Listening to Middle School Students' *Algebraic* Thinking

JOHN P. SMITH III

AND

ELIZABETH A. PHILLIPS

NO PART OF THE K–12 MATHEMATICS curriculum is more fluid and controversial than introductory algebra. Content and assessment issues lie at the core of this debate: What algebra skills and understandings are important? What kind of evidence suggests that students possess these skills? Neither question can be answered in simple terms; in fact, no single "right" answer may exist for either one.

In this article, we take a stand on the first question, then present and discuss student work that illustrates the range, depth, and character of students' algebraic thinking that are possible by the end of middle school. We focus on how students understand linear, as well as nonlinear, relationships and on the mathematical terms that students use to express their understanding. Key concepts, such as slope, equivalence, and intercept, are much more complex than their textbook definitions suggest. Attending carefully to students' language can help us realize both their positive insights and the limitations in their thinking. In that sense, we *can* listen to students' thinking.

The students whose work we present used materials from *The Connected Mathematics Project* (CMP) (Lappan et al. 1995). In grades 6–8, the CMP curriculum is problem centered. Mathematical concepts and procedures grow out of extended work on particular problems and situations. CMP has a rich algebra strand, particularly in the eighth grade. Algebra is presented as a set of tools for analyzing and understanding relationships

JACK SMITH, *jsmith@pilot.msu.edu, and* ELIZABETH PHILLIPS, *ephillip@math.msu.edu, are colleagues at Michigan State University, East Lansing, MI 48824. Smith studies middle and high school students' learning and problem solving, particularly in algebra, and the use of mathematics in the workplace. Phillips's interests include the teaching and learning of mathematics with a focus on the middle grades.*

between covarying quantities. Students study linear and various nonlinear relationships in diverse situations using verbal, tabular, graphical, and symbolic representations. Graphing calculators are an integral part of the curriculum.

Our view of "important" skills and understanding in introductory algebra is closely related to the main features of the CMP curriculum and other curricula, such as the Mathematics in Context curriculum, that attempt to develop meaning for algebraic symbols in realistic contexts. We value students' abilities to *(a)* identify the quantities that vary in problem situations and describe how those variables are related; *(b)* describe the rates of change and *y*-intercepts of those relationships, represented in tables, graphs, and symbolic expressions; *(c)* think "across," and make connections among, those representations; *(d)* understand the equivalence of algebraic expressions in multiple ways; and *(e)* most of all, make sense of algebraic expressions and equations in relation to the contexts in which they appear. We think that this "starter set" of competencies can serve as a solid foundation for developing more advanced algebraic and mathematical abilities.

In the early stages of our work to assess middle school students' algebra learning, we began observing one eighth-grade-mathematics class in May. The school where we conducted our observations is in a small town that is a short distance from a medium-sized city. Because the school does not track students by ability, the twenty-four students in the class had a wide range of ability and motivation. The teacher was experienced and comfortable with the curriculum.

We expected to find some differences in how students reasoned about linear relationships, but the extent of the variation surprised us. When the teacher gave a quiz with the problem shown in **figure 1,** this diversity became clearly evident. The class had recently been working on equivalent expressions but had not yet discussed the distributive property.

Three of the following expressions are equivalent. For the expression that is not equivalent to the others, explain how you can tell, without using your calculator, that it is not equivalent.

(a) $2x - 12x + 10$ (b) $12x - 2x + 10$
(c) $10 - 10x$ (d) $10(1 - x)$

Fig. 1 Equivalent Expressions I

Different Methods for Judging Equivalence

IN GENERAL, THE CLASS WAS SUCCESSFUL WITH this problem. All but three students selected expression (b) and gave explanations for their choices. More significant to us, however, was the range in thinking about equivalence. In the following responses, note the number of questions that are raised or left unanswered.

Anna. "You can predict this, because taking $2x$ from $12x$ is different than taking $12x$ from $2x$."

Chris. "Because (c) and (d) are the same and (a) and (b) are not the same, because it flip-flops the first two numbers around and it messes the answer up."

Steve. "$12x - 2x$ is going to equal –100, and you add 10 to get –90. It would have to be $2x - 12x + 10$ in order to be equivalent to the others."

Rachel. "I put a number for x, and the equation that didn't equal the same number as the other ones is not equivalent."

