

ALGEBRA for All

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Making Sense of Slope

Every natural phenomenon, from the quantum vibrations of subatomic particles to the universe itself, is a manifestation of change. . . . It is of the greatest importance that we should understand and be able to control the changing world in which we live. To do this effectively we must become sensitive to the *patterns* of change.

—IAN STEWART (1990, 183)

First-year algebra has shifted from a course that focused primarily on formal procedures for simplifying symbolic expressions and solving equations to a course that emphasizes applications in which students encounter a wide variety of situations structured by patterns. By observing tables and graphs, they learn to describe and extend patterns, create equations with variables to represent patterns, and make predictions on the basis of these patterns. The study of variability and change is an essential algebraic concept that helps students make sense of our world through mathematics. Hence, understanding slope as a rate of change for variability has become an important concept for all students.

Traditional algebra textbooks introduce the slope of a line as a measure defined by the ratio of vertical change to horizontal change, or “rise over run.” In the traditional course, students practice calculating this ratio by subtracting y -values and x -values for two ordered pairs. When students are asked, What is the slope of a line? a typical response is, Isn’t that when you use the formula $m = (y_2 - y_1)/(x_2 - x_1)$? Although these students can calculate the slope of a line, they often do not understand the concept of slope as a rate of change. This understanding involves much more than merely memorizing and applying a formula.

We have found that students’ first encounters with slope can be presented within real-world applications to give meaning to the concept. Learning is an individual process in which each student builds meaning through interactions with concrete and abstract situations. The role of the teacher is to present mathematical tasks that develop understanding, promote communication about mathematics, and enhance mathematical connections (NCTM 1991). Many students are not abstract learners and benefit from multiple instructional

strategies to develop algebraic concepts. Therefore, we present and discuss examples that can be used to develop the concept of slope as a rate of change through three modes of learning: visualization, verbalization, and symbolization.

INTRODUCING SLOPE AS A RATE OF CHANGE

Imagine proposing the following situation to your algebra class.

Your uncle offers you a summer job managing a produce stand. You need to decide how to price the produce so that you can compete with current grocery store prices.

Students can bring to class from a grocery store various produce-item prices, calculated by the ounce, by the pound, or by the item. For example, a student might find that watermelon is priced at \$0.39 per pound. In cooperative groups, each student can make a table of data for the food item, graph the ordered pairs on graph paper, write a description of the data, and then write an equation for the data. See **figure 1**. Demonstrate to the students how the horizontal increase in the amount sold affects the vertical cost increase. This ratio, cost increase per unit increase, can be described as a rate of change, that is, price per unit. Students can each explain their own graphs to their group. They

Understanding slope as a rate of change requires more than applying a formula

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can pose three questions that can be answered from their graphs. You may want to have a member of each group share a graph with the class.

With this initial understanding of the slope as a rate of change, students can compare the graphs and slopes for two rates. Suggest to your students that they may want to investigate other summer jobs before deciding to run the produce stand. For example, they could work at the local hamburger chain for \$5.25 per hour or work on a construction job for \$8.50 per hour. Students can make a table, graph sample data, and write equations for each rate. See **figure 2**. From the graphs, they can investigate the slope of each line and compare this slope with the equation of the line. They can also observe what happens to the graph of a line as the slope increases.

To continue to build the concept, give your students several other rate equations to graph on a graphing calculator and trace coordinates or obtain ordered pairs from a table generator. Students can explain what the ordered pairs represent in terms of the real-world application and describe the slope of each line as a rate of change. In so doing, the students think about slope in a reversed order, from the symbolic equation to the ordered pairs for the slope.

ADDING A y-INTERCEPT

To introduce visually the concept of y-intercept and allow students to discover the slope-intercept equation, $y = mx + b$, ask students to graph a family of equations on a graphing calculator, for example,

$$y = 2x + 5,$$

$$y = 2x + 10,$$

$$y = 2x - 5.$$

Ask students to explain how the graphs are similar and how they differ. Which value in the equation represents the graph's intersection with the y-axis? Which value represents the slope? Students can then extend the pattern that they have observed by writing an equation for a line with slope 5 and y-intercept -2.

Next, illustrate the concepts with an application by suggesting the following scenario to your students:

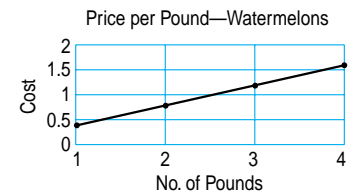
A band is considering playing for the homecoming dance and charging \$100 plus \$2.50 per person.

