

The Art of Problem Posing

Second Edition

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The Far Side



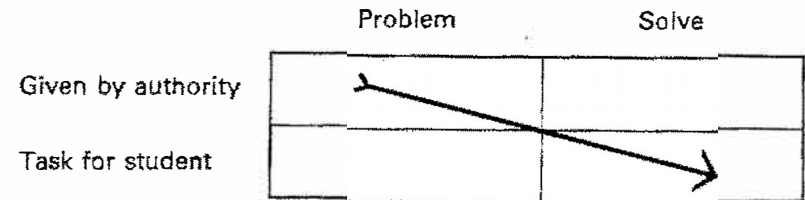
Primitive Think Tanks

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1

Introduction

Where do problems come from, and what do we do with them once we have them? The impression we get in much of schooling is that they come from textbooks or from teachers, and that the obvious task of the student is to solve them. Schematically, we have the following model:



The purpose of this book is to encourage a shift of control from "others" to oneself in the posing of problems, and to suggest a broader conception of what can be done with problems as well.¹ Why, however, would anyone be interested in problem *posing* in the first place? A partial answer is that problem posing can help students to see a standard topic in a new light and provide them with a deeper understanding of it as well. It can also encourage the creation of new ideas derived from any given topic. Although our focus is on the field of mathematics, the strategies we discuss can be applied to activities as diverse as trying

¹We shall not provide a formal definition of "problem" here, for the issue is more complicated than it might appear on the surface. For an effort to define the term see Gene P. Agre "The Concept of Problem," in *Educational Studies*, Vol. 13, No. 2, 1982, 121-142.

1. INTRODUCTION

to create something humorous (as in *The Far Side* cartoon), attempting to understand the significance of the theory of evolution, or searching for the design of a new type of car bumper.

Have you ever thought, for example, of designing car bumpers that make use of liquid or that are magnetized, or shaped like a football, or capable of inflation upon impact? Or, have you thought of the possibility that they may be made of glass, the fragility of which might discourage people from driving recklessly or relying so heavily on the use of the automobile?

In addition to teaching explicit strategies for problem generation, there is an underlying attitude towards "coming to know" something that we would like to encourage. Coming to know something is not a "spectator sport," though standard textbooks, especially in mathematics, and traditional modes of instruction may give that impression. To say, rather, that coming to know something is a *participant* sport is to commit ourselves to a point of view requiring that we operate on and even modify the things we are trying to understand.² This attitude is central to the problem posing activities we shall develop in this book.

Our strategy for presenting ideas is generally an inductive one. Whenever possible, we attempt first to expose some problem posing issue through an activity that gets at it in an implicit and playful manner. After there has been some immersion in an activity we turn towards a reflection on its significance. We believe that it is necessary first to get "caught up in" (and sometimes even "caught" in the sense of "trapped by") the activity in order to appreciate what it is we are after. Such a point of view requires both patience and also an inclination to recover quickly from any embarrassment that results from being "caught." We believe the rewards will be worth the effort.

One way of gaining an appreciation for the importance of problem posing is to relate it to problem solving—a topic that has gained widespread acceptance (or rejuvenation, depending on your point of view). Problem posing is deeply embedded in the activity of problem solving in two very different ways. First of all, it is impossible to solve any novel problem without first reconstructing the task by posing new problem(s) in the very process of solving. Asking questions like, "What is this problem really saying?" or "What if I shift my focus from what seems to be an obvious component of this problem to a part that seems remote?" propels us to generate new problems in an effort to "crack" the original one.

Secondly, it is frequently the case that *after* we have supposedly solved a problem, we do not fully understand the significance of what we have done, unless we begin to generate and try to analyze a completely new set of problems. You have probably had the experience of solving some problem (perhaps of a practical, non-mathematical nature) only to remark, "That was very clever, but what have I really done?" These matters are discussed with examples in chapter 6.

²See John Dewey, *Reconstruction in Philosophy*. Boston, Beacon Press, 1957 for an analysis of such a conception of knowledge.