Melinda. "This equation [b] increases and the others decrease."

Phil. [After some substitutions for x] "All the rest have negative slopes."

The extent of the variation in reasoning surprised us

Gillian. "Not equivalent to the others, because the table of (a), (d), and (c) decrease by negative numbers and (b)'s table increases by 10" (see **fig. 2**).

Thomas. [Eliminates (a)] "Because you will get a $-x$, and in the others you will have a positive x." [Eliminates (c)] "Because as x gets bigger, $10x$ becomes larger, so you begin to go in the negatives, go down instead of up."

These students analyzed the expressions in different ways, using different representations of

x	Value of Expression
0	10
1	0
2	–10
3	–20

For expressions (a), (c), and (d)

x	Value of Expression
0	10
1	20
2	30
3	40

For expression (b)

Fig. 2 Gillian's tables for Equivalent Expressions I

the linear relationships. Anna, Chris, and Thomas examined and compared the algebraic forms of the expressions. Steve and Rachel focused on the value of the expressions for a specific value of x. Phil (and possibly Melinda) thought about their graphs. Gillian (and possibly Melinda) generated tables of values. Many of their explanations however, were unclear or confused. How did Melinda see increases and decreases—in a table, a graph, or both? How were expressions (c) and (d) "the same" for Chris? Why did Thomas fail to report the "$-x$" in (c) that he saw in (a)? We could see that the class as a whole had learned numerical, tabular, graphical, and symbolic approaches to thinking about equivalence, but what about individual students? Was their thinking limited to one representation, or could they move easily among representations? If they knew more than one approach or representation, did they consider one preferable or most convincing?

We needed to get closer to students' thinking

To answer these questions, we needed to get closer to the students' thinking. We quickly developed some interview problems that required making sense of algebraic expressions in different situations. Sixteen of the twenty-four students agreed to participate in the interviews. Some of the volunteers were successful in mathematics, and others were not. The students worked each problem in pairs in a quiet space away from the classroom. Because one volunteer was absent on the interview days, we actually had seven pairs of students. One student worked the problems alone. We provided graphing calculators, graph paper, and blank paper and let the pairs work by themselves for five to ten minutes on each problem. Then we returned and asked them to explain their work. Both their work in pairs while we were not present and the discussions that followed were audiotaped.

Methods That Convince

THE FIRST PROBLEM RETURNED TO THE ISSUE of equivalence but delved more deeply into students' understanding of linear relationships (see **fig. 3**). If the students knew different ways to think about equivalence (including the use of the distributive property that they had recently learned in class), which approach would they find most convincing? We also wanted to know whether they could "model in reverse"—that is, generate a situation from a given algebraic expression. We considered that activity to be more difficult than writing an expression to model a given situation. In developing this first problem, we took an approach that became a general rule of thumb: Begin with a standard algebra task, extend or deepen it, and look for opportunities to reverse the usual direction of students' thinking.

> You and two partners are trying to find expressions equivalent to $2(5 + 3x)$. Don thinks that $2(8x)$ is equivalent; Cathy thinks that $10 + 3x$ is.
>
> (1) Are their expressions equivalent to $2(5 + 3x)$?
> (2) Once you have decided which expressions are equivalent and which are not, how would you explain your reasoning to others in a convincing way?
> (3) Describe a context (problem situation) that you could model with $2(5 + 3x)$ or an expression equivalent to it. What does the variable represent? What does the expression represent?

Fig. 3 Equivalent Expressions II

Every pair of students rejected Don's and Cathy's expressions. All but one pair generated $10 + 6x$ as an equivalent expression. Two students initially said that $2(8x)$ or $10 + 3x$ was equivalent to $2(5 + 3x)$ but were quickly convinced otherwise by their partners. Thomas reminded Allen, "You can't combine $3x$ and 5 by adding"; Beverly substituted a numerical value to persuade Amina. We saw again the diversity in students' methods. Some substituted particular values for x and compared the results, others generated and compared tables of values on their graphing calculators, and a few simply inspected the expressions. In their explanations, the students added more possibilities: applying the distributive property; comparing the graphs on their calculators; and using other manipulation rules, such as, "You cannot add an x number and a regular number." We were able to answer two of our questions definitively. First, we found that individual students knew multiple ways to judge equivalence. Six of the seven pairs used or described two or more methods, and five pairs referred to four or five methods! Second, all but one pair thought that comparing tables of values would be the most effective way to convince their peers. Evidently, comparing specific numerical values was compelling proof for these students.