Students can make a table for revenue for 50, 100, 150, and 200 students; draw a graph of the data; and then write an equation for the income. By beginning with an application and developing concrete data first, students can easily write the abstract equation, $y = 2.50x + 100$. They can then enter the equation in their graphing calculators and use the table generator to investigate further the slope of the line. By setting the table increment

Watermelons are priced at \$0.39 a pound.

A. Complete the chart to investigate the cost.

No. of Pounds	Cost in Dollars
1	0.39
2	0.78
3	1.17
4	1.56



B. Describe the pattern, and write an equation.

C. Graph the equation.

D. Show the slope of the line, and describe the slope as a rate of change.

Fig. 1
Slope as a rate of change

Earnings at
\$5.25 per Hour and \$8.50 per Hour

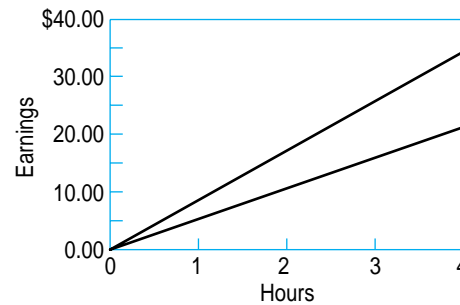


Fig. 2
Comparing rates for $y = \$5.25x$ and $y = \$8.50x$

Students can compare what happens to the graph of a line as the slope increases

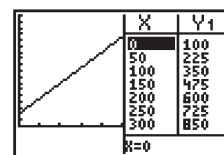
to 50, students observe the constant increase of \$125 in the income as the number of students increases by 50. By changing the increment to 100, the rate of change is represented as \$250 per 100 students, whereas if the increment is set to 1, the rate of change is reduced to \$2.50 to 1. See **figure 3**.

The application can be extended by asking students to write several other methods that the band could use for its charges; for example, the band

TABLE SETUP	
TblStart=0	
ΔTbl=50	
Indent: AUTO	Ask
Depend: AUTO	Ask

(a)

Table generator for investigating rate of change



(b)

Graph of $y = 2.50x + 100$ with the table on a split screen; x-axis and y-axis are scaled by 50 with origin at (0, 0).

Fig. 3

A graph is visual, whereas an equation is symbolic

could charge \$4 per person or it could rent a country club and charge an hourly rate. Students can graph their equations and write a discussion of the method that would produce the maximum income for the number of students that they predict would attend the dance from their school.

SOURCES FOR APPLICATIONS OF SLOPE

The Internet, almanacs, newspapers, environmental brochures, and other media are excellent sources for applications for teaching the concept of slope. For example, the following problem is based on information from a newspaper:

Between 1980 and 1990, the number of cable television subscribers increased by 3.6 million per year. If 17.5 million people subscribed in 1980, how many subscribed in 1990 (North Carolina Department of Public Instruction 1997)?

The students can use 0 for 1980 on the x -axis. Ask students what the 17.5 million and the 3.6 million represent on the graph. Students can graph the y -intercept, use the slope to determine the line, and write an equation for the line using the slope-intercept equation. See **figure 4**. The graph furnishes a visual representation for the application, whereas the equation gives the symbolic generalization. Students can determine the number of subscribers in 1990 by using either the graph or the equation. They can discuss whether the rate of growth in the number of cable subscribers from 1990 to 1998 has continued to be 3.6 million per year. Students can look for data to determine a current growth rate to use for further predictions.

A local business supplied data for the following example.

The River Run Sport Shop can purchase a canoe for \$735, then charge \$25 per day for canoe rental. After how many days' rental will the store recoup the cost of the canoe?

Students can write the equation for the application, graph the equation on a graphing calculator, and investigate the graph and table. Ask them to interpret the slope and the y -intercept. What is the meaning of the x -intercept? Encourage your students to look for additional applications that they can share with the class.

COLLECTED DATA AND LINE OF BEST FIT

Not all linear applications provide "clean" data with points that lie on a line. Collected data often determine a scatterplot in which line of best fit must be calculated. Such sources as the *Statistical Abstract of the United States* (U.S. Department of Commerce 1995) and the *World Almanac and Book of Facts* (Famighetti 1998) can supply data that, when graphed, appear linear in nature. Students can estimate a line that will predict y -values with the least amount of error. They can create this line with a paper-and-pencil graph and calculate the slope-intercept equation, or they can use a graphing calculator with linear regression.