Often our formal education suppresses the relationship between the asking of questions and the coming up with answers. In a book which calls for a new perception of education, D. Bob Gowin comments:

Recently a teacher was overheard to announce: "When I want your questions, I'll give them to you." . . . Much of school practice consists of giving definite, almost concrete answers. Perhaps boredom sets in as answers are given to questions that were never asked.³

More than boredom is at stake, however, when we are robbed of the opportunity of asking questions. The asking of questions or the posing of problems is a much more significant task than we are usually led to believe. The point is made rather poignantly in the story of Gertrude Stein's response to Alice B. Toklas, on Gertrude's death bed. Alice, awaiting Gertrude's legacy of wisdom, asked, "The answers Gertrude, what are the answers?"—Whereupon Gertrude allegedly responded, "The questions, what are the questions?"

The centrality of problem posing or question asking is picked up by Stephen Toulmin in his effort to understand how disciplines are subdivided within the sciences. What distinguishes atomic physics from molecular biology, for example? He points out that our first inclination to look for differences in the specific content is mistaken, for specific theories and concepts are transitory and certainly change over time. On the other hand, Toulmin comments:

If we mark sciences off from one another . . . by their respective "domains," even these domains have to be identified not by the types of objects with which they deal, but rather by the questions which arise about them. . . . Any particular type of object will fall in the domain of (say) "biochemistry," only in so far as it is a topic for correspondingly "biochemical" questions.⁴

An even deeper appreciation for the role of problem generation in literature is expressed by Mr. Lurie to his son, in Chaim Potok's novel *In the Beginning*:

I want to tell you something my brother David, may he rest in peace, once said to me. He said it is as important to learn the important questions as it is the important answers. It is especially important to learn the questions to which there may not be good answers.⁵

Indeed, we need to find out *why* some questions may not have good answers. For example, the questions might seem foolish or meaningless; or it might be that the questions are fundamental human questions that each of us might fight a

³D. Bob Gowin, *Educating*. Ithaca, NY: Cornell University Press, 1981, p. 127.

⁴Stephen Toulmin, *Human Understanding*. Princeton: Princeton University Press, 1977, p. 149.

⁵Chaim Potok, *In the Beginning*. New York: Knopf Publishers, 1976, pp. 295-296.

lifetime to try to understand; it may be that they are unanswerable questions because they are undecidable; it might also be, however, that our perspective on a problem is too rigid and we are blinded in our ability to see how a question might bear on a situation.

The history of every discipline—including mathematics—lends credence to the belief not only that it may be hard to distinguish good questions from bad ones in some absolute sense, but that very talented people may not be capable of seeing the difference even for a period of centuries. For a very long time, people tried to *prove* Euclid's fifth postulate:

Through a given external point, there is exactly one line parallel to a given line.

It was only during the past century that we began to realize that the difficulty in answering the question lay in the assumptions behind the question itself. The implicit question was:

How can you prove the parallel postulate from the other postulates or axioms?

It took hundreds of years to appreciate that the "how" was an unrecognized monster. If you delete the "how," the question is answerable (in the negative it turns out); if you do not do so, the question destroys itself as is the case with the question, "When did you stop beating your spouse"?

So far, we have tried to point out some intimate connections between the asking and answering of questions, and between the posing and solving of problems. There is a sense, however, in which problem generation is an important activity to pursue, even in the absence of uncovering "the right" solution, question or questions. In fact, there are many situations for which the concept of a right question is inappropriate. Imagine being given a situation in which no problem has been posed at all. A reasonable task might be to generate a problem or to ask a question, not for the purpose of *solving* the original situation (a linguistically peculiar formulation), but in order to create a problem that derives from the situation.

