21

An Integrated Approach to Teaching and Learning Geometry

David Wilson

THE NOTION of an integrated course in geometry is certainly not novel. John Swenson's (1936) geometry textbook *Integrated Mathematics with Special Application to Geometry* sought to distinguish his approach from the traditional separation of algebra and geometry, in part through his introduction of the coordinate plane into the traditional Euclidean geometry curriculum. Although many geometry textbooks since have employed the term *integrated,* the "text" presented in this article refers to an approach that weaves topics in algebra, geometry, and introductory trigonometry together seamlessly—that is, without chapters or other separations into content strands. This particular approach is problem-based and emphasizes coordinate geometry as a foundation for the development of many fundamental geometric concepts and as a means to develop students' reasoning and proof abilities. Technology is embedded, and students are expected to use technology as a tool to explore relationships and develop conjectures.

The integrated approach provides continual opportunities for students to make connections among mathematical topics and is central to students' development of geometric knowledge and understanding. The materials wonderfully illustrate recommendations in *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics 2000 [NCTM], p. 289) for—

> instructional materials that are intentionally designed to weave together different content strands … [and] to make sure that courses oriented toward any

particular content area (such as algebra or geometry) contain many integrative problems—problems that draw on a variety of aspects of mathematics, that are solvable using a variety of methods, and that students can access in different ways.

The integrated geometry materials, titled simply Mathematics 2, are written by the mathematics faculty at Phillips Exeter Academy (PEA 2006a) and are available in downloadable PDF files from its Web site. These materials, explored in this article, have the study of plane geometry at their core but integrate several other content strands as well.

Phillips Exeter Academy has a long history of producing curricular materials extending back to the latter part of the nineteenth and early twentieth centuries. The father-and-son faculty members George A. and George Wentworth authored a series of algebra and geometry textbooks that dominated the market at that time. The popularity of the Wentworth geometry textbook was attributed in part to "the abundance of 'original' exercises (proofs left to students' analysis and ingenuity) as opposed to 'book proofs' (full demonstrations to be memorized for reproduction)" (Austin [1919], quoted in Donoghue [2003, p. 335].) The philosophy of leaving much to students' analysis and ingenuity has lasted at Exeter and is evident in the present course.

The materials emphasize students' engagement in problem solving with reflection and communication as essential components of students' learning. These aspects of the materials are reminiscent of ideals promoted in Harold Fawcett's (1938) study of students learning geometry in an environment that provided them with opportunities to explore, derive, and document their own findings in the form of a geometry text. Although the PEA materials do not expect students to produce a geometry "textbook," they do promote students' participation in the exploration, discovery, and justification of significant geometric relationships. The problems help the students develop thinking strategies and reasoning and proof abilities throughout and parallel what Fawcett noted should be a fundamental consideration in teaching geometry:

> If the real purpose of teaching demonstrative geometry is to give the pupil an understanding of the nature of proof, the emphasis should not be placed on the conclusions reached, but rather on the kind of thinking used in reaching these conclusions. (Fawcett [1938, p. 466], quoted in González and Herbst [2006])

The authors (PEA 2006b, Specific comments, para. 1) describe their text as a "mathematical whole" and in their overview emphasize several important characteristics and expectations.

> There is no Chapter 5, nor is there a section on tangents to circles. The curriculum is problem-centered, rather than topic-centered. Techniques and theorems will become apparent as you work through the problems, and you will need to

> keep appropriate notes for your records—there are no boxes containing important theorems. There is no index as such, but the reference section that starts on page 201 should help you recall the meanings of key words that are defined in the problems.

For students, this paragraph is brought into reality as they peruse the Math 2 "text" (PEA 2006a)—nearly eighty pages containing ten to fifteen problems on every page. Diagrams accompany many problems, but students see that the text has no introductory paragraphs and no sample problems—nothing but problem after problem. This format can be slightly intimidating for both students and teachers, particularly since the latter try to imagine how such a text might be implemented and how the day-to-day instruction would occur. To generate some thoughts on how this implementation might be done, the following pages provide a general overview of the Math 2 materials and discuss how some content from a traditional geometry course is developed in the PEA text. In addition, this article discusses the substantive changes in the classroom environment that accompany the implementation of these materials and some personal reflections from the author's experience teaching with them.

Foundations of the Curriculum

A Spiral Approach

The curriculum follows a spiral approach to learning on several levels. On one level it is the embodiment of Jerome Bruner's (1975) reflections on teaching and of a spiral curriculum:

> [S]uccessful efforts to teach highly structured bodies of knowledge like mathematics ...often took the form of metaphoric spiral in which at some simple level a set of ideas or operations were introduced in a rather intuitive way and, once mastered in that spirit, were then revisited and reconstrued in a more formal or operational way.