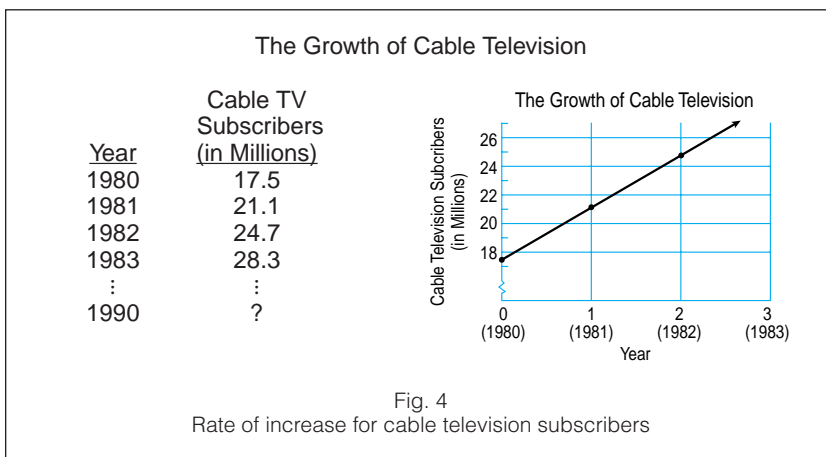
A recent current event inspired the following application:

On 15 October 1997 a British racing team shattered the land speed record by more than 100 MPH with a new speed of 763 MPH (Famighetti 1998). This record event even broke the sound barrier.

A scatterplot for land-speed records, shown in **figure 5**, can be graphed on graph paper and an estimated line determined. From the graph paper, students can estimate a slope and y -intercept for the line. They can interpret the slope, that is, the average increase in the maximum speed of an automobile per year, and the y -intercept. Students can write questions that can be answered from the equation that they have written. Students can then enter the data into a graphing calculator, find the equation of the linear model by using linear regression, and compare the regression equation with their paper-generated version. They can graph their own equation on the same coordinate axes on the calculator and discuss the differences visually. They can discuss how the calculator generates this line to minimize the error in predicted y -values.

Students can investigate reasons for the gaps in the data from 1947 to 1963 and from 1983 to 1997 and the difference between the performance in 1963 and that in 1964. Students can also research and present to the class the effects of aerodynamics, propulsion, and friction on the land speed.

Another source of data, of course, is the Internet. For example, data on housing prices can be obtained from real estate Web sites. Students can use a general Web search with a city name and the



Speed Record	
Year	(MPH)
1906	127.65
1910	131.724
1911	141.732
1919	149.875
1920	155.046
1926	170.624
1927	203.79
1928	207.552
1929	231.446
1931	246.086
1932	253.96
1933	272.109
1935	301.13
1937	311.42
1938	357.5
1939	368.9
1947	394.2
1963	407.45
1964	536.71
1965	600.601
1970	622.407
1983	633.46

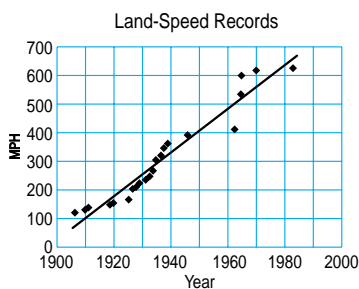
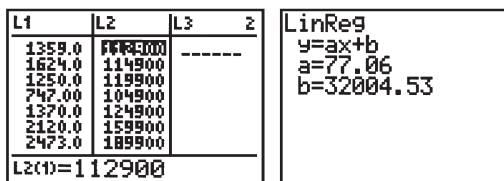


Fig. 5
Line of best fit for land-speed records

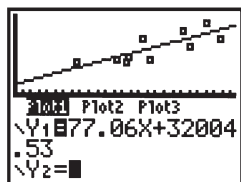
RATES OF CHANGE THAT ARE NOT CONSTANT

To fully understand a concept, both examples and nonexamples are important. A percent increase or decrease illustrates a rate of change that is not constant. Students can investigate the price increase for a \$30 pair of jeans if prices are increasing at a rate of 4 percent per year. They can make a table and use a calculator to find the amount of the increase or apply the percent-increase formula. Ask students to describe how this table differs from other functions that they have investigated. Students can then graph the data and investigate the slope visually as rise over run. See **figure 7**. Questions that you can add include the following: How does this graph differ from graphs that have a constant rate of increase? Describe how the slope of this graph differs from applications that have been studied previously.

words *real estate*, for example, “Raleigh, North Carolina” and “real estate” or a specific Web address. The Web site www.realtor.com allows students to select a state, city, and price range. They can look up data for the cost and the square footage of houses, graph the data, find the line of best fit, and interpret the slope. You can ask students the causes of variation in the data, ways in which they might choose a better sample, and the reasons for outliers in the data. See **figure 6**.



