Addenda & Corrigenda (in progress)

\((X^T \text{ means } X \text{ lines from the top; } Y_B \text{ means } Y \text{ lines from the bottom.})\)

Chapter 1

- Misprint p5, \(5^T\) (thank you Rich Senko): ida should be idea.
- Misprint p24, in reference [6]: 243 should be 243-244.
- Add after Exercise 1.13 an exercise about showing that for any odd integer \(k\), \(\sqrt{k^2+1}\) is irrational [N. Lord: Extending the parity proof that \(\sqrt{2}\) is irrational, Math. Gazette 99 (2015), p155].
- Perhaps of interest are 1. D.M. Bloom: A once-sentence proof that \(\sqrt{2}\) is irrational, Math. Mag 68 (1995)), p286; and follow-up 2. J. Bergen: Is this the easiest proof that \(n\)th roots are always integers or irrational?, Math. Mag. 90 (2017), p225.

Chapter 2

- The paper [C. Mortici: A very elementary proof of Bernoulli’s inequality, College Math. J. 46 (2015), 136-137] contains an excellent simple proof of Bernoulli’s Inequality for \(x > 0\), and extends it somewhat for \(x > -1\). It is also noted there that Isaac Barrow (Newton’s teacher) had a version of Bernoulli’s Inequality 19 years before Bernoulli!

  - Add after Exercise 2.3:
    Show that for \(x_j \geq 0\), \(\prod_{j=1}^{n} (1 + x_j) \geq 1 + \sum_{j=1}^{n} x_j\). (All \(x_j\)’s equal is Bernoulli’s Inequality.)

  - In Exercise 2.24 (thank you Mehdi Hassani, Dragan Banjevic): \(\frac{2+4+6+\ldots+(2n)}{1+3+5+\ldots+(2n-1)}\) should be \(\left(\frac{2+4+6+\ldots+(2n)}{1+3+5+\ldots+(2n-1)}\right)^n\). Without the power \(n\) the limit is 1. With it, the limit is \(e\).

  - In Exercise 2.29, something subtle here needs consideration. See [A. Beardon: The area of a quadrilateral, Math. Gazette 101 (2017), 492-494].

  - Add after Exercise 2.33 [Problem 11751 (C. Kempiak, A. Viejo, B. Suceavă & B. Karaivanov), Amer. Math. Monthly 122 (2015), 905-906]: In a triangle with angles of radian measure \(A, B, C\), show that
    \[
    \frac{\csc A + \csc B + \csc C}{2} \geq \frac{1}{\sin A + \sin B} + \frac{1}{\sin B + \sin C} + \frac{1}{\sin C + \sin A},
    \]
    with equality occurring if and only if the triangle is equilateral.

  - Add after Exercise 2.50 [D.M. Bătinețu-Giurgiù, N. Stanciu, and E. Lampakis: Problem 1050, College Math. J. 47 (2016), 143-144]:
    We saw in Remark 2.14 that the isoperimetric inequality for an \(n\)-sided polygon with area \(T\) and perimeter \(P\) is
    \[
    T \leq \frac{P^2}{4n \tan (\pi/n)}.
    \]
Suppose now that the polygon is convex with sides lengths $a_k$, and let $m \geq 0$. Show that
\[
\left( \sum_{k=1}^{n} a_{2k}^m \right) \left( \sum_{k=1}^{n} \frac{1}{a_k^m} \right) \geq 16T^2 \tan^2\left(\frac{\pi}{n}\right).
\]

• Add after Exercise 2.56 [M. Can, G.E. Bilodeau, & M. Vowe: Problem 1059, College Math. J. 47 (2016), 304-305]: (If you did Exercise 2.54) Let $A, B, C$ be a triangle. Use Chebyshev’s Inequality (and Jensen’s: $\cos(x/2)$ is concave on $(0, \pi)$) to show that
\[
\frac{\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2}}{\sin A + \sin B + \sin C} \geq \frac{1}{\sqrt{3}}.
\]

Chapter 3

• In Exercise 3.37 (thank you Meng Lin Ma): Assume also that $f > 0$.

• Add a Remark at the end of Section 3.3: In [K. Razminia: A short proof of symmetric inequalities, College. Math. J. 46 (2015), 364-366], the Extreme Value Theorem is leaned upon heavily, in proofs of the AGM Inequality (and more) and the Cauchy-Schwarz inequality.

Chapter 4

• Section 4.2: A similar approach to our proof of the Product Rule (and perhaps even slicker!) can be found in P. Josevich: An alternative approach to the product rule, Amer. Math. Monthly 123 (2016), p470.

\[
\lim_{(x,y) \to (x_0,x_0)} \frac{f(x) - f(y)}{x - y}
\]
eexists. Show that $f$ is strongly differentiable at $x_0$ if and only if $f'$ is continuous at $x_0$.