Suppose, for example, that you are given a sugar cube or the statement, "A number has exactly three factors." Strictly speaking, there is no problem in either case. Yet there is an infinite number of questions we can ask about either of the situations—some more meaningful than others, some more significant than others. However, it is rarely possible to tell in the absence of considerable reflection what questions or problems are meaningful or significant. We hope to persuade you in much of what follows, that things like significance and meaningfulness are as much a function of the ingenuity and the playfulness we bring to a situation as they are a function of the questions we ask. Frequently, even a slight turn of phrase will transform a situation that appears dull into one that "glitters."

There is another reason for asking interesting questions about a situation besides wanting to engage in creative activity or searching for answers: In looking at what happens when we pose new questions concerning an object—like a sugar cube or a statement that a number has exactly three different factors—we begin to "see" the object as we had never seen it before. We gain a deeper understanding of what constitutes a sugar cube or what it means for a number to be divisible by three factors. We shall further discuss issues of meaning and significance in chapter 3.

Sometimes the questions we ask about a phenomenon or a situation keep our perceptions of it intact. Sometimes, however, we end up drastically revising what we start with in the modifications we make. It is ironic that we sometimes come to "see" what is staring us in the face only after we "destroy" it in some way—at least mentally. The strategies behind such radical reconstruction of an object are discussed in chapters 4 and 5.

It is no great secret that many people have a considerable fear of mathematics or at least a wish to establish a healthy distance from it. There are many reasons for this attitude, some of which derive from an education which focuses on "right" answers. People tend to view a situation or even a problem as something that is *given* and that must be responded to in a small number of ways. Frequently people fear that they will be stuck or will not be able to come up with what they perceive to be the right way of doing things.

Problem posing, however, can create a totally new orientation towards the issue of who is in charge and what has to be learned. Given a situation in which one is asked to generate problems or ask questions—in which it is even permissible to modify the original thing—there is no *right* question to ask at all. Instead, there are an infinite number of questions and/or modifications and, as we implied earlier, even they cannot easily be ranked in an *a priori* way.

Thus we can break the "right way" syndrome by engaging in problem generation. In addition, we may very well have the beginnings of a mechanism for confronting the rather widespread feelings of mathematical anxiety.

This book then represents an effort on our part to try to understand:

1. What problem posing consists of and why it is important.
2. What strategies exist for engaging in and improving problem posing.
3. How problem posing relates to problem solving.

While problem posing is a necessary ingredient of problem solving, it takes years for an individual—and perhaps centuries for the species—to gain the wisdom and courage to do both of these well. No single book can provide a panacea for improving problem posing and problem solving. However this book offers a first step for those who would like to learn to enhance their inclination to pose problems. While this book does touch upon problem solving, it does so primarily as it relates to problem posing.

AUDIENCE

The Art of Problem Posing is written for a wide audience. It is intended for college math students, interested laypersons, present and future teachers of mathematics in middle school, in secondary school, and in higher levels of education. It also has implications for curriculum writers and for those who wish to do research on the power of problem posing and its relationship to a host of variables ranging from fear of mathematics to new strategies for teaching mathematics. We hope that it suggests directions for educators in fields other than mathematics as well. In fact, we are eager to continue to hear from practitioners in fields such as architecture, medicine, and engineering, who have viewed their work from a problem posing point of view, and who might find it useful to apply some of our techniques.⁶

Much of the book can be appreciated after having completed a high school mathematics program, although we occasionally assume a mathematics background roughly comparable to what is learned through the junior year in college.

Because we hope to appeal to a wide audience, we have intentionally selected a variety of mathematical topics and types of exposition to develop our points. We hope that people without extensive mathematical background will not be put off by what may occasionally appear to be technical approaches. On the other hand, we hope that people with a strong math background will appreciate the fact that material which appears simple may have considerable depth, and that much of what we discuss has implications for the most advanced mathematical material.

WAYS OF READING THE BOOK

While it would be helpful to explore many of the specific mathematical ideas presented in each chapter *before* attempting to follow the more general issues we bring up, it will not be a hindrance to skip a few examples during the first reading, should they seem too difficult.