"Reverse Modeling": Finding a Context for a Linear Expression

MOST ALGEBRA WORD PROBLEMS, INCLUDING many in the CMP materials, present situations and ask students to write expressions or equations that model those situations. Part (3) of Equivalent Expressions II intentionally reversed this line of thinking. The factored form of the expression made this question challenging for most students. They could easily think of situations for $10 + 6x$ and $5 + 3x$ but struggled with the factor 2; however, all but one pair eventually succeeded. Most pairs generated situations that involved purchasing or selling goods or services. They tied the *y*-intercept term, 5, to "flat" or "fixed" costs and the slope term, 3, to "costs per student." The approach for most pairs was to develop a situation for $5 + 3x$, then use 2 as a duplicator. For example, Beverly and Amina described a school trip that required two buses, each with a "set cost" of $5 and variable expenses of "$3 per kid." Similarly, Anna and Susan described a "walkathon" fundraiser involving two groups of participants. Their solution included "fixed costs" of $5 for signing up and "$3 per mile" earned from walking.

Thomas's and Allen's approach was quite different. Thomas, who led the work, thought of "two rooms in a house design, any two things that have dimensions 5 and 3*x*." Generating a design that matched his idea was a challenge. As **figure 4** shows, both boys struggled to generate a shape with the appropriate dimensions, or segments. Their preliminary sketches, such as Thomas's first, suggest that they confused perimeter and area or vacillated between the two concepts. They eventually recognized their mistakes and, led by Thomas, sketched two sets of "rooms" that had areas represented by the original expression. These responses showed that the students could perform "reverse modeling" of a linear relationship but were much more comfortable when the relationship was expressed in the familiar "$mx + b$" form.

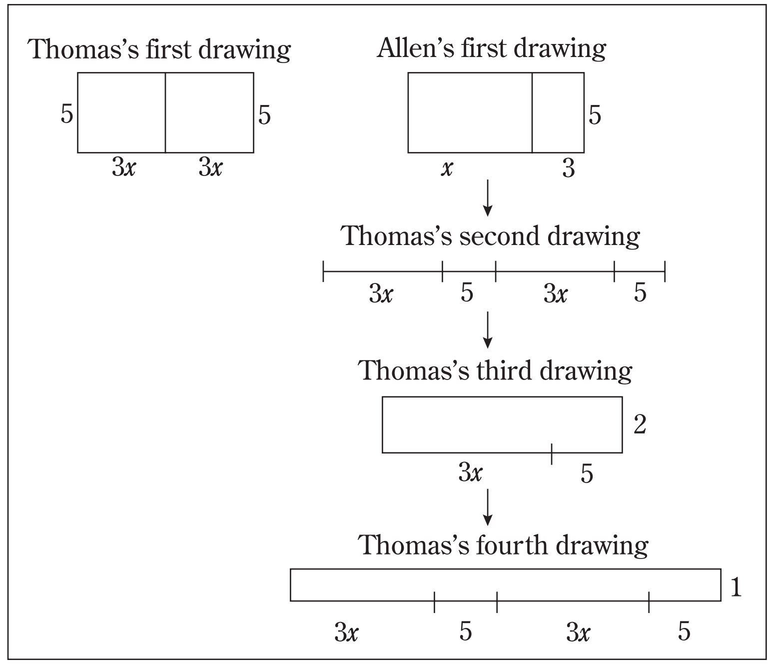


Fig. 4 Thomas's and Allen's geometric modeling

Comparing Linear and Nonlinear Relationships

A SECOND PROBLEM ASKED STUDENTS TO COMpare the growth of three populations, each described by a linear, an exponential, or a quadratic expression (see **fig. 5**). The students had frequently compared two or more linear relationships and had compared a linear with a nonlinear relationship, but they had only rarely confronted a situation with a linear and two different nonlinear relationships.

Initially, population P_3 was represented by the expression $500x + x^2$. Don, in the first interview pair, reacted with surprise: "P_3 is unlike any population that I know about, because it jumps from 0 to 501 in one year." We were chagrined by our failure to check the behavior of our model but impressed by Don's expectation that algebraic expressions should make sense in the situations that they model. We immediately changed $500x + x^2$ to $700 + 10x^2$.

