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Thinking and Writing Mathematically: "Achilles and the Tortoise" as an Algebraic Word Problem

Word problems should be a focus rather than a peripheral part of algebra classes. Meaningful word problems are more effective than traditional exercises at engaging students in comprehensive and active learning. They encourage students to think mathematically and to develop reasoned problem-solving strategies rather than rely on memorized procedures. Schoen (1988) offers an argument for a word-problem–based algebra course.

But for some students, word problems continue to be a plague, threatening doom and ruining mathematics study. An episode of the popular *Foxtrot* cartoon sums up their feelings: cramming for a mathematics test, Paige says, "I hate word problems." Her math-whiz tutor replies, "Without word problems, math would be just some abstract bunch of formulas that live only within the confines of a classroom or a textbook. But in reality, math is everywhere you look! It permeates everything! You can't escape it! And that's what word problems let us in on." To which Paige responds, "And that's *not* a reason to hate them?" (Amend 1995, p. 138).

For some students, word problems continue to be a plague

Like Paige, many students have an aversion to word problems. They may believe that the problems are too hard, contrived, boring, or irrelevant. A study of middle school students found that "nearly every student with whom we worked used an approach to solving word problems that was mechanical rather than based on an attempt to understand the problem" (Bransford et al. [1996], p. 206; see also Charles and Silver [1988]; Silver [1986]; Bransford and Stein [1993]). The 1986 National Assessment of Educational Progress showed that most seventeen-year-olds could perform basic arithmetic operations, but nearly all of them failed to solve multistep word problems (Dossey et al. 1988). And reviewing a series of research studies on understanding and solving word problems, Mayer and Hegarty concluded that the problem persisted through the 1990s (see Hegarty, Mayer, and Green [1992]; Hegarty, Mayer, and Monk [1995]; Mayer [1982]; Lewis [1989]; Lewis and Mayer [1987]). Researchers Mayer and Hegarty (1996, pp. 50–51) write the following:

> Overall our program of research provides converging evidence that students often emerge from K–12 mathematics education with adequate problem execution skills—that is, the ability to accurately carry out arithmetic and algebraic procedures—but inadequate problem representation skills—that is, the ability to understand the meaning of word problems.

Both the research and my own experience as a mathematics teacher point toward two major factors in overcoming negative views of word problems and in making them a focal point of learning algebra. We need to—

- engage the students' imaginations with creative, thought-provoking problems and
- involve the students more directly in evaluating their own word–problem-solving strategies by having them think and write descriptively and critically about their mathematical thinking.

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ENGAGING STUDENTS' IMAGINATIONS

An article in the February 1997 issue of the *Mathematics Teacher* begins, "Do students yawn at problems about upstream and downstream rowboats?" (Appelbaum 1997, p. 96). Some topics for word problems—traveling by rowboat or loading sheep and goats onto a ship—have little relevance for students' lives today. Others—such as traveling by airplane or by automobile or buying butter, bread, or milk—may identify objects and activities in the real world but fail to involve the students themselves in that world. When Bransford et al. (1996) used the first twelve minutes of *Raiders of the Lost Ark* as a context for word problems, students improved dramatically in their ability to understand, solve, and explain problems, as well as to solve analogous transfer problems. The positive results continued with videotapes that the authors filmed to make the students and mathematical thinking an integral part of the adventures.

INVOLVING STUDENTS IN EVALUATION

The NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989) asks students to "reflect on and clarify their thinking" (p. 140), to "validate their own thinking" (p. 81), and to analyze and evaluate their own problem-solving strategies. This analysis encourages them to understand their own thinking processes—an important first step toward developing their metacognitive or thinking-about-thinking skills (Biehler and Snowman 1993).

The written protocol may help clarify and evaluate problem-solving techniques (see Elliot [1996]; Perlstein [1996]; Pugalee [1997]). In this strategy, students describe their thinking as they solve problems—in effect, "thinking aloud" on paper. Initially, writing the protocol helps them focus and direct the problem-solving process. It can later aid reflection and promote self-regulation of thought.

ACHILLES AND THE TORTOISE

To combine an imaginative context with a thinking and writing exercise, I used Hogben's adaptation of Zeno's famous paradox, "Achilles and the Tortoise":

> Achilles runs a race with the tortoise. He runs ten times as fast as the tortoise. The tortoise has 100 yards' start. Now, says Zeno, Achilles runs 100 yards and reaches the place where the tortoise started. Meanwhile the tortoise has gone a tenth as far as Achilles, and is therefore 10 yards ahead of Achilles. Achilles runs this 10 yards. Meanwhile the tortoise has run a tenth as far as Achilles, and is therefore 1 yard in front of him. Achilles runs this 1 yard. Meanwhile the tortoise has run a tenth of a yard, and is therefore a tenth of a yard in front of Achilles. Achilles runs this tenth of a yard. Meanwhile the tortoise goes a tenth of a tenth of a yard. He is now a hundredth of a yard in front of Achilles. When Achilles has caught up this hundredth of a yard, the tortoise is a thousandth of a yard in front. So, argued Zeno, Achilles is always getting nearer the tortoise, but can never quite catch up. (Hogben 1993, p. 11)

Hogben downplays the philosophical dimension of the paradox and makes it a word problem that can be solved at the high school level. By adding specific numbers to Zeno's space and time relationships, he supplies the tools for showing that the paradox is not "a seemingly contradictory statement that may nonetheless be true" but "an assertion that is essentially self-contradictory, although based on a valid deduction from acceptable premises" (*American Heritage Dictionary of the English Language*).