The PEA text begins as Bruner describes—with an intuitive approach. Frequently this approach takes the form of a problem presented in the coordinate plane so that the student has an accurate figure to work from and has available a variety of tools that allow for calculations of slope, distance, intersections, angle measure, and so on. Problems are then revisited with an increasing level of difficulty and gradually embedded in related contexts to allow for deeper understanding to develop and connections to become apparent. For example, several problems (from page 2 in the text) shown in figure 21.1 are designed to develop the notion of a locus of points that satisfy a given condition.

The first two problems permit access for students of varying ability levels to begin intuitively exploring the problem in the coordinate plane. The Pythagorean

1. Two different points on the line $y = 2$ are each exactly 13 units from the point (7, 14). Draw a picture of this situation, and then find the coordinates of these points.

2. Give an example of a point that is the same distance from (3, 0) as it is from (7, 0). Find lots of examples. Describe the configuration of all such points. In particular, how does this configuration relate to the two given points?

3. Verify that $P = (1, -1)$ is the same distance from $A = (5, 1)$ as it is from $B = (-1, 3)$. It is customary to say that P is *equidistant* from A and B. Find three more points that are equidistant from A and B. By the way, to "find" a point means to find its *coordinates*. Can points equidistant from A and B be found in every *quadrant*?

Fig. 21.1. An intuitive beginning. Produced by members of the mathematics department at Phillips Exeter Academy. Used by permission.

theorem is employed in these problems as students develop the notion of distance in the plane through right triangles. Later, problems require the more formalized distance formula, but throughout the text the student is prompted to think of distance as the hypotenuse of a right triangle.

Problem 3 in figure 21.1 has several interesting aspects. First, it is extending problems 1 and 2 to a more challenging level by involving points that are neither horizontally nor vertically aligned. Second, the authors use a scaffolding approach as they give an example of one point that the student has to verify is equidistant from the given points prior to being asked to generate additional points. This type of scaffolding is common throughout the text. Another aspect of problem 3 is the introduction of formal terminology in the problem. Although terminology is frequently defined in problems, the reference section (also downloadable) to the textbook includes a glossary of all the italicized words from the text.

Figure 21.2 displays problems that allow students to revisit the topic of locus and set the stage for further discussions of how to describe the locus of points both verbally and algebraically as well as connect that understanding with related ideas.

Problem 3 occurs a bit later in the text and prompts students to extend their thinking about the perpendicular bisector as a locus of points to the reasoning behind the circumcenter of a triangle. Similarly, problem 4 extends and connects the themes of distance and locus of points with given conditions to parabolas. The many related problems that are completed prior to problem 4 allow students to see its solution as a natural extension of the previous work.

The van Hiele Model

The authors' use of the spiral to provide an intuitive approach, and to revisit

1. Let $A = (1, 5)$ and $B = (3, -1)$. Verify that $P = (8, 4)$ is equidistant from A and B. Find at least two more points that are equidistant from A and B. Describe all such points.

2. Write a formula for the distance from $A = (-1, 5)$ to $P = (x, y)$, and another formula for the distance from $P = (x, y)$ to $B = (5, 2)$. Then write an equation that says that P is equidistant from A and B. Simplify your equation to linear form. This line is called the *perpendicular bisector of AB*. Verify this by calculating two slopes and one midpoint.

3. Let $A = (3, 4)$, $B = (0, -5)$, and $C = (4, -3)$. Find equations for the perpendicular bisectors of segments AB and BC, and coordinates for their common point K. Calculate lengths KA, KB, and KC. Why is K also on the perpendicular bisector of segment CA?

4. Let $F = (0, 4)$. Find coordinates for three points that are equidistant from F and the x-axis. Write an equation that says that $P = (x, y)$ is equidistant from F and the x-axis.

Fig. 21.2. Revisiting the topic. Produced by members of the mathematics department at Phillips Exeter Academy. Used by permission.

and extend topics accompanied by appropriate scaffolding, aligns the text's approach with teaching recommendations based on the van Hiele levels of geometric thinking and phases of learning. The van Hiele levels are theorized levels of geometric knowledge and understanding that students' progress through sequentially; they constitute a framework for the types of tasks that are appropriate for students to engage in as they study geometry.