All the student pairs identified the linear pattern and distinguished it from the others. In their words, species 1 "starts at 10 000" and "goes up by 5" each year. This "what it goes up by" conception of slope (the constant rate of change) was strongly linked to the tabular representation. Students were comfortable with tables and quickly examined the

The growth patterns of three species are given below. *P* represents the number of animals of each species after *x* years in that environment.

Species 1	**Species 2**	**Species 3**
$P_1 = 10\,000 + 5x$	$P_2 = 10(2^x)$	$P_3 = 700 + 10x^2$

(1) How would you describe the pattern of growth of each species?
(2) How do these patterns differ from one another?
(3) Pick two species. Is it possible after some number of years that the populations of these two species would be equal? Explain your answer.

Fig. 5 Populations

population increase or decrease to see if it was constant. The difference in consecutive Y-values (that is, the values in the "Y" column of the table display of their graphing calculators) was "what it goes up by." Some students recognized that species 2 was "exponential," but all described the growth as "doubling" after they examined its table. The quadratic expression was more difficult. Only a few students recognized the quadratic expression and distinguished it from the exponential expression. When they examined their tables, however, most decided that the graph "curved upward." All but one pair identified 10 and 700 as the "starting points" and identified 2^x and $10x^2$ as the factor or term that controlled the growth rates. The students' broad use of the term *starting point* indicated that they had generalized their notion of Y-intercept from linear to nonlinear relationships. This conception was sensible and meaningful in two related ways. *Starting point* named the initial Y-value in the table, and it expressed the basic assumption in the problem that population growth began at some point in time.

The use of the term *starting point* indicated generalization

All groups stated that species 1 started out "ahead," but they were split about whether species 2 or 3 would "catch up." About half expected the exponential expression to overtake the linear; the others thought that this catching up was "possible." The four pairs that explored this question moved back and forth between their tabular and graphical displays and answered correctly. (Because of the time limitations, three pairs did not have the opportunity to work on part [3].) Some also came to the same conclusion for the quadratic expression. Melinda's and Phil's work was particularly impressive. Both students struggled in class, were rated as "weak" in pencil-and-paper computation by their teacher, and were unable to interpret the form of the nonlinear expressions; however, they were very competent with their graphing calculators. They quickly found that species 2 overtook species 1, and they determined that species 3 overtook species 1 "some time between the 30th and 31st year." Their algebraic abilities depended on access to the tabular and graphical representations that the calculator provided.

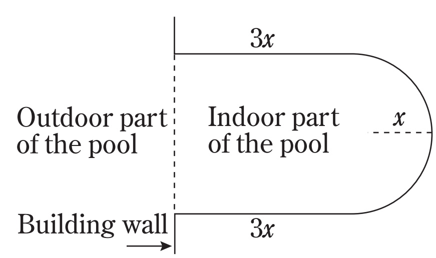
Making Sense of Symbolic Expressions

IN MANY ALGEBRA WORD PROBLEMS, STUDENTS work from a situation to a diagram and, eventually, to a symbolic expression or equation. We transformed this familiar line of thinking by devising a problem in which the situation and the expression were given and students had to complete a diagram. We also moved beyond linear relationships to two-dimensional space, area, and quadratic relationships (see **fig. 6**). Had we been trying to assess symbolic manipulation skills, we would have asked students to simplify or evaluate the expression. We wanted to know, however, if students could make sense of the expression in the situation; specifically, we wondered if they could match quadratic terms to the areas of specific geometric shapes.

This interview problem was the most challenging one we posed, but the students were not deterred. All the pairs successfully matched $6x^2$ and $\pi x^2/2$ to the indoor section of the pool, but in some instances, their language made us cautious about their reasoning. Such statements as "$3x$ and $3x$ is $6x^2$" could indicate either a poor grasp of the area concept or, simply, sloppy mathematical expression. With $\pi x^2/2$ and $6x^2$ identified as indoor sections, it was a simple step to infer that $\pi x^2/4$ and x^2 represented outdoor sections, leaving only the task of placing these regions in the diagram. Most pairs aligned the square and the

Your school is building a pool, part indoors and part outdoors. The plan for the *indoor* part of the pool is shown. The end has the shape of a half-circle, and the rest of the indoor part has the shape of a rectangle. The dimensions of the pool have not yet been set. The area of the surface of the whole pool is given by the expression—

$$\frac{\pi x^2}{2} + 6x^2 + \frac{\pi x^2}{4} + x^2.$$



(1) Which part of the expression represents the area of the indoor part of the pool, and which, the outdoor part?
(2) Sketch a figure that represents the outside of the pool.

