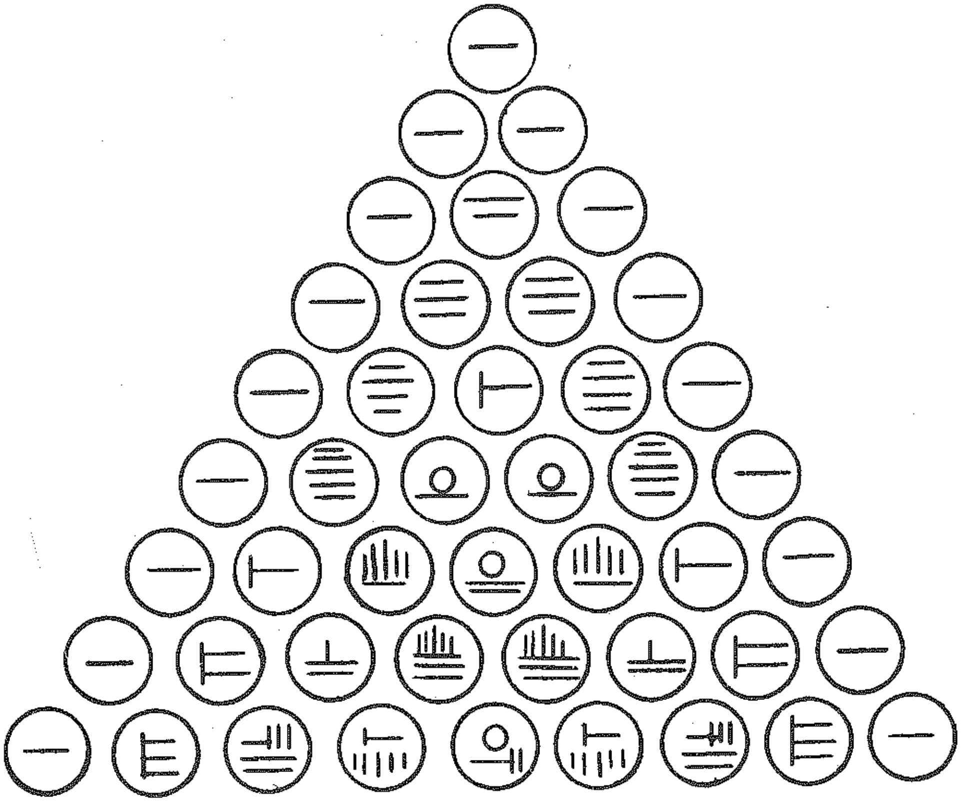
BOSTON COLLEGE MATHEMATICS INSTITUTE

Motivated Math Project Activity

Booklet II

the Incredible PASCAL TRIANGLE



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PREFACE

The series of MOTIVATED MATH PROJECT ACTIVITY BOOKLETS has been written for students and teachers in elementary and secondary schools. Each Booklet treats a topic generally found in the school curriculum or material that is interesting and motivational which may or may not be included in the usual class room textbook. Some topics are treated in two Booklets: one on an elementary level suitable for the middle grades and the other on an advanced level appropriate for junior and senior high schools.

The Booklets can be used in a variety of ways.

Because each Booklet treats a single topic, it is a handy summary and resource unit which can be expanded by both the student and the teacher.

Many of the Booklets, because they summarize and organize a topic in detail, can be used as mini-course modules to supplement standard class instruction or for individualized study. They also provide an invaluable review of a topic and can serve as a criterion for what has been covered on the topic.

The Booklets, unlike a textbook in which a topic may be treated in several nonconsecutive chapters, provide a convenient and readily accessible reference source. The material on a topic can be quickly and easily found.

Each Booklet contains problems which not only reenforce the class room instruction but also provide motivation, interest and challenge. Many problems are open-ended so that all students can achieve some measure of success. These problems are suitable not only for the routine pencil and paper activity but also may be extended by the use of hand electronic calculators or programmed on a computer.

Each Booklet contains solutions to the problems and in many instances comments, explanations and derivations of the key formulas and algorithms. The Booklets are relatively independent of each other and may be studied in any sequence depending on the background and personal preference of the student.

PASCAL'S TRIANGLE

An Incredible Source of Pattern Discovery

Pascal's Triangle is a special array of natural numbers that is introduced to students at different levels of learning in mathematics. Curiously, the triangle was not created by the man, Blaise Pascal, (1623-1662) for whom it is named. Knowledge of the arithmetic triangle, as it is also called, can be traced back to about 1100 A.D. in China and Persia. Eventually the triangle appeared in the works of many mathematicians in Western Europe in the sixteenth and seventeenth centuries. Pascal's particular development of many of the triangle's properties was published in 1665 in " Traité du triangle arithmétique " . It is for this accomplishment that the triangle has become associated with his name.

Pascal worked with the triangle in the format shown in Figure 1 .

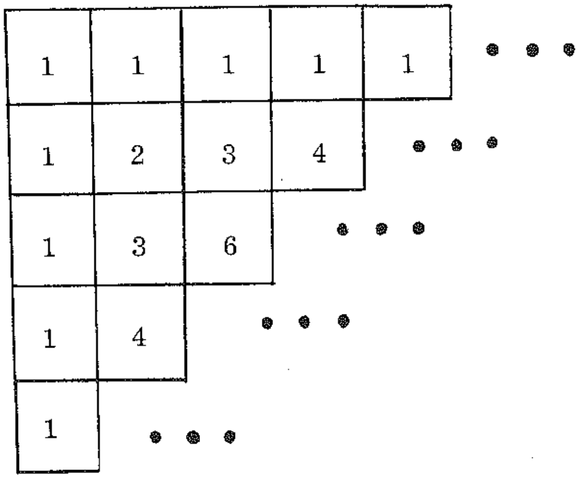


Figure 1

The array from the <u>Precious Mirror</u> of Chu-Shih-Chieh in the year 1303 shows the triangle in the form of Figure 2.

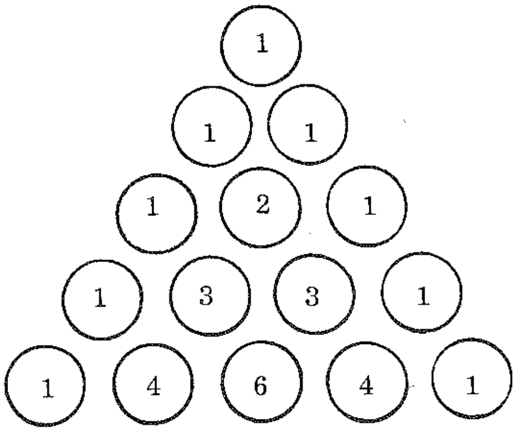


Figure 2

Still another formation that has been studied for patterns is given in Figure 3.

1				
1	1			
1	2	1		
1	3	3	1	
1	4	6	4	1

Figure 3

Figure 2, page 1, is currently the most popular form of the Pascal triangle. We will stress the use of this version in our search for patterns and will be able to uncover many of the well known properties for which the Pascal triangle is famous.

The Pascal triangle is not a mathematical diversion. It deserves a position in the mainstream of mathematical thought. Using it one can review, extend, and unite many aspects of the school mathematics curriculum.

In this presentation our primary objectives are to

1. equip the reader with appropriate techniques and skills for effective problem solving.

   The reasoning used in solving the problems posed in this booklet will be applicable in other contexts.

2. encourage the reader to continue the search beyond the limits of this booklet.

   The path of self-discovery is marked with challenge and frustration but pleasure and personal satisfaction are the rewards of those who choose to cling to the path.

3. convince the reader that separate and distinct mathematical ideas can be united in a single model.

   This fact will become evident as we explore the connections of the triangle with arithmetic, number theory, geometry, set theory, trigonometry, combinatorics, probability, algebra, analysis and mathematical recreations.

In the Pascal triangle, Figure 2 page 1 , the rows are: row 0, row 1, row 2, and so on. The numbers or terms in a given row are the zeroth element, first element, second element , and so on. Thus row 3 consists of zeroth element 1, first element 3, second element 3, and third element 1. Diagonals begin with the zeroth diagonal. In general, it will not be necessary to distinguish between left and right diagonals since they are term by term the same. The first diagonal then is the set of numbers 1, 2, 3, 4, ... For future reference the set of natural numbers is the set N = { 0, 1, 2, 3, 4, ..., n, ... } .

PROBLEMS

1. Complete rows 5 through 10 using the pattern you observe in rows 0 through 4.

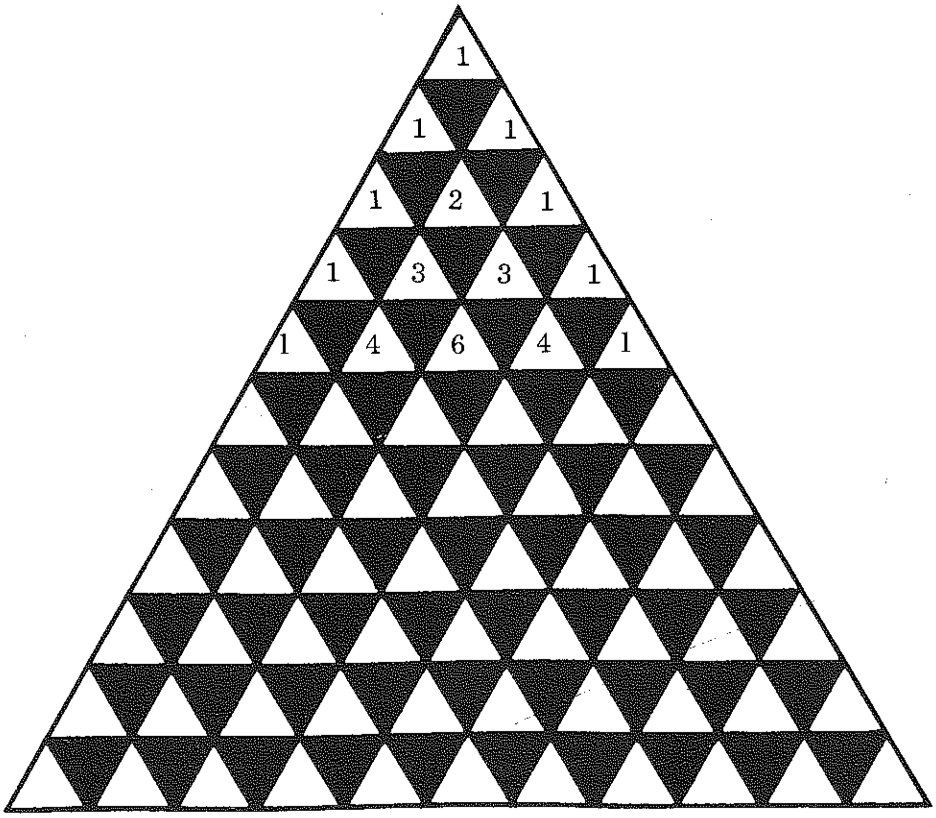


Figure 4

2. Use the completed Figure 4 from problem 1.

   a) What natural number(s) never occur(s) in the Pascal triangle ?

   b) What natural number(s) occur(s) exactly once in the Pascal triangle ?

   c) If the number of a row is odd, how many times does the largest number appear in the row ?

   d) If the number of a row is even, how many times does the largest number appear in the row ?

   e) In rows 0, 1 of the Pascal triangle all entries are odd numbers. List the next three rows in which only odd numbers occur.

   f) In rows 0 to n inclusive what is the total number of times a 1 appears ?

3. Fill in the blanks. In this problem k is any natural number from 0 through n .

   a) If n is an even row, then the kth term in row n is the same as the ____________ term in row n .

   b) If n is an odd row, then the kth term in row n is the same as the ____________ term in row n .

4. In one or two sentences describe how the entire n + 1 st row is formed from the n th row .

5. Using Pascal's form of the triangle

Row	Term					
↓	0	1	2	3	4	...
0	1	1	1	1	1	...
1	1	2	3	4	5	...
2	1	3	6	10	15	...
3	1	4	10	20	35	...
⋮	⋮	⋮	⋮	⋮	⋮	...

describe in a sentence or two how the elements of row n are formed .

6. Use the completed Pascal triangle from Figure 4, page 3.

   a) Give the row sums for each row from 0 through 10.

   b) Give the cumulative row sums for each row from 0 through 10 .

   For example: row 2 sum is 4 and cumulative row 2 sum equals the sum of row 0, row 1, and row 2 which is 7.

c) For row n, give the row sum in terms of n .

d) For row n, give the cumulative row sum in terms of n.

7.

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ \boxed{1 \quad 3 \quad 1} \\ 1 \quad 4 \quad 4 \quad 1 \\ 1 \quad 5 \quad 8 \quad 5 \quad 1 \\ 1 \quad 6 \quad 13 \quad 13 \quad 6 \quad 1 \end{array}$$

Figure 5

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ \boxed{1 \quad 5 \quad 1} \\ 1 \quad 6 \quad 6 \quad 1 \\ 1 \quad 7 \quad 12 \quad 7 \quad 1 \\ 1 \quad 8 \quad 19 \quad 19 \quad 8 \quad 1 \end{array}$$

Figure 6

a) In Figure 5 compute the row sums for rows 0 through 5.

b) If the pattern is continued what is the row sum for row n in terms of n ?

c) In Figure 6 compute the row sums for rows 0 through 5.

d) If the pattern is continued what is the row sum for row n in terms of n ?

e) Let row 2 of a triangular array be given as $\boxed{1 \quad k \quad 1}$.
Assume the usual addition pattern is applied to form row 3 and beyond.
Find the row sum for row n of this triangle in terms of n and k.

8. Let alternating sum S_1 be symbolized as

$$S_1 = a - b + c - d + \dots$$

Let alternating sum S_2 be symbolized as

$$S_2 = a + b - c + d - \dots$$

For example , in row 3:

$$S_1 = 1 - 3 + 3 - 1 = 0 \qquad S_2 = 1 + 3 - 3 + 1 = 2 .$$

a) Find S_1 for each of the rows 0 through 10.

b) Find S_2 for each of the rows 0 through 10.

c) For row n, give alternating sum S_1 .

d) For row n, give alternating sum S_2 .

9. Use the form of the Pascal triangle below .

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
. . .

Let $\boxed{a \quad b \quad c \quad d \quad e \quad f \quad g...}$ represent any row where a is the 0th term, b is the first term, c is the second term , and so on. Then,

1) The left alternating sum for the 3rd term (element d) in the given row is: d - c + b - a .
2) The left alternating sum for the 6th term (element g) in the given row is: g - f + e - d + c - b + a .
3) The left alternating sum for the 2nd term (element c) in the given row is: c - b + a .

a) Complete the chart by getting the left alternating sum for the element specified in given row .

Row	2	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10
Term	1	2	2	3	3	4	2	4	3	5	4	7	3	6	2	7
L.A.S.																

b) Study the Pascal triangle and your results in part (a) . Describe how to find the left alternating sum for an element without doing the adding and subtracting.

10. In row 4 , the sum of the terms in even-numbered positions is 1 + 6 + 1 = 8, while the sum of the terms in odd-numbered positions is 4 + 4 = 8 .

a) Give the sum of the terms in even-numbered positions of row 5.
b) Give the sum of the terms in odd-numbered positions of row 5.
c) Give the sum of the terms in even-numbered positions of row 6.
d) Give the sum of the terms in odd-numbered positions of row 6.
e) Give the sum of the terms in even-numbered positions of row n .
f) Give the sum of the terms in odd-numbered positions of row n .

11. Find the powers below and locate them in the Pascal triangle. (Figure 4)

power of 11	by multiplication	location in the Pascal triangle
11^0		
11^1		
11^2		
11^3		
11^4		
11^5		
11^6		
11^7		
11^8		

12. Find the powers below and locate them in the Pascal triangle. (Figure 4)

power of 11_{five}	in base five	in base ten	location in the Pascal triangle
11^0_{five}			
11^1_{five}			
11^2_{five}			
11^3_{five}			
11^4_{five}			
11^5_{five}			

13 a)

Let set A be	Compute the number of subsets of each A with						
↓	0	1	2	3	4	5	elements
∅							
{ a }							
{ a,b }							
{ a,b,c }							
{ a,b,c,d }							
{ a,b,c,d,e }							

13. b) Let n(A) be Use the Pascal triangle to find the number of subsets of A with

↓	0	1	2	3	4	5	6	7	8	9	10
6											
7											
8											
9											
10											

elements

c) Apply the connection between the number of subsets and the Pascal triangle from above and give the total number of subsets of a set containing m elements.

14. Given two black and two white squares arranged in two groups I, II .

[■ □] [■ □]
I II

Select two squares, one from each group, in as many different ways as possible. The different ways squares can be taken from the groups are recorded in Chart A.

Chart A

	I	II	I	II	I	II
Select:	2 black - 0 white		1 black - 1 white		0 black - 2 white	
Ways:	black	black	black white	white black	white	white

Thus, there are the following combinations of two squares :

1 black-black
2 black-white
1 white-white

a) Consider the case of three black and three white squares arranged in three groups I, II, III .

[■ □] [■ □] [■ □]
I II III

Select three squares, one from each group in as many different ways as possible. Record your results in Chart B.

Chart B

3 black - 0 white			2 black - 1 white			1 black - 2 white			0 black - 3 white		
I	II	III	I	II	III	I	II	III	I	II	III

b) Consider the case of four black and four white squares arranged in four groups I, II, III, IV .

[■ □] [■ □] [■ □] [■ □]

I II III IV

Select four squares, one from each group in as many different ways as possible. Record your results in Chart C.

Chart C

4 black- 0 white	3 black - 1 white	2 black - 2 white	1 black - 3 white	0 black- 4 white
I II III IV	I II III IV	I II III IV	I II III IV	I II III IV

c) Use the totals in (b) directly to get the distribution of five black and five white squares arranged in five groups.

Total number of different ways

1) 5 black-0 white ____________

2) 4 black- 1 white ____________

3) 3 black - 2 white ____________

4) 2 black - 3 white ____________

5) 1 black - 4white ____________

6) 0 black - 5 white ____________

Binomial Theorem

The previous problem is in fact a physical representation of the binomial theorem. The binomial theorem provides us with a recipe for computing $(x+y)^n$ where n is a natural number and $x, y \neq 0$.

By multiplication of polynomials we find

$$(x+y)^0 = 1$$

$$(x+y)^1 = x + y$$

$$(x+y)^2 = (x+y)(x+y) = x^2 + 2xy + y^2$$

$$(x+y)^3 = \left[(x+y)(x+y)\right](x+y) = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^4 = \left[(x+y)(x+y)(x+y)\right](x+y) = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

It appears that the rows of the Pascal triangle give the numerical coefficients in the binomial expansion. That is, for $n = 5$, the coefficients of the terms arranged in decreasing (increasing) powers of x (y) namely,

$$x^5y^0 \quad , \quad x^4y^1 \quad , \quad x^3y^2 \quad , \quad x^2y^3 \quad , \quad x^1y^4 \quad , \quad x^0y^5$$

will be row 5 of the Pascal triangle

$$1 \qquad 5 \qquad 10 \qquad 10 \qquad 5 \qquad 1$$

If this pattern continues the coefficients for the expansion of $(x+y)^n$ will be row n of the triangle.

To expand

$$(x + y)^n = \underbrace{(x+y)(x+y)(x+y) \quad \cdots \quad (x+y)}_{n \quad \text{terms}}$$

note that each term of the expansion must be of combined degree $\underline{n}$ in x and y.

There is no difficulty in writing down all the terms whose combined degree is n. They are in decreasing (increasing) powers of x (y)

$$x^ny^0 \quad , \quad x^{n-1}y^1 \quad , \quad x^{n-2}y^2 \quad , \quad \ldots \quad x^{n-k}y^k \quad \ldots \quad , \quad x^1y^{n-1} \quad , \quad x^0y^n$$

What <u>is difficult</u> is the determination of the <u>coefficients</u> that match these terms. In order to do this the notation $\binom{n}{k}$ is introduced to represent the number of ways in which k y's (and $n - k$ x's) can be selected from the n terms $(x + y)$.

Study Example 1 and refer to problem 14, pages 8 and 9 .

Example 1 $(x + y)^4 = \underset{\text{I}}{(x + y)}\underset{\text{II}}{(x + y)}\underset{\text{III}}{(x + y)}\underset{\text{IV}}{(x + y)}$

		coefficient of	term
$\binom{4}{0}$	the number of ways to pick 4 x's and 0 y's from the 4 groups is	1	x^4y^0
$\binom{4}{1}$	the number of ways to pick 3 x's and 1 y from the 4 groups is	4	x^3y^1
$\binom{4}{2}$	the number of ways to pick 2 x's and 2 y's from the 4 groups is	6	x^2y^2
$\binom{4}{3}$	the number of ways to pick 1 x and 3 y's from the 4 groups is	4	x^1y^3
$\binom{4}{4}$	the number of ways to pick 0 x's and 4 y's from the 4 groups is	1	x^0y^4

Now we develop an explicit representation for $\binom{n}{k}$ in terms of n and k. To do this we will generalize the patterns shown in Example 2.

Example 2

1) Given the letters: a, b, c, d, e .

Write all possible ordered arrangements of these five letters. This will be a total of $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ arrangements since there are five ways to select the first letter, four ways to select the second letter, and so on.

$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5\,!$ in abbreviated form .

Given n distinct letters there will be $n(n - 1) \cdot \cdots \cdot 3 \cdot 2 \cdot 1 = n\,!$ different orderings of these letters.

We define $0! = 1$. Read $n!$ as n factorial .

2) Given the letters : $\underline{a,\ a,}\ b,\ c,\ d$.

Write all possible ordered arrangements of these five letters. This will be $\frac{5!}{2!}$. Since we cannot distinguish between the two a's there will be one half as many as in (1). Note $2! = 2$.

3) Given the letters: $a,\ b,\ \underline{c,\ c,\ c}$.

Write all possible ordered arrangements of these five letters. This will be $\frac{5!}{3!}$. This follows since 3 different letters give 6 distinct arrangements. while 3 same letters produce 1 arrangement. There will be one sixth as many as in (1). Note $3! = 6$.

4) Given the letters : $\underline{a,\ a,}\ \underline{b,\ b,\ b}$

Write all possible ordered arrangements of these five letters. This will be

$$\frac{5!}{2!} \cdot \frac{1}{3!} = \frac{5!}{2!\ 3!} \qquad \text{using (2) and (3)} \ .$$

Generalizing,

Given n letters

$$\underbrace{x\ x\ x\ x\ \cdots\ x}_{n-k \text{ of these}}\ \underbrace{y\ y\ \cdots\ y}_{k \text{ of these}}$$

there will be $\frac{n!}{k!\,(n-k)!}$ different orderings of these letters.

Therefore $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$ where k is any natural number from 0 to n.

We are now in a position to state the binomial theorem. It is

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \cdots$$
$$+ \binom{n}{k} x^{n-k} y^k + \cdots + \binom{n}{1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n \ .$$

In summation notation this is

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \qquad k,\ n \text{ are natural numbers; } x,\ y \text{ are not } 0 \ .$$

15. Compute

a) $\binom{12}{9}$ b) $\binom{12}{3}$ c) $\binom{8}{4}$

d) $\binom{10}{5}$ e) $\binom{n}{0}$ f) $\binom{n}{n}$

16. Use the binomial theorem to expand

a) $(x + y)^6$ b) $(x + y)^8$

17. Find the sum.

a) $\sum_{k=0}^{10} \binom{10}{k}$ b) $\sum_{k=0}^{6} \binom{12}{2k}$ c) $\sum_{k=0}^{3} \binom{6}{2k} \cdot \binom{7}{2k+1}$

Because of its importance in the binomial theorem the term $\binom{n}{k}$ is called the binomial coefficient.

$\binom{n}{k}$ also represents the combinations formula in counting since it is used to determine the total number of different combinations of n distinct objects considered k at a time. This result can be established by studying the close connection between ordered arrangements (permutations) and combinations. Recall the total number of different orderings of n distinct objects (permutations of n objects) taken n at a time is n ! Similarly, the total number of permutations of n distinct objects taken k at a time is

$$n \cdot (n - 1) \cdot (n - 2) \cdot \cdots \cdot (n - (k - 1)) = \frac{n!}{(n - k)!}$$

$$\text{where} \quad 0 \leq k \leq n \quad .$$

For example, it is easy to verify that the number of permutations of 4 different objects taken 2 at a time is $4 \cdot 3$ or $\frac{4\,!}{2\,!}$.

Associated with each combination of k distinct objects is a total of k ! permutations. That is, with 3 objects there are 3 ! = 6 different orderings; with 4 objects there are 4 ! = 24 permutations. If you are in doubt, list them !

The total number of combinations of n distinct objects taken k at a time times the k ! permutations for each one gives the total number of permutations of n distinct objects taken k at a time.

Therefore $$\binom{n}{k} \cdot k\,! = \frac{n\,!}{(n - k)\,!}$$

Again we find $\binom{n}{k} = \frac{n\,!}{k\,!\,(n-k)\,!}$

Thus, each of the terms in the Pascal triangle represents some combination. Now we are in a better position to understand problem 13 which treated the relation between Pascal's triangle and the subsets of a set ! The number of k-element subsets of a set with n elements is the same as the number of combinations of n distinct objects taken k at a time. The combinations form of Pascal's triangle is shown in Figure 7.

$$\begin{array}{ccccccccc} & & & & \binom{0}{0} & & & & \\ & & & \binom{1}{0} & & \binom{1}{1} & & & \\ & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\ & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\ \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \end{array}$$

Figure 7

There are two very basic and important facts that are particularly noticeable in the combinations form of the triangle. One treats the symmetry of the triangle. The other relates to the formation of a new row from the row preceding.

18. Show that the Pascal triangle is symmetric. Verify $\binom{n}{k} = \binom{n}{n-k}$

19. Establish the addition pattern of the Pascal triangle.

Show $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$

20. Jerome Cardan (1501-1576) discovered the rule of succession below in 1570. Show that it is a valid rule.

$$\binom{n}{k-1} \bullet \frac{n-k+1}{k} = \binom{n}{k}$$

21. Can you explain this pattern in the triangle ?

    1) Circle two adjacent numbers in any row r of the Pascal triangle. For example in row 5, pick 5, 10 in that order.

    2) Form the fraction a/b where a is the number on the left and b is the number on the right. This is 5/10 .

3) Give the reduced form of a/b. This is 1/2 .

4) In row 5, the 5 that was selected is 2 numbers from the left while the 10 that was selected is 4 numbers from the right. Form the fraction c/d where c is the count to the left and d is the count to the right. Here we have 2/4.

5) Give the reduced form of c/d. This is 1/2 .

6) The reduced forms are the same in this case.

a) Perform steps (2) through (5) on each pair of numbers and complete the chart.