The PEA text is appropriately designed for students to transition from a late level 2 (descriptive-analytic) and early level 3 (relational-inferential) to level 4 (formal deductive). Furthermore, although possibly not the explicit intent of the authors, the problems in the PEA text are written in ways that align well with the van Hiele's phases of learning and thus support progress within the levels. Mary Crowley (1987) described the five sequential phases of learning that Pierre and Dina van Hiele proposed would assist students in progressing from one level to the next. In Phase 1: Inquiry, Crowley suggests that—

> observations are made, questions are raised, and level-specific vocabulary is introduced. (P. 5)

Phase 2: Directed Orientation should feature carefully sequenced explorations that lead to Phase 3: Explication, where—

> students express and exchange their emerging views about structures that have been observed. (P. 5)

Phase 4: Free Orientation involves—

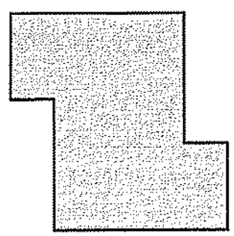
> more complex tasks—tasks with many steps, tasks that can be completed in several ways, and open-ended tasks. (P. 5)

In the last phase, Phase 5: Integration, the students review, summarize, and form an—

> overview of the new network of objects and relations. (P. 6)

The problems in figures 21.1 and 21.2 discussed above illustrate the alignment to a great extent, but an additional sample of problems that look at congruence will further illustrate the style of questions and the challenges posed to students as they deepen their understanding. The notion of congruence is introduced informally early in the text through an interesting problem involving the shape shown in figure 21.3. The problem offers an entry point for students of varying levels of understanding and allows for multiple responses that promote discussion and allow for the introduction of relevant terminology.

1. Some terminology: Figures that have exactly the same shape and size are called *congruent*. Dissect the region shown at right into two congruent parts. How many different ways of doing this can you find?



2. Let $A = (2, 4)$, $B = (4, 5)$, $C = (6, 1)$, $T = (7, 3)$, $U = (9, 4)$, and $V = (11, 0)$. Triangles ABC and TUV are specially related to each other. Make calculations to clarify this statement, and write a few words to describe what you discover.

Fig. 21.3. Congruence—van Hiele Phase 1 (Inquiry). Produced by members of the mathematics department at Phillips Exeter Academy. Used by permission.

Schettino (2003) discussed her students' experience with this problem as she implemented the PEA Math 2 materials and the varied topics that arose, including rotational symmetry and isometries. The second problem shown in figure 21.3 continues Phase 1 as opportunities for exploration and further observations are made possible and additional terminology such as *corresponding* can be introduced. Figure 21.4 displays problems that are presented a bit later in the text and provide additional terminology and structure for the students to explore some implications of congruence. The problems in figure 21.4 are representative of Phase 2: Directed Orientation, and students are guided toward recognizing that corresponding parts of congruent triangles are congruent.

Figure 21.5 presents problems that provide some opportunity for Phase 3: Explication. In the first problem, students are given a set of points that do not

1. Let $A = (-5, 0)$, $B = (5, 0)$, and $C = (2, 6)$; let $K = (5, -2)$, $L = (13, 4)$, and $M = (7, 7)$. Verify that the length of each side of triangle ABC matches the length of a side of triangle KLM. Because of [these] data, it is natural to regard the triangles as being in some sense equivalent. It is customary to call the triangles *congruent*. The basis used for this judgment is called the *side-side-side* criterion. What can you say about the sizes of angles ACB and KML? What is your reasoning? What about the other angles?

2. Let $A = (0, 0)$, $B = (2, -1)$, $C = (-1, 3)$, $P = (8, 2)$, $Q = (10, 3)$, and $R = (5, 3)$. Plot these points. Angles BAC and QPR should look like they are the same size. Find evidence to support this conclusion.

Fig. 21.4. Congruence—van Hiele Phase 2 (Directed Orientation). Produced by members of the mathematics department at Phillips Exeter Academy. Used by permission.

1. Let $A = (0, 0)$, $B = (1, 2)$, $C = (6, 2)$, $D = (2, -1)$, and $E = (1, -3)$. Show that angle CAB is the same size as angle EAD.

2. Using a ruler and protractor, draw a triangle that has an 8-cm side and a 6-cm side, which make a 30-degree angle. This is a *side-angle-side* description. Cut out the figure so that you can compare triangles with your classmates. Will your triangles be congruent?

3. With the aid of a ruler and protractor, draw and cut out three [noncongruent] triangles, each of which has a 40-degree angle, a 60-degree angle, and an 8-cm side. One of your triangles should have an angle-side-angle description, while the other two have angle-angle-side descriptions. What happens when you compare your triangles with those of your classmates?

Fig. 21.5. Congruence—van Hiele Phase 3 (Explication). Produced by members of the mathematics department at Phillips Exeter Academy. Used by permission.

clearly create two congruent triangles, but through the introduction of midpoint F of AC and drawing lines to form triangles ABF and ADE, students are able to apply their knowledge creatively in a novel way and realize the value of the side-side-side (SSS) congruence theorem. As is typical of many problems, students are asked to furnish evidence for their conclusions and discuss their reasoning. This aspect of the tasks gives students the opportunity to share their developing ideas and understanding of the topic. In the second and third problems, students are asked to consider alternative conditions for congruence and to extend and refine their understanding of criteria for congruence. This request represents one aspect of the spiraling nature of the content, but also a brief return to Phase 2: Directed Orientation, as students are specifically focused on completing prescribed tasks.

