + 1/20 + 14/20 + 8/20
= 45/20 = 2 1/4

c.



Check: 1/8 + 1/16 + 1/3 + 15/16 + 7/8 + 3/4 + 1/4
= 6/48 + 3/48 + 16/48 + 45/48 + 42/48 + 36/48 + 12/48
= 160/48 = 3 1/3

55. a. 5/12 = 1/3 + 1/12
b. 8/15 = 1/2 + 1/30
c. 7/20 = 1/3 + 1/60
d. 5/6 = 1/2 + 1/3
e. 2/3 = 1/2 + 1/6
f. 3/5 = 1/2 + 1/10
g. 4/15 = 1/4 + 1/60
h. 13/60 = 1/5 + 1/60
i. 41/60 = 1/2 + 1/6
+ 1/60

56. Project
1. 2/5, (3 x 5) 6/15 = 5/15 + 1/15 = 1/3 + 1/15
2. 2/7, (4 x 7) 8/28 = 7/28 + 1/28 = 1/4 + 1/28
3. 2/9, (2 x 9) 4/18 = 3/18 + 1/18 = 1/6 + 1/18
4. 2/11, (6 x 11) 12/66 = 11/66 + 1/66 = 1/6 + 1/66
5. 2/13, (6 x 13) 12/78 = 6/78 + 6/78 = 6/78 + 3/78 + 2/78 + 1/78
= 1/13 + 1/26 + 1/39 + 1/78

Note: If we went to the next multiple,
14/91 = 13/91 + 1/91 = 1/7 + 1/91
6. 2/15, (2 x 15) 4/30 = 3/30 + 1/30 = 1/10 + 1/30
7. 2/17, (6 x 17) 12/102 = 6/102 + 6/102 = 6/102 + 3/102 + 2/102 + 1/102
= 1/17 + 1/34 + 1/51 + 1/102

Note: The next multiple that gives a decomposition of the desired type is
18/153 = 17/153 + 1/153 = 1/9 + 1/153
8. 2/19, (6 x 19) 12/114 = 6/114 + 6/114 = 6/114 + 3/114 + 2/114 + 1/114
= 1/19 + 1/38 + 1/57 + 1/114

Note: 20/190 = 19/190 + 1/190 = 1/10 + 1/190
9. 2/21, (2 x 21) 4/42 = 3/42 + 1/42 = 1/14 + 1/42
10. 2/23, (6 x 23) 12/138 = 6/138 + 3/138 + 2/138 + 1/138
= 1/23 + 1/46 + 1/69 + 1/138

Note: 24/276 = 23/276 + 1/276 = 1/12 + 1/276
11. 2/25, (3 x 25) 6/75 = 5/75 + 1/75 = 1/25 + 1/75
12. 2/27, (2 x 27) 4/54 = 3/54 + 1/54 = 1/18 + 1/54
13. 2/29, (6 x 29) 12/174 = 6/174 + 3/174 + 2/174 + 1/174
= 1/29 + 1/58 + 1/87 + 1/174

Note: 30/435 = 29/435 + 1/435 = 1/15 + 1/435
14. 2/31, (6 x 31) 12/186 = 6/186 + 3/186 + 2/186 + 1/186
= 1/31 + 1/62 + 1/93 + 1/186

Note: 32/496 = 31/496 + 1/496 = 1/16 + 1/496
15. 2/33, (2 x 33) 4/66 = 3/66 + 1/66 = 1/22 + 1/66
16. 2/35, (3 x 35) 6/105 = 5/105 + 1/105 = 1/21 + 1/105
17. 2/37, (6 x 37) 12/222 = 6/222 + 3/222 + 2/222 + 1/222
= 1/37 + 1/74 + 1/111 + 1/222

Note: 38/703 = 37/703 + 1/703 = 1/19 + 1/703
18. 2/39, (2 x 39) 4/78 = 3/78 + 1/78 = 1/26 + 1/78
19. 2/41, (6 x 41) 12/246 = 6/246 + 3/246 + 2/246 + 1/246
= 1/41 + 1/82 + 1/123 + 1/246

Note: 42/861 = 41/861 + 1/861 = 1/21 + 1/861
20. 2/43, (6 x 43) 12/258 = 6/258 + 3/258 + 2/258 + 1/258
= 1/43 + 1/86 + 1/129 + 1/258

