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Understanding Connections between Equations and Graphs

General consensus holds within the mathematics-education research community that functions are among the most important unifying ideas in mathematics (Romberg, Carpenter, and Fennema 1993). In fact, "it can be argued that functions form the single most important idea in all of mathematics, at least in terms of understanding the subject as well as for using it" (Dubinsky 1993, 527). Further, the introduction of algebraic and graphical representations of functions can be seen as a crucial moment in mathematics learning and represents "one of the earliest points in mathematics at which a student uses one symbolic system to expand and understand another" (Leinhardt, Zaslavsky, and Stein 1990, 2).

Recognizing the important role that multiple representations of functions play in students' mathematical development, the National Council of Teachers of Mathematics emphasizes that students should be able to "translate among tabular, symbolic, and graphical representations of functions" (NCTM 1989, 154). Leading researchers in mathematics education have also acknowledged the significance of multiple representations in secondary school mathematics, identifying the notion as one of the "big ideas" of algebra (Lacampagne, Blair, and Kaput 1995). Because of recent reform initiatives—initiatives that call for schools to implement algebra for all students—their status has been elevated even further (NCTM 1994). Even with the recognized importance of multiple representations in secondary school mathematics, many students leave high school without understanding the connections among these representations (Blume and Heckman 1997).

Many students had a limited understanding of the connection between equations and their graphs

The twofold goal of this article is (1) to present results from a study that examined students' understanding of connections between algebraic and graphical representations of functions, that is, equations and graphs; and (2) to discuss a possible reason for the inadequate, and often absent, connections that students made between them. The study grew out of my concern about my own first-year-algebra students' understandings of the connections between different representations of functions. As I learned more about the nature of my students' understanding in this domain through classroom discourse and various assessment instruments, I began to realize that many of them had limited understanding of the connections, particularly connections between equations and their graphs. Consequently, I decided to see whether my discoveries about my own students might also be true of other high school mathematics students.

THE STUDY

The 284 students who participated in this study attended a large suburban high school and were taking college-preparatory mathematics courses ranging from first-year algebra through Advanced Placement calculus. Each student was asked one of ten different questions, that is, in a class of thirty, three students received the same question. Students were asked to write a response in which they showed their work and explained their thinking. The teacher presented the questions at the beginning of class as a warm-up—part of each participating teacher's regular instructional practice—and students were given ten minutes to work on the question. In addition, to ensure that the requisite concepts had been covered, especially for first-year-algebra students, the study was conducted during spring semester.

The questions for this study were designed to foster insight into students' understandings of connections between equations and graphs. All the problems, with one exception, addressed only linear functions. The rationale for this limitation on the types of functions is that representations of linear functions are a major topic of study in first-year algebra—often the only type of function studied—and their study sets the stage for more advanced work in school mathematics. Thus, the study focused on an area in which all students had expe-

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rience and in which all should have understood the fundamentals. Five of the ten questions in particular were designed to elicit the students' understandings of a fundamental connection—a point satisfies an equation if and only if the point is on the graph of the equation. Those five questions are the focus of this article. The remaining questions did not focus explicitly on this connection, and students' responses are not discussed in this article.

Each of the five questions discussed in this article gave students both an equation and a corresponding graph. For example, question 3 asked students how they would solve an equation that had a missing coefficient. See **figure 3**. Students could answer the question by using either representation; yet perhaps the easiest response was to state that selecting any point on the graph would give a solution, as opposed to first determining the missing coefficient, then substituting a value into the equation for one of the variables, and finally solving for the remaining variable. Students were also asked to furnish an alternative solution method, for example, using the equation instead of the graph, where applicable.

Although the two representations are informationally equivalent, that is, the information available from one representation can be inferred from the other, and vice versa, the two representations are not computationally equivalent (Larkin and Simon 1987). Consequently, the representations differ in their solution efficiency: the graphical representation explicitly provides information—displays an infinite number of points—that is given only implicitly in the algebraic representation, in which points must be found computationally. I hypothesized that if students understood this connection, then they would tend to select the more efficient, and often easier, solution strategy, which for this series of questions entailed using the graphical representation.

I classified students' responses into two categories: algebraic solution methods and graphical solution methods. Responses in which the equation was the primary means for finding a solution were categorized as algebraic solution methods, whereas responses that explicitly used the graph were categorized as graphical solution methods. In some instances, a student used the graph in the solution only to support an algebraic approach; such an approach was categorized as an algebraic solution method. After I classified each student's response within a course, for example, geometry, I aggregated all responses and recorded the result as an overall percent on the basis of the solution method used.

RESULTS

Question 1

Students could solve the first question (see **fig. 1**) by using either representation, although the graph did not require any calculation. Students could simply read the coordinates of any point from the line. In addition, the form of the equation was one that is not typically used, and the coefficients and constant term were not integers. Mentally determining a solution was therefore more difficult. The rationale for this nontypical format was that it led students away from the traditionally emphasized procedure of substituting a value for one variable into the equation and then solving for the other.

