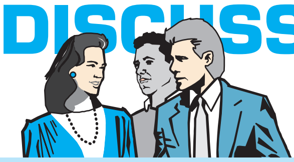
DISCUSS WITH YOUR COLLEAGUES



James E. Schultz and Michael S. Waters

Why Representations?

A growing number of factors seem to influence the teaching and learning of mathematics. These factors include developing conceptual understanding, preparing students for higher-level mathematics, applying mathematics to real-world settings, effectively using more technology, and adapting to different learning styles.

The NCTM's *Principles and Standards for School Mathematics* (2000) document embraces much of the earlier *Curriculum and Evaluation Standards* of 1989. A notable difference is the inclusion of a new process standard that addresses representations (p. 67):

Representation Standard

Instructional programs from prekindergarten through grade 12 should enable all students to—

- create and use representations to organize, record, and communicate mathematical ideas;
- select, apply, and translate among mathematical representations to solve problems;
- use representations to model and interpret physical, social, and mathematical phenomena.

Concern has been growing about the role of representations in teaching mathematics

After discussing the importance of representations from several points of view, this article illustrates the importance of this new process standard with regard to a number of contrasting criteria proposed by the authors:

1. Which representation best promotes conceptual understanding?
2. Which representation best generalizes to higher-level mathematics?
3. Which representation best applies to finding approximate solutions?
4. Which representation best applies to finding exact solutions?
5. Which representation is best for a given type of technology?
6. Which representation best suits the learning style and comfort level of the student?

Although all the listed criteria reflect issues that are likely to be important to teachers, the last two are particularly important for students to consider when learning to make choices about representations.

THE IMPORTANCE OF MULTIPLE REPRESENTATIONS

Concern has been growing about the role of representations in teaching mathematics. For example, writing in *Problems of Representation in the Teaching and Learning of Mathematics*, Kaput (1987, p. 20) argues that "representation is underrepresented," citing research that suggests that representation affects such issues as estimation sense, rote versus meaningful learning, and procedural versus conceptual knowledge.

More recently, Greeno and Hall (1997, p. 362) maintain that "forms of representation need not be taught as though they are ends in themselves. Instead they can be considered as useful tools for constructing understanding and for communicating information and understanding." They underline the importance of students' engaging in *choosing* representations: "If students simply complete assignments of constructing representations in forms that are already specified, they do not have opportunities to learn how to weigh the advantages and disadvantages of different forms of representation or how to use those representations as tools with which to build their conceptual understanding."

Greeno and Hall (1997, p. 365) further suggest several ways in which research shows that representations enhance problem-solving ability. "Representations often match the process of solving the problem, providing a kind of model of the students' thinking as they work. . . . Students often construct representations in forms that help them see patterns and perform calculations, taking advantage of

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the fact that different forms provide different supports for inference and calculation." Citing other research, Brenner et al. (1997, p. 666–67) contend that "attempts to teach problem representation strategies for mathematical problem solving have focused on teaching students ways to translate the words of a problem into other modes of representation using diagrams, pictures, concrete objects, the problem solver's own words, equations, number sentences, and verbal summaries."

Emerging technologies also affect the use of multiple representations in the classroom. Demana and Waits (1990, p. 218) contend that "the ability of students to operate within *and* between different representations of the same concept or problem setting is fundamental in effectively applying technology to enhance mathematics learning."

FIVE REPRESENTATIONS FOR A GIVEN MATHEMATICAL SITUATION

Although the idea of representation applies to almost any topic in the curriculum for grades K–12, this article explores using five different representations to solve two linear equations with two unknowns, a topic that may be developed over a period of years. The reader may wish to reflect on his or her own representations and answers to the six questions above before reading further.

Solve this pair of simultaneous equations:

$$3x + y = 9$$
$$x + 2y = 8$$

Representation A: Concrete

Concrete materials, such as pens and notebooks, shown in **figure 1**, can be used to facilitate the following representation:

> Suppose that three pens and one notebook cost \$9, whereas one pen and two notebooks cost \$8. If each pen costs the same amount and each notebook costs the same amount, find the cost of each.

With the information as shown, the cost of each item is not apparent. One way to solve the problem is to make the number of notebooks the same in both situations and compare the difference in the number of pens and their costs. Begin by doubling the amounts in the first situation, as indicated in **figure 2.** The first situation doubled has the same number of notebooks as the second situation. It also has five more pens, which cost ten more dollars. See **figure 3.** Since each pen costs the same amount, each pen costs \$2. Use this fact and the first situation to find that the cost for each notebook is \$3.