Note: 44/946 = 43/946 + 1/946 = 1/22 + 1/946
21. 2/45, (2 x 45) 4/90 = 3/90 + 1/90 = 1/30 + 1/90
22. 2/47, (6 x 47) 12/282 = 6/282 + 3/282 + 2/282 + 1/282
= 1/47 + 1/94 + 1/141 + 1/282

Note: 48/1128 = 47/1128 + 1/1128 = 1/24 + 1/1128
23. 2/49, (4 x 49) 8/196 = 7/196 + 1/196 = 1/28 + 1/196
24. 2/51, (2 x 51) 4/102 = 3/102 + 1/102 = 1/34 + 1/102
25. 2/53, (6 x 53) 12/318 = 6/318 + 3/318 + 2/318 + 1/318
= 1/53 + 1/106 + 1/159 + 1/318

Note: 54/1431 = 53/1431 + 1/1431 = 1/27 + 1/1431
26. 2/55, (3 x 55) 6/165 = 5/165 + 1/165 = 1/33 + 1/165
27. 2/57, (2 x 57) 4/114 = 3/114 + 1/114 = 1/38 + 1/114

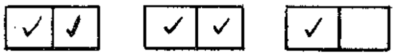
28. 2/59, (6 x 59) 12/354 = 6/354 + 3/354 + 2/354 + 1/354
= 1/59 + 1/118 + 1/177 + 1/354
Note: 60/1770 = 59/1770 + 1/1770 = 1/30 + 1/1770
29. 2/61, (6 x 61) 12/366 = 6/366 + 3/366 + 2/366 + 1/366
= 1/61 + 1/122 + 1/183 + 1/366
Note: 62/1891 = 61/1891 + 1/1891 = 1/31 + 1/1891
30. 2/63, (2 x 63) 4/126 = 3/126 + 1/126 = 1/42 + 1/126
31. 2/65, (3 x 65) 6/195 = 5/195 + 1/195 = 1/39 + 1/195
32. 2/67, (6 x 67) 12/402 = 6/402 + 3/402 + 2/402 + 1/402
= 1/67 + 1/134 + 1/201 + 1/402
Note: 68/2278 = 67/2278 + 1/2278 = 1/34 + 1/2278
33. 2/69, (2 x 69) 4/138 = 3/138 + 1/138 = 1/46 + 1/138
34. 2/71, (6 x 71) 12/426 = 6/426 + 3/426 + 2/426 + 1/426
= 1/71 + 1/142 + 1/213 + 1/426
Note: 72/2556 = 71/2556 + 1/2556 = 1/36 + 1/2556
35. 2/73, (6 x 73) 12/438 = 6/438 + 3/438 + 2/438 + 1/438
= 1/73 + 1/146 + 1/219 + 1/438
Note: 74/2701 = 73/2701 + 1/2701 = 1/37 + 1/2701
36. 2/75, (2 x 75) 4/150 = 3/150 + 1/150 = 1/50 + 1/150
37. 2/77, (4 x 77) 8/308 = 7/308 + 1/308 = 1/44 + 1/308
38. 2/79, (6 x 79) 12/474 = 6/474 + 3/474 + 2/474 + 1/474
= 1/79 + 1/158 + 1/237 + 1/474
Note: 80/3160 = 79/3160 + 1/3160 = 1/40 + 1/3160
39. 2/81, (2 x 81) 4/162 = 3/162 + 1/162 = 1/54 + 1/162
40. 2/83, (6 x 83) 12/498 = 6/498 + 3/498 + 2/498 + 1/498
= 1/83 + 1/166 + 1/249 + 1/498
Note: 84/3486 = 83/3486 + 1/3486 = 1/42 + 1/3486
41. 2/85, (3 x 85) 6/255 = 5/255 + 1/255 = 1/51 + 1/255
42. 2/87, (2 x 87) 4/174 = 3/174 + 1/174 = 1/58 + 1/174
43. 2/89, (6 x 89) 12/534 = 6/534 + 3/534 + 2/534 + 1/534
= 1/89 + 1/178 + 1/267 + 1/534
Note: 90/4005 = 89/4005 + 1/4005 = 1/45 + 1/4005
44. 2/91, (4 x 91) 8/364 = 7/364 + 1/364 = 1/52 + 1/364
45. 2/93, (2 x 93) 4/186 = 3/186 + 1/186 = 1/62 + 1/186
46. 2/95, (3 x 95) 6/285 = 5/285 + 1/285 = 1/57 + 1/285
47. 2/97, (6 x 97) 12/582 = 6/582 + 3/582 + 2/582 + 1/582
= 1/97 + 1/194 + 1/291 + 1/582
Note: 98/4753 = 97/4753 + 1/4753 = 1/49 + 1/4753
48. 2/99, (2 x 99) 4/198 = 3/198 + 1/198 = 1/66 + 1/198
49. 2/101, (6 x 101) 12/606 = 6/606 + 3/606 + 2/606 + 1/606
= 1/101 + 1/202 + 1/303 + 1/606
Note: 102/5151 = 101/5151 + 1/5151 = 1/51 + 1/5151

