# GEOMETRIC MEANING IN THE GEOMETRIC MEAN MEANS MORE MEANINGFUL MATHEMATICS

Teachers can put geometric meaning back into the geometric mean. A first- or second-year algebra student is typically taught the geometric mean in one of two ways: using the ratio-and-proportion form, where, given the proportion

$$\frac{a}{x} = \frac{x}{b},$$

$x$ is the geometric mean of $a$ and $b$; or the geometric-series form with a common ratio, that is, $a$, $ar$, $ar^2$, . . . , $ar^n$, where, given a series of numbers, the geometric mean is any missing element. Neither of these approaches uses mathematics in a manner that is evident as geometric to the student. In the geometry classroom, three geometric means are created when the altitude to the hypotenuse is drawn, thereby creating three similar triangles. Unfortunately, except for the honors student, few understand and yet fewer retain the chain of significant concepts that relate these similar triangles to the mathematical means formed by their corresponding sides.

***Our role is to help students see the implications of their results***

I have created an investigation-based, lecture-free first-year algebra course in which students are responsible for compiling their own textbooks. The core of the course consists of seven chapters, each chapter formed by fourteen laboratory-format investigations. In each investigation, students follow a series of procedures, after which they must determine the purpose of the investigation, as well as draw various conclusions. A daily homework assignment concludes each lesson. With minor alterations, sheets 1–5 in this article constituted lab 11 from chapter 2, as done by three first-year algebra classes in fall 1999.

In **sheet 1,** the purpose is not given. For each of the fourteen labs in chapter 1, the purpose is stated and the students and I discuss the role that the purpose plays. Thereafter, the student determines the purpose after completing the investigation.

Procedure 1 requires that students use their knowledge of square roots to find the lengths of the sides of the two squares and then the area of the rectangle. Many of my students still confuse area with perimeter. During this task, as I roam the classroom and see that particular wrong answer, I ask them, “Now, exactly what is the difference between area and perimeter?” This type of prompt usually elicits a self-correction with no further guidance from me.

Another common difficulty that my students face is multiplying and simplifying irrational numbers. This exercise both illuminates the need for, and gives practice in, such skills. The fraction, decimal, and variable examples challenge different students, depending on their individual strengths and weaknesses. The fraction and decimal examples bother a few students because the numerical value of the side of a square may be larger than the value of its area, an outcome that, of course, occurs whenever the area is less than one. Most students are satisfied when I ask them to think about what happens when they square such a number as 1/2 or 0.7, but this issue is important enough that two homework problems, problems 5 and 6 on **sheet 3,** are devoted to it.

Procedure 1A directs students to observe the values that they obtain in procedure 1, in particular, a property of these values, that is, equivalent ratios; review four possible methods to verify this property; then practice one or all of these methods. Finally, a name, the *geometric mean,* is given to the value that the students obtained. My first-year algebra students are always keen to “get to the answer” and consider me a good teacher when I facilitate this immediate goal. As teachers, however, our role is to help students see the implications of their results.

*“Sharing Teaching Ideas” offers practical tips on teaching topics related to the secondary school mathematics curriculum. We hope to include classroom-tested approaches that offer new slants on familiar subjects for the beginning and the experienced teacher. Of particular interest are alternative forms of classroom assessment. See the masthead page for details on submitting manuscripts for review.*

In this light, procedure 1A is the core of the lesson, and I usually stop the class for a few moments to discuss the implications contained here.

Procedure 2 emphasizes the importance of the fourth verification technique, cross multiplication, as a method to arrive at the geometric mean. Most students like this part of the lab the best because it is the easiest for them, although as a mathematics teacher, I am somewhat dismayed that the students rarely have any sense of the ratios or proportion involved.

Procedure 3 introduces students to the notion of a geometric sequence. Students like the hint that is given but have repeatedly told me that it belongs before the problems, not after. Also, most students in first-year algebra have difficulty dividing fractions, often more difficulty than they have dividing algebraic monomials, even though the rules are the same.

The structure of the practice accorded by procedure 4 creates a dual purpose. As well as practicing a method of solution, the student can compare the difficulty or appropriateness of a particular approach, depending on the circumstance. My experience has indicated that a continuous supply of problems that emphasize alternative approaches enhances critical thinking.

The first five problems in the homework furnish opportunities for review and discussion. The last problem is quite different because it asks students to think about the limitations of using a geometric model for the geometric mean.

Discovery and mastery are important features of a successful mathematics curriculum. This lesson facilitates both these goals while also doing the following:

- It portrays a very basic and real feel that the relationships involved are geometric. What could be more geometric than the areas of squares or rectangles?
- It reviews formulas for the areas of squares and rectangles.
- It furnishes a basis for modeling multiplication, especially variable multiplication, using the areas of rectangles.
- It provides a basis for factoring, both numerical and variable.
- It reviews the relationship that we call *square root* (area → side).
- It facilitates the visualization of simplifying roots, that is, the areas of nonsquare rectangles; multiplying radicals; the lengths of the sides of nonperfect squares; and is an excellent introduction to noninteger, and particularly irrational, factorization.
- It allows teachers to use the geometric mean to teach ratio and proportion, instead of the other way around. This method visually validates an often difficult application of fractions.
- It can easily be used with students in third to fifth grades, for whom it is especially important that we not separate arithmetic, geometry, and algebra.

