Masha Albrecht

The Volume of a Pyramid: Low-Tech and High-Tech Approaches

This lesson came about spontaneously during a geometry unit on volume. I had used the lesson shown here in **activity sheet 1,** in which students use cubic blocks to rediscover the formulas for volumes of right prisms, that is, $V = Bh$ and $V = lwh$. This lesson was a simple review for my tenth-grade class, and they completed it easily before the end of the period. With the wooden cubes still on their desks, most of them used the remaining time to build towers and other objects. I noticed that many students piled the cubes into bumpy pyramidal shapes. Because the next day's lesson involved studying the volume of pyramids, I wondered whether these bumpy shapes could be useful for discovering the volume of a real pyramid with smooth sides. Students could compare the volumes of these "pyramids of cubes" with the volumes of corresponding right prisms and perhaps discover the ratio 1/3 to obtain the formula for the volume of a pyramid, $V = (1/3)Bh$. As it turns out, the ratio of 1/3 does not become evident right away. To my students' delight, we found that using a spreadsheet is an excellent way to investigate this problem. My geometry classes had not used spreadsheets before, and the students enjoyed the experience of using the efficiency of technology to compare hundreds—and even thousands—of shapes with ease.

This lesson came about spontaneously

Prerequisites: Students with only very basic mathematical knowledge can benefit from this lesson. Students should have some skill at describing a pattern with an algebraic equation and some familiarity with a spreadsheet. However, I used this lesson with students who had no previous spreadsheet experience.

Grade levels: Although I originally used this lesson with a regular tenth-grade geometry class, the lesson is appropriate for students at different levels and with different abilities. A prealgebra class could do the low-tech part of the lesson, in which students find patterns by using blocks, but they would need help with the formulas for the spreadsheet. Eleventh-grade or twelfth-grade students with more advanced algebra skills could be left on their own to find the spreadsheet formulas and could be given the difficult challenge of finding the closed formula for the volume in the "pyramid of cubes" column on **activity sheet 2.** A calculus class could find the limit of the ratio column as n goes to infinity before they check this limit on the spreadsheet.

Materials: The entire lesson works well in a two-hour block or in two successive fifty-minute lessons, with the low-tech lesson in the first hour and the high-tech spreadsheet lesson in the second. Cubic blocks are needed for the low-tech lesson. Because approximately forty blocks are needed for each group of four students, large classes will need many blocks. If you do not have enough blocks, groups can share. Simple wooden blocks work best; plastic linking cubes do not work as well, because their extruding joints can get in the way when students build the pyramids.

Spreadsheet software is needed for the high-tech lesson. If you are using a separate computer lab, sign out the lab for the second hour of this activity.

For the low-tech extension lesson, the following additional materials are needed: a hollow pyramid and prism with congruent bases and heights, as well as water, sand, rice, or small pasta.

Masha Albrecht, mashaa@wenet.net, teaches at the Galileo Academy of Science and Technology, her neighborhood school in San Francisco. She is interested in finding appropriate uses of technology for mathematics learning.

*Edited by **A. Darien Lauten,** dlauten@nh.ultranet.com, Rivier College, Nashua, NH 03061*

This section is designed to provide in reproducible formats mathematics activities appropriate for students in grades 7–12. This material may be reproduced by classroom teachers for use in their own classes. Readers who have developed successful classroom activities are encouraged to submit manuscripts, in a format similar to the "Activities" already published, to the senior journal editor for review. Of particular interest are activities focusing on the Council's curriculum standards, its expanded concept of basic skills, problem solving and applications, and the uses of calculators and computers.

Write to NCTM, attention: infocentral, or send e-mail to infocentral@nctm.org, for the catalog of educational materials, which lists compilations of "Activities" in bound form.—Ed.

TEACHING SUGGESTIONS

Sheet 1: Using cubic blocks—volume of prisms

This activity sheet is elementary, and more advanced students can skip it. Have students work in groups, with one set of blocks per group. Often one student quickly sees the answers without needing manipulatives, but the other group members are too shy to admit that they need to build the shapes. Require that each group build most of the solids, even if students protest that this activity seems easy.