I used the paradox as an in-class activity in several courses. Students' backgrounds varied from basic to advanced algebra, and their abilities varied from average to gifted. Although my expectations also varied for the different classes, I asked the students to address the problem in two stages:

1. Keeping in mind that Zeno's tale is a paradox, do you think that Achilles ever catches the tortoise?
2. If not, why not? If so, at what point does Achilles catch the tortoise?

For a record of their thinking, students were to write descriptive and evaluative comments as they worked through the problem. Later we went over their comments as a class to look for strategies that worked and strategies that did not work.

Using Raiders of the Lost Ark _as a word-problem context improved students' performance_

THE PARADOX

The initial difficulty that students confronted was making common sense, as well as mathematical sense, triumph over an apparently airtight, logical argument. "The first barrier to confront in Zeno's tale is the language barrier," wrote one of my students. "Readers must realize that Zeno's tale is a paradox—an apparent contradiction. He attempts to convince his audience that Achilles will never catch the tortoise. Once this seed of thought is planted, it has a certain appeal."

A second student's protocol reveals similar thoughts: "I know that Achilles must pass the tortoise at some point, but it also makes sense that the tortoise can stay a short distance ahead." And a third wrote, "I know Achilles must pass the slower tortoise, but showing it is another matter."

Most students eventually decided that Zeno had tried to trick them into thinking that the tortoise would never be caught. Some students, however, remained caught between rhetoric and their common or mathematical sense. They said that they knew that Achilles would catch the tortoise, but at the same time, they believed that Zeno was correct.

LOGICAL SOLUTIONS

To help students get past the rhetoric, I divided the class into small groups to brainstorm and discuss the problem. My role at this point became primarily that of questioner and prodder to help students discover their own paths through the paradox. “Do you think that Zeno has a valid position?” “Can you see more than one way to look at this problem?” “Are you confused? What do you think is causing the confusion? Are paradoxes supposed to be confusing?” “Could we be looking at more than one answer? Is it possible to have contradictory but equally valid answers?” The comments that I overheard included the following:

One student played Achilles, and another, on hands and knees, the tortoise

Zeno is treating this like movement stops and starts, but this is a race. Achilles isn’t going to wait for the tortoise to do his thing.

Let’s work backwards. Will Achilles have passed the tortoise at 200 yards? Then how about 190 yards?

One small group acted out the paradox for the class, with one student playing Achilles, and another, on hands and knees, playing the tortoise. A second group mapped the race on butcher paper. And students continued to write their ideas and reactions, going back and forth from the cognitive to the affective interpretation, from what they thought to how they felt about it.

One overall effect of the group work was a thorough hashing out of many facets of the problem: an aspect that one student did not think of, another did. A second effect was consensus that Achilles would in fact win the race.

Providing a videotaped context for the problem has also been useful to jump start discussion. In the movie *I.Q.,* with Meg Ryan and Tim Robbins, the race between Achilles and the tortoise is presented primarily as a problem in logic.

MATHEMATICAL SOLUTIONS

When the students realized that Achilles must catch the tortoise, they were ready for the next part of the problem: determining the number of yards traveled when Achilles catches the tortoise. Students’ initial estimates, often derived by trial and error, varied from 200 to 112 yards. Most students then approached the problem using logic and the concept of series. Hogben (1993, pp. 11–14) gives a detailed description of this approach. Excerpts from students’ written protocols show varying levels of sophistication:

Achilles has to be already past the tortoise by the time he’s run 112 yards, therefore, the answer is somewhere between 111 and 112 yards.

What is actually being asked by this problem? The answer is twofold. Does Achilles ever catch the tortoise? is the first question.

The answer can be determined logically. Zeno’s language channels the reader to think that motion is a discrete rather than a continuous phenomenon. The wording leads us to think that Achilles stops after running certain distances while the tortoise covers 1/10 of what distance Achilles has just traversed. This “stop-and-go” maneuver directs the thinking of movement as discrete; remember that this is a race and that both Achilles and the tortoise are moving as fast as they can.

Since Achilles runs 10 times as fast as the turtle, logic and arithmetic suggest that Achilles has traveled 100 yards + 10 yards + 1 yard + 1/10 of a yard + 1/100 of a yard into infinity to have run: $111.111111\overline{1}$.

At this point, more advanced students are ready to discuss the concept of a series that converges to a limiting value: no matter how many 1’s are to the right of the decimal point, the value of $.111111\overline{1}$ never exceeds 1/9.