Row r	Given adjacent pair in order	a/b	Reduced form	c/d	Reduced form
4	6,4				
5	10,10				
6	15,20				
7	7,21				
8	28,56				
9	84,126				
10	210,252				

b) For any $r \geqslant 1$ and any adjacent pair will the reduced forms be equal? Explain your answer. Hint: use combinations.

22. a) Do the computations and fill in each of the blanks.

Row 2: 0 • 1 + 1 • 2 + 2 • 1 = ____(1 + 1) = _____ 2^1

Row 3: 0 • 1 + 1 • 3 + 2 • 3 + 3 • 1 = ____(1 + 2 + 1) = _____ 2^2

Row 4: 0 • 1 + 1 • 4 + 2 • 6 + 3 • 4 + 4 • 1
= ______(1 + 3 + 3 + 1) = __________

Row 5: 0 • 1 + 1 • 5 + 2 • 10 + 3 • 10 + 4 • 5 + 5 • 1
= _______(1 + 4 + 6 + 4 + 1) = ___________

Row 6: 0 • 1 + 1 • 6 + 2 • 15 + 3 • 20 + 4 • 15 + 5 • 6 + 6 • 1
= ________(1 + 5 + 10 + 10 + 5 + 1) = ___________

b) Study the pattern in (a) carefully.
Using combinations notation write a similar statement for row n. Can you prove that your statement is valid ?

23. Row 2 of the Pascal triangle is 1 2 1 and row 4 is 1 4 6 4 1.

a) Consider 121 and 14,641 as numbers in base ten. Show that each is a perfect square.

b) Consider 10,201 and 104,060,401 . Which of these is a perfect square?

c) Consider 1,002,001 and 1,004,006,004,001. Which of these is a perfect square ?

d) Consider 100,020,001 and 10,004,000,600,040,001. Which of these is a perfect square?

e) Consider $1\underbrace{00\cdots0}_{k\text{ zeros}}2\underbrace{00\ldots0}_{k\text{ zeros}}1$ and

$$1\overbrace{00\cdots0}^{k\text{ zeros}}4\overbrace{00\cdots0}^{k\text{ zeros}}6\overbrace{00\ldots0}^{k\text{ zeros}}4\overbrace{00\ldots0}^{k\text{ zeros}}1 .$$

Under what conditions on k will each number be a perfect square ?

24. <u>Guessing the number at the top of the pyramid.</u>

Choose three numbers from 1 through 9, repetitions permitted, and record them in a line. See Figure 8 line (1). On a line above line (1) record the sum of 5, 2 between the 5 and 2; record the sum of 2, 6 between the 2 and 6. Find the sum of 7, 8 . Since 7 + 8 = 15 is greater than 9 record on line (3) $\underline{15 - 9}$ or 6. 6 in this case is the top of the pyramid.

⑥	line (3)
7 8	line (2)
5 2 6	line (1)

Figure 8

Can you predict the number at the top without <u>first building</u> the pyramid ? <u>Remember</u> as you build the pyramid by adding, if the sum S at any stage is greater than 9, use the number S - 9 in its place.

Try to find a scheme. Try it on the following selections. In each case check your choice by doing out the steps as described.

a) 6, 4, 1 b) 4, 9, 7

c) 8, 8, 3 d) 2, 9, 9

Now choose four numbers, repetitions permitted, from 1 through 9. Perform the steps as outlined. There will be one more line in this pyramid. Find a scheme to get the number at the top. Test and check your scheme on these selections.

e) 5, 3, 1, 6 f) 2, 4, 9, 7

f) 9, 9, 3, 8 h) 8, 2, 4, 5

Choose five numbers according to the same rules. Find the number at the top. If you have a valid scheme, it should work for these selections. In each case, check your guess.

i) 5, 3, 1, 6, 2 j) 1, 1, 1, 1, 1

k) 2, 7, 7, 7, 2 l) 4, 2, 9, 3, 6

m) Challenge ! If your scheme has survived the three cases above then you should be able to predict the number at the top of the pyramid formed from <u>ten</u> starting numbers. Try it out on

4, 5, 9, 2, 4, 6, 4, 1, 7, 6.

Hint: Keep your calculations less than or equal to 9 on a step by step basis.

25. Let a, b, c be the bottom line of the pyramid. The sums of two consecutive terms will be the next line of the pyramid. The sum of two consecutive terms in the second line will be the third and here top line of the pyramid.

$$\begin{array}{ccccc} & & \boxed{a + 2b + c} & & \\ & a + b & & b + c & \\ a & & b & & c \end{array}$$

Figure 9

Build a polynomial addition pyramid for

a) 4 terms: a, b, c, d

b) 5 terms: a, b, c, d, e

c) 6 terms: a, b, c, d, e, f

26. Let a, b, c be the bottom line of the pyramid. The differences of two consecutive terms will be the next line of the pyramid. The difference of two consecutive terms in the second line will be the third and here top line of the pyramid.

$$\begin{array}{ccccc} & & \boxed{a - 2b + c} & & \\ & a - b & & b - c & \\ a & & b & & c \end{array}$$

Figure 10

Build a polynomial subtraction pyramid for

a) 4 terms: a, b, c, d

b) 5 terms: a, b, c, d, e

c) 6 terms: a, b, c, d, e, f

27. Four ball bearings are falling down a chute towards a tray. Before they get to the tray, the chute branches out. At each branching point an equal number of ball bearings turn left as right. How many ball bearings reach the tray through each branch ? See Figure 11 bottom row for solution.

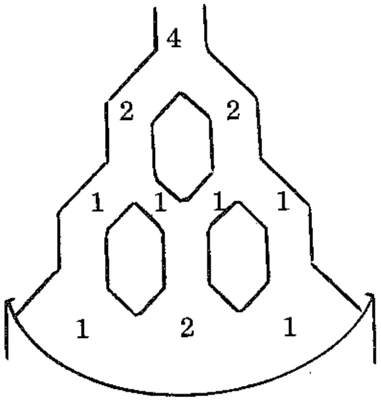


Figure 11

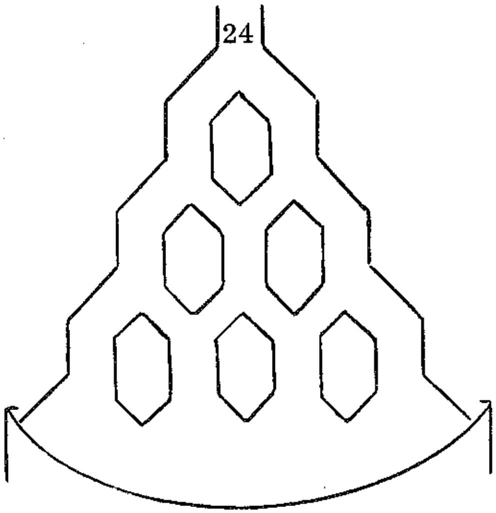


Figure 12

a) Complete Figure 12. How many ball bearings reach the tray through each branch ?

Give the distribution through each branch if there are

b) 40 ball bearings and 4 branches

c) 80 ball bearings and 5 branches

d) 144 ball bearings and 5 branches

e) 224 ball bearings and 6 branches

f) $2^n \cdot m$ ball bearings and $n + 1$ branches

28. Let the grid in Figure 13 represent a street map. Pascal lives at the point marked P; Fermat lives at the point marked E; Bassett lives at B.

a) Pascal is going to visit Fermat. He restricts himself to traveling a northerly or easterly route. How many <u>different</u> routes can he take to Fermat's home ? One possible route is marked in Figure 13.

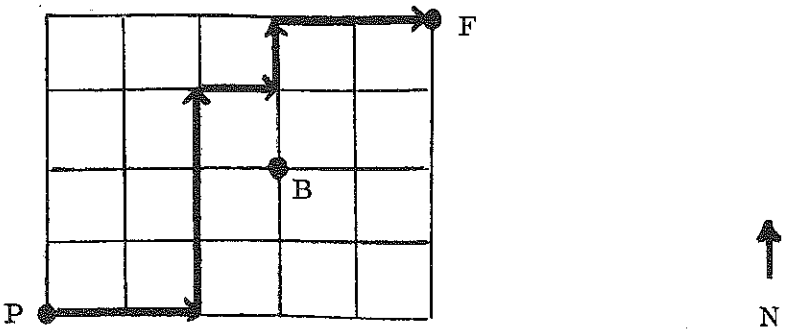


Figure 13

b) On this trip to Fermat's home Pascal must first stop at Bassett's place. In how many different ways can he get to Fermat's still traveling in a northerly or easterly direction ?

c) Suppose the street map is the one in Figure 14.

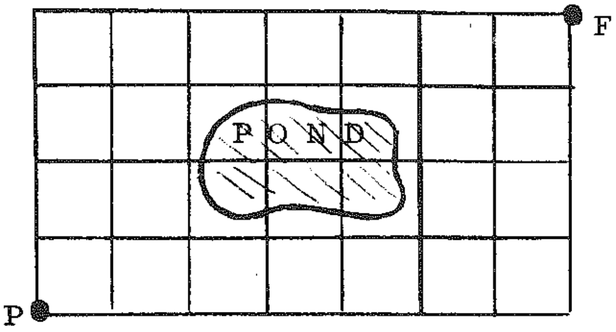


Figure 14

How many routes traveling to the north or east can Pascal take to Fermat's ?

d) If the street map is the one shown in Figure 15, how many trips can Pascal make to Fermat's house traveling to the north or east ?

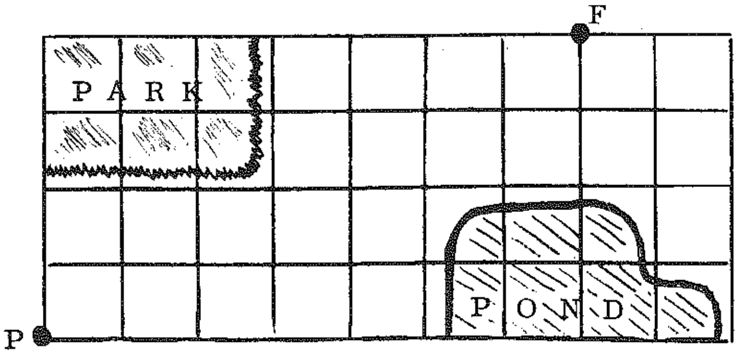


Figure 15

e) At the University Pascal's office is on the first floor of the Mathematics Building. Fermat's office is on the second floor. The floor plans complete with corridors are shown in Figure 16. Note that there is a stairway at each intersection of corridors. If Pascal makes a habit of traveling up or north or east, how many different routes can he take to Fermat's office ?

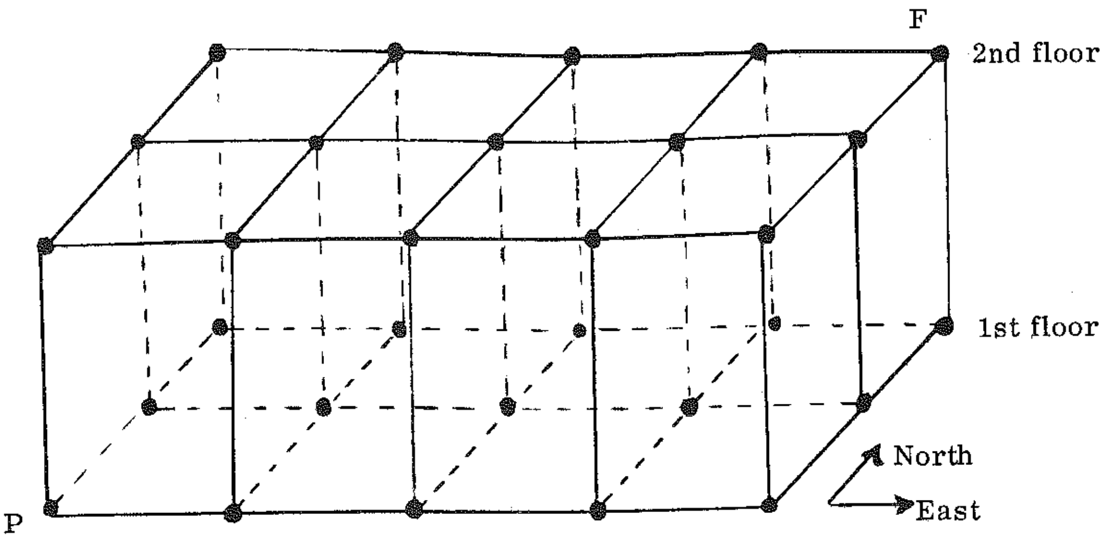


Figure 16

29. a) Study Figure 17. How many ways can you spell ONE by following a continuous path ? You may move from top to bottom or left to right and vice versa. One possible solution path is illustrated .

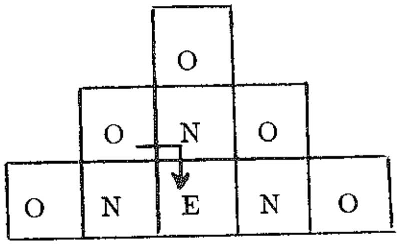


Figure 17

b) How many ways can you spell FOUR by following a continuous path moving from top to bottom or sideways in either direction ?

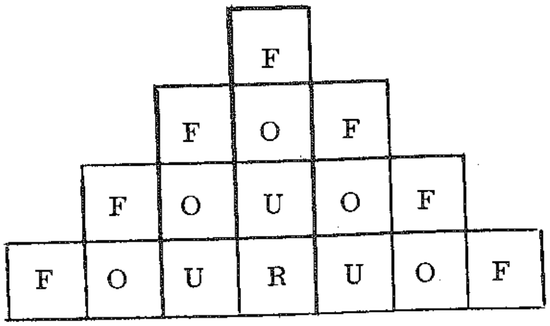


Figure 18

c) How many ways can you spell SEVEN by following a continuous path moving from top to bottom or sideways in either direction ?

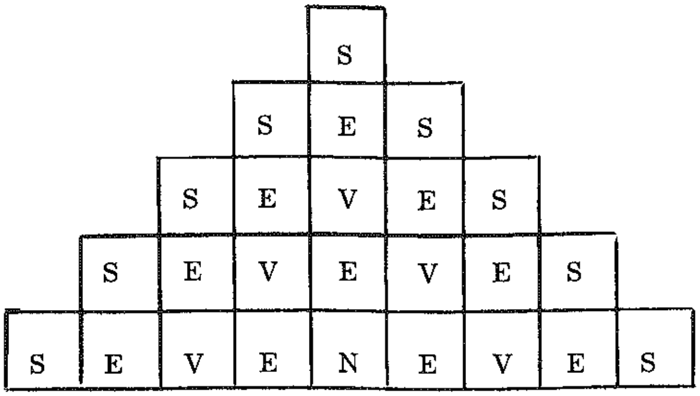


Figure 19

d) How many ways can you spell TWENTY by following a continuous path moving from top to bottom or sideways in either direction.

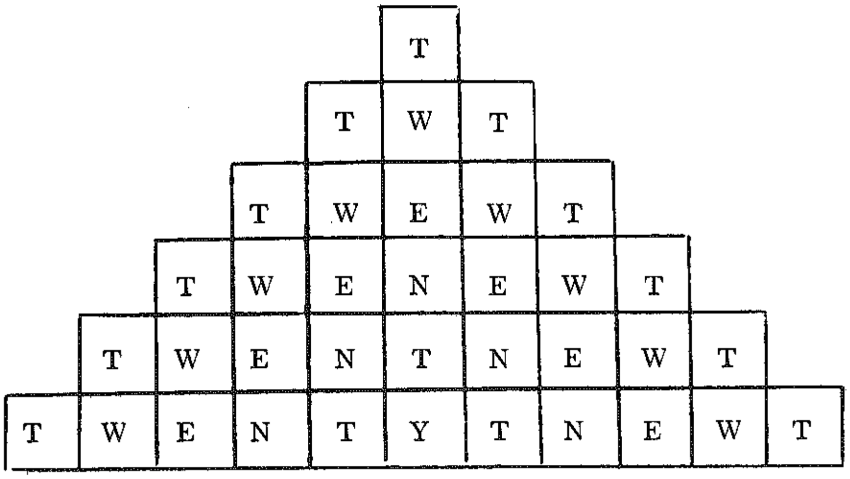


Figure 20

If the same type of diagram is used for each of the words below, how many different ways are there to spell each word ?

e) HUNDRED f) THOUSAND

g) If a word consists of n letters how many different ways can be found to spell it under the given directions ?

30. a) Study Figure 21. How many ways can you spell YOU ?

You may move diagonally left or right from one row to the next.

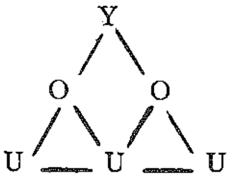


Figure 21

b) Using the same directions as in (a) how many ways can you spell MATH ?

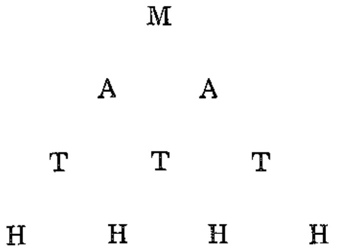


Figure 22

How many ways can you spell

c) STUDY

d) PASCAL

e) a word consisting of n letters ?

31. a) Study Figure 23. How many different ways can you spell POINT ?
You may move diagonally left or right from one row to the next .

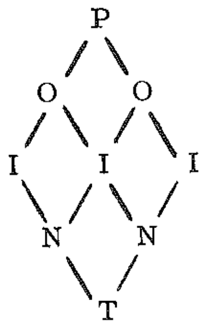


Figure 23

b) How many different ways can you spell ALGEBRA using the rules of (a) ?

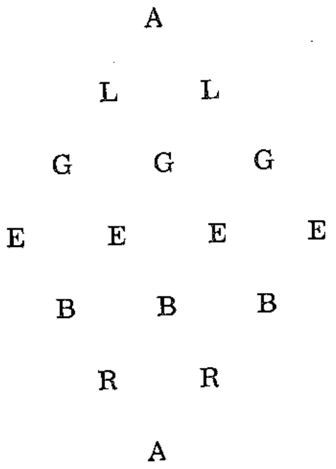


Figure 24

c) How many different ways can you spell CONGRUENT using the rules of (a) ?

C

O O

N N N

G G G G

R R R R R

U U U U

E E E

N N

T

Figure 25

d) How many different ways can you spell MATHEMATICS using the rules of (a) ?

M

A A

T T T

H H H H

E E E E E

M M M M M M

A A A A A

T T T T

I I I

C C

S

Figure 26

32. a) Study Figure 27. How many ways can you spell the palindrome RADAR ? You can move from top to bottom and vice versa. You must move diagonally left or right from one row to the next.

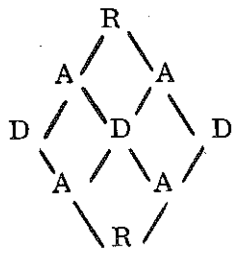


Figure 27

b) Follow the directions in (a) and find the number of ways to spell SPACECAPS.

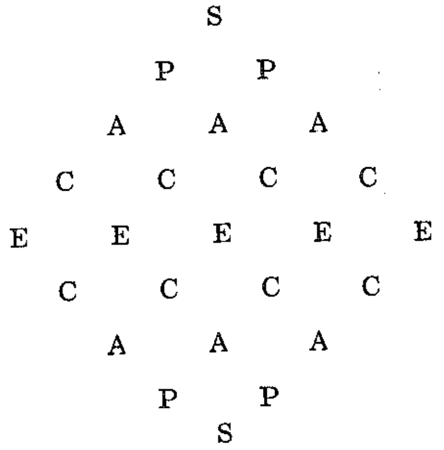


Figure 28

c) Follow the directions in (a) and find the number of ways to spell MADAM I'M ADAM.

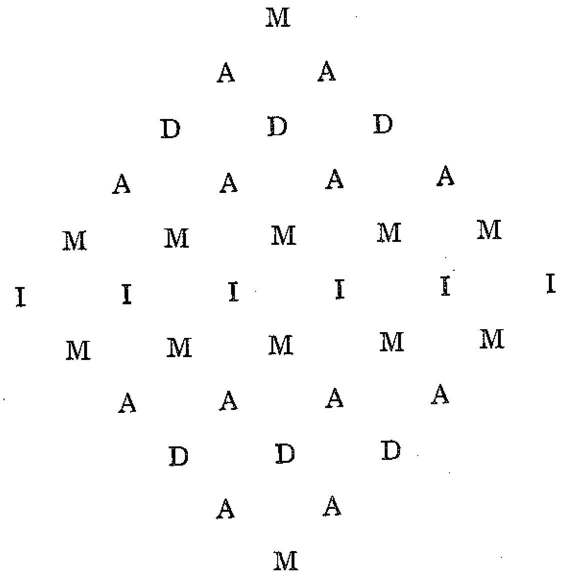


Figure 29

d) Follow the directions in (a) and find the number of ways to spell RISE TO VOTE SIR .

```
            R
          I   I
        S   S   S
      E   E   E   E
    T   T   T   T   T
  O   O   O   O   O   O
V   V   V   V   V   V   V
  O   O   O   O   O   O
    T   T   T   T   T
      E   E   E   E
        S   S   S
          I   I
            R
```

Figure 30

33. Find the number of ways to get a particular sum N using natural number addends. For example, let N = 4. Using 2 addends and the numbers 1, 2, 3 find all sums equal to 4. There are three sums:

$4 = 3 + 1$ $\quad\quad$ $4 = 2 + 2$ $\quad\quad$ $4 = 1 + 3.$

Note the order of terms makes a difference.

a) Using 2 addends and the numbers from 1 through 6 get a total of 7 in as many ways as possible.

b) Using 2 addends and the numbers from 1 through 7 get a total of 8 in as many ways as possible.

c) Using 3 addends and the numbers from 1 through 3 get a total of 5 in as many ways as possible.

d) Using 3 addends and the numbers from 1 through 4 get a total of 6 in as many ways as possible.

e) Using 4 addends and the numbers from 1 through 3 get a total of 6 in as many ways as possible.

f) Using 5 addends and the numbers from 1 through 3 get a total of 7 in as many ways as possible.

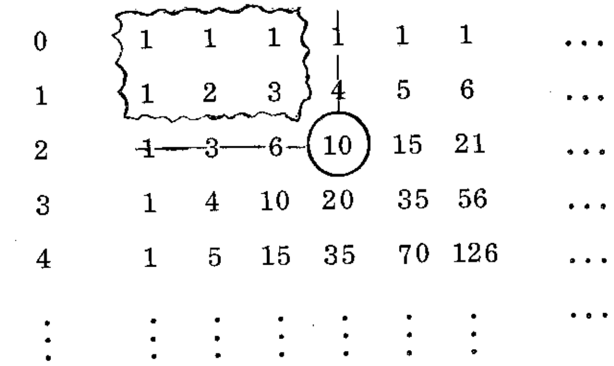
g) Complete the chart summary on the number of ways to sum to N using only natural number addends.

Number of addends

N	1	2	3	4	5	6	7
1							
2							
3							
4							
5							
6							
7							

34. Use Pascal's form of the triangle in Figure 31.

Figure 31



Circle any number in the array except. 1 . Cross out all numbers in the row to the left of and the column above the circled number. Figure 31 illustrates a choice of 10. Sum the numbers in the enclosed region. The sum illustrated is 9. Find the

Find the matching sum if the circled number is

a) 6 in row 1
b) 15 in row 2
c) 56 in row 3
d) 15 in row 4
e) 45 in row 2
f) 70 in row 4
g) $\binom{n}{k}$ in row k

35. In the Pascal triangle locate the sum of

a) $1 + 2 + 3$
b) $1 + 2 + 3 + 4$
c) $1 + 2 + 3 + 4 + 5$
d) $1 + 2 + 3 + 4 + 5 + 6$
e) $1 + 2 + 3 + 4 + 5 + 6 + 7$
f) $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$

g) Describe by row and term number the location of the sum of the numbers 1 through 20. Compute the sum.

h) Describe by row and term number the location of the sum of the numbers 1 through 50. Compute the sum.

i) In terms of n give a formula for the sum of:

$$1 + 2 + 3 + 4 + \dots + n \quad .$$

36. The Buscarloo family are great huggers. Every time members of the family get together they all hug each other once. For example, when Al, Cal and Sal meet:

Al hugs Cal　　Al hugs Sal　　Cal hugs Sal

How many hugs take place when

a) Bim, Jim, Mim, and Tim meet

b) Andy, Candy, Mandy, Randy, and Sandy meet .

Complete the chart ,

c)

Number of Buscarloos	2	3	4	5	6	7	8	9	10 . . .	n
Number of hugs										

37. Study the triangular arrays in Figure 32.

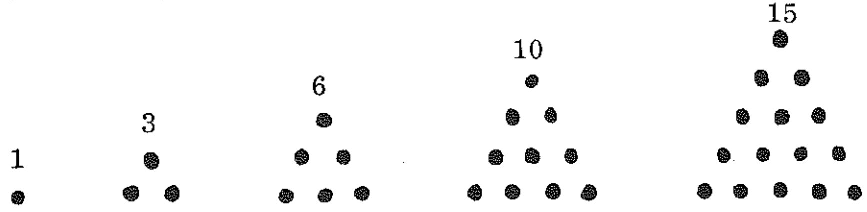


Figure 32

The numbers 1, 3, 6, 10, 15 , . . . were called triangular numbers by the ancient Greeks.

a) Name the next five triangular numbers .

Let $T_1 = 1$, $T_2 = 3$, $T_3 = 6$, . . . T_n = nth triangular number.

b) What are T_{15} , T_{25} , T_{100} ?

c) Find a formula for T_n in terms of n .

d) Add: $T_1 + T_2 + T_3 + T_4 + T_5$.

Use Pascal's triangle to find

e) $T_1 + T_2 + T_3 + \dots + T_{10}$.

f) $T_1 + T_2 + T_3 + \dots + T_n$.

38. It can be shown that any triangular number greater than three is the sum of four triangular numbers.

For example 6 = 3 + 1 + 1 + 1

Find sums to verify this relation for

a) 10 b) 15 c) 21

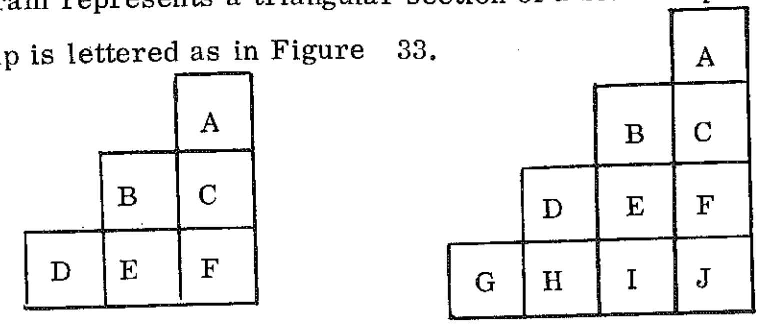
d) 28 e) 36 f) 45

Represent each triangular number given as an array of dots and draw a diagram to show that the relation named above holds for

g) 10 h) 15 i) 21 j) 28

k) If T_n is the triangular number sum, explain how each of the four triangular numbers in the sum are related to T_n .