Sheet 2: Using cubic blocks—volume of pyramids

Students may initially have difficulty understanding what the "pyramids of cubes" look like. Make sure that they build the one with side length 3 correctly. After using the blocks to build a few of the shapes, students recognize the patterns and start filling in the table without using the blocks. Calculating decimal answers for the last column of ratios instead of leaving answers in fraction form helps students look for patterns. Have a whole-class discussion about questions 4, 5, and 6 after students have had a chance to answer these questions in smaller groups, but do not reveal the answers to these questions. Students discover the answers when they continue the table on the spreadsheet.

The last row of the table, where students generalize the results for side length *n,* is optional. On the spreadsheet, students do not need the difficult closed formula for the second column. They can instead use the recursive formula, which is easier and more intuitive. The solutions include more explanation.

Group members may be too shy to admit that they need to build the shapes

Sheet 3: Using a spreadsheet—volume of pyramids

This activity sheet is designed for students who have some spreadsheet knowledge. Having one pair of students work at each computer is useful if at least one student in each pair knows how to use computers and spreadsheets. For students who have no experience with spreadsheets, you can use this activity sheet as the basis for a whole-class discussion while demonstrating the process on an overhead-projection device. Do not bother photocopying **activity sheet 3** for students who are familiar with spreadsheets. Instead ask them to continue the table from **activity sheet 2,** and give them verbal directions as needed.

SOLUTIONS

Sheet 1, part 1

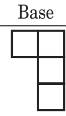
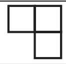
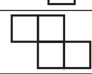
1.

Length	Width	Height	Volume
2 units	2 units	4 units	**16 cubic units**
1 unit	2 units	3 units	**6 cubic units**
2 units	2 units	**2 units**	8 cubic units
0.5 units	2 units	2 units	**2 cubic units**

2. $V = lwh$

Sheet 1, part 2

1.

Base	Area of the base	Height	Volume
	**4 square units**	2 units	**8 cubic units**
	**3 square units**	3 units	**9 cubic units**
	**4 square units**	**3 units**	12 cubic units
	**1 1/2 square units**	4 units	**6 cubic units**

2. $V = Bh,$ where B is the area of the base.

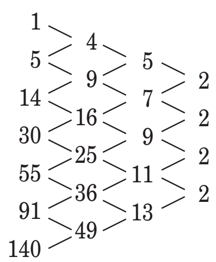
Bumps created by the edges of the blocks become less significant as the "pyramid" grows larger

Sheet 2

1. 8 cubic units
2. 5 cubic units
3.

Length of Side	Volume of Cubic Solid	Volume of "Pyramid of Cubes"	Volume of "Pyramid" Divided by Volume of Cubic Solid
1	1	1	1
2	8	5	5/8 = 0.625
3	27	14	14/27 ≈ 0.518
4	64	30	30/64 ≈ 0.469
5	125	55	55/125 = 0.440
6	216	91	91/216 ≈ 0.421
7	343	140	140/343 ≈ 0.408
8	512	204	204/512 ≈ 0.398
9	729	285	285/729 ≈ 0.391
10	1000	385	385/1000 = 0.385
n (if you can)	n^3	Volume of previous n + n^2 or $(1/3)n^3 + (1/2)n^2 + (1/6)n = (n/6) \cdot (n+1)(2n+1)$	$[(n/6)(n+1) \cdot (2n+1)]/n^3$

Some students may be interested in a derivation of the closed formula in the last cell of the "pyramid of cubes" column. One way to derive the formula from the information in the chart is to begin by establishing that the formula is a cubic function. Students who are familiar with the method of finite differences can see that the relationship is a cubic because the differences become constant after three iterations.



When students know that the formula is a cubic, they know that they can write it in the form $f(n) = an^3 + bn^2 + cn + d$, where n is the side length. Because the four constants a, b, c, and d are unknown, they can be treated as variables for now. Students can use the first four rows of the data in the table to see that $f(1) = 1$, $f(2) = 5$, $f(3) = 14$, and $f(4) = 30$. They can write the following system of equations:

$$a(1)^3 + b(1)^2 + c(1) + d = 1$$
$$a(2)^3 + b(2)^2 + c(2) + d = 5$$
$$a(3)^3 + b(3)^2 + c(3) + d = 14$$
$$a(4)^3 + b(4)^2 + c(4) + d = 30$$

That system is equivalent to the following system:

$$a + b + c + d = 1$$
$$8a + 4b + 2c + d = 5$$
$$27a + 9b + 3c + d = 14$$
$$64a + 16b + 4c + d = 30$$

However students solve this system, they find that $a = 1/3$, $b = 1/2$, $c = 1/6$, and $d = 0$, from which students can obtain the formula shown in the chart. The system is actually not very difficult to solve by hand using linear combinations.