57. a. 2/7
Divide each loaf into 4 equal parts.
Each one gets 1/4 loaf.
Thus, 2/7 = 1/4 + more

Remainder is 1/4, □

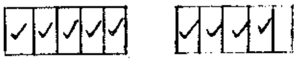
Divide the remainder 1/4 into 7 equal parts.
Each one gets 1/7 of 1/4 = 1/28.
Thus, 2/7 = 1/4 + 1/28.

b. 3/5
Divide each loaf into 2 equal parts.
Then, 3/5 = 1/2 + more.



Divide the remaining 1/2 into 5 equal parts.
Each is 1/5 of 1/2 = 1/10.
Thus, 3/5 = 1/2 + 1/10

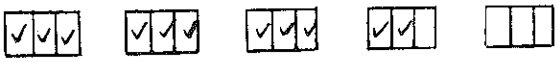
c. 2/9
Divide each loaf into 5 equal parts.
Then, 2/9 = 1/5 + more.



Divide the remaining 1/5 into 9 equal parts.
Each is 1/9 of 1/5 = 1/45.
Thus, 2/9 = 1/5 + 1/45

1/9 of 1/5 = 1/45

d. 5/11
Divide each loaf into 3 parts.
5/11 = 1/3 + more.



Divide each of the remaining four 1/3's into 3 parts.
Each is 1/3 of 1/3 = 1/9.
Thus, 5/11 = 1/3 + 1/9 + more.

1/3 of 1/3 = 1/9

Divide the remaining 1/9 into 11 equal parts.
Each is 1/11 of 1/9 = 1/99.
Thus, 5/11 = 1/3 + 1/9 + 1/99.

1/11 of 1/9 = 1/99

58. a. 1/2 + 1/3 + 1/9 = 9/18 + 6/18 + 2/18 = 17/18
b. No
c. 17/18 x 18 = 17
d. The sum of the fractions is not 1 but 17/18 and 17/18 of 18 = 17. Thus, only 17 horses are given away. Methods for choosing fractions in this way are given on pages 25 and 26.

59. a. If we start with 5 and get 1 more we have 6. Now $D_6 = \{1, 2, 3, 6\}$. We need three numbers which divide 6 and whose sum is 5. In this case the problem cannot be solved. However, it is possible to find 2 numbers which divide 6 and whose sum is 5.
Solution: 5/6 = 2/6 + 3/6 = 1/3 + 1/2
The fractions are 1/3 and 1/2.

b. If we start with 19 and get 1 more we have 20. Now $D_{20} = \{1, 2, 4, 5, 10, 20\}$. We need 3 numbers which divide 20 and whose sum is 19.
Solution: 19/20 = 4/20 + 5/20 + 10/20 = 1/5 + 1/4 + 1/2
The fractions are 1/5, 1/4 and 1/2.

c. If we start with 15 and get 1 more we have 16. Now $D_{16} = \{1, 2, 4, 8, 16\}$. We need 3 numbers which divide 16 and whose sum is 15. No solution is possible. However, it is possible to find 4 such numbers.
15/16 = 1/16 + 2/16 + 4/16 + 8/16 = 1/16 + 1/8 + 1/4 + 1/2
The fractions are 1/16, 1/8, 1/4 and 1/2.

d. If we start with 23 and get 1 more we have 24. Now $D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$. We need 3 numbers which divide 24 and whose sum is 23.
Solution: 23/24 = 12/24 + 8/24 + 3/24 = 1/2 + 1/3 + 1/8
The fractions are 1/2, 1/3 and 1/8.

e. If we start with 9 and get 1 more we have 10. Now $D_{10} = \{1, 2, 5, 10\}$. We need 3 numbers which divide 10 and whose sum is 9. There is no solution to this problem.