Two additional problems shown in figure 21.6 illustrate the last two phases (Free Orientation and Integration). In these problems students are challenged to think broadly about other criteria for congruence and then to refine their thinking once again as they explore a figure that poses problems for using side-side-angle (SSA) as a congruence criterion. As these problems are completed, students can share and summarize their developing knowledge of congruence as it relates to triangles and can set the foundation for further study and application.

1. A triangle has six principal parts—three sides and three angles. The SSS criterion states that three of these items (the sides) determine the other three (the angles). Are there other combinations of three parts that determine the remaining three? In other words, if the class is given three measurements with which to draw and cut out a triangle, which three measurements will guarantee that everyone's triangles will be congruent?
2. Use the diagram to help you explain why SSA evidence is not by itself sufficient to justify the congruence of triangles. The tick marks designate segments that have the same length.

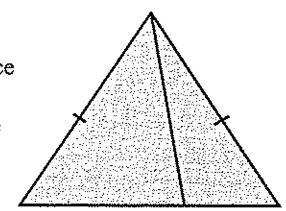


Fig. 21.6. Congruence—van Hiele Phases 4 and 5 (Free Orientation and Integration). Produced by members of the mathematics department at Phillips Exeter Academy. Used by permission.

The PEA Math 2 text engages students in problem solving on multiple topics simultaneously. Thus, as students explore congruence they are also introduced to the concept of a vector and then vector translations and other isometries. Problems within the context of transformations provide opportunities for students to deepen their understanding of congruence. The problems presented in figure 21.7 offer an illustration of this aspect of the content development that again reflects opportunities for students to engage in discussions and Phase 5: Integration activities.

Technology, Cooperative Learning, Reasoning, and Proof

Several other foundations of the PEA materials are the integration of technology, the expectation of students' working together in a cooperative environment,

1. Plot points $K = (0, 0)$, $L = (7, -1)$, $M = (9, 3)$, $P = (6, 7)$, $Q = (10, 5)$, and $R = (1, 2)$. Show that the triangles *KLM* and *RPQ* are congruent. Show also that neither triangle is a vector translation of the other. Describe how one triangle has been transformed into the other.
2. Plot points $K = (-4, -3)$, $L = (-3, 4)$, $M = (-6, 3)$, $X = (0, -5)$, $Y = (6, -3)$, and $Z = (5, 0)$. Show that triangle *KLM* is congruent to triangle *XZY*. Describe a transformation that transforms *KLM* onto *XZY*. Where does this transformation send the point $(-5, 0)$?

Fig. 21.7. van Hiele Phase 5: Integration. Produced by members of the mathematics department at Phillips Exeter Academy. Used by permission.

and embedding reasoning and proof in the problems. Many of the previous problems illustrate these features. Several of the problems are excellent candidates for explorations through such interactive geometry programs as The Geometer's Sketchpad (GSP) (Jackiw 2001). Such an approach would assist students in making initial conjectures and testing to verify or contradict their conjectures. The PEA Math 2 problems vary in their expectation for the use of technology, in that explicit statements may be made regarding technology or the decision about its use may be left to the student. Figure 21.8 presents problems illustrating each of these formats. Problem 1 is from the first pages of the text and clearly relays the

1. Consider the linear equation $y = 3.62(x - 1.35) + 2.74$.
   (a) What is the slope of this line?
   (b) What is the value of y when $x = 1.35$?
   (c) This equation is written in *point-slope* form. Explain the terminology.
   (d) Use your calculator to graph this line.
   (e) Find an equation for the line through $(4.23, -2.58)$ that is parallel to this line.
   (f) Describe how to use your calculator to graph a line that has slope -1.25 and that goes through the point $(-3.75, 8.64)$.
2. The sides of the triangle at right are formed by the graphs of $3x + 2y = 1$, $y = x - 2$, and $-4x + 9y = 22$. Is the triangle isosceles? How do you know?

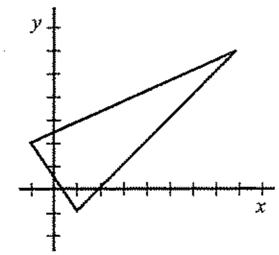


Fig. 21.8. Technology integration. Produced by members of the mathematics department at Phillips Exeter Academy. Used by permission.

expectation of the use of a graphing calculator, whereas problem 2 presents an opportunity for students to decide whether to use a calculator to verify coordinates of vertices.