We have stressed the importance of participation in coming to know and we hope that you will read this book in an active way. So, throughout the book, we ask a number of questions which we hope you will pursue. In fact, quite frequently, what we are after is a set of questions on your part and not primarily a set of answers to our questions.

Chapters 2 through 6 deal primarily with matters of problem generation and

⁶See Donald A. Schön's *The Reflective Practitioner*, New York, Basic Books, 1983 and *Educating The Reflective Practitioner*, San Francisco, Jossey-Bass, 1987 for a discussion of the need to re-conceptualize the preparation of professionals in all fields so as to view problem posing as a central phenomenon. His discussion of the training of architects in particular is quite enlightening.

should be of interest to the mathematics student and the layperson. Chapter 7 creates a pedagogical setting for an application of the point of view developed earlier. There we explore the concept of using an editorial board and suggest methods of incorporating it in the classroom in order to implement some of our problem posing strategies. This approach might be of interest to educators at all levels who would like to investigate a novel scheme for placing the student in a more active role. Especially given the emerging concept of writing across the disciplines (and not just in fields such as history and English), educators in all fields might wish to adapt parts of this approach in their own field of expertise.

ORIGINS OF THE BOOK

The material for this book has grown out of our experience in creating and team teaching courses on problem posing and solving at Harvard Graduate School of Education over two decades ago.⁷ In addition to graduate students whose major concern was mathematics and education on both the elementary and secondary school level, on several occasions we had students at Harvard take our course who were preparing to be lawyers, anthropologists, and historians. Subsequently, we have taught variations of that course to both graduate and undergraduate students at numerous institutions, including Syracuse University, Dalhousie University, the University of Georgia, the University of Oregon, the University at Buffalo, and Hebrew University in Jerusalem. It is interesting for us to reflect on the fact that we did not originally perceive that we were creating something of a paradigm shift in focusing on problem posing. We thought rather that we were adding a new and rather small wrinkle to the already existing body of literature on teaching of problem solving that had been popularized primarily

⁷In addition to teaching courses on problem posing and solving we have published a number of articles dealing either directly or indirectly with several themes of this book. Some of the chapters reflect or incorporate material from these articles. The following co-authored pieces deal directly with the theme of problem posing: "What-If-Not," *Mathematics Teaching* (British Journal), 46, Spring 1969, p. 38-45; "What-If-Not? An Elaboration and Second Illustration," *Mathematics Teaching*, 51, Spring 1970, p. 9-17; "Missing Ingredients in Teacher Training: One Remedy," *American Mathematical Monthly*, April 1971, p. 399-404; "The Roles of the Specific and General Cases in Problem Posing," 59, *Mathematics Teaching*, p. 52-54; "Problem Posing and Solving: An Illustration of Their Interdependence," *Mathematics Teacher*, 70, 1, Fall 1977, p. 4-13.

Modifications of the first two pieces appear as part of chapter 4; a few sections of the third appear in Chapter 7; a small part of the fourth appears in chapter 3; a modified version of the last in chapter 6. The editors of each of the journals within which the material originally was published have granted permission to use what appears herein.

In addition to articles dealing with problem generation explicitly, we have each made implicit use of problem generation in several others. These pieces are mentioned in chapter 5 in the context of the development of relevant ideas.

the work of George Polya. Over time, however, we have come to see that the ability of problem posing is capable of assuming a kind of centrality in its own right. Actually, it was the question we ask the reader to consider at the beginning of chapter 2 (about the Pythagorean equation) which launched us on the venture in the first place.

The emergence of a new interdisciplinary journal in 1987 entitled "Questioning & Exchange" (edited by James T. Dillon from the University of California at Riverside) signals that the primacy of the problem or the question over the solution or the answer has begun to be appreciated in all fields of scholarship.⁸

Before embarking on our actual exploration of problem posing, we should mention that there will be a modicum of repetition of key ideas throughout the text. There are several reasons for this. First, and most importantly, these ideas are novel ones and it will most likely take a while to appreciate their force. Secondly (as we recommended in the previous section), you probably will skim the book at first and will not come across repetition until a second or third reading. Thirdly, we are now 110 years old (between the two of us) and we have become ever so slightly forgetful over the past few years. Fifthly, and most importantly, these ideas are novel ones and it will most likely take a while to appreciate their force!