Student responses ($n = 30$) to part 1(a) indicated an overwhelming reliance on the algebraic representation; all but one of the students used the equation in some manner in finding a solution. The following response, from a first-year-algebra student, was typical: "First I would put the equation in slope-intercept form [often the form most commonly emphasized during instruction], which would solve for y. Then I would substitute into the equation a value for x and then solve the new equation for y." An honors second-year-algebra student's method involved selecting an x-value from the graph, substituting that value into the equation, and then solving the resulting equation for y. Although the method produced a correct solution, that student evidently failed to fully recognize the connection between a point on the line and a solution to the equation—solving for the y-value should have yielded the same value as the y-coordinate of the initially selected point.

One fundamental connection is that a point satisfies an equation if and only if the point is on the graph

Question 1

Given an equation $3.2y + 5.1x = -7.5$ and its graph shown below

a) Explain how you could find a solution to this equation.

b) Could you find a solution by using a method other than the one you used in part (*a*)? Explain your answer.

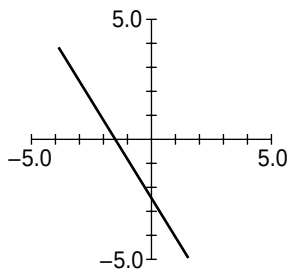

Fig. 1
An equation given in "nontypical" form

Part (*b*) asked students to use a different method than the one that they used in part (*a*). Surprisingly, only 17 percent of the students could give an alternative solution method; thus for the majority of students, the graph appeared to be unnecessary

or even irrelevant in finding a solution. Only three students suggested finding a solution by using the coordinates of any point on the line. The other two students who offered an alternative solution method but had not used a graphical approach suggested writing the equation in a form other than slope-intercept and then using this new equation to solve for x and y. Of those students who apparently could not think of another method, five left the question blank, ten responded no, and the remaining ten gave such inappropriate responses as the following, furnished by a second-year-algebra student: "You can use the addition/multiplication or substitution methods to find where the graph crosses the x-axis."

Given the level of the students, the number of responses that did not make sense was alarming

The failure of so many students to think of an alternative method suggested that many of them lacked a crucial connection concerning equations and graphs. However, in the students' defense for both parts (*a*) and (*b*), having nonintegral coefficients might have suggested to the students that they needed a level of precision that they could not attain by reading the coordinates of a point from the graph, and as a result, they did not perceive that a graphical solution method was valid. Perhaps even more alarming though, given the level of many of these students, was the number of responses that did not make any sense mathematically.

Question 2

Thirty-eight students responded to the second question (see **fig. 2**)—a question that could also be solved using either representation; however, a graphical solution was easier. The previous question asked students how they might determine a solution in general, whereas a particular point was identified as a possible solution to this problem. In part (*a*), only 47 percent of the students responded that any point on the parabola would satisfy the equation. Typical responses included, "Yes. All points on a parabola are a solution to the equation," from a second-year-algebra student, and "Yes, all points on [the] parabola are solutions," from a geometry student. Other correct responses, categorized as using algebraic solution methods, included those from students who may have recognized the relationship between the equation and its graph but thought that they had to verify the coordinates of the point by substituting the coordinates into the equation.

Interestingly, a number of students who used this algebraic method responded that the point was not a solution because the approximate coordinate values that they selected did not satisfy the equation. Apparently, for these students, the fact that the point was on the parabola was not enough to convince them that it was indeed a solution. Other incorrect responses included those from students

Question 2

Given an equation $y = x^2 - 2x - 1$ and its graph shown below

a) Is the point shown by the arrow a solution to the equation? Explain your answer.

b) If the value of x is approximately 2.5, what is the approximate value for y? Explain your answer.

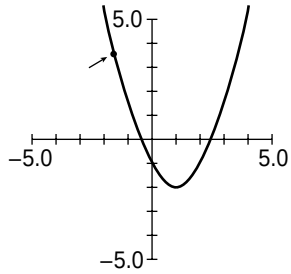

Fig. 2
The graph does not give an exact value.

who had evidently confused a process for determining the roots of a quadratic equation with the question posed; for example, one second-year-algebra student stated "No, because the solution would have to cross the x-axis, and it doesn't" and a first-year-algebra student incorrectly factored the quadratic as $(x - 2)(x + 1) = 0$ and answered "No, not part of quadratic equation."