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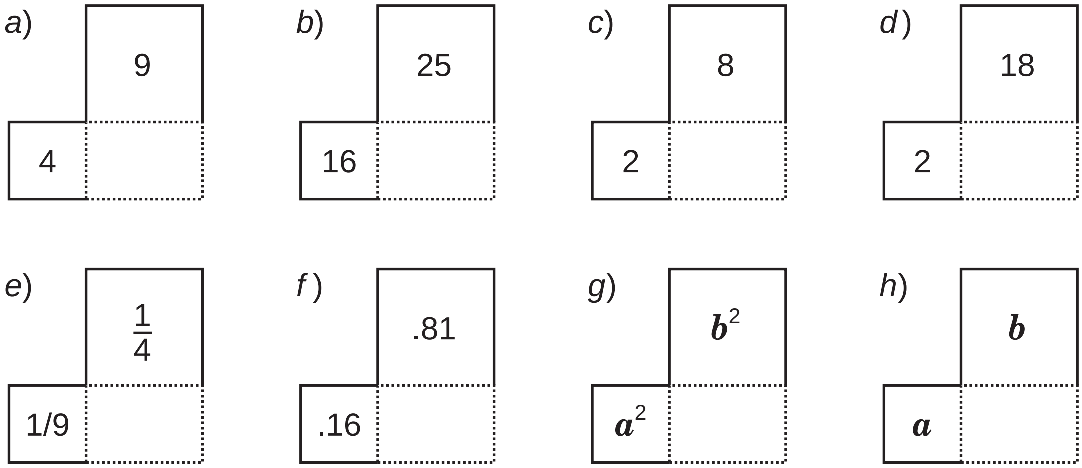
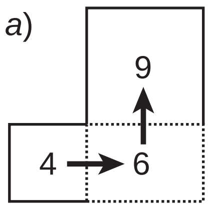
Purpose:

*Procedure 1: Areas of insight*

Determine the area of each dotted rectangle that is bounded on two sides by squares. The areas of the squares are given. The figures are not drawn to scale.




*Procedure 1A: Look into these areas*




Observe the relationship between 4 and 6, that is, 4 : 6 or 4/6, and between 6 and 9, that is, 6 : 9 or 6/9. The ratios—or fractions—are the same. You should be able to use the following four methods to demonstrate that fractions are equal:

| | | | | |
|---|---|---|---|---|
| (i) | Reduce | 4/6 = 2/3 | 6/9 = 2/3 | |
| (ii) | Common denominator | 4/6 = 12/18 | 6/9 = 12/18 | |
| (iii) | Decimal equivalent | 4/6 = $.\overline{6}$ | 6/9 = $.\overline{6}$ | |
| (iv) | Cross multiplication | $\frac{4}{6} \times \frac{6}{9}$ | 4 • 9 = 6 • 6 | 36 = 36 |

Use any of these four techniques to show that the ratios in each figure *b* through *h* are also equivalent.

The values obtained for the areas of these rectangles are called the *geometric means* of the values of the areas of the squares.

*Procedure 2*

The fourth technique, cross multiplication, for verifying the equivalence of ratios can also be used to find the geometric mean, if it is not already known. Let the geometric mean, that is, the area of the rectangle, be called $x$.

$$\frac{4}{x} \quad \frac{x}{9} \qquad 4 \cdot 9 = x \cdot x \qquad 36 = x^2 \qquad 6 = x$$

Find the geometric mean by cross multiplying.

*a*) $\frac{4}{x} = \frac{x}{9}$. $x =$ ____. *b*) $\frac{16}{x} = \frac{x}{25}$. $x =$ ____. *c*) $\frac{2}{x} = \frac{x}{8}$. $x =$ ____. *d*) $\frac{2}{x} = \frac{x}{18}$. $x =$ ____.

*e*) $\frac{\frac{1}{9}}{x} = \frac{x}{\frac{1}{4}}$. $x =$ ____. *f*) $\frac{16}{x} = \frac{x}{.81}$. $x =$ ____. *g*) $\frac{a^2}{x} = \frac{x}{b^2}$. $x =$ ____. *h*) $\frac{a}{x} = \frac{x}{b}$. $x =$ ____.

*i*) $\frac{2}{x} = \frac{x}{4}$. $x =$ ____. *j*) $\frac{3}{x} = \frac{x}{6}$. $x =$ ____. *k*) $\frac{2}{x} = \frac{x}{6}$. $x =$ ____. *l*) $\frac{5}{x} = \frac{x}{10}$. $x =$ ____.

$x$ is called the ______________________________.