Little support is found for how to use the graphing calculator in the materials, as that skill is assumed to be prior knowledge for PEA Math 2 students. The PEA authors (2006b) explicitly describe their expectations for technology use and for students' written responses:

> Many of the problems in this book require the use of technology (graphing calculators or computer software) in order to solve them. Moreover, you are encouraged to use technology to explore, and to formulate and test conjectures. Keep the following guidelines in mind: write before you calculate, so that you will have a clear record of what you have done; store intermediate answers in your calculator for later use in your solution; pay attention to the degree of accuracy requested; refer to your calculator's manual when needed; and be prepared to explain your method to your classmates. (Specific comments, para. 4)

Another foundation of the PEA text is that of group learning. The small-group discussions that occur around a table at Exeter can be effectively emulated in larger classes through small-group work and employing aspects of cooperative learning. This approach should promote an environment similar to that at Exeter, where students are—

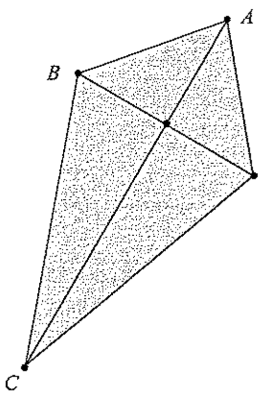
> exposed to problem solving in a very student-centered, discussion-based classroom. Students are held accountable for attempting solutions to homework problems and the class as a whole decides on correct solutions. (Phillips Exeter Academy 2006b, para. 1)

Students are challenged to make and verify conjectures throughout the text. The development of students' problem-solving abilities and reasoning-and-proof abilities are natural outcomes of engagement with the PEA materials. Problems require students to make calculations to verify conjectures and frequently ask for written explanations to justify their conclusions, as shown in many of the problems presented thus far.

The process of justifying responses is connected with, and extended to, the notion of proof about one-quarter of the way through the Math 2 materials. The text presents two formats of proof. The format presented in figure 21.9a, a paragraph format, allows students to demonstrate their ability to construct a cohesive argument with sentences, as opposed to the second, more traditional format of a statement-reason, two-column proof that is presented simultaneously in figure 21.9b.

For the most part, students are not directed toward one form of proof or the other. The introduction of formal proof is accompanied by an increased emphasis on synthetic geometry—that is, outside the coordinate plane in more generalized

a. Here are two examples of proofs that do not use coordinates. Both proofs show how specific *given* information can be used to logically deduce *new* information. Each example concerns a kite *ABCD*, for which $AB = AD$ and $BC = DC$ is the given information. The first proof, which consists of simple text, shows that diagonal *AC* creates angles *BAC* and *DAC* of the same size.



PROOF 1. Because $AB = AD$ and $BC = DC$, and because the segment *AC* is shared by the triangles *ABC* and *ADC*, it follows from the SSS criterion that these triangles are congruent. Thus it is safe to assume that all the corresponding parts of these triangles are congruent as well (often abbreviated to CPCTC, as in proof 2 below). In particular, angles *BAC* and *DAC* are the same size.

b. Now let *E* mark the intersection of diagonals *AC* and *BD*. The second proof, which is an example of a two-column proof, is written symbolically in outline form. It shows that the diagonals intersect perpendicularly. This proof builds on the first proof, which thus reappears as the first five lines.

PROOF 2.

$AB = AD$	given
$BC = DC$	given
$AC = AC$	shared side
$\Delta ABC \cong \Delta ADC$	SSS
$\angle BAC = \angle DAC$	CPCTC
E = intersection of AC and BD	
$AB = AD$	given
$\angle BAE = \angle DAE$	preceding CPCTC
$AE = AE$	shared side
$\Delta ABE \cong \Delta ADE$	SAS
$\angle BEA = \angle DEA$	CPCTC
$\angle BEA$ and $\angle DEA$ supplementary	E is on BD
$\angle BEA$ is right	definition of right angle

Fig. 21.9. Formal proof. Produced by members of the mathematics department at Phillips Exeter Academy. Used by permission.

terms. For example, problems such as those shown in figure 21.10 are mixed among problems that continue to use the coordinate plane as an avenue for the development of concepts.

1. In quadrilateral $ABCD$, it is given that $AB = CD$ and $BC = DA$. Prove that angles ACD and CAB are the same size. N.B. If a polygon has more than three vertices, the *labeling convention* is to place the letters around the polygon in the order that they are listed. Thus AC should be one of the diagonals of $ABCD$.

2. If the diagonals of a quadrilateral bisect each other, then any two nonadjacent sides of the figure must have the same length. Prove that this is so.

3. Triangle ABC is isosceles, with AB congruent to AC. *Extend* segment BA to a point T (in other words, A should be between B and T). Prove that angle TAC must be twice the size of angle ABC. Angle TAC is called one of the *exterior angles* of triangle ABC.