Chapter 5


• Add after Exercise 5.36 [Problem 11788 (S. Andriopoulos & B. Karaivanov), Amer. Math. Monthly 123 (2016), p619] Let $n$ be a positive integer and let $0 < y_j \leq x_j < 1$ for $1 \leq j \leq n$. Show that
\[
\frac{\ln x_1 + \cdots + \ln x_n}{\ln y_1 + \cdots + \ln y_n} \leq \sqrt{\frac{1 - x_1}{1 - y_1} + \cdots + \frac{1 - x_n}{1 - y_n}}, \quad (and\ this\ is\ strict\ for\ n > 1).
\]

Hint (suggestion): Use induction; show first that $f(x) = \frac{\ln x}{\sqrt{1 - x}}$ is increasing on $(0, 1)$.

• In reference [54]: 1980 should be 1989.
Chapter 6

• p122, 6β in Section 6.2: Perhaps also of interest is the paper [I. Patyi: On some elementary functions, Math.
Gazette 99 (2015), 263-275].

• For some collateral reading, see also the excellent paper [N. Lord: The versatile exponential inequality
\(e^x \geq 1 + x\), Math. Gazette 101 (2017), 470-475].

p300]: Let \(a > b > 1\) and \(x > y > 0\). Show that
\[
\frac{a^x - b^y}{x - y} > a^{(x+y)/2} \ln(a) > \left(\frac{a + b}{2}\right)^{(x+y)/2} \ln\left(\frac{a + b}{2}\right).
\]

• Misprint p157, in reference [7]: Should be p651 here, not p615.

• Oversight p157, in reference [29]: This paper is authored by Hansheng, Y. and Lu, B.

• Oversight p157, in reference [36] (thank you Eunjeong Yi): This paper is authored by Kan, C.X and Yi, E.

• Misprint p157, in [39] (thank you Fuad Kittaneh): This paper is authored by Kittaneh, F. & Hirzallah, O.

Chapter 7

• Add to general remarks at the end of Section 7.2 the recent paper:

• Misprint in Exercise 7.14, p168: 5.39 should be 5.10.

• Add after Exercise 7.14 p169 [C. Mortici: Funny forms of the Mean value theorem, Amer. Math. Monthly 122
(2015), p780]: (a) Let \(a < b\) with \(a \neq -b\). Let \(f\) be continuous on \([a, b]\) and differentiable on \((a, b)\), with \(af(b) = bf(a)\).
Show that there is \(c \in (a, b)\) such that \(f'(c) = \frac{f(a) + f(b)}{a + b}\).
(b) Let \(a < b\) with \(ab > 0\). Let \(f\) be continuous on \([a, b]\) and differentiable on \((a, b)\), with \(1/f(b) - 1/f(a) = 1/b - 1/a\).
Show that there is \(c \in (a, b)\) such that \(f'(c) = \frac{f(a)f(b)}{ab}\).

Chapter 8

Here’s another proof of the Binomial theorem: (a) Show that \(\frac{d}{dt} \frac{t^k}{(1 + t)^n} = \frac{kt^{k-1} - (n - k)t^k}{(1 + t)^{n+1}}\).
(b) Multiply through by \(\frac{n!}{k!(n-k)!}\) and sum from 0 to \(n\) to show that
\[
\frac{d}{dt} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \frac{t^k}{(1 + t)^n} = \cdots = 0.
\]
(c) Make a couple of observations which allow you to conclude that \(\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} t^k = (1 + t)^n\).
• Add after Exercise 8.39 [G. Apostolopoulos, the Hofstra Univ. Problem Solvers & Á. Plaza: Problem 1039, College. Math. J. 46 (2015), 143-144]: Let $ABC$ be an acute triangle. Show that
\[
\sum_{A,B,C \text{ cyclic}} \frac{\sec A}{\sqrt{\cos A + \cos B}} \geq 6.
\]

• Add after Exercise 8.41 [M. Bence & J.C. Smith: Problem 11843, Amer. Math. Monthly 124 (2017), p374]: Show that $f(t) = \frac{1}{1+e^t}$ is convex on $[0, \infty)$. Show that
\[
\sum_{j=1}^{M} \frac{1}{1 + z_j} \geq \frac{M}{1 + (z_1 z_2 \cdots z_M)^{1/M}}.
\]

• Add after Exercise 8.48 [J.L. Diaz-Barrero & B. Bradie: Problem 1039, College. Math. J. 46 (2015), p373]: Let $x_1, x_2, \ldots, x_n > 0$ with $x_1 x_2 \cdots x_n = 1$ (and $n > 1$). Show that for each $m \geq 2$, we have
\[
\frac{n - 1}{n} \sum_{\text{cyclic}} \frac{x_1^m}{x_2 + x_3 + \cdots + x_n} \geq 1.
\]
Hint: Cauchy-Schwarz Inequality, AGM Inequality, Power Mean Inequality (see exercise above).