*Procedure 3*

If a sequence of numbers is generated by starting with an initial number and repeatedly multiplying by a common ratio to create new elements, then the sequence is called a *geometric sequence*. Determine the common ratio and the missing element, which is also the geometric mean of the numbers on each side of it.

| Sequence | Ratio |
|---|---|
| *a*) 2, (4), 8, 16, 32 . . . | (2) |
| *b*) 2, ___, 18, 54, 162 . . . | __________ |
| *c*) 2, ___, 4, $4\sqrt{2}$, 8 . . . | __________ |
| *d*) 8, ___, 2, 1, 1/2 . . . | __________ |
| *e*) 1, ___, 1/9, 1/27, 1/81 . . . | __________ |
| *f*) 9, ___, 4, 8/3, 16/9 . . . | __________ |
| *g*) 4, ___, 9, 27/2, 81/4 . . . | __________ |
| *h*) 1/4, ___, 1/9, 2/27, 4/81 . . . | __________ |
| *i*) $a^2$, ___, $b^2$, $b^3/a$, $b^4/a^2$ . . . | __________ |

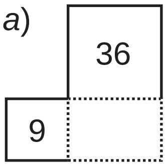
If determining the ratio is difficult for you, divide any element by the one that comes before it. For example, in the first sequence, 16/8 = 2, so the ratio is 2.

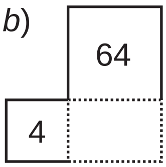
Again, these sequences are called *geometric sequences* and the missing elements are called *geometric means*.

*Procedure 4: Two many*

For each of the figures, use two methods to determine the area of the dotted rectangle. First, find the length of each side and multiply, then set up the proportion and solve for $x$ with cross multiplication.

| | Find the Length of Each Side, and Multiply | Solve for $x$ with Cross Multiplication |
|---|---|---|

*a*)



*b*)
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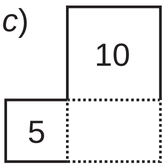
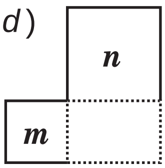
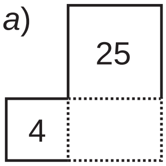
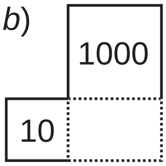
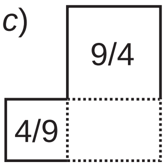
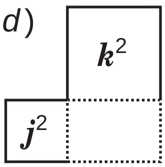
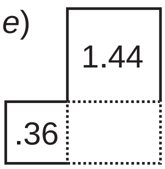
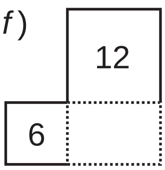
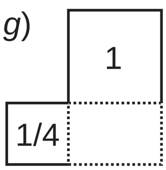
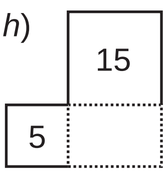



*d* )



*Analysis and conclusion*

1. Determine the geometric mean in two ways, first by setting up the proportion and solving, second by finding the length of each side of the rectangle and multiplying.

| | Set Up the Proportion, and Solve | Find the Length of Each Side of the Rectangle, and Multiply |
|---|---|---|
| a)   | ________ | ________ |
| b)   | ________ | ________ |
| c)   | ________ | ________ |
| d)   | ________ | ________ |
| e)   | ________ | ________ |
| f)   | ________ | ________ |
| g)   | ________ | ________ |
| h)   | ________ | ________ |

From the *Mathematics Teacher,* March 2001

2. Determine the missing geometric means, and give the common ratio:

   *a*) 81, 27, ____, 3

   *b*) 1/2, 1/4, ____, 1/16

   *c*) 3.6, ____, 0.9, 0.45

   *d*) $a^2b^2$, ____, $a^2$, $a^2/b$

   *e*) $x/y$, $x$, ____, $xy^2$

   *f*) 1, ____, 9/16, 27/64

3. Look for the pattern, and determine the geometric mean:

   *a*) 5, ____, 10, $10\sqrt{2}$, 20

   *b*) 6, $6\sqrt{3}$, ____, $18\sqrt{3}$, 54

4. When asked which procedure in the investigation required them to make the most mathematical connections, a group of students debated between procedure 1A and procedure 3. What might have been the major points for each side of the argument?

5. Graph the three functions $f(x) = \sqrt{x}$, $f(x) = x$, and $f(x) = x^2$ on the same axis. Choose values of $x$ that allow you to clearly distinguish the behavior of the three functions on the domain [0,1].

6. Draw two large congruent squares on graph paper, label one 1 m $\times$ 1 m, and label the other 100 cm $\times$ 100 cm. Divide each square into quarters, find the area of each quarter, and discuss how a 50 cm side results in a 2500 $cm^2$ area, a seeming numerical increase, whereas a 1/2 m side results in a 1/4 $m^2$ area, a seeming numerical decrease.

7. Observe the geometric sequence (2, –4, 8, –16, . . . ), which contains a common ratio that is negative. What does this example say about our geometric model? Look up the definition of a principal square root, and discuss its relevance to our geometric model of the geometric mean.