I then asked students to try an alternative solution that uses geometry, algebra, or both. Although their approaches vary, the two examples that follow demonstrate a solid grasp of concepts and in some ways even parallel the solutions of experts whose work the students had never read. For instance, the first student presents a solution that is nearly identical to that of Hogben (1993, pp. 263–64), and the second student appears to follow Pólya’s approach to problem solving: understand the problem; devise a plan; carry out the plan; examine the solution (1973, pp. xvi–xvii).

Student 1

The key equation to work with is:

1) distance = rate × time

From this equation, the process unfolds:

2) The distance Achilles travels is: $D = RT$
3) The distance the Tortoise travels is: $d = rt$
4) Achilles’ speed can be quantified: $10s = S$
5) Achilles covers the distance: $100 + d = D$
6) The distance the tortoise travels before he is caught is divided by the time he runs in order to obtain his speed.

$$s = d/t$$

7) The same logic applies to Achilles to yield:

$$S = D/T$$

8) Combining #4 and #6 we obtain: $S = 10(d/t)$
9) Combining #5 and #7 we obtain: $S = (100 + d)/t$
10) Combining #8 and #9 we obtain:

$$10d/t = (100 + d)/t$$

11) Solving for d gives us:

$$9d = 100$$
$$d = 11\ 1/9 \text{ yards.}$$

Student 2

To begin with, what does Zeno communicate to us that can be used to solve the problem of where Achilles catches the Tortoise? He tells us two important details: the fact that Achilles runs 10 times as fast as the Tortoise; and that the Tortoise has a 100 yard head start. These two facts can be put into an algebraic equation in the following manner:

$$A: 10T - 100 = T$$

What this equation tells us is that first of all, what is true for Achilles is put in terms of the Tortoise. That is, Achilles runs 10 times as fast as the Tortoise, or $10T$. Achilles also was 100 yards in the hole at the beginning of the race, hence –100. These two facts can be put into the equation with reference to the Tortoise to give us one unknown to solve for, T.

The solution to this equation is, of course, 11 1/9, which is the total distance the Tortoise traveled before being caught by Achilles.

Because the second student did not label variables, the work needs some correction. Although T is usually used for time, this student uses T for the distance traveled by the tortoise at any time. The explanation would be clearer if the student had used D for distance. If D stands for the distance in yards traveled by the tortoise at a given time in the race, then after Achilles has started, his distance in yards is $10D - 100$. If the tortoise and Achilles meet, they are at the same place at the same time, so the equation $D = 10D - 100$ is true and its solution describes the meeting point.

Presented as an algebraic word problem, the tale of Achilles and the tortoise engaged my students' imaginations. They discussed the story's context in class, argued about their answers, and continued to think about the problem outside of class—all indications of active involvement in the problem-solving process. The tale also required them to confront directly an important difficulty that many students have in solving word problems: the tendency to apply operations and execute procedures without regard to real-world standards or experiences (Bransford et al. 1996, p. 207). Real-world observations told them that Achilles had to catch the tortoise at some point. Finding a carryover of real-world sense into the mathematical world of word problems helped convince them that solving word problems can be both possible and meaningful.

Approaching this problem as a thinking and writing exercise also increased students' involvement. The students believed that writing about their own mathematical thinking helped them organize their thoughts, identify dead ends and productive paths, and learn something about themselves. One student remarked, "Doing this taught me more than I ever thought possible. I didn't think that I could enjoy solving a math problem." Another wrote, "I knew that I could stick to a task and not be a quitter. Now, I have my protocol to support this self-evaluation." All students were more confident that, with practice, they would be able to accomplish both tasks—solving word problems and describing what they are doing to solve word problems—more effectively.

THINKING ABOUT THINKING

Moving beyond the *what, when,* and *how* of the written protocols to the *why* and *why not* of metacognition can begin as individual or collaborative learning experiences. I use open-ended questions to encourage students to rethink and reflect on their problem solving.

- Why did you do this?
- Why did you not do that?
- What were you thinking when you . . . ?
- What would have happened if you had . . . ?
- Is this way better (or more effective, efficient, and so forth) than that?
- Would you do anything differently?

I also ask students to pose and answer their own questions. The result may fall short of formal metacognitive analysis, but students are thinking about thinking in a productive, systematic way.

CONCLUSION

Although the Achilles-and-the-tortoise activity has been effective with students at the high school and college levels, the specific context for the student examples presented here was a modeling exercise in a secondary-level mathematics-methods course. Student-teachers were learning how to teach solving word problems by role-playing class sessions. Several of them later used the activity in their high school classes. Generally, the college students and the more advanced high school students achieved better solutions than less advanced students, but all students responded positively to the context and method.

Students with a background in calculus enjoy discussing this problem in terms of an infinite series. They look at the race between Achilles and the tortoise by using two relevant ideas: the limit of a sequence and the sum of a series. Given that not all infinite series converge, high school students and freshmen and sophomores in college, as well as secondary mathematics-methods students, have had lively discussions about whether or not a convergent sequence does exist in this problem.

The tale of Achilles and the tortoise engaged students' imaginations

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