Chapter 9

• p213: Computation of $\int_0^1 \frac{1}{1+x} \, dx = \ln(2)$, as in [F. Sánchez & J.M. Sanchis: Darboux sums and the alternating harmonic series, Math. Mag. 91 (2018), p96], could be added here as well (perhaps before Example 9.7).

• p215. At the end of Section 9.1, add a remark or two about the cool paper [I.C. Bivens & B.G. Klein: The median value of a continuous function, Math. Mag. 88 (2015), 39-51].

• p229, 8T. After [33], add also: E. Omey: On Xiang’s observations concerning the Cauchy-Schwarz Inequality, Amer. Math. Monthly 122 (2015), 696-698. — which extends [33] very nicely.

• Misprint in Exercise 9.7, p235 (thank you Felipe Filho): The exponent on the right-hand side of the inequality to be shown should be $-xf(x)$, not $-yg(x)$.

• Add after Exercise 9.22 [Problem 1064 (M. Merca & B. Dunn), College. Math. J. 48 (2017), 140-141]: Let $n$ be a positive integer. Show that
\[
0 < \frac{1}{n} 2^{2n} \left( \frac{2n}{n} \right) - \sum_{k=1}^{n} \cos \left( \frac{k\pi}{2n+1} \right)^{2n+1} < 1.
\]

• Add after Exercise 9.36 [Problem 1024 (O. Furdui & E. Herman), College. Math. J. 46 (2015), p146.]: Let $f$ be continuous on $[a,b]$ with
\[
\int_a^b f^2(x) \, dx = \left( \int_a^b f(x) \, dx \right)^2.
\]
(a) Show that $b - a = 1$. (b) Show that $f$ is constant.
• Add after Exercise 9.36 [Problem 11780 (C. Lupu, T. Lupu, & R. Tauraso), Amer. Math. Monthly 123 (2016), 614-615]: Let \( f \) be positive and concave on \([0, 1]\). Show that
\[
\frac{3}{4} \left( \frac{1}{0} \int f(x) \, dx \right)^2 \leq \frac{1}{8} + \frac{1}{0} \int (f(x))^3 \, dx.
\]

• Add after Exercise 9.37 [C. Lupu & R. Boukharfane: Problem 11819, Amer. Math. Monthly 123 (2016), p1054]: Let \( f \) be continuous and nonnegative on \([0, 1]\). Show that
\[
\int_0^1 f(x)^3 \, dx \geq 4 \int_0^1 x^2 f(x) \, dx \int_0^1 x f(x)^2 \, dx.
\]

• Add after Exercise 9.40 [Problem 1069 (Á. Plaza & M. Andreoli) College Math. J. 48 (2017), 60-61]: Let \( f_1, f_2, \ldots, f_n \) be positive and continuous on \([0, 1]\). Use Exercise 9.40 to show that
\[
\int_0^1 f_1(x) \frac{1}{f_2(1-x)} \, dx \cdot \int_0^1 f_2(x) \frac{1}{f_3(1-x)} \, dx \cdots \int_0^1 f_n(x) \frac{1}{f_1(1-x)} \, dx \geq 1.
\]

• Add after Exercise 9.51(c): Show that
\[
\left( e^2 \int_1 \left( \frac{\ln x}{x} \right)^n \, dx \right)^{1/n} \to 1/e \quad \text{as} \quad n \to \infty.
\]


Chapter 10

• Add at the end of Section 10.1: Perhaps also of interest is the reference: [J.R. Nurcombe: Rearranging the signs of the alternating harmonic series, Math. Gazette 98 (2014), 321-324].

• Add to Exercise 10.21:
Show that \( \int \sec(x) \, dx = \frac{1}{2} \ln \left| \frac{1+\sin(x)}{1-\sin(x)} \right| + C \), and that \( \int \csc(x) \, dx = \frac{1}{2} \ln \left| \frac{1-\cos(x)}{1+\cos(x)} \right| + C \).

• Example 10.12, p258:
This curious fact was apparently first observed by D.P. Dalzell (On 22/7, J. London Math. Soc. 19 (1944), 133-134.)