4. Accept any reasonable answer at this point. Such answers might be similar to, "The ratio gets smaller as the shapes get bigger." In fact, the ratio in the last column approaches 1/3, or 0.33333. . . .
5. Again, accept any reasonable answer. The ratio approaches 1/3 because the pyramid of cubes becomes a closer approximation of an actual smooth-sided pyramid. The size of the cubic blocks does not change as the pyramids become larger, so bumps created by the edges of the blocks are less significant as the "pyramid" becomes larger. If students are familiar with the notion of a limit, you can discuss how the limit of these larger and larger shapes is an infinitely large pyramid with completely smooth sides.
6. Although the ratio in the last column keeps getting smaller, it never reaches 0. Let students discuss this result, but do not reveal the answer.

Sheet 3

1. and 2.

Length of Side	Volume of Cube	Volume of "Pyramid"	Volume of "Pyramid" Divided by Volume of Cube
1	1	1	
2	8	5	

3. Although the formulas are displayed here, numbers should show in the cells on the students' spreadsheets.

	A	B	C	D
1	Length of Side	Volume of Cube	Volume of "Pyramid"	Volume of "Pyramid" Divided by Volume of Cube
2	1	1	1	=C2/B2
3	2	8	5	

4.

	A	B	C	D
1	Length of Side	Volume of Cube	Volume of "Pyramid"	Volume of "Pyramid" Divided by Volume of Cube
2	1	1	1	=C2/B2
3	2	8	5	=C3/B3

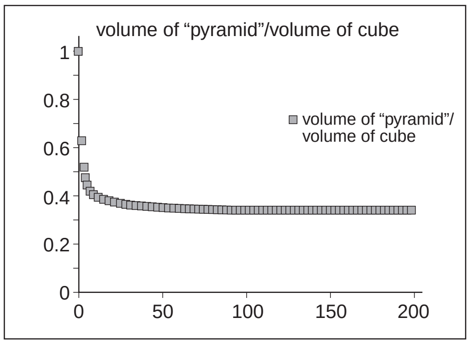
5.

	A	B	C	D
1	Length of Side	Volume of Cube	Volume of "Pyramid"	Volume of "Pyramid" Divided by Volume of Cube
2	1	1	1	=C2/B2
3	2	8	5	=C3/B3
4	=A3+1	=A4^3	=C3+A4^2	=C4/B4

6. A few sample rows are shown here.

Length of Side	Volume of Cube	Volume of "Pyramid"	Volume of "Pyramid Divided by Volume of Cube
196	7 529 536	2 529 086	0.335888692
197	7 645 373	2 567 895	0.335875699
198	7 762 392	2 607 099	0.335862837
199	7 880 599	2 646 700	0.335850105
200	8 000 000	2 686 700	0.3358375

7. How students create this graph varies depending on the spreadsheet software and the platform. To select the side-length column and the non-adjacent ratio column, first select one column, then select the other while holding down the control key. Excel users should look for the Chart Wizard icon on the menu bar, click on this icon after selecting the side length and ratio column, and follow the menu choices until the appropriate graph appears.



8. The numbers in the last column get closer and closer to 1/3.
9. No. The ratio will always be higher than 1/3.
10. $V = (1/3)Bh$.

Possible extensions

My students enjoyed moving away from the computers for this low-tech finale. If you have a hollow pyramid-and-prism set that has congruent bases and congruent heights, have students use the pyramid as a measuring device to fill the prism with water, sand, rice, or pasta. They should find that three pyramids of water or sand fill the prism exactly to the brim.

I ended the lesson by giving students a picture of some Egyptian pyramids from a book on architecture. The caption to the picture includes measurements, so students can calculate the volume of one of the actual pyramids.

The pyramid of Cheops, the biggest of the three pyramids at Giza, measures 230.5 meters (756 feet) at its base and is 146 meters high. The slope is 51° 52′. At the center is the pyramid of Chephren. Although it is 215 meters (705 feet) at its base and 143 meters (470 feet) high, it appears higher because of its steeper slope (52° 20′). The pyramid of Mycerinus, in the foreground, is the smallest of the three. It measures 208 meters (354 feet) at its base and 62 meters (203 feet) in height, with a slope of 51°.

REFERENCE

Norwich, John Julius. *World Atlas of Architecture.* New York: Crescent Books, 1984.