Other examples using this concrete representation can be found in such curricula as "Comparing Quantities," a sixth-grade algebra unit from *Mathematics in Context* (Kindt et al. 1998). For a thorough discussion of several different strategies developed by sixth graders to solve systems using concrete representations, see "Multiple Strategies = Multiple Challenges" (Meyer 1999).

Discuss: Concrete representations seem to appear in elementary textbooks but rarely in high school algebra textbooks. Should high school textbooks give more concrete examples and illustrate the use of manipulatives to solve problems?

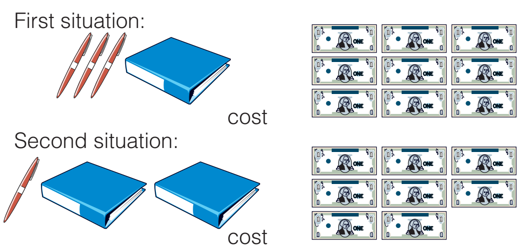

Fig. 1
Pictorial representation for solving the system $3x + y = 9$ and $x + 2y = 8$

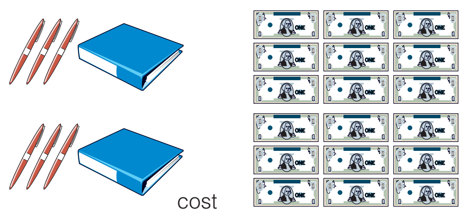

Fig. 2
The first situation in figure 1, doubled

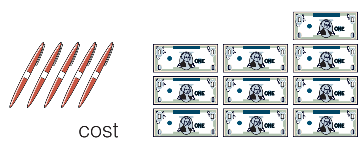

Fig. 3
The difference between the first situation doubled and the second situation

Representation B: Tables

Tables facilitate the guess-and-check, or trial-and-error, approach. **Figure 4** shows the table created by an eighth-grade student, Bethany, who arrived at a correct solution in four tries. **Figure 5** shows a similar approach using a spreadsheet. Tables can also be generated using graphing calculators. Begin by solving the original equations for y and creating a table of values:

x	y	3x + y	x + 2y
1	1	4	3
1	4	7	9
3	2	11	7
2	3	9	8

Bethany

Fig. 4
Handwritten table representation
for solving the system $3x + y = 9$ and $x + 2y = 8$

C3 = 3 * A3 + B3

	A	B	C	D	E
1	x	y	3x + y	x + 2y	
2	1	1	4	3	
3	1	4	7	9	
4	3	2	11	7	
5	2	3	9	8	
6					

Fig. 5
Spreadsheet table representation
for solving the system $3x + y = 9$ and $x + 2y = 8$

$3x + y = 9$, so $y = 9 - 3x$.
Use Y1 = 9 – 3X.

$x + 2y = 8$, so $y = 4 - 0.5x$.
Use Y2 = 4 – 0.5X.

Make a table showing X, Y1, and Y2. Try various values of X to find when Y1 = Y2. See **figure 6.** The solution is $x = 2$ and $y = 3$.

The use of the table feature of graphing calculators to solve simultaneous equations can be found in such textbooks as *Algebra 1* (Schultz, Kennedy, et al. 2001).

F1 F2 Setup

x	y1	y2
-3.	18.	5.5
-2.	15.	5.
-1.	12.	4.5
0.	9.	4.
1.	6.	3.5
2.	3.	3.
3.	0.	2.5
4.	-3.	2.

x=2.
MAIN RAD AUTO FUNC

Fig. 6
TI-92 table representation
of Y1=9–3X and Y2=4–0.5X

Representation C: Graphs

Graph Y1 = 9 – 3X and Y2 = 4 – 0.5X, obtained as in representation B. The solution is the point of intersection of the two lines. See **figure 7.** The solution is $x = 2$ and $y = 3$. The traditional graphical and algebraic representations can readily be found in most algebra textbooks.

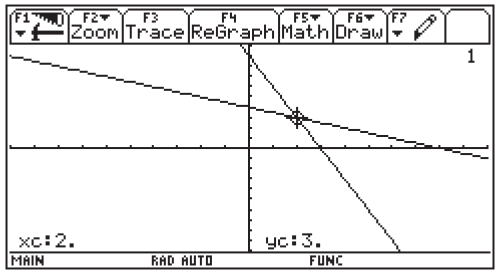

Fig. 7
TI-92 graphical representation of
Y1=9–3X and Y2=4–0.5X

Representation D: Algebraic

The reader is undoubtedly familiar with several approaches to solve this system algebraically. The addition-and-subtraction-with-multiplication method, which was used informally in the solution given in representation A, can be shown algebraically, as in **figure 8.** Another method is to use the substitution method for the equations obtained in representation 5.