What problems can you pose starting with the number 110?

2

Two Problem Posing Perspectives: Accepting and Challenging

A FIRST LOOK; WHAT ARE SOME ANSWERS?

$$x^2 + y^2 = z^2$$

After looking at the above equation, respond to the following:

"What are some answers?"

We have asked this question of our students and colleagues over the years. Before reading on, you might wish to answer it yourself. Jot down a few responses on a piece of paper, and we will then discuss the significance of the above question.

First Question Revisited

What was your reply to the question which opened this chapter? Answers will depend in part on your level of mathematical experience. People who have had very little experience with mathematics frequently respond, "Oh, that reminds me of some statement about right triangles, but I can't remember it exactly."

Those who have had more experience with mathematics sometimes respond with a list like the following:

3, 4, 5
5, 12, 13
8, 15, 17.

⁸Though an important and ambitious undertaking, the journal was unfortunately short-lived. The last issue was published two years after the first.

Then they remark that they know there are a few other such number triples but they cannot recall them.

Among those who have had a great deal of experience, we have frequently received the above and, in addition, a comment about the potential length of such a list. They suggest how many triples there are and, occasionally, either recall or attempt to generate a formula for them.

People who know more about real or imaginary numbers are often pleased when, almost in a sense of amusement, they produce the following sets of numbers:

$$2, 3, \sqrt{13} \quad \text{or} \quad i, 1, 0.$$

Now look back at all the above responses. Notice that something very significant has happened. All the responses to, "What are some answers?" assume that a question has been asked by the equation itself. Furthermore, they assume that the specific question asked when we wrote " $x^2 + y^2 = z^2$ " was: "What are some integer solutions (or perhaps real or imaginary ones)?"

Notice, however, that " $x^2 + y^2 = z^2$ " is not in itself a question at all. If anything, it begs for you to *ask a question* or to *pose a problem* rather than to answer a question.

It may look as if we are splitting hairs or that we have pulled your leg by setting a trap. Our experience indicates, on the contrary, that we are getting at something important. Perceiving $x^2 + y^2 = z^2$ only as an equation that requires solving for x , y , and z reveals a very limited perspective. As you read on, you will see that the issues we are getting at open up vast new possibilities for learning in general and for learning mathematics in particular.

There is a myth that it is the role of the expert or authority (textbook, teacher, research mathematician) to ask the questions and for the student merely to answer them. Of course, it is considered good pedagogy to encourage students to ask questions, but they are usually questions of an instrumental nature—questions that enable teachers to pursue their pre-conceived agendas.

Frequently students are encouraged to ask questions which enable them to better follow well-trodden terrain that has been laid out not only by their teacher, but by the mathematics community at large. In grade school, for example, teachers encourage children to ask questions to make sure that they understand existing procedures.

A typical interchange might be:

Teacher: "Do you understand how to add two-digit numbers?"

Johnny: "Did you say we have to add from right to left?"

Teacher: "Good question. Yes. Let me show you what happens if you did it from left to right. Let's do a problem.

$$\begin{array}{r} 95 \\ + 87 \\ \hline \end{array}$$

"Let's find the sum by adding from left to right but carrying in the way I showed you.

$$\begin{array}{r} 95 \\ + 87 \\ \hline 713 \end{array}$$

"You see, you get the answer 713, but you know it must be wrong because you can tell the answer must be less than 200."

Note that the teacher has not approached the question in a completely arbitrary way, for a reason has been offered other than the teacher's authority for preferring one method over another. Furthermore, it is possible that Johnny's question could lead to some interesting exploration. For example, a teacher could encourage a student to investigate modified strategies for which we could get the correct answer by adding from left to right. Or the teacher and student could explore the extent to which the notation itself imposes one algorithm over another. Nonetheless, students and teachers do not usually ask questions for such purposes; rather, they are interested in making sure that their students understand and can execute what is expected of them.