Fig. 21.10. Synthetic geometry. Produced by members of the mathematics department at Phillips Exeter Academy. Used by permission.

The PEA Math 2 emphasis on reasoning and proof, the focus on independent and cooperative learning opportunities, and the rich connections that arise in this problem-based approach offer great potential for providing students with a very strong background to build on. However, as noted previously, some fairly large impediments arise when it comes to implementing such a text, and those warrant discussion here as well.

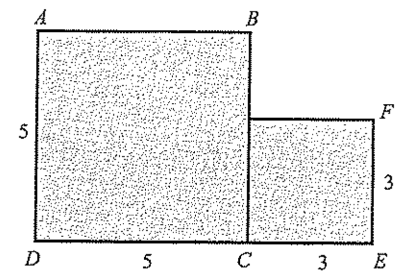
Classroom Implementation

The challenge for the teacher implementing the PEA materials is multidimensional. To a great extent, previous experiences for students and teachers with a student-centered instructional format will ease the transition. Many schools that have implemented *Standards*-based curricula will already have struggled with some of the same issues, although the *Standards*-based curricula tend to have themes or topics that focus the students' work and are often defined by chapters or units. The lack of any such separations in the PEA materials presents diverse challenges for the teacher. Issues of how classroom time is structured, how students' learning is assessed, and the student's role and responsibility all require time and effort to develop and coordinate into a successful teaching and learning environment. Furthermore, districts must allot adequate time, support, and professional development opportunities to accompany any implementation effort.

The Classroom Environment

The problem-based nature of the PEA materials, and the interwoven strands of content, may elicit some discomfort from students and teachers as they adjust their expectations for what the teaching and learning environment entails. Students may need support in bringing to bear the necessary skills and confidence to approach problems. Careful selection of items for classroom work, as well as for homework assignments, will help facilitate the development of such skills and confidence. The first problem in the Math 2 text is displayed in figure 21.11 and may be challenging for many students if approached as it is posed.

1. A 5×5 square and a 3×3 square can be cut into pieces that will fit together to form a third square.
   (a) Find the length of a side of the third square.
   (b) In the diagram at right, mark P on segment DC so that $PD = 3$, then draw segments PA and PF. Calculate the lengths of these segments.



   (c) Segments PA and PF divide the squares into pieces. Arrange the pieces to form the third square.

2. (Continuation) Change the sizes of the squares to $AD = 8$ and $EF = 4$, and redraw the diagram. Where should point P be marked this time? Form the third square again.

3. (Continuation) Will the preceding method *always* produce pieces that form a new square? If your answer is *yes*, prepare a written explanation. If your answer is *no*, provide a counterexample—two specific squares that can *not* be converted to a single square.

Fig. 21.11. First problem. Produced by members of the mathematics department at Phillips Exeter Academy. Used by permission.

Bill Campbell (1997), an instructor from Phillips Exeter Academy, shared an approach he used with this problem to allow greater initial access. The problem can be posed as a puzzle task for which the figure is drawn on a grid as shown in figure 21.12. Students cut along the lines connecting A to P and P to F, as well as on the outline of the squares, to form five pieces. The example shown uses side lengths of 12 and 5 for the two squares rather than 5 and 3. Students are challenged to arrange the five pieces into a square.

The puzzle can be presented without the original problem and thus provides all students with a place to begin. Students' initial attempts to complete the puzzle frequently yield a "solution" of the 12×14 rectangle shown in figure 21.13. The

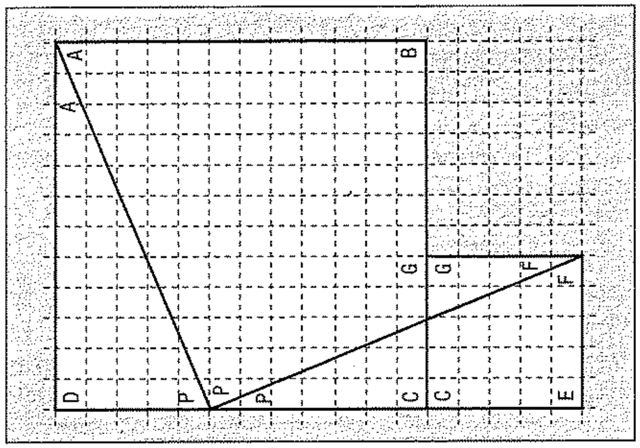


Fig. 21.12. An alternative approach. Produced by members of the mathematics department at Phillips Exeter Academy. Used by permission.

rectangle provides an opportunity to direct students' attention to the areas of the two original squares, and the convenient choice of dimensions of 5 and 12 allows many students to see what the dimensions of the solution square must be. Thus, the approach to the problem through the puzzle offers an opportunity to use part (a) of the problem as a hint toward the solution. Since the puzzle involves a grid, the students are able to look at distance easily, and once they realize that the side length must be 13, they are able to consider which pieces have such a length.