60. a. Begin with the divisors of 6, $D_6 = \{1, 2, 3, 6\}$.

Then		Number Selection
1/2 x 6 = 3	⟶	3
1/3 x 6 = 2	⟶	2
1/6 x 6 = 1		
		5

The only combination which adds up to 5 is 3 + 2; the only solution uses two unit fractions, 1/2 and 1/3. There is no solution using three unit fractions.

b. Begin with the divisors of 20, $D_{20} = \{1, 2, 4, 5, 10, 20\}$

Then	Number Selection
1/2 x 20 = 10	10
1/4 x 20 = 5	5
1/5 x 20 = 4	4
1/10 x 20 = 2	
1/20 x 20 = 1	
	19

The only combination which adds up to 19 is 10 + 5 + 4; the only solution uses three unit fractions, 1/2, 1/4 and 1/5.

c. Begin with the divisors of 16, $D_{16} = \{1, 2, 4, 8, 16\}$

Then	Number Selection
1/2 x 16 = 8	8
1/4 x 16 = 4	4
1/8 x 16 = 2	2
1/16 x 16 = 1	1
	15

The only combination which adds up to 15 is 1 + 2 + 4 + 8; the only solution uses four unit fractions, 1/2, 1/4, 1/8 and 1/16; there is no solution using three unit fractions.

d. Begin with the divisors of 24, $D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$

Then	Number Selection				
1/2 x 24 = 12	12	12	12	12	
1/3 x 24 = 8	8	8			8
1/4 x 24 = 6			6	6	6
1/6 x 24 = 4			4		4
1/8 x 24 = 3	3			3	3
1/12 x 24 = 2		2		2	2
1/24 x 24 = 1		1	1		
	23	23	23	23	23

There is one solution with three unit fractions: 1/2, 1/3, 1/8.
There are three solutions with four unit fractions:
1) 1/2, 1/3, 1/12, 1/24   2) 1/2, 1/4, 1/6, 1/24   3) 1/2, 1/4, 1/8, 1/12
There is one solution with five unit fractions: 1/3, 1/4, 1/6, 1/8, 1/12

e. Begin with the divisors of 10, $D_{10} = \{1, 2, 5, 10\}$
Then
1/2 x 10 = 5
1/5 x 10 = 2
1/10 x 10 = 1
No combination of 1, 2, 5 adds up to 9. There is no solution of the desired type for 9 horses + 1 horse.

61. a. Since all the exact divisors of 6 are 1, 2, 3, 6, $1/1 + 1/2 + 1/3 + 1/6 = \frac{1+2+3+6}{6} = \frac{12}{6}$ (= 2). Check: 6 will serve as a common denominator.

$$1/1 + 1/2 + 1/3 + 1/6 = (1 \times 6)/(1 \times 6) + (1 \times 3)/(2 \times 3) + (1 \times 2)/(3 \times 2) + 1/6$$
$$= 6/6 + 3/6 + 2/6 + 1/6 = \frac{6+3+2+1}{6} = \frac{12}{6} \ (= 2)$$

b. Since all the exact divisors of 9 are 1, 3, 9, $1/1 + 1/3 + 1/9 = \frac{1+3+9}{9} = \frac{13}{9}$. Check: 9 will serve as a common denominator.

$$1/1 + 1/3 + 1/9 = (1 \times 9)/(1 \times 9) + (1 \times 3)/(3 \times 3) + 1/9 = 9/9 + 3/9 + 1/9$$
$$= \frac{9+3+1}{9} = \frac{13}{9}$$

c. Since all the exact divisors of 12 are 1, 2, 3, 4, 6, 12,

$$1/1 + 1/2 + 1/3 + 1/4 + 1/6 + 1/12 = \frac{1 + 2 + 3 + 4 + 6 + 12}{12} = \frac{28}{12} \quad (= \frac{7}{3})$$

Check: 12 will serve as a common denominator. Follow the procedure in (a), (b).

d. Since all the exact divisors of 27 are 1, 3, 9, 27, $1/1 + 1/3 + 1/9 + 1/27 = \frac{1 + 3 + 9 + 27}{27} = \frac{40}{27}$. Check: 27 will serve as a common denominator. Follow the procedure in (a), (b).