Such an atmosphere neither leads to understanding the significance of an activity or a procedure, nor does it contribute to the development of a sense of autonomy and independence. In contrast, problem posing can give one a chance to develop independent thinking processes. Most people have been denied this ingredient in their learning—especially in their mathematics classes. It is the ability to pose problems rather than merely to ask procedural questions, such as the one above, that is a central component in the development of an autonomous person.

A SECOND LOOK AT $x^2 + y^2 = z^2$

Reconsider $x^2 + y^2 = z^2$. Now let us ask not, "What are some *answers*?" but, "What are some *questions*?" **Before reading on, list some of your own questions.**

Here are some of the responses people have given at this early stage:

1. Who first discovered it?
2. Are the solutions always integers?
3. How do you prove it?
4. What's the geometric significance of this?

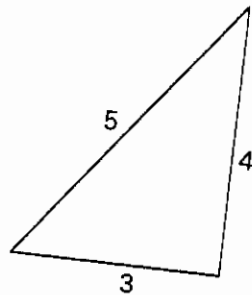
Loosening Up

In order to loosen up your own thinking processes further at this point and to give them free reign, write down any ideas, not necessarily questions, that occur to you when you look at and think about $x^2 + y^2 = z^2$. No holds barred! We are asking you to free associate and write down any ideas you have.

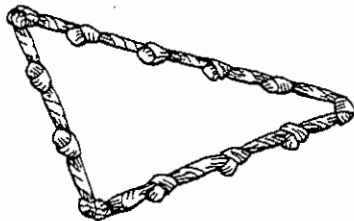
Some Typical Observations

Did you write down any statements or questions of the following types?

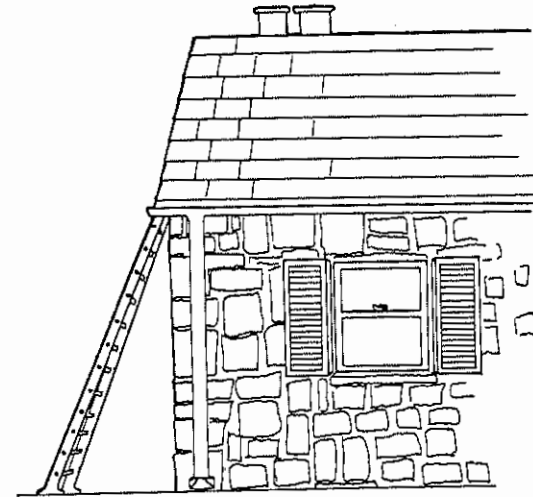
1. Some famous right triangles are 3, 4, 5; 5, 12, 13; and 8, 15, 17.



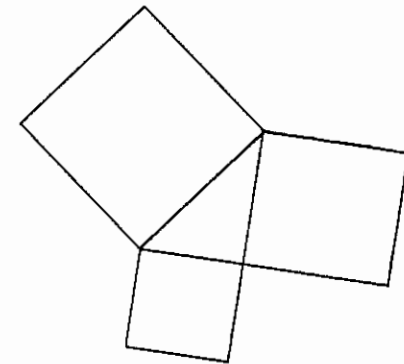
2. The Greeks used knots on a rope to make a right triangle.



3. You can use it to introduce irrational numbers.
4. This is associated with the Pythagorean Theorem.
5. It's the only thing I remember from geometry.
6. It reminds me of ladders against a wall.



7. How do you find more triplets satisfying $x^2 + y^2 = z^2$?
8. Is 3, 4, 5 considered different from 6, 8, 10 or from 30, 40, 50?
9. It has to do with squares on the sides of a right triangle.



10. What good is any of this?

Notice that even in this small sample of observations, we have a few new ideas for $x^2 + y^2 = z^2$: Namely, that it can deal with triangles; that it has a history; and that there are applications of it.