(Worksheets begin on page 62)

Part 1: Volume of a rectangular box

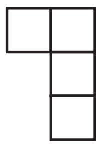
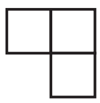
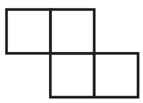
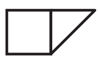
1. Construct each solid with your cubic blocks, and complete the chart. Use your imagination for the last answer.

Length	Width	Height	Volume
2 units	2 units	4 units	
1 unit	2 units	3 units	
2 units	2 units		8 cubic units
0.5 units	2 units	2 units	

2. Write a formula for the volume of a rectangular box. ________________________________

Part 2: Volume of a right prism

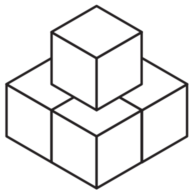
1. Construct each solid with your cubic blocks, and complete the chart. Use your imagination for the last answer.

Base	Area of Base	Height	Volume
		2 units	
		3 units	
			12 cubic units
		4 units	

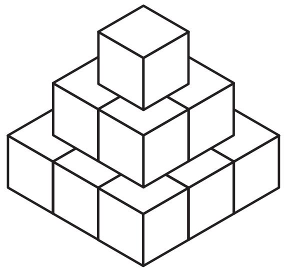
2. Write a formula for the volume of any right prism. ________________________________

From the *Mathematics Teacher,* January 2001

Although we cannot build exact pyramids with cubes, we can approximate them by building "pyramids of cubes" such as the two pictured below. You will compare the volume of a "pyramid of cubes" with the volume of the prism having the same base and height.



"Pyramid of cubes" with a square base of side length 2 and height of 2



"Pyramid of cubes" with a square base of side length 3 and height of 3

1. Find the volume of a cubic solid with a side of length 2. ________________
2. Find the volume of the "pyramid of cubes" with a square base of side length 2 and a height of 2 (pictured above). ________________
3. Complete the chart below. In the last column, compute the ratio of the number in the third column divided by the number in the second column.

Length of Side	Volume of Cubic Solid	Volume of "Pyramid of Cubes"	Volume of "Pyramid" Divided by Volume of Cubic Solid
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
n (if you can!)			

4. What happens to the ratio in the last column as your solids become larger?

5. Why do you think that you obtain this result?

6. Does the ratio in the last column ever become 0?

From the *Mathematics Teacher,* January 2001

As you can tell, finding the pattern in the last column of your table is difficult unless you continue the table. You can create a spreadsheet to do the work for you instead of doing the work by hand.

1. In a spreadsheet, type the headings for the four columns of your table, as shown. You may want to abbreviate the headings.

Length of Side	Volume of Cube	Volume of “Pyramid”	Volume of “Pyramid” Divided by Volume of Cube

2. Enter the values for the first two rows into your spreadsheet. Do not enter numbers for the last column, because you will use a formula to cause the spreadsheet to calculate these values.

Length of Side	Volume of Cube	Volume of “Pyramid”	Volume of “Pyramid” Divided by Volume of Cube
1	1	1	
2	8	5	

3. Enter a formula for ratio into the first empty cell in the last column. Remember that the formulas in a spreadsheet begin with an “=.” Do not just type in the number 1.

4. Copy the ratio formula that you just wrote into the cell below it. Your spreadsheet should look something like the following:

Length of Side	Volume of Cube	Volume of “Pyramid”	Volume of “Pyramid” Divided by Volume of Cube
1	1	1	1
2	8	5	0.625

5. The next row of your spreadsheet will contain only formulas. Enter all four appropriate formulas for the next row. For help, use the patterns that you noticed when you built the shapes with blocks. You can also work with other students.

6. Select the row of formulas that you just created, and copy them into the next row. Continue to copy down into more and more rows. Use any shortcut that your software allows, such as Fill Down, until your table is long enough that you are sure of a pattern in the last column.

7. Use the graphing feature of your spreadsheet to make a graph of the ratio numbers in the last column.

Use your spreadsheet to answer the following questions. Some of them are repeated from **sheet 2.**

8. What happens to the ratio in the last column as the solids become larger?

9. Will the ratio in this column ever be 0? Why or why not?

10. You can use your experience with the “bumpy” pyramids that you made with blocks to generalize the outcome for any pyramid. If a pyramid has a base of area B and a height of h, write a formula for its volume.