62. a.

			N	1	2	3	$\boxed{4}$	5	6	7	8	...
D	1	2	3	4	5	6	$\boxed{7}$	8	9	10	11	...

4/7 is greater than 3/6, 2/5, 1/4. 4/7 is less than 5/8, 6/9, 7/10, 8/11, ...

b.

N	1	2	3	4	5	6	$\boxed{7}$	8	9	10	...
			D	1	2	3	$\boxed{4}$	5	6	7	...

7/4 is greater than 8/5, 9/6, 10/7, ... 7/4 is less than 6/3, 5/2, 4/1.

c.

					N	1	$\boxed{2}$	3	4	5	6	...
D	1	2	3	4	5	6	$\boxed{7}$	8	9	10	11	...

2/7 is greater than 1/6. 2/7 is less than 3/8, 4/9, 5/10, 6/11, ...

d.

				N	1	2	3	4	$\boxed{5}$	6	7	8	9	...
D	1	2	3	4	5	6	7	8	$\boxed{9}$	10	11	12	13	...

5/9 is greater than 4/8, 3/7, 2/6, 1/5. 5/9 is less than 6/10. 7/11, 8/12, ...

63. a.

$$\frac{1/2 + 3/5}{2} = \frac{5/10 + 6/10}{2} = \frac{11/10}{2} = 1/2 \times 11/10 = 11/20$$

Check: 11/20 < 3/5 since 5 x 11 < 20 x 3 = 55 < 60
1/2 < 11/20 since 20 x 1 < 2 x 11 = 20 < 22

b.

$$\frac{3/2 + 5/3}{2} = \frac{9/6 + 10/6}{2} = \frac{19/6}{2} = 1/2 \times 19/6 = 19/12$$

Check: 19/12 < 5/3 since 3 x 19 < 12 x 5 = 57 < 60
3/2 < 19/12 since 12 x 3 < 2 x 19 = 36 < 38

c.

$$\frac{2/7 + 1/3}{2} = \frac{6/21 + 7/21}{2} = \frac{13/21}{2} \; 1/2 \times 13/21 = 13/42$$

Check: 13/42 < 1/3 since 3 x 13 < 42x 1 = 39 < 42
2/7 < 13/42 since 42 x 2 < 7 x 13 = 84 < 91

d.

$$\frac{1/4 + 1/3}{2} = \frac{3/12 + 4/12}{2} = \frac{7/12}{2} = 1/2 \times 7/12 = 7/24$$

Check: 7/24 < 1/3 since 3 x 7 < 24 x 1 = 21 < 24
1/4 < 7/24 since 24 x 1 < 4 x 7 = 24 < 28

e. We can show that 3/4 is half-way between 2/3 and 5/6 by showing that 3/4 - 2/3 = 5/6 - 3/4.
3/4 - 2/3 = 9/12 - 8/12 = 1/12
5/6 - 3/4 = 10/12 - 9/12 = 1/12

64. a. 1/4 < (1 +1)/(4+3) < 1/3 or 1/4 < 2/7 < 1/3
Check: 1/4 < 2/7 since 7 x 1 < 4 x 2 = 7 < 8
2/7 < 1/3 since 3 x 2 < 7 x 1 = 6 < 7

b. $2/3 < (2+3)/(3+4) < 3/4$ or $2/3 < 5/7 < 3/4$
Check: $2/3 < 5/7$ since $7 \times 2 < 3 \times 5 = 14 < 15$
$5/7 < 3/4$ since $4 \times 5 < 7 \times 3 = 20 < 21$

c. $3/8 < (3+5)/(8+13) < 5/13$ or $3/8 < 8/21 < 5/13$
Check: $3/8 < 8/21$ since $21 \times 3 < 8 \times 8 = 63 < 64$
$8/21 < 5/13$ since $13 \times 8 < 21 \times 5 = 104 < 105$