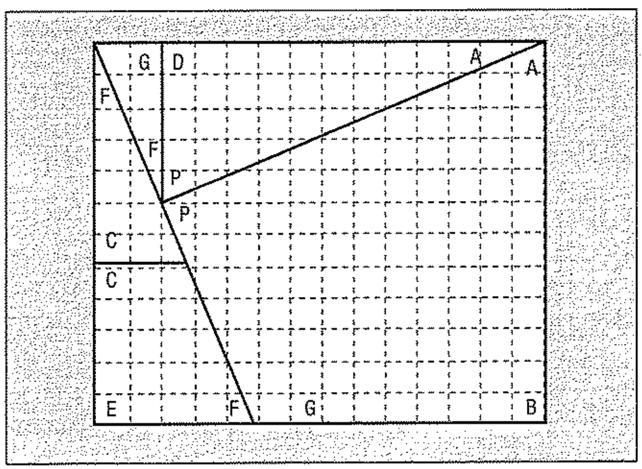


Fig. 21.13. An incorrect attempt. Produced by members of the mathematics department at Phillips Exeter Academy. Used by permission.

The solution shown in figure 21.14 provides opportunities for multiple solution paths to be discussed, including transformations of triangles *ADP* and *PEF*, and of course provides students with a visual connection to the Pythagorean theorem as they respond to problem 3. The shifting of the problem to a more familiar form—that of a puzzle task—engineers a comfort zone for the students and allows them to engage in the task. The original problem can be returned to as a follow-up to the puzzle task, with students now having the requisite knowledge and understanding to proceed.

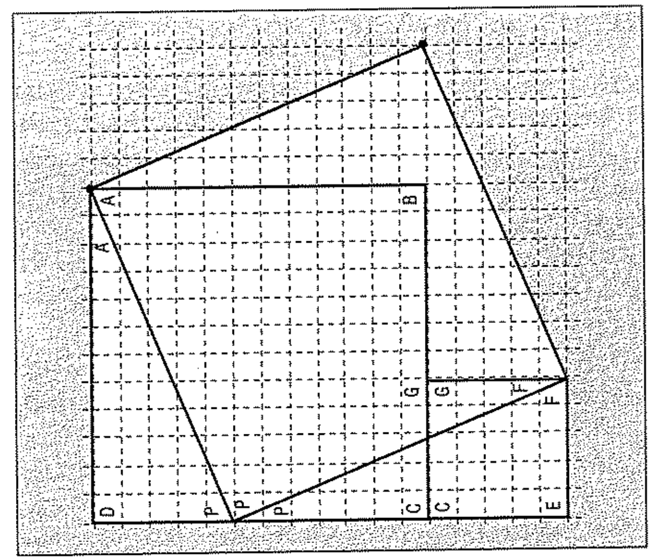


Fig. 21.14. A solution. Produced by members of the mathematics department at Phillips Exeter Academy. Used by permission.

Students will become comfortable with an environment in which they are expected to solve novel problems on a regular basis over time, but the teacher plays a vital role in this adjustment phase as he or she provides appropriate scaffolding for the students' problem-solving efforts and establishes a new "contract" with the students that shifts their role to that of active thinkers with an increased responsibility for learning mathematics.

The notion of a "contract" between the teacher and the students that defines their respective roles is helpful. The metaphor offers a way of anticipating some of the feelings of apprehension and uneasiness that may manifest themselves early on. Patricio Herbst (2006) wrote of one teacher's classroom experiences as she attempted to implement a problem-centered approach to teaching a geometry unit and the "negotiations in the contract" that took place. The teacher was attempting to use a problem-based approach to help students develop the ability to reason and make and test conjectures. Herbst documented the struggles and adjustments that were made as the shifts in the expectations for the learning environment were taking place. He noted that part of the difficulty may have stemmed from limited opportunities for the students to engage in deductive reasoning previously in the year. He concluded that (p. 344)—

> if in order to afford students an opportunity to grasp the meaning of mathematical ideas we recognize the need to afford access to how those ideas connect to and warrant each other, students should be exposed to tasks that require them to reason deductively to find things out … even if the product is not yet fully written as a mathematical proof.

The PEA Math 2 content is certainly well-aligned with such a vision. The difficulties in implementing a shift in practice can be limited in part by beginning the school year with new "contract" expectations that are aligned with the text and by fostering a sense of ownership of the learning. The PEA materials provide a sound foundation for this change.