39. Each diagram represents a triangular section of a block of postage stamps. Each stamp is lettered as in Figure 33.



(a) Figure 33 (b)

a) Study Figure 33 (a) . List the different ways in which 3 stamps connected on at least one side can be separated from the given piece. Note that the stamps that are left in the block need not be joined.

b) Study Figure 33 (b). List the different ways in which 3 stamps connected on at least one side can be separated from the given piece.

c) Challenge ! How many different ways can you separate 3 stamps connected on at least one side from a triangular section of 15 stamps?

d) Challenge ! How many different ways can you separate 3 stamps connected on at least one side from a triangular section of 21 stamps?

f) Challenge ! Can you find a formula that gives the total number of ways 3 stamps connected at least on one side can be separated from a triangular section of T_n stamps ?

40. The sequence of numbers 1, 4, 10, 20, 35, ... is known as the sequence of tetrahedral numbers. A tetrahedron is a pyramid with a triangular base.

Try building the tetrahedral numbers using styrofoam spheres. You will find out for example that 4 is a layer of 3 spheres in the form of a triangle with the 4th sphere perched on the top of the 3 .

Figure 34

a) Use the Pascal triangle to name the next three terms of the sequence.

b) Find a formula which will generate the sequence of tetrahedral numbers.

c) Let $P_1 = 1$, $P_2 = 4$, $P_3 = 10$, ... P_n = nth tetrahedral number. Find a formula for the sum

$$P_1 + P_2 + P_3 + \dots + P_n .$$

41. a) Use the Pascal triangle to name the next three terms of the sequence 1, 5, 15, 35, 70 .

b) Find a formula which will generate the sequence in (a) .

c) Let $Q_1 = 1$, $Q_2 = 5$, $Q_3 = 15$, ... Q_n = nth term of sequence in (a). Find a formula for the sum

$$Q_1 + Q_2 + Q_3 + \dots + Q_n .$$

42. In row 4 of the Pascal triangle, excluding the first and last 1, the elements are 4, 6, 4. 4 divides 4 exactly but 4 does not divide 6. Thus 4 <u>does not</u> divide each number except 1 in row 4.

In row 5, excluding the first and last 1, the elements are 5, 10, 10, 5. 5 divides 5, 10 exactly. Thus 5 <u>does</u> divide each number except 1 in row 5.

a) Study rows 6 through 12 of the Pascal triangle. Name the rows in which the row number divides each term except the first and last 1.

b) Using (a) make a conjecture concerning when row number n divides each term of row n (except for the first and last 1) .

43. Study the set of numbers on the "∨" in Figure 35. The "∨" is formed by the intersection of the second diagonals. With the exception of the two 1's 3 divides each of the numbers on the "∨"

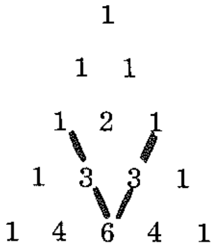


Figure 35

a) Complete the chart below by listing the numbers and checking the appropriate column.

n	"$\vee$" formed by intersection of (n-1)st diagonals consists of the numbers	excepting 1's n	
		does	does not
		divide each number on "$\vee$"	
3	1 3 6 3 1	✓	
4			
5			
6			
7			
8			
9			

b) Make a conjecture concerning when n will divide each number on the "$\vee$" formed by the intersection of the (n-1)st diagonals.

44. The Whooligans are scrappers and the Halligans are peaceful folk. Whenever the clans attend a concert together the seating plans are arranged so that no two Whooligans sit next to each other. In how many different ways can you seat a collection of Whooligans and/or Halligans in a row of n seats.

Example: If n = 2 there are 3 possibilities

WH HW HH .

Complete the seating chart.

Number of seats	n	1	2	3	4	5	6	7	8	9	10
Number of arrangements	A	2	3								

45. Fibonacci's famous rabbits. Fibonacci started with a pair of rabbits on January 1st that produced a second pair of rabbits on February 1st. Assuming

1) each new pair of rabbits become adult at the age of one month,
2) at the age of two months and each month thereafter every pair of adult rabbits produce a new pair on the first day of the month,
3) no rabbit dies during the period January 1st through January 1st,

how many pairs of rabbits does Fibonacci have the following January 1st ? To answer this question, study the diagram and complete the chart. Fibonacci, or Leonardo of Pisa, was an Italian mathematician. This problem appeared in his book the Liber Abaci in the year 1202.

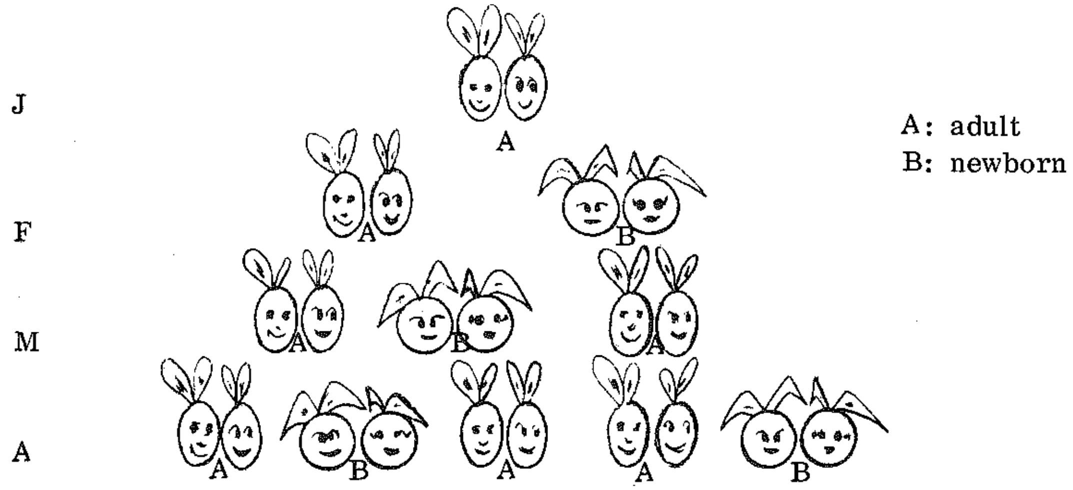


Figure 36

First day of	J	F	M	A	M	J	J	A	S	O	N	D	J
Number of adult pairs	1	1											
Number of newborn pairs	0	1											
Total	1	2											

46. Use the Pascal triangle in the form shown in Figure 37.

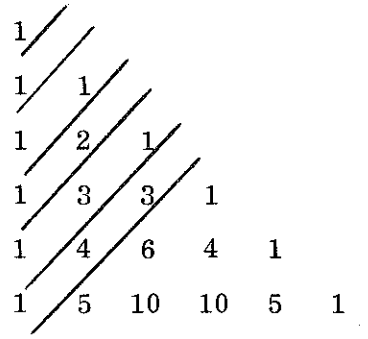


Figure 37

a) Let F_j , $j = 0, 1, 2, \ldots, n$ be the sum of the numbers on the jth rising diagonal . Compute the sums F_0 , ..., F_{12} .

F_j	F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}
Sum													

b) Assume F_0 up through F_{n-1} have been determined. Describe a quick way to find F_n .

c) Describe an easy way to sum $F_0 + F_1 + F_2 + \ldots + F_n$.

d) $F_4 = \binom{4}{0} + \binom{3}{1} + \binom{2}{2}$.

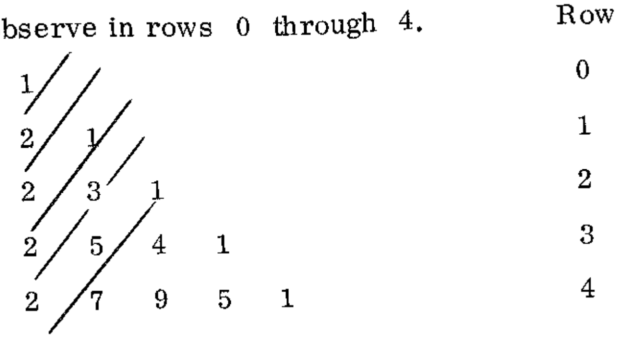
Express F_6 , F_8, F_{10}, F_{2n} as sums of combinations.

e) $F_5 = \binom{5}{0} + \binom{4}{1} + \binom{3}{2}$

Express F_7, F_9, F_{11}, F_{2n+1} as sums of combinations.

47. Given the triangular array in Figure 38. Complete rows 5 through 10 using the pattern you observe in rows 0 through 4.

Figure 38



48. Let D_k, $k = 0, 1, 2, \ldots, n$ be the sum of the numbers on the kth rising diagonal.

a) Find $D_0 + D_2 + D_4 + D_6 + D_8 + D_{10}$.

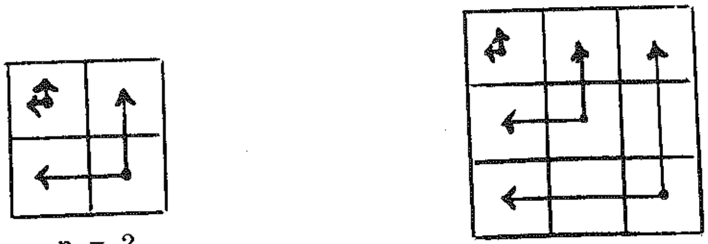
b) Find $\sum_{j=0}^{n} D_{2j}$ c) Find $\sum_{j=1}^{n} D_{2j-1}$

49. Study the complete triangle from problem 47. Assume the patterns continue. In row n give

a) the zeroth element b) the first element

c) the second element d) the third element

50.



n = 2 n = 3

(a) Figure 39 (b)

a) In the 2 x 2 array there are 3 unit squares on the second gnomon.
In the 3 x 3 array there are 5 unit squares on the third gnomon.
In an n x n array how many unit squares are on the nth gnomon. ?

b) What is the total number of unit squares in an n x n array ?

c) In the 2 x 2 array there are a total of 5 squares.
In the 3 x 3 array there are a total of 14 squares.

In an n x n array how many squares are there in all. ?

d) What is the connection between this problem and the triangle in Figure 38 ?

51. Use the completed triangle from Figure 38.

a) Give the row sums for each row from 0 through 10.

b) Give the cumulative row sums for each row from 0 through 10.

c) For row n, give the row sum in terms of n .

d) For row n, give the cumulative row sum in terms of n .

52. Use the completed triangle from Figure 38. Let $\boxed{a \quad b \quad c \quad d \ldots}$ be any row.

Let alternating sum S_1 be symbolized as $S_1 = a - b + c - d + \ldots$

Let alternating sum S_2 be symbolized as $S_2 = a + b - c + d - \ldots$

a) Find S_1 for each of the rows 0 through 10.

b) Find S_2 for each of the rows 0 through 10.

c) For row n, give alternating sum S_1 .

d) For row n, give alternating sum S_2 .

53. Study the triangle in Figure 40. Except for row 0 which has been deleted, each row is the reverse of the corresponding row of the triangle in Figure 38.

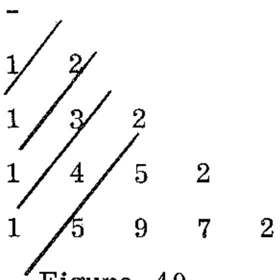


Figure 40

a) Let L_k be the sum of the numbers on the kth rising diagonal.

Compute the sums. The L_k numbers are known as Lucas numbers.

L_k	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
Sum										

b) Assume L_1 through L_{n-1} have been determined. Describe a quick way to get L_n .

c) Find a easy way to sum $L_1 + L_2 + L_3 + \ldots + L_n$.

d) Calculate $\sum_{k=1}^{n} L_{2k}$ e) Calculate $\sum_{k=1}^{n} L_{2k-1}$

54. In the Pascal triangle, row n is the set of coefficients for the expanded form of $(a + b)^n$. Study the triangle in Figure 40. Find an expression which in expanded form has row n for coefficients.

55. a) Given the triangular array in Figure 41. Complete rows 5 through 10 using the pattern you observe in rows 0 through 4.

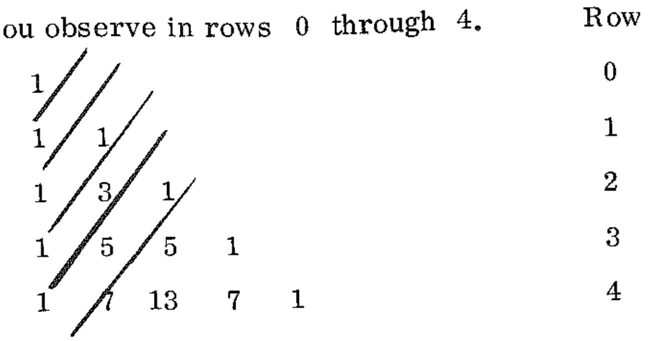


Figure 41

b) Let T_k be the sum of the numbers on the kth rising diagonal. Compute the sums.

T_k	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
Sum										

c) Assume T_1 through T_{n-1} have been determined. Describe a quick way to get T_n

56. a) Form a new triangle in which the elements are the <u>squares</u> of the terms in the Pascal triangle. Complete rows 0 through 7.

b) Find the row sums of each row of this triangle from row 0 through 7.

c) Locate the row sums in the Pascal triangle and complete the following:

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \dots + \binom{n}{n}^2 = \underline{\qquad\qquad} .$$

57. Use the completed triangle from problem 56 (a). Let $\boxed{a^2\ b^2\ c^2\ \dots}$ be any row. Let alternating sum A_1 be symbolized as $A_1 = a^2 - b^2 + c^2 - d^2 + \dots$.

a) Find A_1 for each of the rows 0 through 10 .

b) If n is any odd number, what is alternating sum A_1 ?

c) If n is any even number, what is alternating sum A_1 ?

58. Use the completed triangle from problem 56 (a). Let $\boxed{a^2\ b^2\ c^2\ \dots}$ be any row. Let alternating sum A_2 be symbolized as $A_2 = a^2 + b^2 - c^2 + d^2 - \dots$.

a) Find A_2 for each of the rows 0 through 10 .

b) If n is any odd number, what is alternating sum A_2 ?

c) If n is any even number, what is alternating sum A_2 ?

59. Rows 0 through 3 of Fibonacci's odd number triangle are shown in Figure 42.

1

3 5

7 9 11

13 15 17 19

Figure 42

a) Continue the triangle through row 10 .

b) Compute the row sums for rows 0 through 10.

c) For row n , give the row sum in terms of n.

d) Even numbered rows have a middle element. In row n describe the middle element in terms of n .

e) Find the alternating sum $S_1 = a - b + c - d + \dots$ for rows 0 through 10.

f) If n is any even number, what is alternating sum S_1 ?

g) If n is any odd number, what is alternating sum S_1 ?

60. a) Figure 43 gives the rows 0 through 3 of a triangular array. Determine how the triangle is formed and produce rows 4 and 5.

1 1

1 2 1

1 3 2 3 1

1 4 3 5 2 5 3 4 1

Figure 43

b) Compute the number of terms in each of the rows 0 through 5. Can you predict the number of terms for row n ?

c) Compute the row sums for each of the rows 0 through 5. Can you predict the sum of the terms in row n ?

d) Study the triangle carefully. Name some properties of this triangle based on rows 0 through 5 .

61. Consider the triangle in Figure 44.

1

1 1

1 1 1

1 1 2 1

1 1 3 2 1

Figure 44

a) Rows 0 through 4 are given . Produce rows 5 through 10 .

b) Describe the procedure for forming a row of this triangle.

c) Compute the row sums for rows 0 through 10. Predict the row sum for row n.

62. Pascal Mod Art. Use a black dot to represent an odd number or 1 modulo 2. and a white dot in place of an even number or 0 modulo 2. You will need the basic addition facts for modulo 2. These are in the tables below.

+	0	1
0	0	1
1	1	0

OR

+	even	odd
even	even	odd
odd	odd	even

OR

+	white	black
white	white	black
black	black	white

Complete a Pascal triangle composed of white and black dots. To proceed from one line to the next use the addition pattern of the triangle and the facts in the tables above. Build rows 0 through 31 inclusive. Rows 0 through 3 should look like Figure 45.

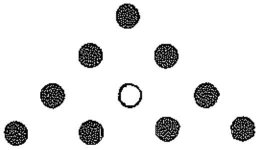


Figure 45

63. Use rows 0 through 31 as constructed in problem 62.

Complete the chart.

Number of black dots per row	Rows having this number of black dots

64. From row 32 onward, predict the next five rows of the Pascal triangle in which

a) only odd numbers appear,

b) exactly two odd numbers appear,

c) odd and even numbers alternate.

65. Make a conjecture about $(a + b)^n$ for $n = 2^m$. Hint: study problems 62 - 64.

66. Study the completed figure from problem 62.

a) Examine the white dot pattern in your design. What triangular numbers do the white dots form ? How many times does each different triangular number appear ?

b) Examine the black dot pattern in your design. What triangular numbers do the black dots form ? How many times does each different triangular number appear ?

c) How many dots are in your design ? How many are white ? black ?

d) A perfect number p is a number that equals the sum of all of its divisors except p. Which of the numbers in your answer to (a) are perfect ?

67. <u>More Pascal Mod Art.</u> Use a black dot to represent 1 or 2 mod 3 and a white dot for 0 mod 3. You will need the basic addition facts for modulo 3. These are

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Complete a Pascal triangle composed of white and black dots. Build rows 0 through 27. Rows 0 through 3 should look like Figure 46.

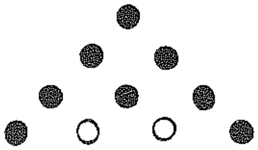


Figure 46

68. Use rows 0 through 27 as constructed in problem 67.

Complete the chart.

Number of black dots per row	Rows having this number of black dots.

______________ ______________________

______________ ______________________

______________ ______________________

______________ ______________________

______________ ______________________

______________ ______________________

______________ ______________________

69. From the information in the chart in problem 68, predict the next three rows of the Pascal triangle in which

a) all black dots appear,

b) exactly two black dots appear,

c) exactly three black dots appear.

70. Make a conjecture about $(a + b)^n$ for $n = 3^m$. Hint: study problems 67 - 69.

71. Study the completed figure from problem 67.

a) Examine the white dot pattern in your design. What triangular numbers do the white dots form ? How many times does each different triangular number appear ?

b) Examine the black dot pattern in your design. What triangular numbers do the black dots form ? How many times does each different triangular number appear ?

c) How many dots are in your design ? How many are white ? black ?

72. Use the Pascal triangle to get the expansion of $(a + b)^n$ in modulo n.

For example in modulo 2,

$$(a + b)^2 = a^2 + 2ab + b^2 = a^2 + 0 \cdot ab + b^2 = a^2 + b^2$$

$$\text{since} \quad 2 \equiv 0 \pmod 2$$

Do this for n equal to

a) 3 b) 4 c) 5 d) 6 e) 7 f) 8 g) 9

73. a) Make a conjecture about the expansion of $(a + b)^p$ in modulo p where p is a prime.

b) Make a conjecture about the expansion of $(a + b)^{p^n}$ in modulo p where p is a prime.

Pascal and Probability

Suppose we perform an experiment using a fair or unbiased coin. This means that when the coin is tossed, it is just as likely to land on one side as on the other. Toss the coin twice. The possible outcomes for this experiment are

Table A

First Toss	Second Toss	Number of Heads
T	T	0
T	H	1
H	T	1
H	H	2

Study the outcomes. Each is just as likely to occur as any other. This data is the basis for many statements in probability. For example:

1) The probability of throwing two heads is 1/4 since exactly one out of four outcomes consists of two heads. This is the outcome HH .

   Write $P(\text{2 heads}) = P(2) = 1/4$.

2) The probability of throwing exactly one head is $2/4 = 1/2$ since exactly two out of four outcomes consist of one head. These are the outcomes TH and HT .

   Write $P(\text{1 head}) = P(1) = 1/2$.

3) The probability of throwing zero heads is 1/4 since exactly one out of four outcomes consists of 0 heads. This is the outcome TT.

   Write $P(\text{0 heads}) = P(0) = 1/4$.

4) The probability of throwing one or two heads is

   $$P(1) + P(2) = 1/2 + 1/4 = 3/4 .$$

   In this case $P(1) + P(2)$ also represents the probability of throwing at least one head.

5) Any possible outcome consists of 0, 1, or 2 heads. The probability of getting 0, 1 or 2 heads is

   $$P(0) + P(1) + P(2) = 1/4 + 1/2 + 1/4 = 1$$

6) The probability of throwing at least one head can be calculated directly or indirectly. Here we have

   direct: $P(1) + P(2) = 1/2 + 1/4 = 3/4$

   indirect: $1 - P(0) = \left[P(0) + P(1) + P(2)\right] - P(0)$

   $= 1 - 1/4 = 3/4$

In summary, the probability of throwing exactly k heads is

$$\boxed{P(k) = P(k \text{ heads}) = \frac{\text{number of outcomes containing } k \text{ heads}}{\text{total number of possible outcomes}}}$$

74. Toss a fair coin three times.

a) List the possible outcomes of this experiment as in Table A.

b) Find $P(0)$, $P(1)$, $P(2)$, $P(3)$

c) Compute $P(0) + P(1) + P(2) + P(3)$.

d) What is the probability of getting at least one head ?

e) What is the probability of getting at most two heads ?

75. Toss a fair coin four times.

a) List the possible outcomes of this experiment as in Table A.

b) Find $P(0)$, $P(1)$, $P(2)$, $P(3)$, $P(4)$.

c) Compute $P(0) + P(1) + P(2) + P(3) + P(4)$.

d) What is the probability of getting at least one head ?

e) What is the probability of getting at most three heads ?

f) For the number of heads k, where $k = 0, 1, 2, 3, 4$, determine if $P(k) = P(4 - k)$

76. Use the Pascal triangle to answer the questions below.

a) In six tosses of a fair coin, what are the chances of getting exactly 3 heads ?

b) In seven tosses of a fair coin, what are the chances of getting exactly 4 heads ?

c) In eight tosses of a fair coin, what are the chances of getting 3 or 4 heads ?

d) In five tosses of a fair coin, what are the chances of getting at most 4 heads ?

e) In five tosses of a fair coin, what is the probability of getting two tails?

f) In nine tosses of a fair coin, what is the probability of getting three tails?

g) In eight tosses of a fair coin, what is the probability of getting six tails ?

h) In ten tosses of a fair coin, what is the probability of getting at most 9 tails?

i) In n tosses of a fair coin, find $P(h)$ where h is the number of heads.

j) In n tosses of a fair coin, what is the probability of getting t tails ?

k) In n tosses of a fair coin, what is the probability of getting at least 1 head ?

l) In n tosses of a fair coin, what is the probability of getting at most $n - 1$ heads ?

77. Use the Pascal triangle to find the solution to each of these questions.

a) What is the probability that the third head appears on the seventh toss of a fair coin ?

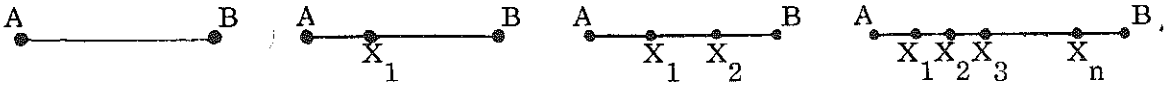
b) What is the probability that the sixth head appears on the eighth toss of a fair coin ?

c) What is the probability that the fourth tail appears on the sixth toss of a fair coin ?

d) What is the probability that the k th head appears on the n th toss of a fair coin ?

Can you locate the solutions to these geometry counting problems in the Pascal triangle ? Be careful . Some are very tricky .

78. Study Figure 47.



(a) (b) (c) (d)

Figure 47

How many line segments are in

a) Figure 47 (a)

b) Figure 47 (b)

c) Figure 47 (c)

d) Figure 47 (d)

79. Given a collection of n points, $n \geqslant 2$, no three of which are collinear. Compute the total number of line segments containing exactly two of these points.

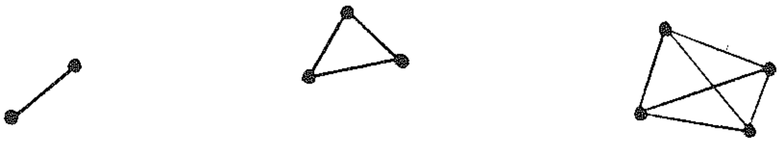


Figure 48

80. Given a collection of n points, $n \geqslant 3$, no three of which are collinear. Compute the total number of triangles that can be formed whose vertices are 3 of the points.

81. Compute the number of diagonals in a convex n - gon, $n \geqslant 3$.

82. How should a circle be cut straight across to maximize the number of pieces obtained from n cuts ? One, two, three cuts are shown in Figure 49.

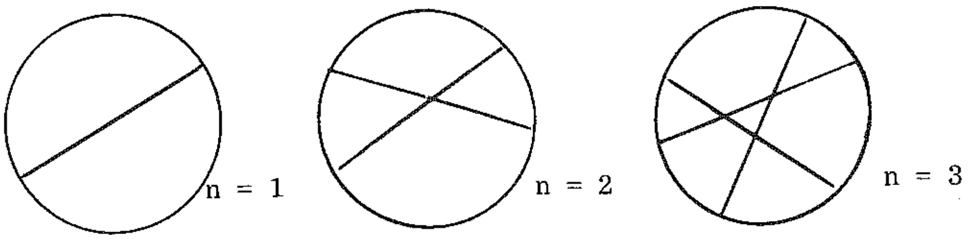


Figure 49

Complete the chart .

number of cuts n	1	2	3	4	5	6	7	8	9	10	n
max number of pieces											

83. Study the intersection points of the diagonals of the polygons in Figure 50. Note that excluding the vertices of the polygon only two diagonals intersect at a point. Find the number of diagonal intersection points (excluding vertices) for a convex n - gon .