d. $3/7 < (3+4)/(7+9) < 4/9$ or $3/7 < 7/16 < 4/9$
Check: $3/7 < 7/16$ since $16 \times 3 < 7 \times 7 = 48 < 49$
$7/16 < 4/9$ since $9 \times 7 \quad 16 \times 4 = 63 < 64$

e. To show that 7/9 is not half-way between 2/3 and 5/6, we find the differences $7/9 - 2/3$ and $5/6 - 7/9$. $7/9 - 2/3 = 7/9 - 6/9 = 1/9$
$5/6 - 7/9 = 15/18 - 14/18 = 1/18$
Since the differences are unequal, 7/9 is not half-way between 2/3 and 5/6. In fact, 7/9 is closer to 5/6 than to 2/3.

65. a.

Begin with	1/3	2/5
Get same denominators	5/15	6/15
Multiply numerator and denominator by 2	10/30	12/30

Therefore, 11/30 is between 1/3 and 2/5.

b.

Begin with	5/6	11/12
Same denominators	10/12	11/12
Multiplier is 3	30/36	33/36

Therefore, 31/36 and 32/36 are between 5/6 and 11/12.

c.

Begin with	23/15	8/5
Same denominators	23/15	24/15
Multiplier is 3	69/45	72/45

Therefore, 70/45 and 71/45 are between 23/15 and 8/5.

d.

Begin with	1/2	5/6
Same denominators	3/6	5/6 *
Multiplier is 5	15/30	25/30

Note that at the step marked * the numerators differed by 2 rather than by 1 as in the previous examples. Thus, multiplying by 5 yields $9 = 2 \times 5 - 1$ fractions between 1/2 and 5/6 rather than just $4 = 1 \times 5 - 1$. The fractions are: 16/30, 17/30, 18/30, 19/30, 20/30, 21/30, 22/30, 23/30, and 24/30.

66. a. Yes. Use 1,000,001 as the multiplier. The fractions would be

$$\underbrace{\frac{2,000,003}{6,000,006}, \frac{2,000,004}{6,000,006}, \cdots, \frac{3,000,001}{6,000,006}, \frac{3,000,002}{6,000,006}}_{\text{1 million fractions}}$$

b. Yes. Use 1,000,000,001 as the multiplier. The fractions would be:

$$\underbrace{\frac{2,000,000,003}{6,000,000,006}, \frac{2,000,000,004}{6,000,000,006}, \cdots, \frac{3,000,000,001}{6,000,000,006}, \frac{3,000,000,002}{6,000,000,006}}_{\text{1 billion fractions}}$$

c. Yes. Let N be the desired number of fractions between 1/3 and 1/2. Use N + 1 as a multiplier.

1/3	1/2
2/6	3/6
$\frac{2(N+1)}{6(N+1)} = \frac{2N+2}{6N+6}$	$\frac{3(N+1)}{6(N+1)} = \frac{3N+3}{6N+6}$

Fractions between $1/3 = 2N + 2/6N + 6$ and $1/2 = 3N + 3/6N + 6$ are:

$$\underbrace{\frac{2N+3}{6N+6}, \frac{2N+4}{6N+6}, \cdots, \frac{3N+1}{6N+6}, \frac{3N+2}{6N+6}}_{\text{N fractions}}$$

67. From problem 66 we see that we can find any number of fractions between any two given fractions. No matter what fraction we might guess as the "immediate successor" of 1/2, we have a method for finding distinct fractions between 1/2 and our candidate. Therefore we have a contradiction to the definition of "immediate successor", and so 1/2 (or any other fraction) does not have an immediate successor in the set of fractions.