The classroom environment will also change regarding assessment practices. The students' work during class time, the problems they complete outside class, and their contributions during cooperative learning and during whole-class summary discussions are integral parts of the learning and assessment processes. The traditional quiz and test are harder to place in the schedule because the materials do not have the conventional chapter separations.

Lastly, appropriate structure of the cooperative process can greatly enhance the success of implementing this problem-based material. The fundamental aspects of cooperative learning—including group selection, positive interdependence, individual accountability, social skills, and group processing—are elements that enhance the learning process.

Successful use of the PEA materials is tied to many factors. Substantial time, effort, and commitment, accompanied by appropriate professional development, contribute to successful implementation. The type of implementation will vary and will largely determine the need for the supportive elements mentioned above. For example, one or two teachers implementing the PEA materials for an honors group will require less support than a broad-based adoption for all geometry classes. Either one requires extensive time for the teachers to work through the problems to gain a sense of the how content is developed and how previous work is revisited and built on to introduce new themes and topics. Invitations to serve as coaches may be appropriate to extend to individuals with experience and expertise in implementing problem-based curricula and cooperative learning or in fostering student-centered learning environments. Perhaps an invitation to an Exeter faculty member should be considered as well. The PEA materials are available to be used in any way a teacher sees appropriate and are a wonderful resource, but the full implementation of the text requires extensive planning and support.

The following section provides a glimpse of challenges and rewards the author had experienced during implementation of the PEA Math 2 materials and offers a few additional suggestions.

Some Personal Reflections on Implementation

The PEA text comes with neither a solutions manual nor any guidelines for implementation, although it does include an introduction for students written by former students. My initial experience in using the material was as a source of weekly problem sets for my geometry students that were presented each Monday and due on Friday. Several times each week, students would share ideas and ask questions that would help them progress in their work. The discussions surrounding the problems gradually consumed more and more time, and I found the students' work on the problems to be more beneficial than many of my lectures and other assignments. I began to get a sense of the spiral that was embedded in the PEA text and how work on early problems was often laying the foundation for the more formal development and discussion to come later. I also watched my students' abilities and confidence in problem solving grow dramatically.

During several years of using the problems as a supplemental source, my instructional approach underwent significant changes as I attempted to make my classroom more student-centered and align my practices with the vision put forth in the various NCTM *Standards* documents (NCTM 1989, 1991, 1995, 2000). Part of this change involved my decision to use the PEA Math 2 text as the primary source of material with an honors geometry class. This choice posed a significant challenge to my students as well as to me. Many of the students preferred listening to lectures because their high ability had always allowed for "successful" experiences in that environment. Having to come to class each day and work on problems was not what they had come to expect, and this format evinced some resistance. The unstructured nature of the text bothered some students because they wanted to know what topic they were studying and be able to label notes appropriately. Parents also raised questions regarding the classroom environment and the method of instruction. Most students and parents, however, were quite excited about the challenging work and appreciated the approach. The largest source of questions and anxiety from the students stemmed from the disruption of their expectations of their role in the classroom. For many, this anxiety was resolved by the end of the first quarter.

A second fundamental change involved assessment. Tests became a quarterly event with several quizzes dispersed over the quarter as well, but submitted problems became the primary source of assessment. Part of the students' evolving comfort level stemmed from their ease with the state examination questions that we occasionally attempted. They found those questions to be trivial compared with what they were working on daily, and they became more confident that they were learning much more than they perceived. The lack of a sense of learning "content" may have been partly due to the lack of tests that had so often been their measure of success in previous courses. Working on rich problems each day, and reaching solutions to them, did not seem to equate to achieving a high test score.

Accompanying my transition to student-centered, problem-based classroom was an attempt at cooperative learning. I later realized how valuable a few basic practices of cooperative learning were. For example, group assessment and random selection of students to present solutions would have enhanced the small-group experience and fostered a more successful learning environment.

In spite of my novice efforts at shifting instructional practices and the venture into an unconventional text, the end results were highly successful. The students had become such strong problem solvers that as we reviewed for the New York State examination, those questions, as mentioned earlier, seemed simple. The students' scores reflected this success with many 100 percents and certainly the best state examination scores a group of my students had experienced. The nature of the learning experience allowed them to have a much higher retention rate and to achieve a deeper understanding of the content than they had previously experienced.

Conclusions

The many problems contained in this article reflect a small sample of the PEA Math 2 materials. Actively engaging in the problems as you read will help develop a sense of the expectations and spiraling nature of the curriculum. You may wish to return to different sections and work through the problems as you reread to truly gain some insight into how the PEA problems provide an opportunity for students to attain a rich and deep understanding of geometry.

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A PDF file of the Phillips Exeter Academy's Mathematics 2 course and a link to Phillips Exeter Academy's Web site are found on the CD-ROM disk accompanying this Yearbook.