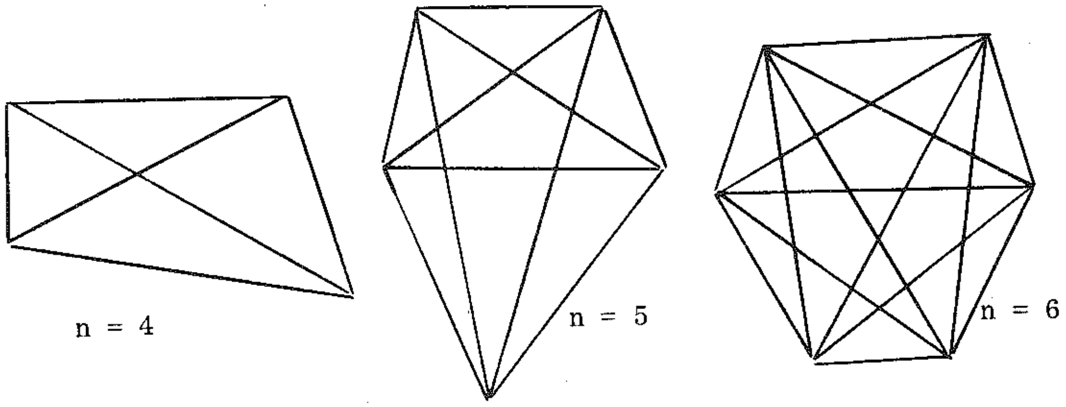


Figure 50

number of sides of n - gon	4	5	6	7	8	n
number of diagonal intersection points						

84. Now concentrate on the regions determined by the diagonals of the polygons in Figure 50. Find the maximum number of interior regions for a convex n-gon. That is, continue to use polygons in which only two diagonals intersect at an interior point.

number of sides of n - gon	3	4	5	6	7	8	n
number of diagonal determining regions							

85. Figure 51 gives rows 1 through 4 of a triangle known as the harmonic triangle. The development of this triangle is due to the German mathematician, Gottfried Leibniz (1646 - 1716). Study the formation of the rows given and continue the pattern to complete rows 6 through 10.

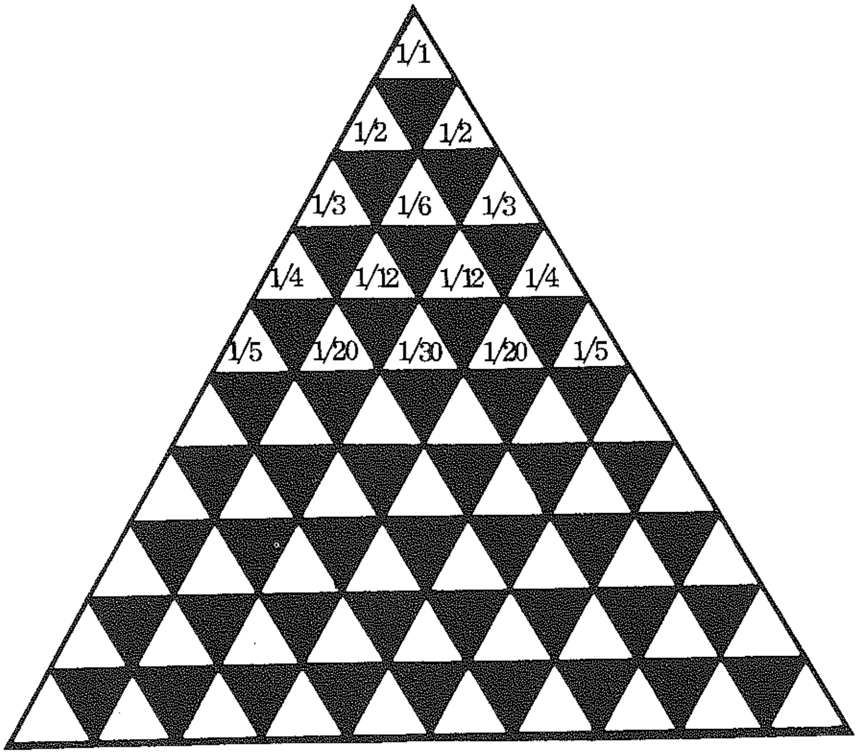


Figure 51

86. Unravel the fraction pattern below.

a) Find A, B in $\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$

b) Find A, B, C in $\frac{2}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$

c) Find A, B, C, D in $\frac{6}{x(x+1)(x+2)(x+3)}$

$$= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2} + \frac{D}{x+3}$$

d) Find C in $\frac{24}{x(x+1)(x+2)(x+3)(x+4)}$

$$= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2} + \frac{D}{x+3} + \frac{E}{x+4}$$

e) Find D in $\dfrac{120}{x(x+1)(x+2)(x+3)(x+4)(x+5)}$

$$= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2} + \frac{D}{x+3} + \frac{E}{x+4} + \frac{F}{x+5} .$$

87. $(a+b)^{-1} = \dfrac{1}{a+b} = a^{-1} - a^{-2}b + a^{-3}b^{2} - a^{-4}b^{3} + \ldots$

Use the division method outlined to determine $(a+b)^{-n}$ for each n given and then uncover the absolute values of each set of coefficients in the Pascal triangle.

a) $(a+b)^{-2}$ b) $(a+b)^{-3}$

c) $(a+b)^{-4}$ d) $(a+b)^{-n}$ n, a natural number $>$ 0.

88. a) Use the addition pattern illustrated below to complete the table. You should note that in any row two adjacent numbers a, b have a sum c located in the row above in b's column .

r n	0	1	2	3	4	5	6	7
7	1							
6	1							
5	1							
4	1							
3	1							
2	1	2						
1	1	1	0	0	0	0	0	0
0	1							
-1	1							
-2	1							
-3	1							
-4	1							
-5	1							
-6	1							
-7	1							

b) What do you observe in rows 7 through 0 ?

c) What do you observe in rows -1 through -7 ?

d) Label the entry in the nth row and r th column with the symbol $\binom{n}{r}$.

If $n \geqslant 0$ and $0 \leqslant r \leqslant n$ then $\binom{n}{r}$ is the familiar combinations symbol representing $\frac{n!}{r!(n-r)!}$. If $n \geqslant 0$ and $r > n$ what is $\binom{n}{r}$? If $n < 0$ find a formula for $\binom{n}{r}$ using factorials.

e) Use the table to give the expansion of

1) $(a + b)^{-4}$ 2) $(a + b)^{-5}$

3) $(a + b)^{-6}$ 4) $(a + b)^{-7}$

89.

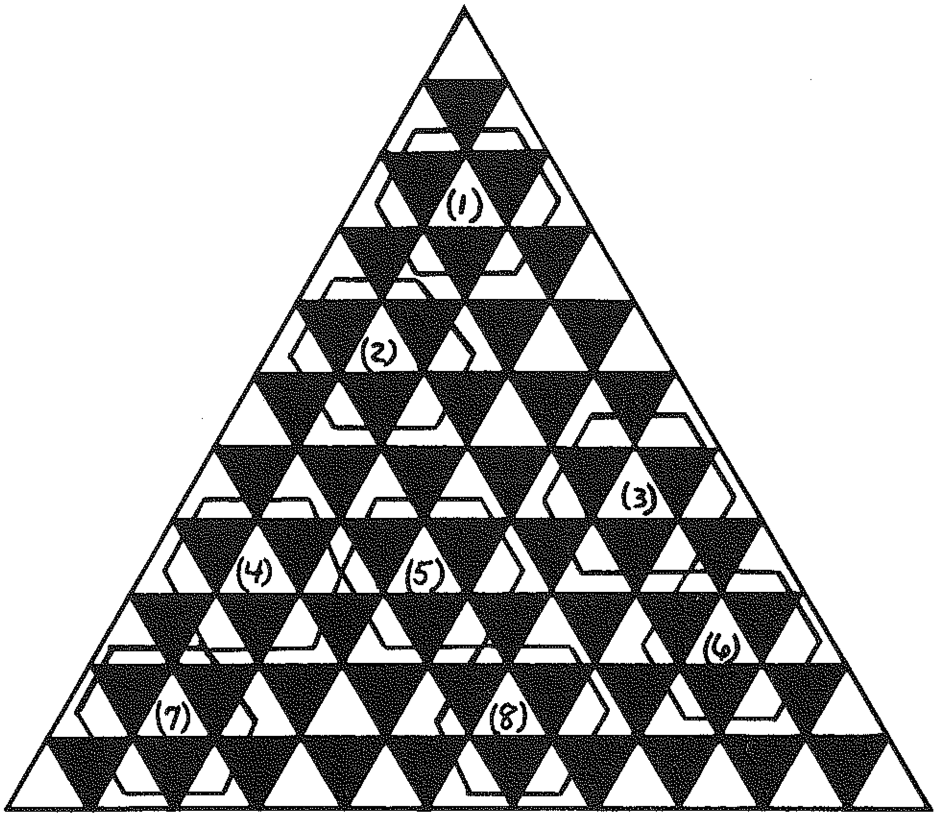


Figure 52

a) Write each of the elements in the Pascal triangle in prime factored form in rows 0 through 10.

b) For each hexagon shown in Figure 52 write the product of the six vertex numbers in prime factored form only.

c) Make a conjecture based on your results in (b). Give the conjecture in combinations form.

d) Prove your conjecture is valid.

90. $a + ib$ is a complex number in which a is the real part and ib is the imaginary part. $i = \sqrt{-1}$ and $i^2 = -1$. Use the Pascal triangle

to expand the binomial $(\cos\ \phi$ and $i\ \sin\ \phi)^n$. In each case write your final answer in $a + ib$ form. Do the expansion for the cases

a) $n = 2$ b) $n = 3$ c) $n = 4$

91. DeMoivre's Theorem states that

$(\cos\ \phi + i\ \sin\ \phi)^n = \cos n\phi + i\ \sin n\phi$ for n, a natural number.

Use DeMoivre's Theorem and the solutions to problem 90 to find a trigonometric identity in terms of ϕ for

a) $\cos\ 2\phi$ b) $\sin\ 2\phi$

c) $\cos\ 3\phi$ d) $\sin\ 3\phi$

e) $\cos\ 4\phi$ f) $\sin\ 4\phi$

92. Recall $\tan\ \phi = \dfrac{\sin\ \phi}{\cos\ \phi}$

Using your solutions to problem 91, derive $\tan n\phi$ in terms of $\tan\ \phi$.

a) $n = 2$ b) $n = 3$ c) $n = 4$

d) Study the coefficient pattern in the numerator and denominator of your your answer to (a), (b), (c) . Continue the pattern and write the identity for $\tan 5\phi$ and $\tan 6\phi$ in terms of $\tan\ \phi$.

Matrix Connections

A matrix is a rectangular array of numbers or elements. For example

$$\begin{pmatrix} 6 & 2 \\ 3 & 0 \\ -5 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \quad \text{are matrices.}$$

A matrix that has n rows and n columns is a square matrix. Every square matrix of numbers has a number associated with it known as the determinant of the matrix. For information on matrix multiplication and determinants the reader is advised to consult The Mathematics of Matrices by Philip J. Davis, New York: Blaisdell Publishing Co. , 1965 or other similar reference.

93. Perform the matrix products.

a) $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Make a conjecture !

e) $\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & \cdots \\ 1 & 3 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 2 & 3 & \cdots \\ 0 & 0 & 1 & 3 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}$

94. Compute the determinant of each matrix.

a) $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{pmatrix}$

e) Consider the n x n matrix D_n whose rows (columns) are the diagonals of the Pascal triangle. Complete this statement:
For each natural number $n \geqslant 1$, it appears that the determinant of D_n is ____________________ .

95. For each matrix A, find its transpose A^t and compute $A A^t$.

a) $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -2 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -3 & 3 & -1 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 1 & -3 & 3 & -1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix}$

Make a conjecture !

e) $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & 0 & \cdots \\ 1 & -3 & 3 & -1 & 0 & \cdots \\ 1 & -4 & 6 & -4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}$

96. Compute the determinant of each matrix .

a) $\begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}$

b) $\begin{pmatrix} 2 & 3 & 4 \\ 3 & 6 & 10 \\ 4 & 10 & 20 \end{pmatrix}$

c) $\begin{pmatrix} 2 & 3 & 4 & 5 \\ 3 & 6 & 10 & 15 \\ 4 & 10 & 20 & 35 \\ 5 & 15 & 35 & 70 \end{pmatrix}$

d) $\begin{pmatrix} 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 10 & 15 & 21 \\ 4 & 10 & 20 & 35 & 56 \\ 5 & 15 & 35 & 70 & 126 \\ 6 & 21 & 56 & 126 & 252 \end{pmatrix}$

e) Consider the n x n matrix D_n whose rows (columns) are the diagonals (excepting the zero diagonal) of the Pascal triangle. Complete this statement: For each natural number $n \geqslant 1$, it appears that the determinant of D_n is ________.

97. Compute the determinant of each matrix.

a) $\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$

b) $\begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}$

c) $\begin{pmatrix} 3 & 6 \\ 4 & 10 \end{pmatrix}$

d) $\begin{pmatrix} 4 & 10 \\ 5 & 15 \end{pmatrix}$

e) $$\begin{pmatrix} \binom{k}{1} & \binom{k+1}{2} \\ \binom{k+1}{1} & \binom{k+2}{2} \end{pmatrix}$$

98. Compute the determinant of each matrix.

a) $$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{pmatrix}$$

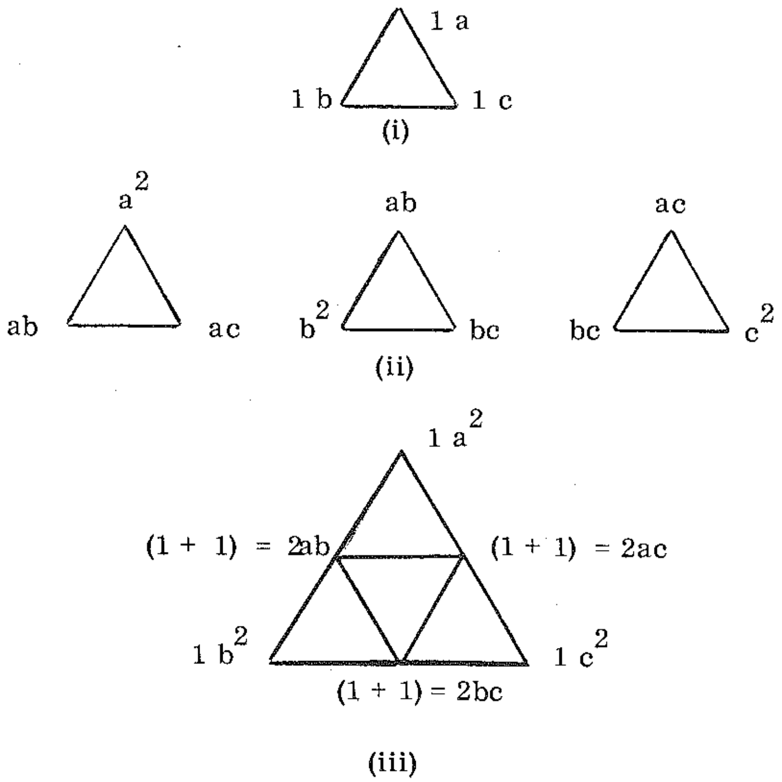
b) $$\begin{pmatrix} 2 & 3 & 4 \\ 3 & 6 & 10 \\ 4 & 10 & 20 \end{pmatrix}$$

c) $$\begin{pmatrix} 3 & 6 & 10 \\ 4 & 10 & 20 \\ 5 & 15 & 35 \end{pmatrix}$$

d) $$\begin{pmatrix} 4 & 10 & 20 \\ 5 & 15 & 35 \\ 6 & 21 & 56 \end{pmatrix}$$

e) $$\begin{pmatrix} \binom{k}{1} & \binom{k+1}{2} & \binom{k+2}{3} \\ \binom{k+1}{1} & \binom{k+2}{2} & \binom{k+3}{3} \\ \binom{k+2}{1} & \binom{k+3}{2} & \binom{k+4}{3} \end{pmatrix}$$

99.



(i)

(ii)

(iii)

Figure 53

Figure 53 illustrates the expansion of $(a + b + c)^2$.

In i) begin with $(a + b + c)$

ii) multiply $(a + b + c)$ in turn by a, by b , by c.

iii) combine like terms by matching vertices with same label. The sum of the terms at each vertex in (iii) is the expansion.

a) Develop an illustration for the expansion of $(a + b + c)^3$. Start with Figure 53 (iii) . Multiply each term by a, by b, by c, obtaining three triangles. Put the three triangles together so that matching vertices meet . Assign appropriate numbers to each vertex of your final figure. Check your work by expanding $(a + b + c)^3$ algebraically.

b) Perform the same steps as in (a) to illustrate the expansion of $(a + b + c)^4$. You must start with your answer to (a). Check your work by expanding $(a + b + c)^4$ algebraically .

c) Study Figure 53 and your answers to (a) and (b) above. Discover the pattern of formation and put down the final array for $(a + b + c)^5$ without any preliminary steps.

d) Do the same for $(a + b + c)^6$.

100. a) Describe how each of the triangular arrays in problem 99 is related to the Pascal triangle.

b) Find the sum of the numbers in Figure 53 (i), Figure 53(iii) and in each part of problem 99. Make a conjecture about the sum of the coefficients of $(a + b + c)^n$.

****

Many properties relating to the Pascal triangle and the binomial expansion have parallels in the trinomial case. See how many you can uncover !

We have by no means exhausted the supply of Pascal related problems. Be on the alert for evidence of the incredible triangle in all branches of mathematics !

As you read over the problems in this booklet remember the words of the French mathematician Rene Descartes (1596 - 1650)

" I hope that posterity will judge me kindly not only as to the things which I have explained but also as to those which I have omitted so as to leave to others the pleasure of discovery . "

SOLUTIONS TO PASCAL TRIANGLE PROBLEMS

1.

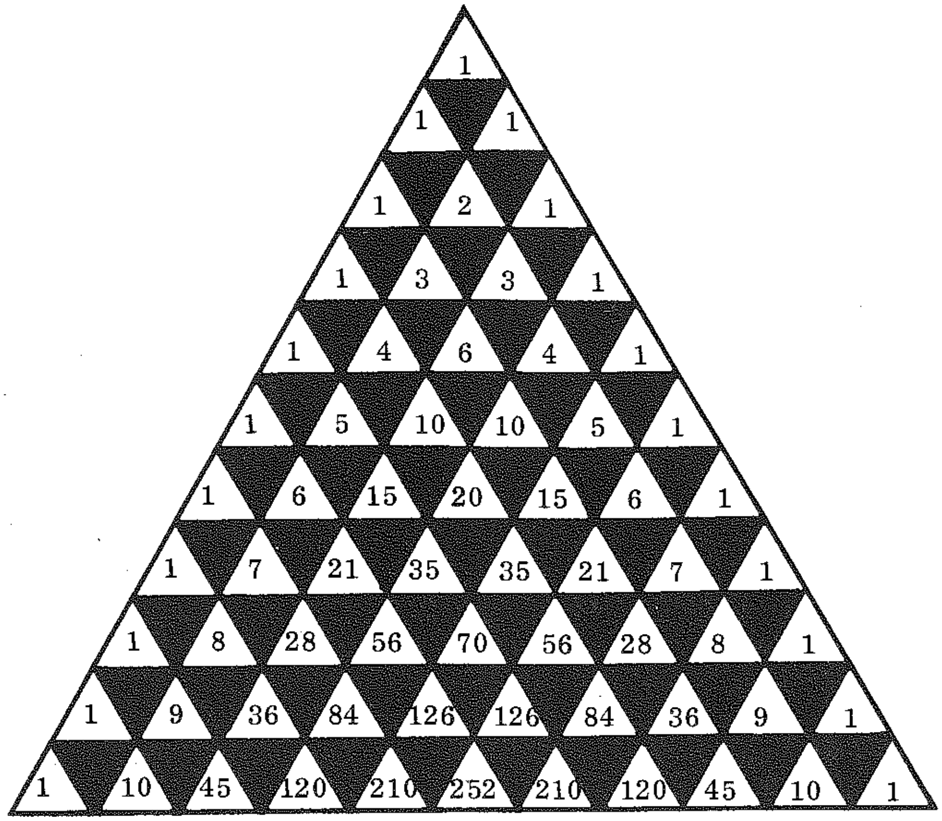


Figure 4

2. a) 0 b) 2 c) twice d) once

e) Rows 0, 1, 3, 7, 15. It appears that rows $2^n - 1$ contain only odd numbers. We will learn later that in rows 2^n only the two end terms are odd. Consequently we can argue that to produce this effect all terms in row preceding (namely $2^n - 1$) must be odd.

f) One occurs $1 + \underbrace{2 + 2 + 2 + \ldots + 2}_{n \text{ of these}} = 2n + 1$ times

3. a) n - k th b) n - k th

4. To form the n + 1 st row: the 0 th term is 1, the 1 st term is the sum of the 0 th and 1 st terms in the n th row, the 2 nd term is the sum of the 1 st and 2 nd terms in the n th row, ... , the k th term is the sum of the k - 1 st term and k th term in the n th row , ... and so on until the n + 1 st term which is 1.

5. To form the n th row : the 0 th term is 1, the first term is the sum of the 0 th term in row n and the 1 st term in row n - 1, the second term is the sum of the 1 st term in row n and the 2 nd term in row n-1, ... , the k th term is the sum of the k - 1 st term in row n and the kth term in row n - 1, ... Note each row is non-finite and that $n \neq 0$, i.e. start with the 0 row all 1's.

6.

	Row	0	1	2	3	4	5	6	7	8	9	10
a)	Row Sum	1	2	4	8	16	32	64	128	256	512	1024
b)	Cum Row Sum	1	3	7	15	31	63	127	255	511	1023	2047

c) The n th row sum is 2^n. (This is a consequence of the Binomial Theorem in which $x = y = 1$.)

d) The cumulative row sum for rows 0 through n is the geometric series, $1 + 2^1 + 2^2 + 2^3 + \dots + 2^n$, Applying the formula for the sum,

$S = \dfrac{a(r^{n+1} - 1)}{a(r - 1)}$ with $a = 1$, $r = 2$, the total is $2^{n+1} - 1$.

7. a)

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ \boxed{1 \quad 3 \quad 1} \\ 1 \quad 4 \quad 4 \quad 1 \\ 1 \quad 5 \quad 8 \quad 5 \quad 1 \\ 1 \quad 6 \quad 13 \quad 13 \quad 6 \quad 1 \end{array}$$

Row	Row Sum
0	1
1	2
2	5
3	$10 = 2 \cdot 5$
4	$20 = 4 \cdot 5$
5	$40 = 8 \cdot 5$

b) $2^{n-2} \cdot 5$

c)

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ \boxed{1 \quad 5 \quad 1} \\ 1 \quad 6 \quad 6 \quad 1 \\ 1 \quad 7 \quad 12 \quad 7 \quad 1 \\ 1 \quad 8 \quad 19 \quad 19 \quad 8 \quad 1 \end{array}$$

Row	Row Sum
0	1
1	2
2	7
3	$14 = 2 \cdot 7$
4	$28 = 4 \cdot 7$
5	$56 = 8 \cdot 7$

d) $2^{n-2} \cdot 7$

e)

$$\begin{array}{c} 1 \\ 1 \qquad 1 \\ \boxed{1 \qquad k \qquad 1} \\ 1 \quad k+1 \quad k+1 \quad 1 \\ 1 \quad k+2 \quad 2k+2 \quad k+2 \quad 1 \\ 1 \quad k+3 \quad 3k+4 \quad 3k+4 \quad k+3 \quad 1 \end{array}$$

Row	0	1	2	3	4	5	n
Row Sum	1	2	k + 2	2k + 4	4k + 8	8k + 16	$2^{n-2} \cdot (k + 2)$

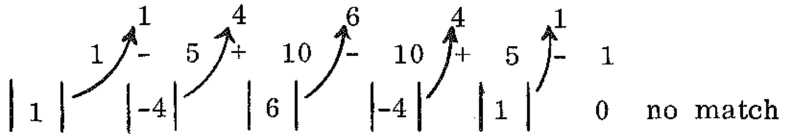
8.

	Row	0	1	2	3	4	5	6	7	8	9	10		If n ≠ 0
a)	Alt Sum S_1	1	0	0	0	0	0	0	0	0	0	0	c)	$S_1 = 0$
b)	Alt Sum S_2	1	2	2	2	2	2	2	2	2	2	2	d)	$S_2 = 2$

Alt Sum S_1 is a consequence of the Binomial Theorem in which $x = 1$, $y = -1$.

Alt Sum S_2 can be established using the binomial coefficients and considering the cases n-odd and even separately.

As you do out an alternating sum S_1 for a row , note that the absolute value of the partial sums on a step by step basis appear in the row above one position to the right. For example,



9.

1										
1	1									
1	2	1								
1	3	3	1							
1	4	6	4	1						
1	5	10	10	5	1					
1	6	15	20	15	6	1				
1	7	21	35	35	21	7	1			
1	8	28	56	70	56	28	8	1		
1	9	36	84	126	126	84	36	9	1	
1	10	45	120	210	252	210	120	45	10	1

a)

Row	2	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10
Term	1	2	2	3	3	4	2	4	3	5	4	7	3	6	2	7
L.A.S.	1	1	3	1	4	1	10	5	20	6	35	1	56	28	36	36

b) The left alternating sum for the kth element in row n is the kth element in row n - 1. This is an immediate consequence of the addition pattern.

10. a) $1 + 10 + 5 = 16 = 2^4$

b) $5 + 10 + 1 = 16 = 2^4$

c) $1 + 15 + 15 + 1 = 32 = 2^5$

d) $6 + 20 + 6 = 32 = 2^5$

e), f) In each case the sum is 2^{n-1}. This is a consequence of the addition pattern and is illustrated below for $n = 5, 6$.

$n = 5$

1	5	10	10	5	1	comes from
1	1 + 4	4 + 6	6 + 4	4 + 1	1	
e	o	e	o	e	o	

Note that for terms marked e, o respectively, the sum is

$1 + 4 + 6 + 4 + 1 = 2^4$.

$n = 6$

1	6	15	20	15	6	1	comes from

1	1 + 5	5 + 10	10 + 10	10 + 5	5 + 1	1
e	o	e	o	e	o	e

Note that for terms marked e, o respectively, the sum is

$1 + 5 + 10 + 10 + 5 + 1 = 2^5$.

One can prove these patterns using appropriate sums and differences of binomial coefficients.

11.

power of 11	by multiplication	location in Pascal triangle
11^0	1	Row 0
11^1	11	1
11^2	121	2
11^3	1,331	3
11^4	14,641	4
11^5	161,051	5 *
11^6	1,771,561	6 *
11^7	19,487,171	7 *
11^8	214,358,881	8 *

* 11^5 is row 5 of the Pascal triangle with a slight adjustment. In expanded form

$$\begin{aligned} 1\ 5\ 10\ 10\ 5\ 1 &= 1\cdot 10^5 + 5\cdot 10^4 + 10\cdot 10^3 + 10\cdot 10^2 + 5\cdot 10 + 1 \\ &= 1\cdot 10^5 + 5\cdot 10^4 + 11\cdot 10^3 + 0\cdot 10^2 + 5\cdot 10 + 1 \\ &= 1\cdot 10^5 + 6\cdot 10^4 + 1\cdot 10^3 + 0\cdot 10^2 + 5\cdot 10 + 1 \\ &= 161,051 = 11^5 \end{aligned}$$

The reader should verify the cases for $n = 6, 7,$ and 8 in a similar manner. In summary, since each position represents a power of ten only the digits 0 to 9 may be used. Thus, changes following usual conventions of expanded notation determine 11^n from row n of the Pascal triangle. The validity of this pattern is a consequence of the Binomial Theorem in which $x = 10$, $y = 1$.