Since less than half the students seemed to recognize the relationship between the graph and the equation in part (*a*), not surprisingly did only 11 percent of the students' responses for part (*b*) describe a method that used the graph. The majority of students substituted 2.5 into the equation to determine y, as opposed to reading an approximate y-coordinate directly from the graph. Three of the four students who actually used the graph were calculus students, whereas the fourth student, an honors second-year-algebra student, solved for y algebraically but then mentioned, "You can also look at the graph and estimate." Although the majority of students did not give responses indicating that the point is a solution because it is on the curve for part (*a*), the nature of the question may have led the students to believe that they needed to empirically verify the point as a solution. However, in part (*b*), assuming that the students would have been led to use the graph seemed just as reasonable, given that only an approximate value was required for an answer. An overwhelming majority of the students, including many of those who responded appropriately in part (*a*), still relied on an algebraic approach in finding the approximate y-value.

Question 3

The third question (see **fig. 3**) was similar in design

to 1(*a*), but the coefficient of *x* was missing. This missing coefficient was irrelevant if a graphical method was used to solve the problem, but it was essential if the solution used an algebraic method. Only a fifth of the thirty-five students responded that finding a solution without the missing coefficient was possible, although the most obvious response was that the coordinates of any point on the graph would yield a solution. The following response from a precalculus student was representative of those from students who thought that solving the problem was possible: "Any point on the curve can solve the equation." Typical responses from students who thought that solving the problem was impossible included a first-year-algebra student's "No, you need all parts of the equation before finding a solution" and a response of "No, it is not possible to find a solution until you find the missing value" from a precalculus student. Evidently, the majority of students did not perceive that the graph had any relevant information that would directly contribute to finding a solution. Rather, students needed the equation to be complete to determine a solution. That result was not surprising, given the number of students who used an algebraic approach on the other questions.

Question 3

The graph below represents the equation $?x + 3y = -6$. We do not know the value of the coefficient of *x*.

a) Is it possible to find a solution to the equation without the missing coefficient? Explain your answer.

b) How could you find the missing coefficient? Explain your answer.

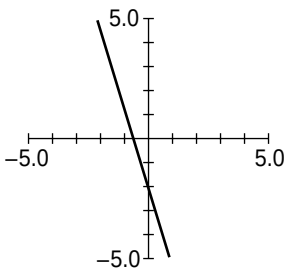

Fig. 3
The graph supports students' algebraic manipulations.

Students solved part (*b*) primarily in two ways: (1) rewriting the equation in slope-intercept form and using the slope found from the graph to find the missing coefficient; or (2) substituting an *x*-value and its corresponding *y*-value, both found from the graph, into the equation and solving for the missing coefficient. Seventy-five percent of the students selected the former method, whereas only three students used the latter method. Other students described an incorrect method. Ironically, a number of students responded to part (*a*) by stating that finding a solution was impossible without the missing coefficient; however, these same students responded to part (*b*) by stating that using the graph would allow them to find the missing coefficient. In other words, these students could find the missing coefficient by either calculating the slope of the line—a calculation that involved using the graph to identify two sets of ordered pairs, both of which were solutions—or selecting points from the graph—points whose coordinates satisfied the equation—to substitute into the equation. Presumably, these students did not recognize that the procedures that they used relied on the fact that any point on a line is a solution to the equation of the line.

Question 4

The fourth question (see **fig. 4**) required that students match one of three equations to a corresponding graph. The equations were in standard form rather than slope-intercept form, and both the *x*- and *y*-intercepts of the given line were integers. I expected that the students would determine the correct equation of the line by using the relatively simple procedure of substituting the coordinates of the intercepts into each equation instead of the more lengthy procedure of converting each equation into slope-intercept form, using the graph to calculate the slope and to identify the *y*-intercept, and then comparing the slope and intercept with the corresponding parameters of the converted equations. However, seventeen students out of the nineteen used the latter procedure to arrive at a solu-

Question 4

Given the equations $6x + 3y = 6$, $-4x + 2y = 4$, and $-5x + 2y = 4$

a) Which equation is represented by the graph shown below?

b) Explain how you decided on your answer in part (*a*).

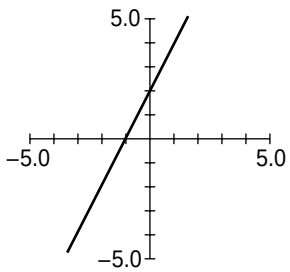

Fig. 4
Students convert the given equations into slope-intercept form.

The majority did not perceive that the graph had information that might help find a solution

tion. A typical response, from a precalculus student, was "I set the equations in $y = mx + b$ form and then saw which equation matched the graph." The two students who did not convert the equation into slope-intercept form chose the former method: "I plotted [i.e., substituted] x-values on the graph into the equation and determined if the y-value was correct as compared with the graph," said a precalculus student. Again, much like students' responses to the other questions, students chose a more involved algebraic method over a somewhat less involved graphical method.