$$\left.\begin{array}{l} 3x + y = 9 \\ x + 2y = 8 \end{array}\right\} \Rightarrow \left.\begin{array}{rl} 6x + 2y &= 18 \\ x + 2y &= 8 \\ \hline 5x &= 10 \end{array}\right\} \Rightarrow \begin{array}{l} x = 2 \\ y = 3. \end{array}$$

Fig. 8
The addition-and-subtraction-with-multiplication
method, shown algebraically

Representation E: Matrices

To represent this system of equations using matrix methods, use matrices

$$a = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix},$$

$$z = \begin{bmatrix} x \\ y \end{bmatrix},$$

and

$$b = \begin{bmatrix} 9 \\ 8 \end{bmatrix}.$$

Then the solution to $az = b$ is $z = a^{-1}b$, which a graphing calculator handles easily. See **figure 9.**

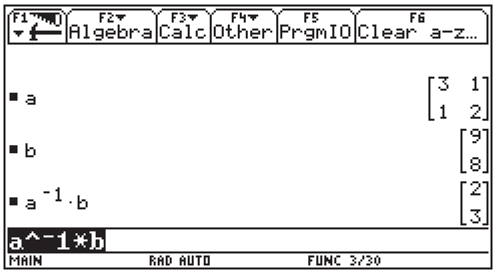

Fig. 9
TI-92 matrix representation
for solving the matrix equation $az = b$

The use of the matrix feature of graphing calculators to solve simultaneous equations can be found in such textbooks as *Algebra 2* (Schultz, Ellis, et al. 2001).

CHOOSING THE REPRESENTATION ACCORDING TO THE CRITERIA

Each of these representations has its place in sixth through twelfth grades, depending on the criteria that are applied. The six criteria given at the beginning of the article are considered in the following:

Criterion 1: Which representation best promotes conceptual understanding?

The five representations were given in order from the most concrete to the most abstract, as judged by the authors. In particular, the first representation can be thought of in terms of real-world objects familiar to the student. Such representations, in which students actually manipulate concrete objects—possibly algebra tiles—or refer to the pictures of objects, could be given in sixth grade or earlier. Representation in a table to summarize guess-and-check methods also serves as a bridge to the more abstract methods. Unfortunately, far too many students never experience anything but the more abstract representations while learning mathematics.

Criterion 2: Which representation best generalizes to higher-level mathematics?

Two types of generalizations are considered:

- Solving more than two linear equations with more than two unknowns
- Solving two nonlinear equations with two unknowns

Interestingly, each of these generalizations is best approached by a different representation. Solving more than two linear equations using matrix methods and technology is an easy matter. Using graphing technology to solve two nonlinear equations is also relatively simple. By contrast, solving more than two linear equations by graphing techniques involves graphing in three or more dimensions, and solving two nonlinear equations by using matrices involves more advanced methods. See Peressini, Sullivan, and Uhl (1988).

Criterion 3: Which representation best applies to finding approximate solutions?

Somewhat ironically, the algebraic representation that is commonly emphasized in mathematics classes is typically limited to the make-believe world of textbooks and often does not lend itself to equations whose coefficients are generated by real-world data. But equations with "unfriendly" coefficients can readily be handled using graphs or matrices. Examples abound in everyday situations. For example, consider the following problem:

> A consumer estimates that her long-distance company charges an average of 10.24 cents per minute with no monthly fee. Another long-distance company has a plan in which all long-distance calls are 7 cents per minute, with a monthly fee of $4.95. Should the consumer switch long-distance companies?

Of course, the answer depends on how many minutes of long-distance calls the consumer makes each month. To obtain the answer, solve the system of equations $y = 0.1024x$ and $y = 4.95 + 0.07x$. A graphing calculator can handle this problem easily by graphing Y1 = 0.1024X and Y2 = 4.95 + 0.07X with an appropriate viewing rectangle and the intersection feature. See **figure 10.** So if the consumer makes 153 minutes or more of long-distance calls each month, then she would be wise to switch long-distance companies.