12.

power of 11_{five}	in base five	in base ten	location in Pascal triangle
11^0_{five}	1	1	Row 0
11^1_{five}	11	6	1
11^2_{five}	121	$36 = 6^2$	2
11^3_{five}	1,331	$216 = 6^3$	3
11^4_{five}	20,141	$1,296 = 6^4$	4 *
11^5_{five}	211,101	$7,776 = 6^5$	5 *

* 11^4_{five} is row 4 of the Pascal triangle with a slight adjustment. In expanded form

$$1\ 4\ 6\ 4\ 1 = 1\cdot5^4 + 4\cdot5^3 + 6\cdot5^2 + 4\cdot5 + 1$$

$$= 1\cdot5^4 + 5\cdot5^3 + 1\cdot5^2 + 4\cdot5 + 1 = 2\cdot5^4 + 0\cdot5^3 + 1\cdot5^2 + 4\cdot5 + 1$$

$$= 20,141_{\text{five}} = 11^4_{\text{five}}$$

The reader should verify the case for n = 5 in a similar manner. In summary, since each position represents a power of five only the digits 0 to 4 may be used. Thus, changes following usual conventions of expanded notation determine 11^n_{five} from row n of the Pascal triangle. The validity of this pattern is also a consequence of the Binomial Theorem in which $x = 10_{\text{five}}$, $y = 1$.

In fact row n with appropriate modifications as described is 11^n in any base b.

13. a)

Set A	Number of subsets of A with						
	0	1	2	3	4	5	elements
ϕ	1						
{ a }	1	1					
{ a,b }	1	2	1				
{ a,b,c }	1	3	3	1			
{ a,b,c,d }	1	4	6	4	1		
{a,b,c,d,e}	1	5	10	10	5	1	

b)

n(A)	0	1	2	3	4	5	6	7	8	9	10
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1

c) Using the results of 6 (c) the total number of subsets is 2^m.

14. a)

3 black - 0 white			2 black - 1 white			1 black - 2 white			0 black - 3 white		
I	II	III	I	II	III	I	II	III	I	II	III
b	b	b	b	b	w	b	w	w	w	w	w
			b	w	b	w	b	w			
			w	b	b	w	w	b			

b)

4 black - 0 white				3 black - 1 white				2 black - 2 white				1 black - 3 white				0 black - 4 white			
I	II	III	IV	I	II	III	IV	I	II	III	IV	I	II	III	IV	I	II	III	IV
b	b	b	b	b	b	b	w	b	b	w	w	w	w	w	b	w	w	w	w
				b	b	w	b	b	w	b	w	w	w	b	w				
				b	w	b	b	b	w	w	b	w	b	w	w				
				w	b	b	b	w	b	b	w	b	w	w	w				
								w	b	w	b								
								w	w	b	b								

c)

			Total number of different ways
1)	5 black- 0 white		1
2)	4 black - 1 white	= 4 b - 0 w + w + 3 b - 1 w + b	1 + 4 = 5
3)	3 black - 2 white	= 3 b - 1 w + w + 2 b - 2 w + b	4 + 6 = 10
4)	2 black - 3 white	= 2 b - 2 w + w + 1 b - 3 w + b	6 + 4 = 10
5)	1 black - 4 white	= 1 b - 3 w + w + 0 b - 4 w + b	4 + 1 = 5
6)	0 black - 5 white		1

15. a) $\binom{12}{9} = \frac{12\,!}{3\,!\;\;9\,!} = \frac{12 \cdot 11 \cdot 10}{6} = 220$

e) $\binom{n}{0} = \frac{n\,!}{0\,!\,(n-0)\,!} = \frac{n\,!}{1 \cdot n\,!} = 1$

b) $\binom{12}{3} = \frac{12\,!}{9!\;\;3\,!} = 220$

f) $\binom{n}{n} = \frac{n\,!}{n\,!\;(n-n)\,!} = \frac{n\,!}{n\,!\;0\,!} = 1$

c) $\binom{8}{4} = \frac{8\,!}{4\,!\;\;4\,!} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{24} = 70$

d) $\binom{10}{5} = \frac{10\,!}{5\,!\;\;5\,!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 252$

16. a) $(x + y)^6 = x^6 + 6\,x^5y + 15\,x^4y^2 + 20\,x^3y^3 + 15\,x^2y^4 + 6\,xy^5 + y^6$

b) $(x + y)^8 = x^8 + 8\,x^7y + 28\,x^6y^2 + 56\,x^5y^3 + 70\,x^4y^4 + 56\,x^3y^5 + 28\,x^2y^6 + 8xy^7 + y^8$.

17. a) $\sum_{K=0}^{10} \binom{10}{k} = 2^{10}$

b) $\sum_{K=0}^{6} \binom{12}{2k} = 2^{11}$

c) $\binom{6}{0} \bullet \binom{7}{1} + \binom{6}{2} \bullet \binom{7}{3} + \binom{6}{4} \bullet \binom{7}{5} + \binom{6}{6} \bullet \binom{7}{7} = 848$

18. $\binom{n}{k} = \frac{n\,!}{k\,!\;(n-k)\,!} = \frac{n!}{(n-k)\,!\;\;k!} = \binom{n}{n-k}$

19. $\binom{n}{k} + \binom{n}{k+1} = \frac{n\,!}{k\,!\;\;(n-k)\,!} + \frac{n\,!}{(k+1)\,!\;(n-k-1)\,!}$

$$= \frac{n!\,(k+1)}{k!\,(k+1)(n-k)!} + \frac{n!\,(n-k)}{(k+1)!\,(n-k-1)!\,(n-k)}$$

$$= \frac{n!\,(k+1)}{(k+1)!\,(n-k)!} + \frac{n!\,(n-k)}{(k+1)!\,(n-k)!}$$

$$= \frac{n!\,(k+1) + n!\,(n-k)}{(k+1)!\,(n-k)!}$$

$$= \frac{n!\,(n+1)}{(k+1)!\,(n-k)!} = \frac{(n+1)!}{(k+1)!\,(n-k)!} = \binom{n+1}{k+1}$$

20. $$\binom{n}{k-1}\bullet\frac{n-k+1}{k} = \frac{n!}{(k-1)!\,(n-k+1)!}\bullet\frac{n-k+1}{k}$$

$$= \frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

21. a)

Row r	Given adjacent pair in order	a/b	Reduced form	c/d	Reduced form
4	6,4	6/4	3/2	3/2	3/2
5	10,10	10/10	1/1	3/3	1/1
6	15,20	15/20	3/4	3/4	3/4
7	7,21	7/21	1/3	2/6	1/3
8	28,56	28/56	1/2	3/6	1/2
9	84,126	84/126	2/3	4/6	2/3
10	210,252	210/252	5/6	5/6	5/6

b) Yes. This statement is true. Using the combinations form a/b for any adjacent pair is $\binom{n}{k} / \binom{n}{k+1} = \frac{n!}{k!\,(n-k)!} \bullet \frac{(k+1)!\,(n-k-1)!}{n!}$

$$= k+1 / n-k$$

where $k+1$ is the number of numbers to the left and $n-k$ is the number of numbers to the right.

22. a) $0\cdot1 + 1\cdot2 + 2\cdot1 = 2(1+1) = 2\cdot2^1$

$0\cdot1 + 1\cdot3 + 2\cdot3 + 3\cdot1 = 3(1+2+1) = 3\cdot2^2$

$= 4(1+3+3+1) = 4\cdot2^3$

$= 5(1+4+6+4+1) = 5\cdot2^4$

$= 6(1+5+10+10+5+1) = 6\cdot2^5$

b) The statement for row n

$$0\cdot\binom{n}{0} + 1\cdot\binom{n}{1} + 2\cdot\binom{n}{2} + \ldots + n\cdot\binom{n}{n}$$

$$= n\left[\binom{n-1}{0} + \binom{n-1}{1} + \dots + \binom{n-1}{n-1}\right] = n\cdot 2^{n-1}$$

Using sigma notation

$$0\binom{n}{0} + 1\binom{n}{1} + \dots + n\binom{n}{n} = \sum_{k=0}^{n} k\binom{n}{k}$$

$$= \sum_{k=0}^{n} \frac{k\ n\,!}{k\,!\ (n-k)\,!}$$

$$= \sum_{k=1}^{n} \frac{k\ n\,!}{k(k-1)!\,(n-k)\,!}$$

$$= \sum_{k=1}^{n} \frac{n\,!}{(k-1)\,!\,(n-k)\,!}$$

$$= \sum_{k=1}^{n} \frac{n\,(n-1)\,!}{(k-1)\,!\ (n-k)\,!}$$

$$= n \sum_{k=1}^{n} \binom{n-1}{k-1}$$

$$= n \bullet 2^{n-1}$$

23. All the numbers are perfect squares.

a) $121 = (10+1)^2 = 11^2$ $\qquad 14{,}641 = (10+1)^4 = \left[(10+1)^2\right]^2$

b) $10{,}201 = (100+1)^2 = 101^2$

$104{,}060{,}401 = (100+1)^4 = \left[(100+1)^2\right]^2$

c) $1{,}002{,}001 = (1000+1)^2 = 1001^2$

$1{,}004{,}006{,}004{,}001 = (1000+1)^4 = \left[(1000+1)^2\right]^2$

d) $100{,}020{,}001 = (10{,}000+1)^2 = 10001^2$

$10{,}004{,}000{,}600{,}040{,}001 = (10{,}000+1)^4 = \left[(10{,}000+1)^2\right]^2$

e) $1\underbrace{00\ldots0}_{k\ 0's}2\underbrace{00\ldots0}_{k\ 0's}1 = (10^{k+1}+1)^2$

$100\ldots0400\ldots0600\ldots0400\ldots01 = (10^{k+1}+1)^4 = \left[(10^{k+1}+1)^2\right]^2$

The numbers are perfect squares for all nonnegative integers k.

Try to create other number patterns such as these.

24. a)
6
1 5
6 4 1

b)
2
4 7
4 9 7

c)
9
7 2
8 8 3

d)
2
2 9
2 9 9

e)
5
3 2
8 4 7
5 3 1 6

f)
3
1 2
6 4 7
2 4 9 7

g)
8
3 5
9 3 2
9 9 3 8

h)
4
7 6
1 6 9
8 2 4 5

i)
4
5 8
3 2 6
8 4 7 8
5 3 1 6 2

j)
7
8 8
4 4 4
2 2 2 2
1 1 1 1 1

k)
3
6 6
5 1 5
9 5 5 9
2 7 7 7 2

l)
3
4 8
8 5 3
6 2 3 9
4 2 9 3 6

m) The scheme for making correct predictions based on Pascal's triangle is described in the solution to problem 25. To guess the number at the top, starting with a set of 10 numbers, use row 9 of the triangle as this has 10 terms. It is

1 9 36 84 126 126 84 36 9 1

Reduce each of these terms to a number less than 9. To do this just divide each one by 9 and keep the remainder. This will produce

1 0 0 3 0 0 3 0 0 1. Take the given set of numbers and form the sum of the products

$$\underline{4\cdot1} + 5\cdot0 + 9\cdot0 + \underline{2\cdot3} + 4\cdot0 + 6\cdot0 + \underline{4\cdot3} + 1\cdot0 + 7\cdot0 + \underline{6\cdot1} .$$

The top number is the sum of $4 + 6 + 12 + 6 = 1$ (after reducing to a final total of less than or equal to 9.) Given any set of ten numbers $a_1, a_2, \ldots a_{10}$. The number at the top will be

$$a_1 + 3a_4 + 3a_7 + a_{10} .$$

25. a)

$a + 3b + 3c + d$

$a + 2b + c$ $\quad$ $b + 2c + d$

$a + b$ $\quad$ $b + c$ $\quad$ $c + d$

a $\quad$ b $\quad$ c $\quad$ d

In 24 (e) to get the number 5 without forming the pyramid form the sum of the products $5\cdot\underline{1} + 3\cdot\underline{3} + 1\cdot\underline{3} + 6\cdot\underline{1} = 5$

b)

$$\begin{array}{c}
a + 4b + 6c + 4d + e \\
a + 3b + 3c + d \qquad b + 3c + 3d + e \\
a + 2b + c \qquad b + 2c + d \qquad c + 2d + e \\
a + b \qquad b + c \qquad c + d \qquad d + e \\
a \qquad b \qquad c \qquad d \qquad e
\end{array}$$

In 24 (i) to get the number 4 without forming the pyramid form the sum of the products $5\cdot\underline{1} + 3\cdot\underline{4} + 1\cdot\underline{6} + 6\cdot\underline{4} + 2\cdot\underline{1} = 4$ (after reducing to a final total less than or equal to 9).

c)

$$\begin{array}{c}
a + 5b + 10c + 10d + 5e + f \\
a + 4b + 6c + 4d + e \qquad b + 4c + 6d + 4e + f \\
a + 3b + 3c + d \qquad b + 3c + 3d + e \qquad c + 3d + 3e + f \\
a + 2b + c \qquad b + 2c + d \qquad c + 2d + e \qquad d + 2e + f \\
a + b \qquad b + c \qquad c + d \qquad d + e \qquad e + f \\
a \qquad b \qquad c \qquad d \qquad e \qquad f
\end{array}$$

In summary, the polynomial pyramids actually generalize the previous problem: naming the top of the pyramid (which is quite effective as a card trick) .

If we select $\underline{\ 3\ }$ numbers a, b, c we use the scheme $a + 2b + c$ to get the top number where the coefficients of a, b, c are row $\underline{\ 2\ }$ of the Pascal triangle; if we select $\underline{\ 4\ }$ numbers a, b, c, d we use the scheme $a + 3b + 3c + d$ to get the top number where the coefficients of a, b, c, d are row $\underline{\ 3\ }$ of the Pascal triangle, and so on.

26. a)

$$\begin{array}{c}
a - 3b + 3c - d \\
a - 2b + c \qquad b - 2c + d \\
a - b \qquad b - c \qquad c - d \\
a \qquad b \qquad c \qquad d
\end{array}$$

b)

$$\begin{array}{c}
a - 4b + 6c - 4d + e \\
a - 3b + 3c - d \qquad b - 3c + 3d - e \\
a - 2b + c \qquad b - 2c + d \qquad c - 2d + e \\
a - b \qquad b - c \qquad c - d \qquad d - e \\
a \qquad b \qquad c \qquad d \qquad e
\end{array}$$

c)

a - 5b + 10c - 10d + 5e - f

a - 4c + 6c - 4d + e   b - 4c + 6d - 4e + f

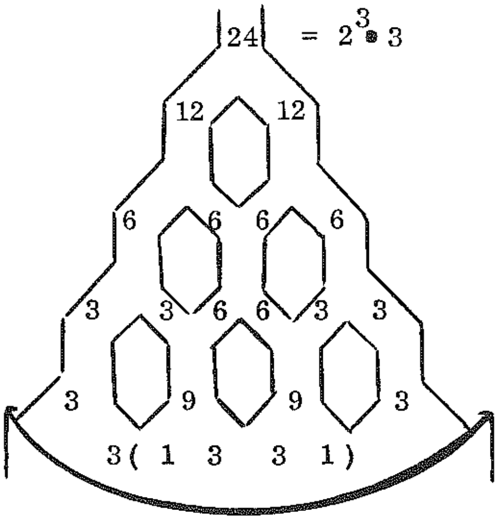
a - 3b + 3c - d   b - 3c + 3d - e   c - 3d + 3e - f

a - 2b + c   b - 2c + d   c - 2d + e   d - 2e + f

a - b   b - c   c - d   d - e   e - f

a   b   c   d   e   f

27. a)



b) 5 15 15 5   5 (1 3 3 1)
and 40 = $2^3 \bullet 5$

c) 5 20 30 20 5   5 (1 4 6 4 1)
and 80 = $2^4 \bullet 5$

d) 9 36 54 36 9   9 (1 4 6 4 1)
and 144 = $2^4 \bullet 9$

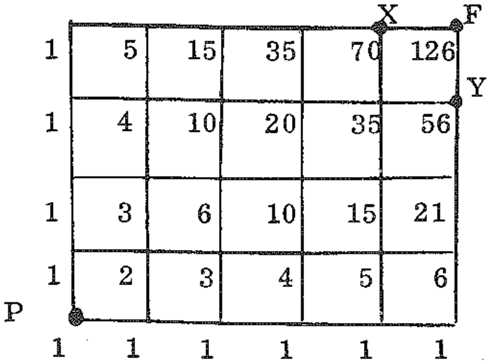
e) 7 35 70 70 35 7   7 (1 5 10 10 5 1)
and 224 = $2^5 \bullet 7$

f) The distribution through each branch would be found by taking each term of row n in the Pascal triangle and multiplying it by m. Using the combinations form this is

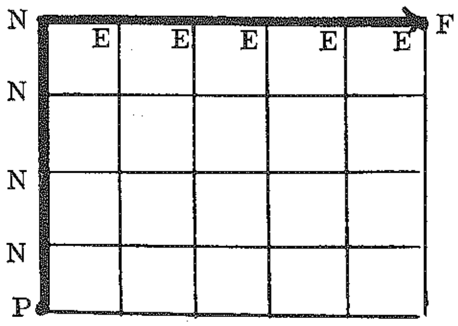
$$m\binom{n}{0} \quad m\binom{n}{1} \quad m\binom{n}{2} \quad \ldots \quad m\binom{n}{n-1} \quad m\binom{n}{n}$$

The pattern in this problem is due to the fact that the sum of the numbers in row n is 2^n .

28. a) One possible method of solution is to recognize that the number of ways to reach F will be the sum of the number of ways to reach X and Y. Thus starting at P we determine the number of ways to reach each point in the grid according to the given rules and eventually reach F. It soon becomes apparent the numbers involved are from the Pascal triangle.

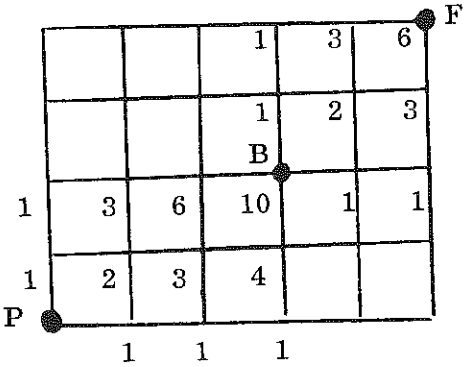


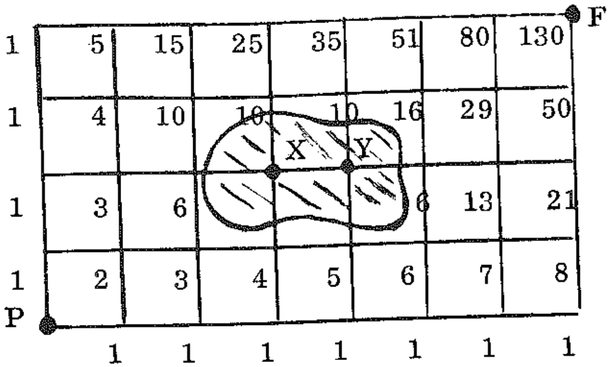
1	5	15	35	70	126
1	4	10	20	35	56
1	3	6	10	15	21
1	2	3	4	5	6
1	1	1	1	1	1



A second method of solution is to notice that each trip can be described by a sequence of 9 letters, four N's and five E's. The one illustrated on page 61 is NNNNEEEEE. The total number of trips is the same as the total number of different ways to arrange 4 N's, 5 E's in an ordered sequence. But this is $\binom{9}{5} = \binom{9}{4} = \frac{9\,!}{5\,!\;4\,!} = 126$.

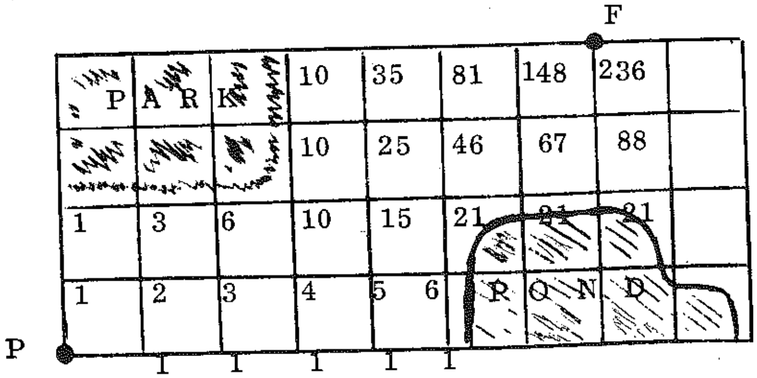
b) Find the number of ways to get from P to B, then the number of ways to get from B to F. The solution is the product of these two numbers. Using the grid we have $10 \cdot 6 = 60$. Using combinations $\binom{5}{2} \cdot \binom{4}{2} = 60$.



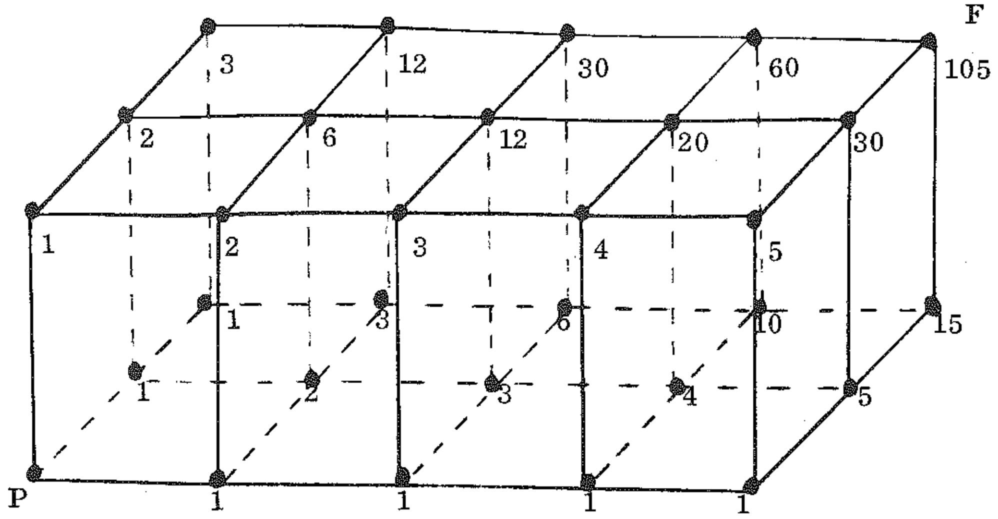


c) To use the addition scheme just avoid the pond and notice that some numbers do not change from point to point. The solution is 130. The problem can also be solved by computing the total number of ways to get from P to F and from this subtracting the number of ways to get from P to X to F and P to Y to F. This is $\binom{11}{7} - \binom{6}{4} \cdot \binom{5}{3} - \binom{6}{4} \cdot \binom{5}{3} = 330 - 150 - 150 = 130.$

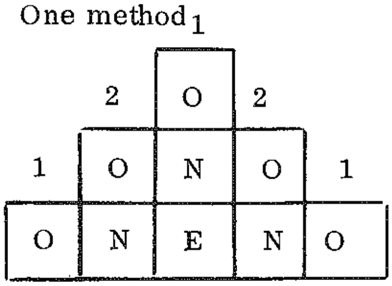
d) Use the addition scheme and avoid the points enclosed in park or pond. The solution is 236 .



e) This problem is an extension of the previous parts. There are 3 directions to consider. One method of solution is to get the numbers for the first floor as before by adding two ways. Then complete the second floor by adding three ways. A second method is to observe each trip consists of 4 E's 2 N's 1 U. One such arrangement is EEEENNU . The different ordered arrangements of these letters total $\frac{7\,!}{2!\;4!\;1!} = 105$.



29. a) One method

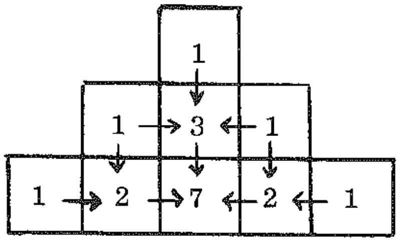


Total is

$2\ (1 + 2 + 1) - 1$

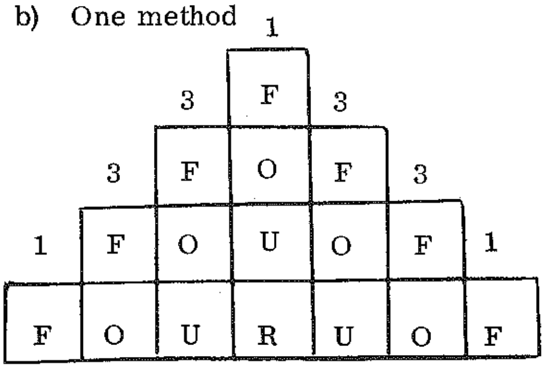
$= \ 8 - 1 = 7 \ = \ 2^3 - 1$

Alternate

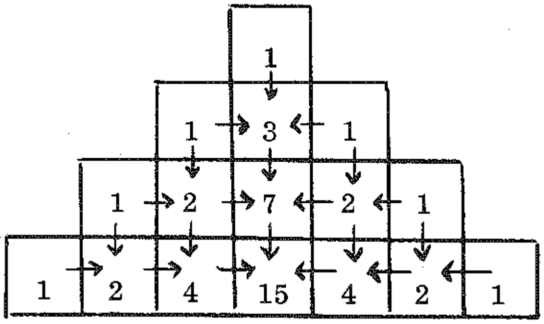


Use the addition scheme of the path problem. The number in 'E' box will be the total .

b) One method



Alternate



Total is $2(1 + 3 + 3 + 1) - 1 = 16 - 1 = 15 = 2^4 - 1$.

c) $2(1 + 4 + 6 + 4 + 1) - 1 = 32 - 1 = 31 = 2^5 - 1$

d) $2(1 + 5 + 10 + 10 + 5 + 1) - 1 = 64 - 1 = 63 = 2^6 - 1$

e) $2^7 - 1$ f) $2^8 - 1$ g) $2^n - 1$

30. In each case replace a letter by the number of moves that can be made to that letter. The total number of spellings of the word will be the sum of the numbers in the bottom row.

a)
$$\begin{array}{ccccc} & & Y & & \\ & O & & O & \\ U & & U & & U \end{array} \longrightarrow \begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 2 & & 1 \end{array} \longrightarrow 1 + 2 + 1 = 4 = 2^2$$

b)
$$\begin{array}{ccccccc} & & & M & & & \\ & & A & & A & & \\ & T & & T & & T & \\ H & & H & & H & & H \end{array} \longrightarrow \begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array} \longrightarrow 1 + 3 + 3 + 1 = 8 = 2^3$$

c) $1 + 4 + 6 + 4 + 1 = 16 = 2^4$

d) $1 + 5 + 10 + 10 + 5 + 1 = 32 = 2^5$

e) 2^{n-1} different ways to spell a n letter word .