68. a. 1/2 < ?/16 < 2/3. 48 is the least common denominator for 1/2, 2/3 and fractions with denominator 16. 24/48 < ?/48 < 32/48.
All the fractions 25/48, 26/48, . . . , 31/48 are between 1/2 and 2/3. Those which may be reduced to a fraction with denominator 16 are:
27/48 = 9/16 and 30/48 = 10/16.

b. 3/7 < ?/21 < 1/2. 42 is the least common denominator for 3/7, 1/2 and fractions with denominator 21. 18/42 < ?/42 < 21/42. The fractions 19/42 and 20/42 are between 3/7 and 1/2. 20/42 = 10/21 reduces to a fraction with denominator 21.

c. 3/2 < ?/15 < 5/3. 30 is the least common denominator for 3/2, 5/3 and fractions with denominator 30. 45/30 < ?/30 < 50/30. The fractions 46/30, 47/30, 48/30 and 49/30 are between 3/2 and 5/3. The fractions 46/30 = 23/15 and 48/30 = 24/15 reduce to fractions with denominator 15.

d. 2/5 < ?/25 < 1/2. 50 is the least common denominator for 2/5, 1/2 and fractions with denominator 25. 20/50 < ?/50 < 25/50. 21/50, 22/50 23/50 and 24/50 are between 2/5 and 1/2. 22/50 = 11/25 and 24/50 = 12/25 reduce to fractions with denominator 25.

69. a.

10/21
10/(21 - 10) = 10/11
10/(11 - 10) = 10

13/27
13/(27 - 13) = 13/14
13/(14 - 13) = 13

Therefore 10/21 < 13/27
The above could be compared using the fraction slide rule.

b.

5/31
5/(31 - 5) = 5/26
5/(26 - 5) = 5/21
5/(21 - 5) = 5/16
5/(16 - 5) = 5/11
5/(11 - 5) = 5/6
5/(6 - 5) = 5

2/13
2/(13 - 2) = 2/11
2/(11 - 2) = 2/9
2/(9 - 2) = 2/7
2/(7 - 2) = 2/5
2/(5 - 2) = 2/3
2/(3 - 2) = 2

Therefore 5/31 > 2/13
The above could be compared using the fraction slide rule.

c.

11/23
11/(23 - 11) = 11/12
11/(12 - 11) = 11/1

23/48
23/(48 - 23) = 23/25
23/(25 - 23) = 23/2 = 11 1/2

Therefore 11/23 < 23/48

d.

15/68
15/(68 - 15) = 15/53
15/(53 - 15) = 15/38
15/(38 - 15) = 15/23
15/(23 - 15) = 15/8

5/27
5/(27 - 5) = 5/22
5/(22 - 5) = 5/17
5/(17 - 5) = 5/12
5/(12 - 5) = 5/7

Therefore 15/68 > 5/27.

Note: After a few examples we should realize that it is not necessary to subtract a single multiple of the numerator at each step. If we want to compare 15/68 and 5/27, find the largest value of N such that 68 - 15N > 0 and at the same time, 27 - 5N > 0. Here N = 4. Then we compare 15/68 and 5/27 by comparing 15/(68 - 4 x 15) = 15/8 and 5/(27 - 4 x 5) = 5/7.

70\. a.

4/5	44/55
4/(5 - 4)= 4/1	44/(55 - 44)= 44/11 = 4

Therefore 4/5 = 44/55

b.

44/55	444/555
44/(55 - 44)= 44/11 = 4	444/(555 - 444)= 444/111 = 4

Therefore 44/55 = 444/555

c.

4/5	444/555
4/(5 - 4)= 4/1	444/(555 - 444)= 444/111 = 4/1 = 4

Therefore 4/5 = 444/555

d.

444/555	4444/5555
444/(555 - 444)= 444/111 = 4/1 = 4	4444/(5555 - 4444)= 4444/1111 = 4/1 = 4

71\.

7/4	11/5
(7 - 4)/4 = 3/4	(11 - 5)/5 = 6/5

Therefore 7/4 < 11/5

Check: 7 x 5 < 11 x 4 and so 7/4 < 11/5.

The technique works in general. Consider

a/b	a'/b' improper

To compare by cross products we must compare the numbers

ab'	a'b

Using the technique above we obtain

(a - b)/b	(a' - b')/b'

Since the original fractions were improper, a - b and a' - b' are positive. Therefore we compare these fractions by comparing the cross products

	(a - b)b'	(a' - b')b
or	ab' - bb'	a'b - bb'

Thus, ab' - bb' is < (respectively =, >) a'b - bb' iff ab' is < (respectively =, >) a'b. The same relation holds between the derived fractions as holds between the original fractions.

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