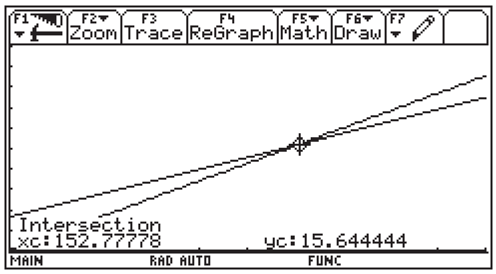

Fig. 10
TI-92 graphical representation
for solving $y = 0.1024x$ and $y = 4.95 + 0.07x$
using the trace or intersection feature

Many students never experience the concrete representations while learning mathematics

A graphing calculator's zoom feature enables the user to apply a graphic representation rather effortlessly to find approximate solutions with a high degree of accuracy. Alternatively, its matrix feature can also be used with minimal effort. ➔

Each of these methods has prerequisites. The graphics approach requires that the user represent each equation in "Y= " form and select an appropriate viewing rectangle, whereas the matrix approach requires that the user represent equations in matrix form. When using technology, new skills emerge to replace old skills, as in rewriting an equation in a certain form, selecting an appropriate viewing rectangle, or representing equations in matrix form.

Discuss: Under criterion 3, the authors mention that using technology for the graphing approach requires that students have such new skills as algebraic manipulation and the ability to estimate range and domain. What skills become less important when using graphing calculators instead of pencil-and-paper graphing?

Criterion 4: Which representation best applies to finding exact solutions?

An exact solution, such as $x = 2/7$ and $y = 5/6$, is generally best found when using algebraic or matrix representations, whereas solutions generated by tables or graphs are often approximate. Examples of systems of linear equations for which exact answers are required do not readily present themselves; however, nonlinear systems requiring exact answers occur frequently, such as in solving optimization problems in calculus.

Discuss: How do you handle approximate answers from students using graphs, as opposed to exact answers obtained by algebraic manipulations? Are either acceptable? Do you impose a level of accuracy for approximate answers?

Criterion 5: Which representation is best for a given type of technology?

Technology greatly enhances our ability to solve equations. Technology that can solve systems of linear equations has been present for many years. Symbolic manipulators that readily handle virtually all symbolic manipulation are now commonplace and can be accessed using the World Wide Web (for example, Maple is available at **www.maplesoft.com**) or by using handheld devices (for example, Derive on the TI-89).

Depending on the choice, technology can also be used as indicated in the following examples:

- If a spreadsheet or a graphing calculator with a table feature is used, then representation B (tables) is a logical choice. See **figures 5** and **6.** Some graphing calculators facilitate generating tables by both using an initial value and an increment and using a more flexible "ask" feature.
- If a graphing package on a calculator or computer is used, then representation C (graphs) is the natural choice. See **figures 7** and **10.**
- If technology having matrix capabilities is used, then representation E (matrices) is the choice. See **figure 9.**
- Finally, if a symbolic manipulator (computer-algebra system) such as Maple, Derive, or Mathematica—some of which are now available in handheld versions—is used, then representation D (algebraic) is a suitable choice. See **figure 11.**

Discuss: What are the implications for representations when computer-algebra systems that do all the algebraic manipulations come into greater use?

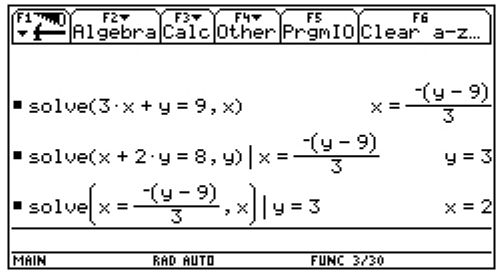

Fig. 11
TI-92 symbolic-manipulator representation for the solution to $3x + y = 9$ and $x + 2y = 8$

Criterion 6: Which representation best suits the learning style and comfort level of the student?

The choice here, of course, rests with the student. But unless students are familiar with the various representations and have had opportunities to learn to choose and create suitable representations—much like problem solving involves knowing how to apply various computational techniques and choosing the ones that apply to a given situation—their ability to make appropriate choices is limited.

Discuss: Do students with different ability levels prefer different representations, or do all ability levels benefit from the same representations? In other words, do higher-ability students prefer abstract representations over visual or concrete representations? What representations do your students prefer?

CONCLUSION

Thus, representations are significant not only in the realm of skills but in developing concepts and problem-solving ability as well. Moreover, students benefit by learning to choose the representations that apply to a given situation. This article illustrates how several important criteria may influence a choice with regard to just one of many topics in the school mathematics curriculum. The ideas from this article could also be applied to a wide range of other topics, such as representing fractions, percents, areas, and probabilities. In addition, other benefits from familiarizing students with alternative representations range from checking their work to developing a deeper appreciation for connections within mathematics.

This topic and these criteria support the notion that as part of attaining true mathematical power, students should be acquainted with multiple representations for a given situation. In summary, the authors suggest that these illustrations support the representation standard as a key feature of the NCTM's *Principles and Standards for School Mathematics* (NCTM 2000).

Discuss: How can the authors' suggestions be applied to teaching such other topics as quadratic equations, logarithms, and trigonometry?

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