Fig. 6 The School Pool Problem. Students understood "surface of the whole pool" to mean the top of the body of water.

quarter circle side by side along the dotted segment but were bothered that this placement violated the pool's overall symmetry. At the interviewer's suggestion, some pairs experimented with different shapes and placements that conserved the total area of the outdoor sections and were more symmetrical.

Steve and Julie worked well individually and together. Steve looked at the rectangular section and said, "We need to find the missing dimension." He extended the dotted radius to a length of $2x$ and concluded that the width of the rectangle was "just the diameter of the circle, so it must be $2x$." He then matched the term $6x^2$ to the rectangle and sketched the outdoor section **(fig. 7a).** When the interviewer asked if another design was possible, Steve was skeptical but Julie quickly drew a second solution **(fig. 7b).** She also decided that x^2 could be broken into two rectangles of dimensions $.5x$ and x **(fig. 7c).** Steve questioned Julie's approach ("I think they want a square") but soon convinced himself that the rectangle, which was $2x$ by $.5x$, produced an area of x^2.

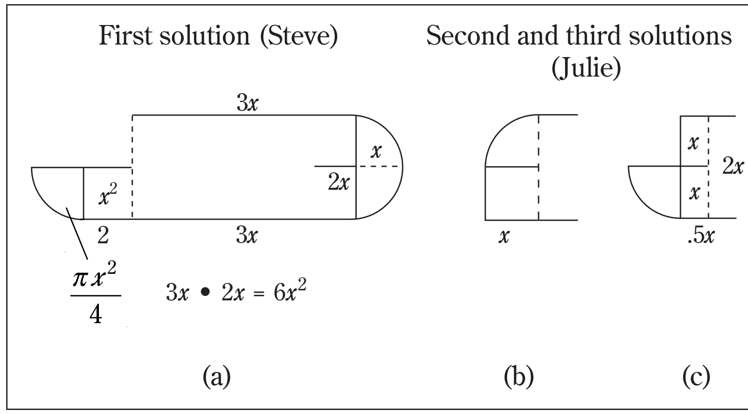


Fig. 7 **Steve's and Julie's solutions to the School Pool Problem**

What Did We Learn?

THE STUDENTS' WORK ON THESE PROBLEMS taught us two important lessons about middle school students' algebra learning. First, powerful algebraic ideas are accessible to middle school students. The students whose work we presented were well on the way to mastering important algebraic understanding and skills. Their knowledge included these five items: (1) a solid grasp of linear functions and constant rate of change; (2) the ability and flexibility to analyze functional relationships with tabular, graphical, and symbolic representations; (3) analytic skills with graphing calculators; (4) an understanding of equivalence in each representation; and (5) a beginning understanding of exponential and quadratic relationships. These abilities did not emerge in response to a few good assessment questions; they required years of experience and instruction in a carefully sequenced curriculum. The problem-centered approach in the CMP curriculum is one way to support this kind of learning, but not the only way. Success in algebraic understanding also has consequences. Students who move into their high school years with an understanding of these introductory algebraic ideas will have a firmer conceptual foundation than their predecessors on which to build. Will we be ready for them?

We also learned that in introductory algebra, as with all challenging mathematics, students' initial ideas are neither flawless nor useless. Helping students explore the world of algebra—where symbolic expressions have meaning in situations—means appreciating and supporting their early insights. That work begins with listening for, hearing, and appreciating the ways in which students express their thinking, knowing that their words may not be the ones we would choose. Learning algebra is a complex, multiyear process that involves many intellectual challenges. The students with whom we worked had successfully built solid foundations for making sense of algebra, but more work lay ahead for them. For example, their view of slope as a constant difference between consecutive Y-values (e.g., "what it goes up by") may or may not have been connected with the concept of constant ratio (e.g., "rise over run"). The students' conception of Y-intercept as a "starting point" may need revision when the domain of functions includes both positive and negative values.

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