31. a) This type of spelling pattern can be directly related to the path problem. The number of ways to spell POINT is the same as the number of ways to get from point P to point T traveling to the left or right. To solve use

i) the addition scheme

$$\begin{array}{ccccc} & & P & & \\ & O & & O & \\ I & & I & & I \\ & N & & N & \\ & & T & & \end{array} \longrightarrow \begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 2 & & 1 \\ & 3 & & 3 & \\ & & 6 & & \end{array}$$

ii) combinations

$$\binom{1 + r}{r} = \binom{4}{2} = 6 .$$

b) i) the addition scheme

$$\begin{array}{ccccccc} & & & A & & & \\ & & L & & L & & \\ & G & & G & & G & \\ E & & E & & E & & E \\ & B & & B & & B & \\ & & R & & R & & \\ & & & A & & & \end{array} \longrightarrow \begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \\ & 4 & & 6 & & 4 & \\ & & 10 & & 10 & & \\ & & & 20 & & & \end{array}$$

ii) combinations

$$\binom{1 + r}{r} = \binom{6}{3} = 20 .$$

c) i) the addition scheme for CONGRUENT

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ & 5 & & 10 & & 10 & & 5 & \\ & & 15 & & 20 & & 15 & & \\ & & & 35 & & 35 & & & \\ & & & & 70 & & & & \end{array}$$

ii) combinations

$$\binom{1 + r}{r} = \binom{8}{4} = 70 .$$

d) i) the addition scheme for MATHEMATICS

ii) combinations

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\ 6 \quad 15 \quad 20 \quad 15 \quad 6 \\ 21 \quad 35 \quad 35 \quad 21 \\ 56 \quad 70 \quad 56 \\ 126 \quad 126 \\ 252 \end{array}$$

$$\binom{1 + r}{r} = \binom{10}{5} = 252$$

32. As in problem 31 the number of spellings can be determined using the Pascal triangle path method or by combinations. However since each word or phrase is a palindrome the answer will be doubled.

a)

$$\begin{array}{c} R \\ A \quad A \\ D \quad D \quad D \\ A \quad A \\ R \end{array} \longrightarrow \begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 3 \quad 3 \\ 6 \end{array} \qquad \text{or} \qquad \binom{2 + 2}{2} = 6$$

The number of ways then is $2 \cdot 6 = 2\binom{4}{2} = 12$.

b) The number of ways is $2 \cdot 70 = 2\binom{8}{4} = 140$

c) The number of ways is $2 \cdot 252 = 2\binom{10}{5} = 504$

d) The number of ways is $2\binom{6+6}{6} = 2\binom{12}{6} = 2 \cdot 924 = 1848$

33. a) $7 = 6 + 1 = 5 + 2 = 4 + 3 = 3 + 4 = 2 + 5 = 1 + 6$

b) $8 = 7 + 1 = 6 + 2 = 5 + 3 = 4 + 4 = 3 + 5 = 2 + 6 = 1 + 7$

c) $5 = 2 + 2 + 1 = 2 + 1 + 2 = 1 + 2 + 2$
$= 3 + 1 + 1 = 1 + 3 + 1 = 1 + 1 + 3$

d) $6 = 4 + 1 + 1 = 1 + 4 + 1 = 1 + 1 + 4$
$= 2 + 2 + 2 = 1 + 2 + 3 = 1 + 3 + 2$
$= 2 + 3 + 1 = 2 + 1 + 3 = 3 + 2 + 1 = 3 + 1 + 2$

e) $6 = 2 + 2 + 1 + 1 = 2 + 1 + 2 + 1 = 1 + 1 + 2 + 2 = 1 + 2 + 1 + 2$
$= 1 + 2 + 2 + 1 = 2 + 1 + 1 + 2 = 3 + 1 + 1 + 1 = 1 + 3 + 1 + 1$
$= 1 + 1 + 3 + 1 = 1 + 1 + 1 + 3$

f) $7 = 2 + 2 + 1 + 1 + 1 = 2 + 1 + 2 + 1 + 1 = 2 + 1 + 1 + 2 + 1$
$= 2 + 1 + 1 + 1 + 2 = 1 + 2 + 2 + 1 + 1 = 1 + 1 + 2 + 2 + 1$
$= 1 + 1 + 1 + 2 + 2 = 1 + 2 + 1 + 2 + 1 = 1 + 2 + 1 + 1 + 2$
$= 1 + 1 + 2 + 1 + 2 = 3 + 1 + 1 + 1 + 1 = 1 + 3 + 1 + 1 + 1$

$= 1 + 1 + 3 + 1 + 1 = 1 + 1 + 1 + 3 + 1 = 1 + 1 + 1 + 1 + 3$

g) Number of addends

N	1	2	3	4	5	6	7
1	1						
2	1	1					
3	1	2	1				
4	1	3	3	1			
5	1	4	6	4	1		
6	1	5	10	10	5	1	
7	1	6	15	20	15	6	1

34. a) 5 b) 14 c) 55 d) 14 e) 44 f) 69

g) $\binom{n}{k} - 1$. The sum is 1 less than the given number regardless of its position. This is a direct consequence of the basic addition relation.

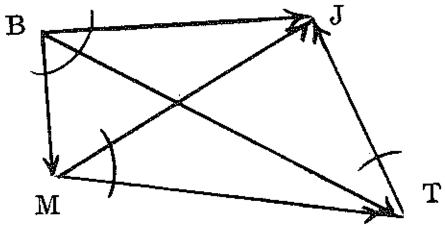
35. a) 6 b) 10 c) 15 d) 21 e) 28 f) 36

g) row 21 term 2 (or 19). Sum is $\frac{20 \cdot 21}{2} = 210$

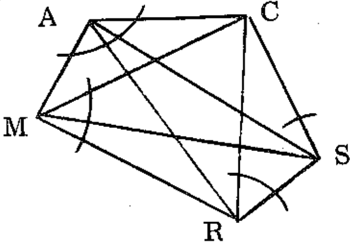
h) row 51 term 2 (or 49). Sum is $\frac{50 \cdot 51}{2} = 1275$

i) The sum is $\binom{n+1}{2} = \binom{n+1}{n-1} = \frac{(n+1)!}{2!\,(n-1)!} = \frac{n(n+1)}{2}$

36. a) Number of hugs: 3 + 2 + 1 = 6



b) Number of hugs: 4 + 3 + 2 + 1 = 10



c)

B	2	3	4	5	6	7	8	9	10	...	n
H	1	3	6	10	15	21	28	36	45		$\frac{n(n-1)}{2}$

It is often helpful to use a geometric model to solve a problem here. Note we have also learned that the number of sides and diagonals of a n - gon is n(n - 1)/2 .

37. a) 21 28 36 45 55

b) $T_{15} = 15 \cdot 16/2 = 120$ $T_{25} = 25 \cdot 26/2 = 325$ $T_{100} = 100 \cdot 101/2 = 5050$

c) The triangular numbers are located on the second diagonal. The n th triangular number is term 2 of row n + 1 . Thus $T_n = n(n + 1)/2$.

d) $1 + 3 + 6 + 10 + 15 = 35$ e) $1 + 3 + \ldots + 55 = 1540$

f) The sum is term 3 (or n - 1) of row n + 2 . It is

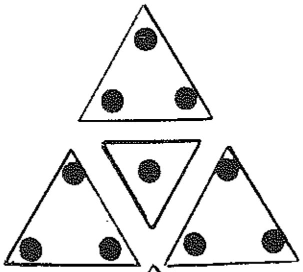
$$\binom{n+2}{3} = \binom{n+2}{n-1} = \frac{(n+2)!}{3!\,(n-1)!} = \frac{n(n+1)(n+2)}{6}$$

38. a) $10 = 3 + 3 + 3 + 1$ b) $15 = 6 + 3 + 3 + 3$

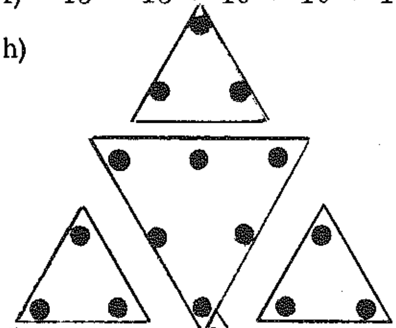
c) $21 = 6 + 6 + 6 + 3$ d) $28 = 10 + 6 + 6 + 6$

e) $36 = 10 + 10 + 10 + 6$ f) $45 = 15 + 10 + 10 + 10$

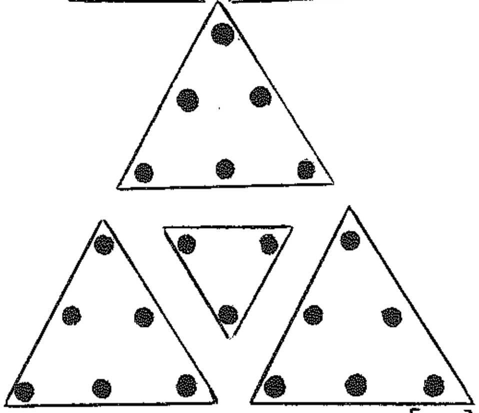
g)



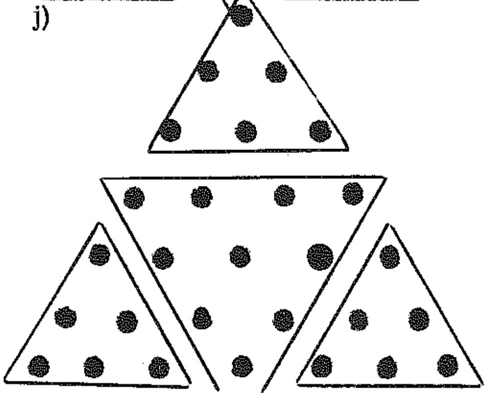
h)



i)



j)



k) $T_n = 3\,T_x + T_y$ $x = [n/2]$,that is, x is the greatest integer less than or equal to n/2

$y = [n/2] + (-1)^{n-1}$

39. a) ABC ACF BCE BCF BDE BEF CEF DEF

b) The eight ways in (a) plus

BEI CFJ DEH DEI DGH DHI EFI EFJ EHI EIJ FIJ GHI HIJ

c) $40 = 21 + 19 = 1 + 7 + 13 + 19$

d) $65 = 40 + 25 = 1 + 7 + 13 + 19 + 25$

e) $96 = 65 + 31 = 1 + 7 + 13 + 19 + 25 + 31$

f) There are several ways to arrive at a formula. One possibility is to rewrite the accumulated data in chart form as shown .

n	Number of stamps in block	Number of groups of 3 connected
1	1	0
2	3	$1 = 1 + 6 \bullet 0$
3	6	$8 = 2 + 6 \bullet 1$
4	10	$21 = 3 + 6 \bullet 3$
5	15	$40 = 4 + 6 \bullet 6$
6	21	$65 = 5 + 6 \bullet 10$

On the basis of the data the formula for the number of groups of 3 connected on at least 1 side can be easily determined. With a block of T_n stamps we have a total of $(n - 1) + 6\,T_{n-2}$ groups. This is $(n - 1) + \frac{6(n - 2)(n - 1)}{2}$ or $3n^2 - 8n + 5$.

40. a) 56 84 120 b) $\binom{n}{3} = \frac{n(n - 1)(n - 2)}{6}$ for $n \geqslant 3$ or $\frac{n(n + 1)(n + 2)}{6}$ for $n \geqslant 1$

c) Observe $P_1 + P_2 + P_3 = \binom{6}{2} = \binom{3 + 3}{3 - 1}$

$$P_1 + P_2 + P_3 + P_4 = \binom{7}{3} = \binom{4 + 3}{4 - 1}$$

implying $P_1 + P_2 + \dots + P_n = \binom{n + 3}{n - 1} = \frac{(n + 3)(n + 2)(n + 1)\,n}{24}$ $n \geqslant 1$

41. a) 126 210 330 b) $\binom{n}{4} = \frac{n(n - 1)(n - 2)(n - 3)}{24}$ for $n \geqslant 4$ which is

$$\frac{n(n + 1)(n + 2)(n + 3)}{24} \quad \text{for} \quad n \geqslant 1$$

c) Observe $Q_1 + Q_2 + Q_3 = \binom{7}{2} = \binom{3 + 4}{3 - 1}$

$$Q_1 + Q_2 + Q_3 + Q_4 = \binom{8}{3} = \binom{4 + 4}{4 - 1}$$

implying $Q_1 + Q_2 + \dots + Q_n = \binom{n + 4}{n - 1}$

$$= \frac{(n + 4)(n + 3)(n + 2)(n + 1)\,n}{120} \quad n \geqslant 1$$

42. a) 7 11

b) n divides each number in row n if and only if n is a prime number. This is a test, though not very practical, for primality. It is a fact since $\binom{n}{k} \Big/ n = \frac{n\,!}{k\,!\,(n - k)!\,n}$ is an integer for all $k \neq 0$, n if and only if n is a prime.

$$\frac{n!}{k!\,(n-k)!\,n} = \frac{\cancel{n}\,(n-1)(n-2)\;\ldots\;(n-(k-1))}{k!\;\cancel{n}}$$

Every group of k numbers contains a multiple of k. Since n is a prime the multiples of k must be one of the numbers

$(n-1)\,(n-2)\;\ldots\;(n-(k-1))$.

43. a)

n	numbers on "$\bigvee$" formed by intersection of (n - 1)st diagonals
3	1 3 6 3 1
4	1 4 10 20 10 4 1
5	1 5 15 35 70 35 15 5 1
6	1 6 21 56 126 252 126 56 21 6 1
7	1 7 28 84 210 462 924 462 210 84 28 7 1
8	1 8 36 120 330 792 1716 3432 1716 792 330 120 36 8 1
9	1 9 45 165 495 1287 3003 6435 12870 6435 3003 1287 495 165 45 9 1

Excepting 1's n = 3,5,7 does divide each number on the "$\bigvee$".

b) n divides each number on the "$\bigvee$" formed by the intersection of the (n - 1)st diagonals if and only if n is a prime. This is a direct consequence of the addition pattern of the triangle and the fact noted in problem 42.

44. The totals can best be determined by counting arrangements according to the number of Whooligans being seated. Notice the Pascal triangle and Fibonacci (to follow) numbers that result.

	Whooligans						Total number of arrangements
n	0	1	2	3	4	5	
1	1	1					2
2	1	2					3
3	1	3	1				5
4	1	4	3				8
5	1	5	6	1			13
6	1	6	10	4			21
7	1	7	15	10	1		34
8	1	8	21	20	5		55
9	1	9	28	35	15	1	89
10	1	10	36	56	35	6	144

45.

First day of	J	F	M	A	M	J	J	A	S	O	N	D	J
Number of adult pairs	1	1	2	3	5	8	13	21	34	55	89	144	233
Number of newborn pairs	0	1	1	2	3	5	8	13	21	34	55	89	144
Total	1	2	3	5	8	13	21	34	55	89	144	233	377

46. a)

F_j	F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}
Sum	1	1	2	3	5	8	13	21	34	55	89	144	233

b) $F_n = F_{n-1} + F_{n-2} \qquad n \geqslant 2$

c) $F_0 + F_1 + \dots + F_n = F_{n+2} - 1 \qquad n \geqslant 0$

d) $F_6 = 1 + 5 + 6 + 1 = \binom{6}{0} + \binom{5}{1} + \binom{4}{2} + \binom{3}{3}$

$F_8 = 1 + 7 + 15 + 10 + 1 = \binom{8}{0} + \binom{7}{1} + \binom{6}{2} + \binom{5}{3} + \binom{4}{4}$

$F_{10} = 1 + 9 + 28 + 35 + 15 + 1$

$= \binom{10}{0} + \binom{9}{1} + \binom{8}{2} + \binom{7}{3} + \binom{6}{4} + \binom{5}{5}$

$$F_{2n} = \sum_{i=0}^{n} \binom{n+i}{n-i}$$

e) $F_7 = 1 + 6 + 10 + 4 = \binom{7}{0} + \binom{6}{1} + \binom{5}{2} + \binom{4}{3}$

$F_9 = 1 + 8 + 21 + 20 + 5 = \binom{9}{0} + \binom{8}{1} + \binom{7}{2} + \binom{6}{3} + \binom{5}{4}$

$F_{11} = 1 + 10 + 36 + 56 + 35 + 6$

$= \binom{11}{0} + \binom{10}{1} + \binom{9}{2} + \binom{8}{3} + \binom{7}{4} + \binom{6}{5}$

$$F_{2n+1} = \sum_{i=0}^{n} \binom{n+1+i}{n-i}$$

47.

1							
2	1						
2	3	1					
2	5	4	1				
2	7	9	5	1			
2	9	16	14	6	1		
2	11	25	30	20	7	1	
2	13	36	55	50	27	8	1

2 15 49 91 105 77 35 9 1
2 17 64 140 196 182 112 44 10 1
2 19 81 206 336 378 294 156 54 11 1

48. a) The sum of the numbers along the rising diagonals are the Fibonacci numbers.

$D_0 = 1$, $D_1 = 2$, $D_3 = 3$, ...

$D_0 + D_2 + D_4 + \ldots D_{10} = D_{11} - 1 = 233 - 1 = 232$

b) $\sum_{j=0}^{n} D_{2j} = D_{2n+1} - 1$ c) $\sum_{j=1}^{n} D_{2j-1} = D_{2n} - 1$

49. a) In row n - 0th element is 2
b) In row n - 1st element is $2n - 1$
c) In row n - 2nd element is $(n - 1)^2$
d) In row n - 3rd element is $1^2 + 2^2 + \ldots + (n-2)^2$

50. a) $2n - 1$ b) n^2 c) $1^2 + 2^2 + \ldots + n^2$

d) The solutions to the counting problems here are in the triangle in Figure 38. The solution to (a) is the first element of row n. The solution to (b) is the second element of row n + 1 . The solution to (c) is the third element of row n + 2 .

In order to see the pattern for problems 51 and 52 clearly rewrite the triangle of Figure 40 as

1
1 + 1 1
1 + 1 2 + 1 1
1 + 1 3 + 2 3 + 1 1
1 + 1 4 + 3 6 + 3 4 + 1 1
1 + 1 5 + 4 10 + 6 10 + 4 5 + 1 1

Figure 40

The terms are found by the usual addition pattern. But any sum (a + b) + (c + d) is written as (a + c) + (b + d) . The result is that two Pascal triangles are involved in the structure of the triangle. The patterns known for the Pascal triangle will determine the patterns asked for here.

51. a)

Row	0	1	2	3	4	5	6	7	8	9	10
Row sum	1	2+1	4+2	8+4	16+8	32+16	64+32	128+64	256+128	512+256	1024+512

b)

Row	0	1	2	3	4	5	6
Cum Row Sum	1	3+1	7+3	15+7	31+15	63+31	127+63

Row	7	8	9	10
Cum Row Sum	255+127	511+255	1023+511	2047+1023

c) The nth row sum is $2^n + 2^{n-1} = 2^{n-1}(2 + 1) = 3 \cdot 2^{n-1}$

d) The nth cum row sum is $2^{n+1}-1 + 2^n-1 = 2^{n+1} + 2^n - 2 = 2^n(2 + 1) -2$

$= 3 \cdot 2^n - 2$

52.

	Row	0	1	2	3	4	5	6	7	8	9	10		n ($n \neq 0, 1$)
a)	S_1	1	1	0	0	0	0	0	0	0	0	0	c)	0
b)	S_2	1	3	4	4	4	4	4	4	4	4	4	d)	4

Observe in Figure 40 row 5 consists of the coefficients of $(x + y)^5$ and $(x + y)^4$. In row n the terms are the coefficients of $(x + y)^n$ and $(x + y)^{n-1}$. Since the alternating sums S_1, S_2 for each separate binomial are 0, 2 respectively (c) and (d) follow .

53. a)

L_k	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
Sum	1	3	4	7	11	18	29	47	76	123

b) $L_n = L_{n-1} + L_{n-2}$

c) $\sum_{k=1}^{n} L_k = L_{n+2} - 3$

d) $\sum_{k=1}^{n} L_{2k} = L_{2n+1} - 1$

e) $\sum_{k=1}^{n} L_{2k-1} = L_{2n} - 2$

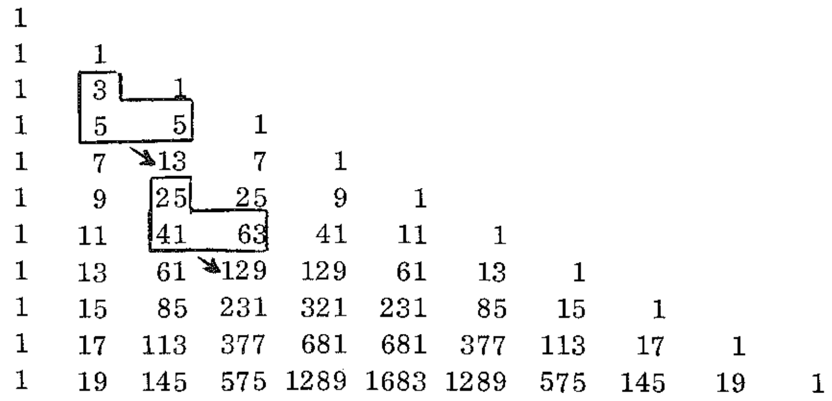
54. Consider the array of Figure 40 in the form below.

row	1	a	$2b$			
	2	a^2	$3ab$	$2b^2$		
	3	a^3	$4a^2b$	$5ab^2$	$2b^3$	
	4	a^4	$5a^3b$	$9a^2b^2$	$7ab^3$	$2b^4$

row	1	is the expanded form of	$a + 2b$
	2		$(a + b)(a + 2b)$
	3		$(a + b)^2(a + 2b)$
	4		$(a + b)^3(a + 2b)$
	n		$(a + b)^{n-1}(a + 2b)$

55. The triangular array in Figure 41 is challenging but it is also formed by an addition pattern.

a)



1										
1	1									
1	3	1								
1	5	5	1							
1	7	13	7	1						
1	9	25	25	9	1					
1	11	41	63	41	11	1				
1	13	61	129	129	61	13	1			
1	15	85	231	321	231	85	15	1		
1	17	113	377	681	681	377	113	17	1	
1	19	145	575	1289	1683	1289	575	145	19	1

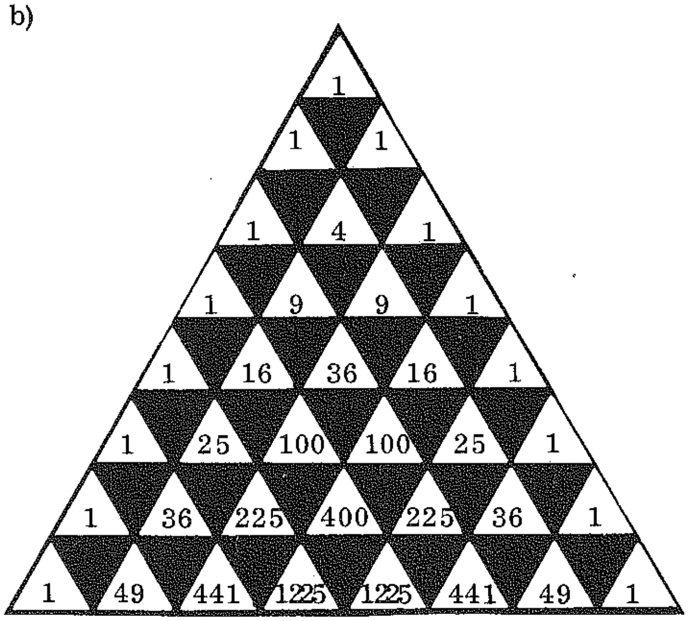
Figure 41

b)

T_k	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
Sum	1	1	2	4	7	13	24	44	81	149

c) $T_n = T_{n-1} + T_{n-2} + T_{n-3}$. These are the Tribonacci numbers.

56. a) b)



Row Sum

$1 = \binom{0}{0}$

$2 = \binom{2}{1}$

$6 = \binom{4}{2}$

$20 = \binom{6}{3}$

$70 = \binom{8}{4}$

$252 = \binom{10}{5}$

$924 = \binom{12}{6}$

$3432 = \binom{14}{7}$

c) $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \dots + \binom{n}{n}^2 = \binom{2n}{n}$

To prove this consider $(a + b)^{2n}$. Using the Binomial Theorem the coefficient of $a^n b^n$ is $\binom{2n}{n}$.

But $(a + b)^{2n} = (a + b)^n (a + b)^n$

$$= \left[\sum_{k=0}^{n} \genfrac{}{}{0pt}{}{n}{k} a^{n-k} b^k\right]\left[\sum_{k=0}^{n} \genfrac{}{}{0pt}{}{n}{k} a^{n-k} b^k\right]$$

After multiplication is carried out, all the terms containing $a^n b^n$ are included in

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} \qquad a^{n-k}b^{k}\ a^{n-(n-k)}b^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k}^{2} \ a^{n}b^{n} \qquad \therefore \sum_{k=0}^{n} \binom{n}{k}^{2} = \binom{2n}{n}$$

57. a) Alternating Sum A_1

Row	Expansion	Sum A_1
0	1^2	1
1	$1^2 - 1^2$	0
2	$1^2 - 2^2 + 1^2$	-2
3	$1^2 - 3^2 + 3^2 - 1^2$	0
4	$1^2 - 4^2 + 6^2 - 4^2 + 1^2$	6
5	$1^2 - 5^2 + 10^2 - 10^2 + 5^2 - 1^2$	0
6	$1^2 - 6^2 + 15^2 - 20^2 + 15^2 - 6^2 + 1^2$	-20
7	$1^2 - 7^2 + 21^2 - 35^2 + 35^2 - 21^2 + 7^2 - 1^2$	0
8	$1^2 - 8^2 + 28^2 - 56^2 + 70^2 - 56^2 + 28^2 - 8^2 + 1^2$	70
9	$1^2 - 9^2 + 36^2 - 84^2 + 126^2 - 126^2 + 84^2 - 36^2 + 9^2 - 1^2$	0
10	$1^2 - 10^2 + 45^2 - 120^2 + 210^2 - 252^2 + 210^2 - 120^2 + 45^2 - 10^2 + 1^2$	-252

b) If n is an odd number the alternating sum A_1 is 0 .

c) If n is an even number the alternating Sum A_2 varies as:

1) if $n/2$ is even then the alternating sum is the square root of the middle term of row n of this triangle .