(a) Verify that for \( t > 1 \), we have
\[
\frac{4}{(t+1)^2} < \frac{1}{t} < \frac{1}{2\sqrt{t}} + \frac{1}{2t\sqrt{t}}.
\]

(b) For \( 0 < a < b \), integrate over \([1, b/a] \) and do some tidying, to obtain (again!) \( G < L < A \).

• Exercise 10.15b (thank you Dragan Banjevic) is clearly nonsense. (My word, not Dragan's!) Omit it post-haste. Also, in Exercise 10.15c, the \( a > 1 \) there must be an integer.

• Misprint p247, in reference [1]: Wilkins, E.J. Jr. should be Wilkins, J. E., Jr.

• Misprint p248, in reference [28]: Sieffert should be Seiffert.
Let \( \phi \), defined on \([0, 1]\), have a continuous nonzero derivative, with \( \phi(0) = 0 \) and \( \phi(1) = 1 \). Let \( f \), continuous on \([0, 1]\), satisfy \( \int_0^1 f(x) \, dx = \int_0^1 \phi(x)f(x) \, dx \). Show that there is \( t \in (0, 1) \) such that \( \int_0^t \phi(x)f(x) \, dx = 0 \).

In the references, p280, item 39: The volume number 20 there should be volume 29.

**Chapter 11**


(a) Let \( f \) be continuous on \([0, 1]\) with \( \int_0^1 f(x) \, dx = 0 \). Let \( \phi \) be differentiable on \([0, 1]\) with \( \phi(0) = 0 \) and \( \phi'(x) > 0 \) for \( x \in (0, 1) \). Show that there exists \( x_0 \in (0, 1) \) such that

\[
\int_0^{x_0} f(x)\phi(x) \, dx = 0.
\]

Misprint p310, in reference [35]: Sieffert should be Seiffert.

**Chapter 12**


Daners’ ideas in [5], which we follow in Section 12.4, are extended nicely in [B.D. Sittinger: Computing \( \zeta(2m) \) by using telescoping sums, *Amer. Math. Monthly* 123 (2016), 710-715].

A wonderful approach to Theorem 12.7 in Section 12.4 appears in [S.G. Moreno: A short and elementary proof of the Basel problem, *College Math. J.* 47 (2016), 134-135]. This approach is arguably simpler than the one in the book!

Another excellent and very elementary approach to Theorem 12.7 in Section 12.4 appears in [N. Lord: The most elementary proof that \( \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \), *Math. Gazette* 100 (2016), 429-434]. This paper is definitely worth a good look.


\[
\lim_{n \to \infty} \sum_{k=1}^{n-1} \frac{n^2}{k^2(n-k)^2} = \frac{\pi^2}{3}.
\]
• Add after Exercise 12.9 [J. Gräter and K.J. Withs: On elementary bounds for \(\sum_{k=n}^{\infty} k^{-s}\), Amer. Math. Monthly 122 (2015), 155-158]:

(a) Show that for \(s > 1\), \(h_s(x) = (1 - x)^{1-s} - (1 + x)^{1-s} - 2x(s - 1)\) is strictly positive on \((0, 1)\).

(b) Use (a) to show that
\[
\left(1 - \frac{1}{2k}\right)^{1-s} - \left(1 + \frac{1}{2k}\right)^{1-s} > \frac{s-1}{k}.
\]

(c) Use (b) to show that
\[
k^{-s} < \frac{(k - \frac{s}{2})^{1-s}}{s - 1} - \frac{(k + \frac{s}{2})^{1-s}}{s - 1}.
\]

(d) Conclude that
\[
\sum_{k=n}^{\infty} k^{-s} < \frac{(n - \frac{s}{2})^{1-s}}{s - 1}.
\]

• Oversight p329, in reference [26]: This paper is authored by Sondow, J. and Yi, H.

Chapter 13

• Misprint p338, line 13\(^T\): The 0 there should be a 1. That is, \(e^{\frac{1}{n}} \to 1\) as \(n \to +\infty\). (Blushes.)

• Add in (or after) Example 13.10 p337: In [M. Shauo: Bounding the Euler-Mascheroni constant, College. Math. J. 46 (2015), p347], the Midpoint Rule is used to obtain also \(\gamma < 2(1 - \ln(2)) \approx 0.6137\). I should have thought of this!

Chapter 14

Appendix

• p 405, 8\(^T\): To references [2,7,8], add the interesting paper [R. Kantrowitz & M. Neumann: Another face of the Archimedean property, College. Math. J. 46 (2015), 139-141], in which the Archimedean Property (in any ordered field) is shown to be equivalent with conditions involving various simple geometric sequences, including the geometric series test (cf. Example 2.3 in Chapter 2).