Question 5

The equation in the last question (see **fig. 5**) was in slope-intercept form, the form with which the students were most familiar from both instruction and practice. They could find the correct answer either by substituting $y = 9$ into the equation and solving for x or by reading the x-coordinate directly from the graph. In part (a), 70 percent of the twenty-two students solved the problem algebraically. Of those students, only 27 percent used the graph as their alternative solution method. Other students, also approaching part (a) algebraically, used alternative solution methods that were incomplete or incorrect. For example, a geometry student stated, "You can use the y-intercept or the slope of the line to find x," and a first-year-algebra student commented that x could be found "by using the distance formula." In addition, all seven of the students who answered part (a) by using a graphical approach could suggest an alternative solution method—a method that used the equation.

The questions encouraged a graphical approach, but responses showed reliance on algebraic methods

Question 5

Given an equation $y = 2x + 3$ and its graph shown below

a) What is the value for x if $y = 9$?

b) Can you find the value for x in another way? Explain your answer.

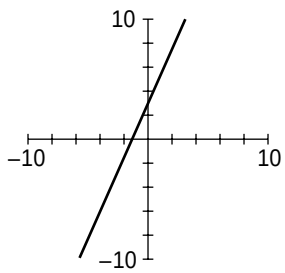

Fig. 5
An equation given in a more recognizable form

DISCUSSION OF RESULTS

Many mathematics educators would consider these questions to be routine exercises, exercises that students should not have had any difficulty answering appropriately by using either representation. Accordingly, little instructional time, other than the initial introduction and subsequent practice in first-year algebra, is spent on the aforementioned connection. After students have been exposed to this connection, teachers generally assume that little or no review is needed (Schoenfeld, Smith, and Arcavi 1993). The findings of this study suggest otherwise. The results are particularly distressing considering that more than half the students tested were enrolled in second-year algebra and above—a population often thought to be representative of our best mathematics students.

The nature of the instruction that students receive, in both the representations that are emphasized and the kinds of translation tasks that are presented, may significantly contribute to the difficulties that many students have in connecting equations and graphs. In the former situation, difficulties are related to curricular and instructional emphases in secondary school mathematics. "The nature of the algebra curriculum is such that the problems we offer students are for the most part limited to those problems that can be readily solved within the framework of symbolic representations alone. As a result, visual representation is not perceived as necessary by most students when engaged in mathematics problem solving" (Yerushalmy and Schwartz 1993, 43). The data certainly seem to support this conclusion: the questions in the study were designed to encourage a graphical solution method, but the students' responses illustrated an overwhelming reliance on algebraic solution methods, often at the expense of, and with apparent unawareness of, a simpler graphical solution method. Further, many students seemed to perceive that the graph was unnecessary or that it served only to support their algebraic solution methods rather than be a means to a solution in and of itself.

In addition, many students seem to have developed a ritualistic procedure for solving problems of the type presented in this study. Traditionally, beginning in first-year algebra, students are introduced to the slope-intercept form of an equation, and it remains the predominant form throughout their work with linear functions. In fact, when equations are given in a different form, students are often instructed to change them into slope-intercept form. This tendency was certainly evident in a majority of students' responses to several of the questions—for example, 1, 3, and 4—questions that should have led students away from an algebraic solution method. Even in their successful use

of this algebraic solution method, many students failed to realize—after converting the equations into slope-intercept form and calculating the slope of each line by selecting points from each line—that the points used in calculating each slope were themselves solutions to the equations.

Finally, the nature of the translation tasks—tasks that require moving from one mode of representation to another—may also play a role in limiting the connections that students make between the representations. Students are routinely given tasks that require translations in the equation-to-graph direction; students initially produce a table of values that satisfies the equation, which is typically in slope-intercept form, and then plot the values on a coordinate graph (Leinhardt, Zaslavsky, and Stein 1990). As a consequence, students may have difficulty on tasks in which they must proceed in the graph-to-equation direction. Indeed, all the questions described in this article required just such a translation, and students' responses indicate that the nature of the translation tasks may have contributed to their difficulties.

CONCLUDING REMARKS

Students often *appear* to understand connections between equations and graphs, particularly given the nature of the tasks that they typically encounter. As the results of this study suggest, for many students, their actual understanding of the connections is often superficial at best. An important aspect of developing a robust understanding of the notion of function means not only knowing which representation is most appropriate for use in different contexts but also being able to move flexibly between different representations in different translation directions. As teachers, we need to recognize this goal as being important for instruction. We must give students opportunities to interact with, and to build connections between, graphical and algebraic representations.

In my own classroom, I often encourage students to use different representations in their solution methods; present different forms of equations, for example, point-slope and standard; emphasize graphical representations whenever appropriate, for example, when using a graphing calculator; and pose tasks that require translations in the graph-to-equation direction. Perhaps most important, however, I give students opportunities to share and discuss their different solution approaches and the advantages or disadvantages of each approach, even on such routine problems as the ones presented in this article.

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