2) if $n/2$ is odd then the alternating sum is the opposite of the square root of the middle term of row n of this triangle .

58. a) Alternating Sum A_2

Row	Expansion	Sum A_2
0	1^2	1
1	$1^2 + 1^2$	2
2	$1^2 + 2^2 - 1^2$	4
3	$1^2 + 3^2 - 3^2 + 1^2$	2
4	$1^2 + 4^2 - 6^2 + 4^2 - 1^2$	-4
5	$1^2 + 5^2 - 10^2 + 10^2 - 5^2 + 1^2$	2
6	$1^2 + 6^2 - 15^2 + 20^2 - 15^2 + 6^2 - 1^2$	22

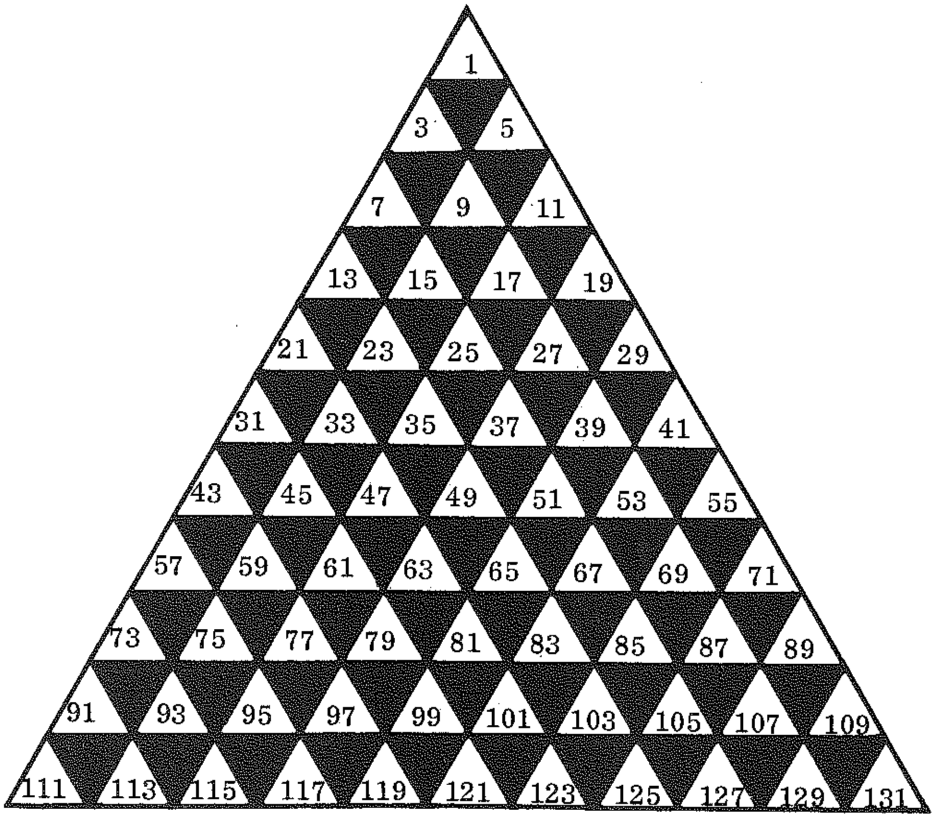
7	$1^2 + 7^2 - 21^2 + 35^2 - 35^2 + 21^2 - 7^2 + 1^2$	2
8	$1^2 + 8^2 - 28^2 + 56^2 - 70^2 + 56^2 - 28^2 + 8^2 - 1^2$	-68
9	$1^2 + 9^2 - 36^2 + 84^2 - 126^2 + 126^2 - 84^2 + 36^2 - 9^2 + 1^2$	2
10	$1^2 + 10^2 - 45^2 + 120^2 - 210^2 + 252^2 - 210^2 + 120^2 - 45^2 + 10^2 - 1^2$	254

b) If n is an odd number the alternating sum A_2 is 2 .

c) If n is an even number the alternating sum A_2 varies as:

1) if n/2 is odd then the alternating sum is 2 greater than the square root of the middle term of row n of this triangle.
2) if n/2 is even then the alternating sum is the opposite of 2 less than the square root of the middle term of row n.

59. a)



b)

Row	0	1	2	3	4	5	6	7	8	9	10
Row Sum	1	8	27	64	125	216	343	512	729	1000	1331

c) $(n + 1)^3$　　d) $(n + 1)^2$

e)

Row	0	1	2	3	4	5	6	7	8	9	10
Alt Sum	1	-2	9	-4	25	-6	49	-8	81	-10	121

f) If n is even $S_1 = (n + 1)^2$　　g) If n is odd $S_1 = -2(n+1)/2 = -(n+1)$

60. a) This triangle has an addition pattern in combination with a bringing down of elements as they are from the preceding row. Each row begins with and ends in 1 . To form the row following the one that starts

1 a b c d ...

write

1 1 + a a a + b b b + c c c + d d ...

Row 4: 1 5 4 7 3 8 5 7 2 7 5 8 3 7 4 5 1

Row 5: 1 6 5 9 4 11 7 10 3 11 8 13 5 12 7 9 2
9 7 12 5 13 8 11 3 10 7 11 4 9 5 6 1

b) The number of terms in rows 0 through 5 respectively are:

2, 3, 5, 9, 17, 33 .

Since $2 = 2^0 + 1$, $3 = 2^1 + 1$, $5 = 2^2 + 1$, $9 = 2^3 + 1$, $17 = 2^4 + 1$ and $33 = 2^5 + 1$ we predict that the number in row n is $2^n + 1$.

c) The row sums for rows 0 through 5 respectively are:

2, 4, 10, 28, 82, 244 .

Since $2 = 3^0 + 1$, $4 = 3^1 + 1$, $10 = 3^2 + 1$, $28 = 3^3 + 1$, $82 = 3^4 + 1$, and $244 = 3^5 + 1$ we predict that the row sum for row n is $3^n + 1$.

d) Many solutions are possible. Here are a few.

1) Any consecutive entries in a row are relatively prime.

2) n is a prime number if and only if it occurs n - 1 times in row n - 1.
That is, 2 occurs once in row 1, 3 occurs twice in row 2, 5 occurs four times in row 4 .

3) The entries in columns 2^n $n = 0, 1, 2, \ldots$ are 1, 2, 3 ... .

61. a)

1
1 1
1 1 1
1 1 2 1
1 1 3 2 1
1 1 4 3 3 1
1 1 5 4 6 3 1
1 1 6 5 10 6 4 1
1 1 7 6 15 10 10 4 1
1 1 8 7 21 15 20 10 5 1
1 1 9 8 28 21 35 20 15 5 1

Figure 44

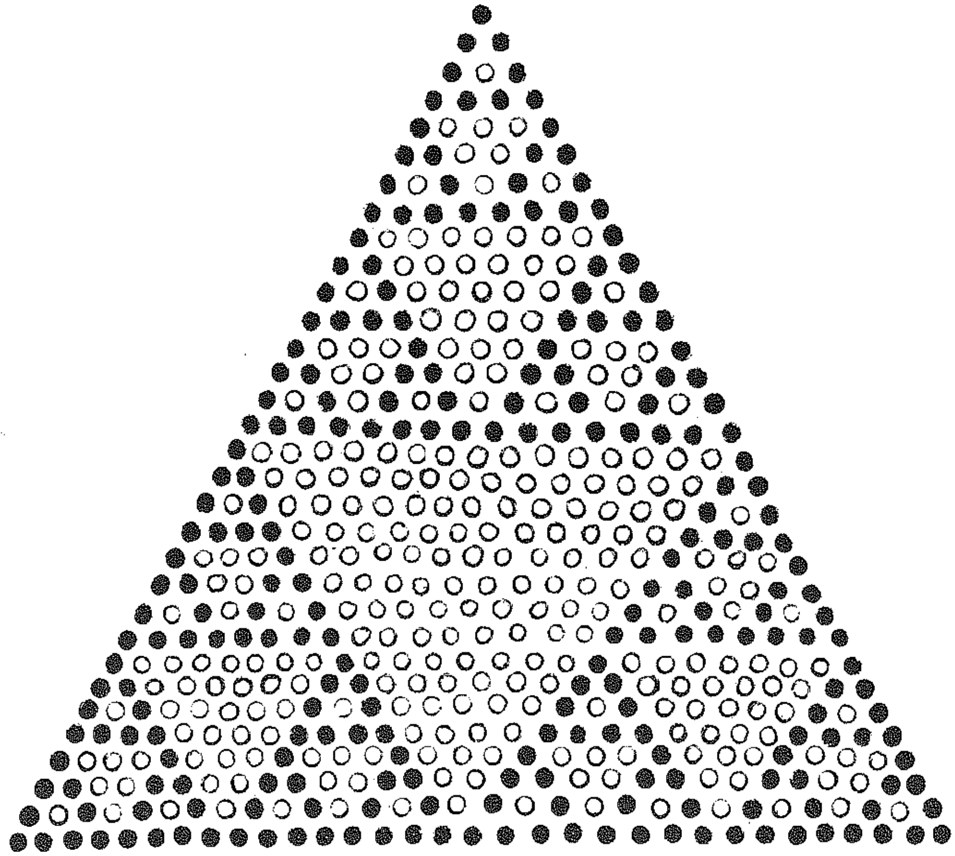
b) Given 1 1 a b c ... as the start of a row .

To form the next row:

1) start with a 1 2) bring down a 1, 3) put in sum 1 + a,
4) bring down a 5) put in sum b + c, 6) bring down c, and so on.
That is, alternate bringing down a sum with bringing down the second addend in the sum. End with a 1. Note the diagonal patterns !

c) The row sums for rows 0 through 10 respectively are: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144. These are the Fibonacci numbers. The sum of the terms in row n is F_{n+1}.

62.



63.

Number of black dots per row	Rows having this number of black dots
1	0
2	1 2 4 8 16
4	3 5 6 9 10 12 17 18 20 24
8	7 11 13 14 19 21 22 25 26 28
16	15 23 27 29 30
32	31

64. a) 63,127,255,511,1023 b) 32,64,128,256,512 c) 62,126,254,510,1022

65. If $n = 1,2,4,8,16$, $(a + b)^n = a^n + b^n$. This is valid for all n, a power of 2.

66. a) $T_1 = 1$, 27 times : $T_3 = 6$, 9 times : $T_7 = 28$, 3 times: $T_{15} = 120$, once.

b) The triangular number $T_2 = 3$ appears 81 times.

c) The total number of dots is $T_{32} = 32 \bullet 33/2 = 528$.

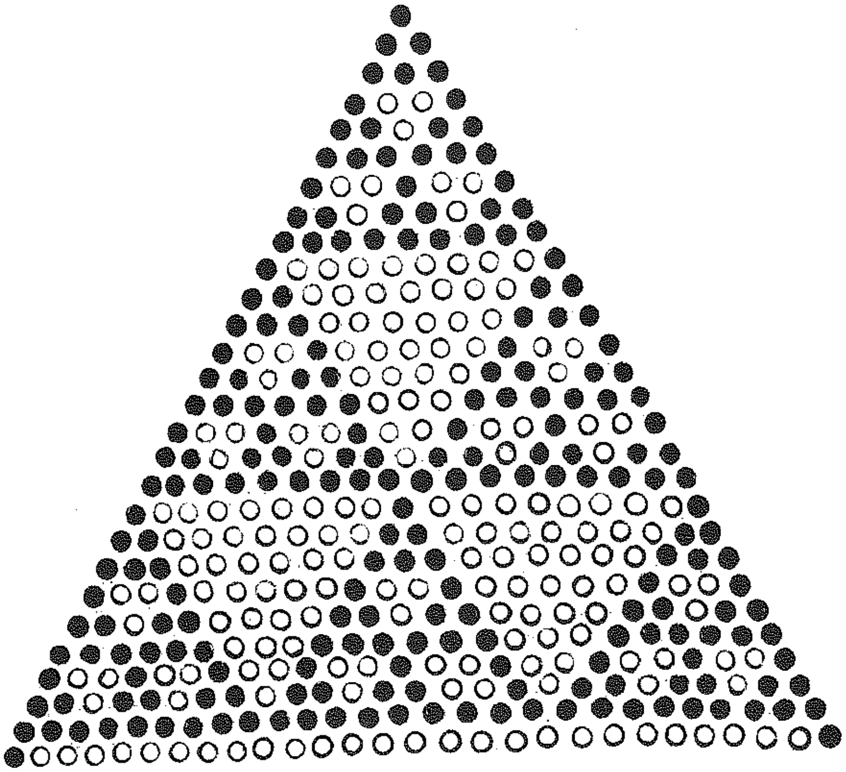
The total for the white dots is 27 + 54 + 84 + 120 = 285 .

The total for the black dots is 243.

d) The formula for even perfect numbers is $2^{n-1}(2^n - 1)$ where $2^n - 1$ is a prime number. Since $2^{n-1}(2^n - 1) = 2^n(2^n - 1)/2$ it is clear that an even perfect number is also triangular. In other words if the number of lines of dots forming the triangle is a prime of the form $2^n - 1$ then the corresponding triangular number will be perfect.

$1 = 2^{1-1}(2^1 - 1)$ but $2^1 - 1$ is not a prime and 1 is not perfect.
$6 = 2^{2-1}(2^2 - 1)$ but $2^2 - 1$ _is_ a prime and thus 6 _is_ perfect.
$28 = 2^{3-1}(2^3 - 1)$ but $2^3 - 1$ _is_ a prime and thus 28 _is_ perfect.
$120 = 2^{4-1}(2^4 - 1)$ but $2^4 - 1$ is not a prime and 120 is not perfect.

67.



It is easier to construct a modulo 3 triangle first. Then replace 1, 2 by a black dot and 0 by a white dot. The reader is encouraged to try other mods for patterns and/or extend the mod 2 and mod 3 designs on a computer.

68.

Number of black dots per row	Rows having this number of black dots
1	0
2	1 3 9 27
3	2 6 18
4	4 10 12
6	5 7 11 15 19 21
8	13
9	8 20 24
12	14 16 22
18	17 23 25
27	26

69. a) $2\cdot3^3 - 1,\ 3^4 - 1,\ 2\cdot3^4 - 1$ b) $3^4,\ 3^5,\ 3^6$

c) $2\cdot3^3,\ 2\cdot3^4,\ 2\cdot3^5$

70. If $n = 1, 3, 8, 27$ $(a + b)^n = a^n + b^n$. This is valid for all n, a power of 3.

71. a) $T_2 = 3$, 18 times: $T_8 = 36$, 3 times.

b) The triangular number $T_3 = 6$ appears 36 times.

c) The total number of dots is $T_{28} = 28\cdot29/2 = 406$.

The total for the white dots is $54 + 108 + 26 = 188$.

The total for the black dots is $216 + 2 = 218$.

72. a) 1 3 3 1 is 1 0 0 1 modulo 3.

$\therefore (a + b)^3 = a^3 + b^3$

b) 1 4 6 4 1 is 1 0 2 0 1 modulo 4.

$\therefore (a + b)^4 = a^4 + 2a^2b^2 + b^4$

c) 1 5 10 10 5 1 is 1 0 0 0 0 1 modulo 5.

$\therefore (a + b)^5 = a^5 + b^5$

d) 1 6 15 20 15 6 1 is 1 0 3 2 3 0 1 modulo 6.

$\therefore (a + b)^6 = a^6 + 3a^4b^2 + 2a^3b^3 + 3a^2b^4 + b^6$.

e) 1 7 21 35 35 21 7 1 is 1 0 0 0 0 0 0 1 in modulo 7.

$\therefore (a + b)^7 = a^7 + b^7$

f) 1 8 28 56 70 56 28 8 1 is 1 0 4 0 6 0 4 0 1 in mod 8.

$\therefore (a + b)^8 = a^8 + 4a^6b^2 + 6a^4b^4 + 4a^2b^6 + b^8$.

g) 1 9 36 84 126 126 84 36 9 1 is

1 0 0 3 0 0 3 0 0 1 in modulo 9.

$\therefore (a + b)^9 = a^9 + 3a^6b^3 + 3a^3b^6 + b^9$.

73. Using some facts about divisibility both of the conjectures that follow can be proved.

a) $(a + b)^p = a^p + b^p$ in modulo p, where p is a prime.

b) $(a + b)^{p^n} = a^{p^n} + b^{p^n}$ in modulo p, where p is a prime.

74. a)

Toss (1)(2)(3)	Number of heads
T T T	0
T T H	1
T H T	1
T H H	2
H T T	1
H T H	2
H H T	2
H H H	3

1) Note the pattern in the columns
col. 1: 4 T's then 4 H's
col. 2: 2 T's, 2 H's, 2 T's, 2 H's
col. 3: T, H alternate in that order
Using this system, we can extend to more tosses quite easily.

2) Note the pattern for the number of heads

Heads	0	1	2	3
Outcomes	1	3	3	1

Sum is 8

This is row 3 of the Pascal triangle.

b) $P(0) = 1/8$ $P(1) = 3/8$ $P(2) = 3/8$ $P(3) = 1/8$

c) $P(0) + P(1) + P(2) + P(3) = 1$

d) $P(1) + P(2) + P(3) = 1 - P(0) = 1 - 1/8 = 7/8$

e) $P(0) + P(1) + P(2) = 1 - P(3) = 1 - 1/8 = 7/8$

75. a)

Toss (1)(2)(3)(4)	Number of heads
T T T T	0
T T T H	1
T T H T	1
T T H H	2
T H T T	1
T H T H	2
T H H T	2
T H H H	3
H T T T	1
H T T H	2
H T H T	2
H T H H	3
H H T T	2
H H T H	3
H H H T	3
H H H H	4

1) Note the pattern in the columns
col. 1: 8 T's then 8 H's
col. 2: 4 T's, 4 H's, 4T's, 4 H's
col. 3: 2T's, 2 H's alternate in that order
col. 4: T, H alternate in order

2) Note the pattern for the number of heads

H	0	1	2	3	4
O	1	4	6	4	1

Sum is 16

This is row 4 of the Pascal triangle.

b)

P(0)	P(1)	P(2)	P(3)	P(4)
1/16	4/16	6/16	4/16	1/16

At this point the reader should realize how to use the Pascal triangle to compute the probabilities in a coin toss problem.

c) The sum is 1. There are 16 outcomes. The numerators of the fractions are row 4 of the triangle. The sum of terms in row 4 is $2^4 = 16$.

d) $P(1) + P(2) + P(3) + P(4) = 1 - P(0) = 1 - 1/16 = 15/16$

e) $P(0) + P(1) + P(2) + P(3) = 1 - P(4) = 1 - 1/16 = 15/16$

f) $P(0) = P(4)$, $P(1) = P(3)$, $P(2) = P(2)$, a consequence of the row symmetry in the Pascal triangle.

76. a) $P(3) = \dfrac{\binom{6}{3}}{2^6} = 20/32 = 5/8$ b) $P(4) = \dfrac{\binom{7}{4}}{2^7} = 35/128$

c) $P(3) + P(4) = \dfrac{\binom{8}{3}}{2^8} + \dfrac{\binom{8}{4}}{2^8} = (56 + 70)/256 = 126/256 = 63/128$

d) $1 - P(5) = 1 - \dfrac{\binom{5}{5}}{2^5} = 31/32$

e) In order to get 2 tails, one must get three heads ! $P(3) = \dfrac{\binom{5}{3}}{2^5} = 10/32 = 5/16$

f) To get 3 tails, one must get 6 heads ! $P(6) = \dfrac{\binom{9}{6}}{2^9} = 84/512 = 21/128$

g) To get 6 tails, one must get 2 heads ! $P(2) = \binom{8}{2} / 2^8 = 28/256 = 7/64$

h) To get from 0 to 9 tails, one must get from 10 to 1 heads.

$P(1) + \ldots + P(10) = 1 - P(0) = 1023/1024$

i) $P(h) = \dfrac{\binom{n}{h}}{2^n} = \dfrac{n!}{h!(n-h)!2^n}$

j) To get t tails, one must get n - t heads. $P(n-t) = \dfrac{\binom{n}{n-t}}{2^n} = \dfrac{n!}{(n-t)!\ t!\ 2^n}$

k) $P(1) + \ldots + P(n) = 1 - P(0) = 1 - \dfrac{\binom{n}{0}}{2^n} = 1 - \dfrac{1}{2^n} = \dfrac{2^n - 1}{2^n}$

l) $P(0) + \ldots + P(n-1) = 1 - P(n) = 1 - \dfrac{\binom{n}{n}}{2^n} = 1 - \dfrac{1}{2^n} = \dfrac{2^n - 1}{2^n}$

77. a) To answer this question we need to know the chances of getting two heads in six tosses (since the last toss is fixed as a head) . In other words we first find the number of outcomes for six tosses that have two heads. This is $\binom{6}{2}$. Note that in seven tosses there are 2^7 out comes. Exactly one half end in H. Thus there are 2^6 outcomes ending in H. The required probability is $\binom{6}{2} / 2^6$.

b) Applying the reasoning in (a) the probability is $\binom{8}{4} / 2^8$.

c) To answer this question we need to know the chances of getting three tails in five tosses. This is the same as getting two heads in five tosses. The probability is $\binom{5}{2} / 2^5$.

d) $P(\text{kth head on nth toss}) = \binom{n-1}{k-1} / 2^{n-1}$

78. a) 1 b) $1 + 2 = 3$ c) $1 + 2 + 3 = 6$

d) $1 + 2 + \ldots + (n+1) = (n + 1)(n + 2)/2$

The solutions are on diagonal 2 of the Pascal triangle, i.e. the triangular numbers. In combinations notation and noting that we begin with $n = 0$ this is $\binom{n+2}{2}$.

79. It is not surprising that the solution is in the Pascal triangle since it is an exercise in combinations. All possible line segments are found by taking all combinations of n points 2 at a time.

In chart form

Points n	2	3	4	5	...
Lines	1	3	6	10	...

In the Pascal triangle this is the second diagonal. Starting with $n = 2$, the formula is $\binom{n}{2} = n(n - 1)/2$.

80. All possible triangles are found by taking all combinations of n points 3 at a time. In the Pascal triangle this is diagonal 3 - 1, 4, 10, 20, 35, ... Starting with $n = 3$ the formula is $\binom{n}{3} = n(n - 1)(n - 2)/3\,!$ This problem is related to problem 79 in that the number of triangles N is such that

$$N = \frac{n(n-1)}{2} \bullet n - 2 \bullet \frac{1}{3}$$

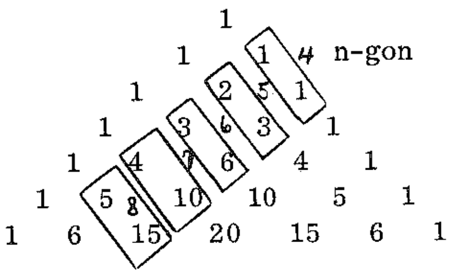
number of line segments | number of possible 3rd vertices

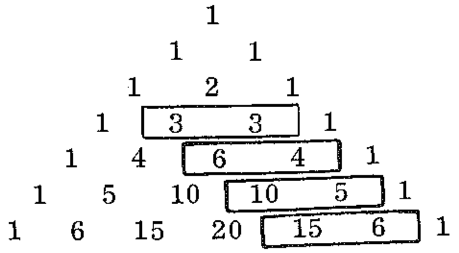
to eliminate duplications as $\overline{AB}, C$; $\overline{AC}, B$; $\overline{BC}, A$ all represent triangle ABC .

81. The solution here is related to problem 79. The number of diagonals of an n-gon is the same as the number of line segments between n points less n, the number of sides of an n-gon. This is $\frac{n(n-1)}{2} - n, \quad n \geqslant 3$.

In the Pascal triangle the solution is the difference of the first diagonal from the second diagonal : $\binom{n}{2} - \binom{n}{1}$.

The pattern appears in at least 2 ways in the Pascal triangle.



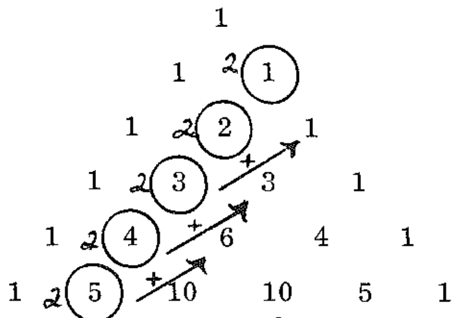


The pattern on the left gives $\binom{n}{1} + \binom{n+1}{2} = \frac{n^2 + 3n}{2}$, the number of diagonals for an n + 3 - gon. For a n -gon replace n in the combinations expression by n - 3. This is $\binom{n-3}{1} + \binom{n-2}{2} = (n^2 - 3n)/2$

The pattern on the right gives $\binom{n}{2} - \binom{n}{n-1} = \binom{n}{2} - \binom{n}{1} = (n^2 - 3n)/2, n \geqslant 3.$

82.

n	Max number of pieces
1	$2 = 2 \cdot 1$
2	$4 = 2 \cdot 2$
3	$7 = 2 \cdot 3 + 1$
4	$11 = 2 \cdot 4 + 3$
5	$16 = 2 \cdot 5 + 6$
6	$22 = 2 \cdot 6 + 10$
n	$2n + T_{n-2} = 2n + \frac{(n-1)(n-2)}{2} = \frac{n^2 + n + 2}{2}$



As the kth cut is taken k regions double and the balance from the previous step are added to give the new total. In the Pascal triangle this is twice the 1st diagonal plus the second diagonal as illustrated. Formula is also $2\binom{n}{1} + \binom{n-1}{2}$.