(a) Data for square footage and housing costs
(b) Linear-regression calculation on the TI-83



(c) Scatterplot and line of best fit viewed on a split screen

Fig. 6

No. of Years	Cost of Jeans
0	30.00
5	36.50
10	44.41
15	54.03
20	65.73
25	79.98
30	97.30
35	118.38
40	144.03
50	213.20
60	315.59

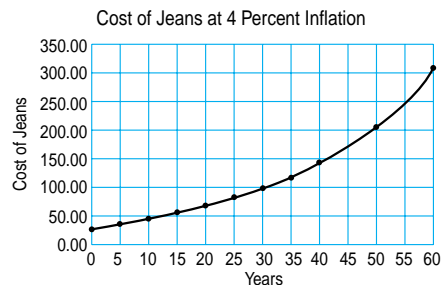


Fig. 7
Students investigate a slope for a nonlinear function.

ASSESSING THE UNDERSTANDING OF SLOPE

From our own experiences teaching algebra, we have observed that many students can calculate slopes and write equations for lines without understanding the concept of slope. To know whether your students truly understand the concept, assessing each student’s reasoning about slope is essential. Assessment can be done in a variety of ways, through student oral or written explanations. We have found a “thinking critically” multiple-choice writing response to be an effective assessment task. A practical application is given in the question

stem, which is followed by a set of four statements to analyze as either true or false. See **figure 8**. The student must write a rationale explaining why each statement is either true or false. This question can be graded using a total of four points, one point for each correct rationale. Students receive the point only if the *rationale* is correct, not for whether the answer chosen is correct. The thinking-critically assessment can be used at the beginning or end of a class, on a quiz, or as a question on a test. The rationales can easily be read and scored, and they supply rich information about each individual's understanding of slope.

Students receive the point only if the rationale is correct, not for whether the answer chosen is correct

Thinking Critically . . .

For the following problem, one choice is correct. Write your reasoning for accepting or rejecting each statement. Your reasoning for rejecting choices is just as important as your reasoning for choosing the correct answer.

Printing Costs for School Yearbooks

The data below represent the printing costs for school yearbooks, where the cost is dependent on the number of books printed. Graph the data.

No.	Cost
100	\$ 5 000
200	\$ 6 300
300	\$ 7 600
400	\$ 8 900
500	\$10 200

- According to the data, the cost to print 0 yearbooks is \$0.
- If 1000 yearbooks are printed, the total cost for printing will be \$20 400.
- According to the graph, the minimum charge is \$3 700, with a cost per book of \$13.
- The equation for the total cost is $c = 13n + 5000$, where n is the number of books and c is the total cost.

Fig. 8
Thinking-critically assessment

CONCLUSION

The development of slope as a rate of change contributes to an application-based curriculum that builds understanding of the concepts and insight into the importance of mathematics. By using multiple representations—words, tables, graphs, and equations—students experience the patterns of change that lead to connections and sense-making. The process of assessing how well students have internalized the concept of slope involves students'

communicating their understanding orally or in writing. By building a foundation in all students that mathematics is a study of patterns and of change, teachers will better prepare the next generation to understand and predict a changing world.

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DISCUSSION QUESTIONS FOR "MAKING SENSE OF SLOPE"

- How does the treatment of slope in this article differ from the way that you learned about it in school? What are the advantages of the methods proposed in this article? Can you think of any disadvantages?
- Some students just want to memorize a formula without understanding how or why it works. What strategies can you use to discourage students from blind memorization?
- What good ways can you use to show the connection among the patterns in a table, the graph, and the equation?
- At what point do you think that the slope formula would be helpful for students?
- Look at your mathematics curriculum. If you present slope as it is presented here, what other changes in the curriculum might be appropriate?
- Write a sample quiz for students who have learned slope in this way. What would students need to do well to succeed on the quiz? (A)