A NEW PERSPECTIVE

We are not yet done with the equation $x^2 + y^2 = z^2$. In fact, we have hardly begun. Compare the kinds of questions just mentioned with the following list of questions also using $x^2 + y^2 = z^2$ as a point of departure:

1. For what integral values of x , y , z is it true that $x^2 + y^2 < z^2$?
2. For what values is it true that $x^2 + y^2 = 1 + z^2$?
3. What happens to the Pythagorean Theorem if the triangle is not a right triangle? That is, suppose the right angle is replaced by a 60° angle. How is $x^2 + y^2 = z^2$ affected?
4. For what 60° triangles (i.e., triangles with at least one 60° angle) can you find three sides that are all integers?
5. If you replace the squares on each side of the triangle by rectangles that are not squares, do the areas on the legs ever add up to the area of the rectangle on the hypotenuse?
6. Is there a three-dimensional analogue for the Pythagorean Theorem?
7. What analogy is there for four-sided figures?

A Reflection on the Preceding Questions

What are the differences between the thinking shown in the preceding set of questions and the thinking shown in questions and observations we made in the previous subsections?

There are many ways of answering this question. In making such comparisons what we see is very much a function of our idiosyncratic experiences and of the kinds of lenses we are accustomed to use. We could, for example, try to compare the questions with regard to their degree of generality or the sense in which they call for algorithms. Since you do not know where we are headed at this point, it is probably very difficult to get a grasp on what we see as a *salient* difference between the above seven questions and those that precede them. Nevertheless we urge you to take a stab. Compare the section "Some Typical Observations" with the section "A New Perspective" and see if you can find some differences that mean something to you.

Perhaps you have observed differences of the following kinds:

1. Degree to which you strive only for *solutions* of the equation $x^2 + y^2 = z^2$.
2. Degree to which you search for literal or narrow interpretation of the equation $x^2 + y^2 = z^2$.
3. The degree to which you risk asking questions for which you may not have a method of solution.

4. The style in which you interpret the mathematics, for example, geometrically or algebraically.

In addition to these four categories (and many others), there is one that has significantly influenced our thinking about problem posing. It is a bit subtle and took us a while to appreciate, although after realizing it, it is hard to imagine that there was a time we did not see it.

5. The degree to which one "accepts" the given.

Notice that sometimes we have essentially accepted the given—in this case $x^2 + y^2 = z^2$ and its relationship to the Pythagorean Theorem—and sometimes we have *challenged* the given in order to ask new questions. We do not, in the latter case, take the given for granted. Rather, the given is a starting point for investigations that modify it. For example, instead of considering $x^2 + y^2 = z^2$ and squares on the sides of right triangles, we asked about such forms as $x^2 + y^2 = 1 + z^2$ and we replaced squares with rectangles. Though problem posing using each of these perspectives is valuable and necessary, it is *challenging* the given which frequently opens up new vistas in the way we think. Only after we have looked at something, not as it "is" but as it is turned inside out or upside down, do we see its essence or significance.

We have refined this notion of challenging the given into a strategy that can be learned, and we have coined the phrase "What-If-Not?" to describe it. It is a second phase of problem posing after the earlier one of *accepting* the given. Ironically, we first begin to gain a deeper understanding of something when we modify it into something else. It is worth keeping in mind that though it is possible to learn a strategy for challenging the given, it is not possible to guarantee (by using any particular procedure) that we will capture those phenomena we later perceive to be most significant. It takes not only insight but courage and sometimes totally new world views for anyone to find significant challenges. Merely looking for something to challenge will not guarantee that we will find it. The fact that it has taken such a long time even to *realize* that there were attitudes to be challenged concerning full-time employment, leisure time, aging, race, and sex roles suggests how difficult it is to perceive that there are things to challenge.

In the next chapter we explore the first phase of problem posing—where we accept the given—and in chapter 4, in which we introduce the second phase, the "What-If-Not" approach.