83.

n-gon	Number of diagonal intersection points
4	1
5	5
6	15
7	35
8	70
n	$\binom{n}{4}$

To explain why it is $\binom{n}{4}$ consider any four vertices. Arrange them in a clockwise order around the n - gon. The diagonal joining first and third points will intersect the diagonal joining second and fourth points. There will be no other pair of diagonals joining <u>these</u> 4 points intersecting within the n-gon. Thus, we have a one-to-one match between a diagonal intersection point and a set of 4 points. There are as many diagonal intersection points as there are sets of 4 points. But the number of these is $\binom{n}{4}$.

$$\binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{24} = \frac{n^4 - 6n^3 + 11n^2 - 6n}{24}$$

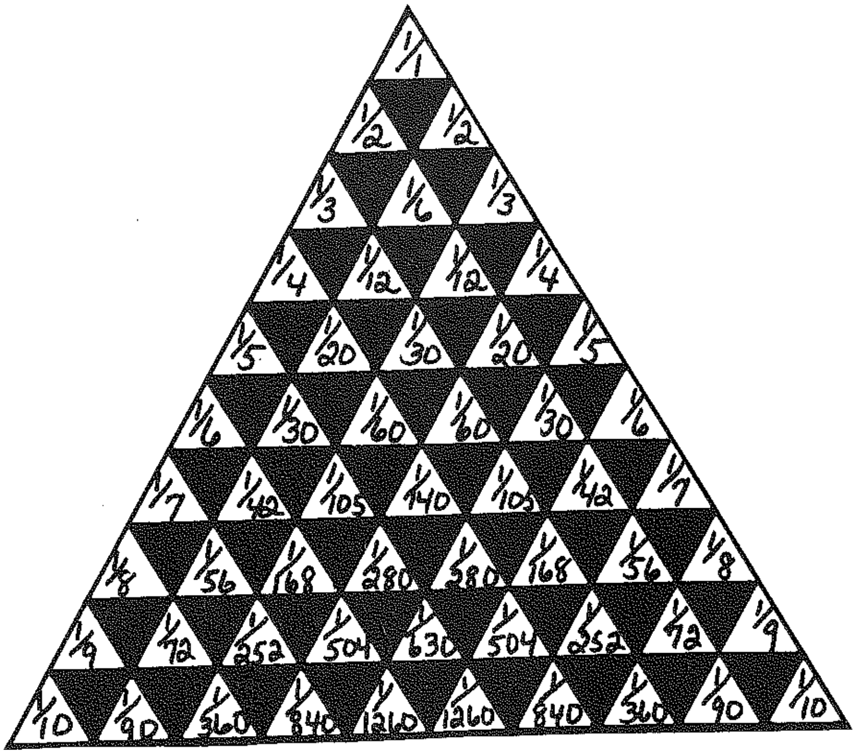
84. Perhaps the simplest way to develop the relation is by means of a recursion. For example show that adding a fifth point to a quadrilateral with four regions leads to a pentagon with four plus seven regions.

n-gon	Number of diagonal determining regions	
3	1	1
4	4	1 + 3
5	11	1 + 4 + 6
6	25	5 + 10 + 10
7	50	15 + 20 + 15
8	91	35 + 35 + 21
n	$\binom{n-1}{n-3} + \binom{n-1}{n-4} + \binom{n-1}{n-5} = \frac{(n-1)(n-2)(n^2-3n+12)}{24}$, $n \geqslant 3$.	

In the Pascal triangle the pattern is

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ \boxed{1} \quad 2 \quad 1 \\ \boxed{1 \quad 3} \quad 3 \quad 1 \\ \boxed{1 \quad 4 \quad 6} \quad 4 \quad 1 \\ 1 \quad \boxed{5 \quad 10 \quad 10} \quad 5 \quad 1 \\ 1 \quad 6 \quad \boxed{15 \quad 20 \quad 15} \quad 6 \quad 1 \\ 1 \quad 7 \quad 21 \quad \boxed{35 \quad 35 \quad 21} \quad 7 \quad 1 \end{array}$$

85.



The reader should investigate the Leibniz harmonic triangle and search for patterns.

86. a) $A = 1$ $B = -1$ b) $A = 1$ $B = -2$ $C = 1$

c) $A = 1$ $B = -3$ $C = 3$ $D = -1$ d) $C = 6$ e) $D = -10$

The easiest way to get the replacements in this problem is to

1) clear denominators
2) choose strategic values of x to get the unknowns.

For example to find A, B, C in (b)

1) simplify and get

$$2 = A(x+1)(x+2) + Bx(x+2) + Cx(x+1)$$

which is valid for <u>all</u> x so let

$x = 0$ this implies $2 = 2A$ and $A = 1$

$x = -1$ this implies $2 = -B$ and $B = -2$

$x = -2$ this implies $2 = 2C$ and $C = 1$.

87. a) $(a+b)^{-2} = a^{-2} - 2a^{-3}b + 3a^{-4}b^{2} - 4a^{-5}b^{3} + \dots$

The absolute values of these coefficients are the terms of the first diagonal in the Pascal triangle (Figure 2). Each coefficient in absolute value is of the form $\binom{k+1}{k}$, $k = 0, 1, 2, \dots$.

b) $(a+b)^{-3} = a^{-3} - 3a^{-4}b + 6a^{-5}b^{2} - 10a^{-6}b^{3} + \dots$

The absolute values of these coefficients are the terms of the second diagonal in the Pascal triangle (Figure 2). Each coefficient in absolute value is of the form $\binom{k+2}{k}$, $k = 0, 1, 2, \dots$.

c) $(a+b)^{-4} = a^{-4} - 4a^{-5}b + 10a^{-6}b^{2} - 20a^{-7}b^{3} + \dots$

The absolute values of these coefficients are the terms of the third diagonal in the Pascal triangle (Figure 2) . Each coefficient in absolute value is of the form $\binom{k+3}{k}$ $k = 0, 1, 2, \dots$.

d) $$(a+b)^{-n} = \binom{n-1}{0}a^{-n} - \binom{n}{1}a^{-(n+1)}b + \binom{n+1}{2}a^{-(n+2)}b^{2}$$

$$- + \dots + (-1)^{k}\binom{n-1+k}{k}a^{-(n+k)}b^{k} + - \dots$$

The absolute values of the coefficients of $(a+b)^{-n}$ are the terms of the n-1 st diagonal in the Pascal triangle. Each coefficient in absolute value is of the form $\binom{k+n-1}{k}$ $k = 0, 1, 2, \dots$

88. a)

n \ r	0	1	2	3	4	5	6	7
7	1	7	21	35	35	21	7	1
6	1	6	15	20	15	6	1	0
5	1	5	10	10	5	1	0	0
4	1	4	6	4	1	0	0	0
3	1	3	3	1	0	0	0	0
2	1	2	1	0	0	0	0	0
1	1	1	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0
-1	1	-1	1	-1	1	-1	1	-1
-2	1	-2	3	-4	5	-6	7	-8
-3	1	-3	6	-10	15	-21	28	-36
-4	1	-4	10	-20	35	-56	84	-120
-5	1	-5	15	-35	70	-126	210	-330
-6	1	-6	21	-56	126	-252	462	-792
-7	1	-7	28	-84	210	-462	924	-1716

b) The Pascal triangle appears upside down in rows 7 through 0 .

c) The absolute values of the entries in the columns in rows - 1 through -7 are the diagonals of the Pascal triangle.

Note that ignoring the negative signs each column is reflected about a different horizontal line.

d) If $n \geqslant 0$ and $r > 0$ $\binom{n}{r} = 0$.

If $n < 0$ then $\binom{n}{r} = (-1)^r \binom{r-n-1}{r}$

For example, $\binom{-7}{3} = (-1)^3 \binom{3+7-1}{3} = -\binom{9}{3} = -84$.

e) $(a+b)^{-4} = a^{-4} - 4a^{-5}b + 10a^{-6}b^2 - 20a^{-7}b^3 + 35a^{-8}b^4 - 56a^{-9}b^5 + \ldots$

$(a+b)^{-5} = a^{-5} - 5a^{-6}b + 15a^{-7}b^2 - 35a^{-8}b^3 + 70a^{-9}b^4 - 126a^{-10}b^5 + \ldots$

$(a+b)^{-6} = a^{-6} - 6a^{-7}b + 21a^{-8}b^2 - 56a^{-9}b^3 + 126a^{-10}b^4 - 252a^{-11}b^5 + \ldots$

$(a+b)^{-7} = a^{-7} - 7a^{-8}b + 28a^{-9}b^2 - 84a^{-10}b^3 + 210a^{-11}b^4 - 462a^{-12}b^5 + \ldots$

89. The pattern that is developed in this problem is one of the more recent discoveries pertaining to the triangle. There are many interesting extensions of perfect square patterns and the reader is encouraged to look for others.

89. a)

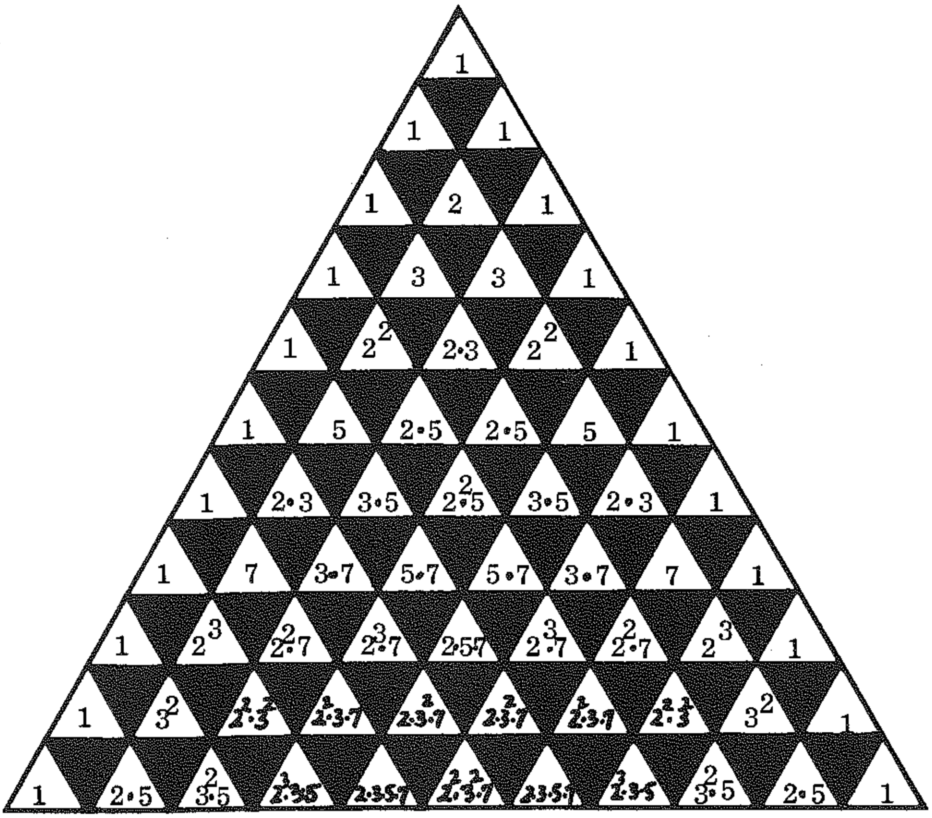


Figure 52

b) (1) 3^2 (2) $2^2\bullet3^2\bullet5^2$ (3) $3^2\bullet5^2\bullet7^2$

(4) $2^6\bullet3^2\bullet7^2$ (5) $2^6\bullet3^2\bullet5^4\bullet7^4$ (6) $2^4\bullet3^4\bullet7^2$

(7) $2^6\bullet3^4\bullet5^2$ (8) $2^{10}\bullet3^6\bullet5^2\bullet7^6$

c) The product of the 6 numbers is a perfect square.

$$\binom{n-1}{r-1}\bullet\binom{n-1}{r}\bullet\binom{n}{r-1}\bullet\binom{n}{r+1}\bullet\binom{n+1}{r}\bullet\binom{n+1}{r+1} \quad \text{is a perfect square .}$$

d) $$\frac{(n-1)\,!}{(r-1)!\,(n-r)!}\bullet\frac{(n-1)!}{r!\,(n-1-r)!}\bullet\frac{n\,!}{(r-1)!\,(n-r+1)!}\bullet\frac{n\,!}{(r+1)!\,(n-r-1)!}\bullet\frac{(n+1)!}{r!\,(n+1-r)!}\bullet\frac{(n+1)!}{(r+1)!\,(n-r)\,!}$$

$$=\left[\frac{(n-1)\,!\;n\,!\;(n+1)\,!}{r!\,(r-1)!\,(r+1)!\,(n-1-r)!\,(n-r)!\,(n+1-r)!}\right]^2 \quad \text{a perfect square .}$$

90. a) $$(\cos\phi + i\sin\phi)^2 = \cos^2\phi + i(2\cos\phi\sin\phi) + i^2\sin^2\phi$$
$$= (\cos^2\phi - \sin^2\phi) + i(2\cos\phi\sin\phi)$$

b) $$(\cos\phi + i\sin\phi)^3 = \cos^3\phi + i(3\cos^2\phi\sin\phi) + i^2(3\cos\phi\sin^2\phi) + i^3\sin^3\phi$$
$$= (\cos^3\phi - 3\cos\phi\sin^2\phi) + i(3\cos^2\phi\sin\phi - \sin^3\phi)$$

c) $$(\cos\phi + i\sin\phi)^4 = \cos^4\phi + i(4\cos^3\phi\sin\phi) + i^2(6\cos^2\phi\sin^2\phi) + i^3(4\cos\phi\sin^3\phi) + i^4(\sin^4\phi)$$

$$= (\cos^4\phi - 6\cos^2\phi\sin^2\phi + \sin^4\phi) + i(4\cos^3\phi\sin\phi - 4\cos\phi\sin^3\phi)$$

91. a) $\cos 2\phi = \cos^2\phi - \sin^2\phi$

b) $\sin 2\phi = 2\cos\phi\sin\phi$

c) $\cos 3\phi = \cos^3\phi - 3\cos\phi\sin^2\phi$

d) $\sin 3\phi = 3\cos^2\phi\sin\phi - \sin^3\phi$

e) $\cos 4\phi = \cos^4\phi - 6\cos^2\phi\sin^2\phi + \sin^4\phi$

f) $\sin 4\phi = 4\cos^3\phi\sin\phi - 4\cos\phi\sin^3\phi$

92. a) $$\tan 2\phi = \frac{\sin 2\phi}{\cos 2\phi} = \frac{2\cos\phi\sin\phi}{\cos^2\phi - \sin^2\phi} = \frac{2\,\frac{\sin\phi}{\cos\phi}}{1 - \frac{\sin^2\phi}{\cos^2\phi}}$$

$$= \frac{2\tan\phi}{1 - \tan^2\phi} \qquad \boxed{\begin{array}{ccc} & 2 & \\ 1 & & 1 \end{array}}$$

Similarly making substitutions from problem 91 we can show that

b) $$\tan 3\phi = \frac{3\tan\phi - \tan^3\phi}{1 - 3\tan^2\phi} \qquad \boxed{\begin{array}{cccc} & 3 & & 1 \\ 1 & & 3 & \end{array}}$$

c) $$\tan 4\phi = \frac{4\tan\phi - 4\tan^3\phi}{1 - 6\tan^2\phi + \tan^4\phi} \qquad \boxed{\begin{array}{ccccc} & 4 & & 4 & \\ 1 & & 6 & & 1 \end{array}}$$

d) $$\tan 5\phi = \frac{5\tan\phi - 10\tan^3\phi + \tan^5\phi}{1 - 10\tan^2\phi + 5\tan^4\phi} \qquad \boxed{\begin{array}{cccccc} & 5 & & 10 & & 1 \\ 1 & & 10 & & 5 & \end{array}}$$

e) $$\tan 6\phi = \frac{6\tan\phi - 20\tan^3\phi + 6\tan^5\phi}{1 - 15\tan^2\phi + 15\tan^4\phi - \tan^6\phi} \qquad \boxed{\begin{array}{ccccccc} & 6 & & 20 & & 6 & \\ 1 & & 15 & & 15 & & 1 \end{array}}$$

93. a) $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{pmatrix}$ e) $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots \\ 1 & 3 & 6 & 10 & 15 & \cdots \\ 1 & 4 & 10 & 20 & 35 & \cdots \\ 1 & 5 & 15 & 35 & 70 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}$

94. a) b) c) d) e) For each n, $n \geqslant 1$ the determinant of D_n is 1.

95. a) $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \bullet \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -2 & 1 \end{pmatrix} \bullet \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -3 & 3 & -1 \end{pmatrix} \bullet \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 1 & -3 & 3 & -1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix} \bullet \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{pmatrix}$

e) The conjecture is: For each matrix A, $A\,A^t =$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots \\ 1 & 3 & 6 & 10 & 15 & \cdots \\ 1 & 4 & 10 & 20 & 35 & \cdots \\ 1 & 5 & 15 & 35 & 70 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}$$

96. a) $\begin{vmatrix} 2 & 3 \\ 3 & 6 \end{vmatrix} = 2 \bullet 6 - 3 \bullet 3 = 3$ b) 4 c) 5 d) 6

e) For each natural number $n \geqslant 1$, it appears that the determinant of the $n \times n$ matrix D_n is $(n+1)$.

97. a) $3 - 2 = 1$ b) $12 - 9 = 3$ c) $30 - 24 = 6$ d) $60 - 50 = 10$

e) $\binom{k}{1} \bullet \binom{k+2}{2} - \binom{k+1}{2} \bullet \binom{k+1}{1} = \dfrac{\cancel{k!}}{1!\,(k-1)!} \bullet \dfrac{(k+2)!}{2!\,\cancel{k!}} - \dfrac{(k+1)!}{2!\,(k-1)!} \bullet \dfrac{(k+1)!}{1!\;k!}$

$$= \frac{(k+2)(k+1)k}{2} - \frac{(k+1)k(k+1)}{2} = \frac{k(k+1)}{2} = \binom{k+1}{2}$$

98. a) $30 + 12 + 12 - 9 - 24 - 20 = 1$

b) $240 + 120 + 120 - 200 - 180 - 96 = 4$

c) $1050 + 600 + 600 - 500 - 900 - 840 = 10$

d) $3360 + 2100 + 2100 - 1800 - 2800 - 2940 = 20$

e) $\binom{k}{1} \bullet \binom{k+2}{2} \bullet \binom{k+4}{3} + \binom{k+1}{2} \bullet \binom{k+3}{3} \bullet \binom{k+2}{1} + \binom{k+2}{3} \bullet \binom{k+1}{1} \bullet \binom{k+3}{2}$

$$- \binom{k+2}{1} \bullet \binom{k+2}{2} \bullet \binom{k+2}{3} - \binom{k+3}{2} \bullet \binom{k+3}{3} \bullet \binom{k}{1} - \binom{k+4}{3} \bullet \binom{k+1}{1} \bullet \binom{k+1}{2}$$

$$= \frac{k(k+1)(k+2)^2(k+3)(k+4)}{12} + \frac{k(k+1)^2(k+2)^2(k+3)}{12} + \frac{k(k+1)^2(k+2)^2(k+3)}{12}$$

$$- \frac{k(k+1)^2(k+2)^3}{12} - \frac{k(k+1)(k+2)^2(k+3)^2}{12} - \frac{k(k+1)^2(k+2)(k+3)(k+4)}{12}$$

$$= \frac{k(k+1)(k+2)}{12}\Big[(k+2)(k+3)(k+4) + 2(k+1)(k+2)(k+3) - (k+1)(k+2)^2 - (k+2)(k+3)^2 - (k+1)(k+3)(k+4)\Big]$$

$$= \frac{k(k+1)(k+2)}{12}\Big[2\Big] = \binom{k+2}{3}$$

99. a)

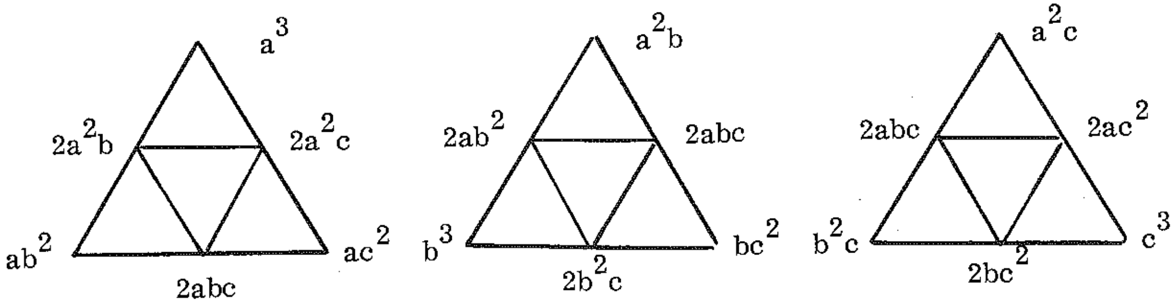
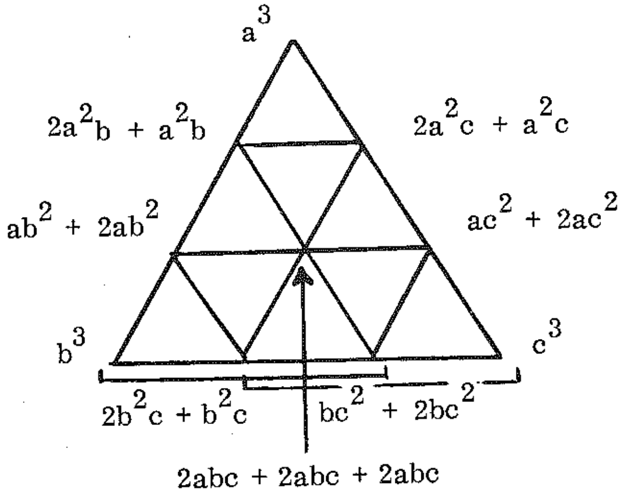
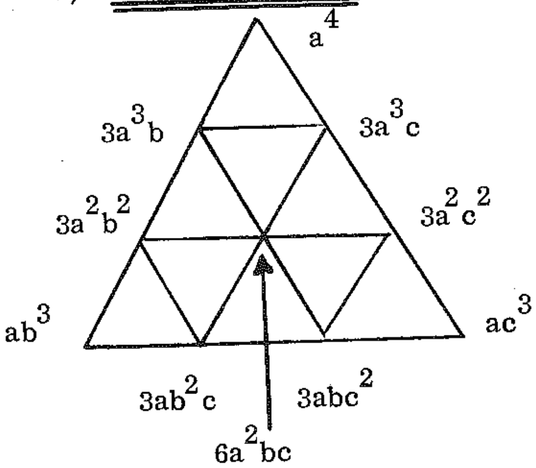
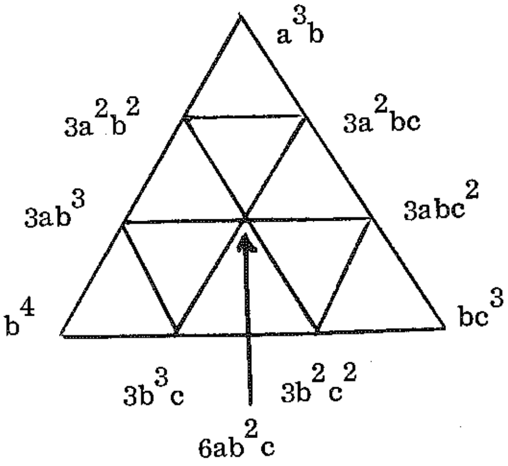


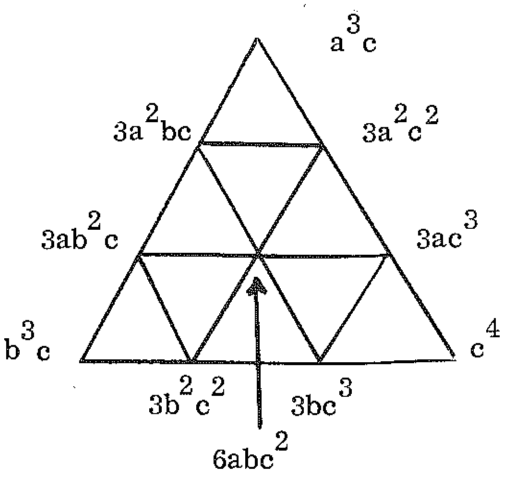
Fig. 53 (iii) x a x b x c

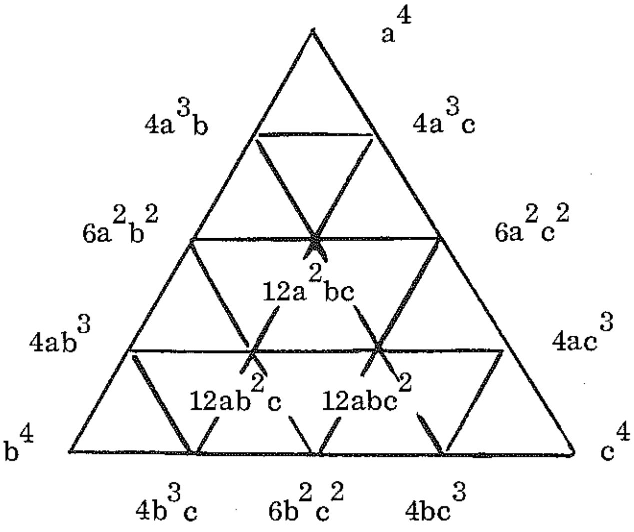


b) Solution to 99 (a) x a , x b respectively .









Solution to 99(a) x c

c) $(a + b + c)^5$ is the sum of the terms in triangular array below .

Note that row 5 of the Pascal triangle appears on the 3 sides of the array.

a^5
$5a^4b \quad 5a^4c$
$10a^3b^2 \quad 20a^3bc \quad 10a^3c^2$
$10a^2b^3 \quad 30a^2b^2c \quad 30a^2bc^2 \quad 10a^2c^3$
$5ab^4 \quad 20ab^3c \quad 30ab^2c^2 \quad 20abc^3 \quad 5ac^4$
$b^5 \quad 5b^4c \quad 10b^3c^2 \quad 10b^2c^3 \quad 5bc^4 \quad c^5$

d) $(a + b + c)^6$ is the sum of the terms in triangular array below .

Note that row 6 of the Pascal triangle appears on the 3 sides of the array .

a^6
$6a^5b \quad 6a^5c$
$15a^4b^2 \quad 30a^4bc \quad 15a^4c^2$
$20a^3b^3 \quad 60a^3b^2c \quad 60a^3bc^2 \quad 20a^3c^3$
$15a^2b^4 \quad 60a^2b^3c \quad 90a^2b^2c^2 \quad 60a^2bc^3 \quad 15a^2c^4$
$6ab^5 \quad 30ab^4c \quad 60ab^3c^2 \quad 60ab^2c^3 \quad 30abc^4 \quad 6ac^5$
$b^6 \quad 6b^5c \quad 15b^4c^2 \quad 20b^3c^3 \quad 15b^2c^4 \quad 6bc^5 \quad c^6$

100. a) The formation in the solution to problem 99 illustrates just how the Pascal triangle appears in the expansion of $(a + b + c)^n$.

Briefly: (1) the Pascal triangle for row n appears on each side of the triangular array; (2) the terms inside the array are also rows 0 to n of the Pascal triangle, each being multiplied by an appropriate factor.

b) The sum of the numbers in the triangular arrays are : 9, 27, 81, 243, and 729. The conjecture is : the sum of the coefficients is 3^n. This is reasonable since the sum of the coefficients of $(a+b)^n$ is 2^n from a = b = 1. Here in $(a+b+c)^n$ a = b = c = 1 implies a total of 3^n .

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