

The Common Core State Standards in Mathematics

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Background

In 2009, 48 out of the 50 states in the U.S. came together under the leadership of the National Governors Association (NGA) and the Council of Chief State School Officers (CCSSO) to write school standards for Mathematics and English Language Arts that would ensure students leaving high school are ready for college and career. The Common Core State Standards (CCSS) were released on 2 June 2010, and they have been adopted by 45 states.¹

Twenty years earlier, in 1989, the National Council of Teachers of Mathematics made the first move in the modern era towards a common understanding of school mathematics [7]. Before that time curriculum varied among the nation's numerous school districts. The NCTM standards were not themselves an act of government, but in response to them the governments of the 50 states started developing their own standards, bringing a measure of consistency to the mathematics curriculum within states. However, consistency between states proved elusive and in the years after 1989 state standards diverged greatly. For example, Table 1 shows the 2006 distribution of grade levels at which 42 state standards introduced addition and subtraction of fractions:

Grade	1	2	3	4	5	6	7	8
Number of states	2		7	22	9	1	1	

Table 1: Distribution of grade levels where state standards introduce addition and subtraction of fractions[12].

The situation was famously described as follows in 1997:

There is no one at the helm of mathematics and science education in the U.S. . . . No single coherent vision of how to educate today's children dominates U.S. educational practice

These splintered visions produce unfocused curricula and textbooks . . . [that] emphasize familiarity with many topics rather than concentrated attention to a few . . . [and] likely lower the academic

¹A 46th state, Minnesota has adopted the English Language Arts Standards.

performance of students who spend years in such a learning environment. *Our curricula, textbooks, and teaching all are “a mile wide and an inch deep.”* [emphasis added] [14]

The mile wide inch deep curriculum is partly a natural result of the U.S. system of local control, in which there is no central Ministry of Education, authority over education is delegated to the states and, on many questions of policy, to the 16,000 school districts. Another cause was the math wars, an ideological conflict about almost every question related to mathematics education: curriculum, assessment, methods of teaching, the nature of mathematics itself. The fragmented system of local control allowed this debate to rage unchecked, drawing school boards, parent groups, curriculum reformers, policy makers, and university faculty into a draining conflict about curriculum materials at the expense of work on other important problems in mathematics education.

Various efforts in the last decade have had a primary or secondary goal of improving this situation by bringing the different sides together and aligning state standards with each other and with international standards: the American Diploma Project, Finding Common Ground in K–12 Mathematics Education, Adding it Up, the NCTM publications *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics* and *Focus in High School Mathematics*, reports by ACT and College Board, and the report of the National Mathematics Advisory Panel ([11], [6], [5], [8], [9], [1], [2], [10]).

These efforts came to a head with the surprisingly rapid consensus around developing common standards in 2009. The initiative was spearheaded by NGA and CCSSO, two new actors in the world of standards writing, who showed agility in bypassing old stalemates. Key to the success of the endeavor in political terms was that unlike previous efforts, the Common Core effort was led by state policymakers, not the federal government. The standards drew on many sources, including the reports and publications mentioned earlier, and also including standards of U.S. states and high achieving countries, particularly in East Asia. A recent analysis has shown that the standards are closely aligned with the standards of the A+ countries, a group of countries that formed a statistically significant group of top achievers on TIMSS 1995, and that state achievement in mathematics on the National Assessment of Education Progress is correlated with the closeness of previous state standards to the Common Core[13].

With the great majority of states adopting CCSS, the U.S has entered an unprecedented time of opportunity, with long-standing arrangements in mathematics education now open to renegotiation. For over 20 years discourse in mathematics education has been dominated by the math wars, in which apparently divergent views of mathematics competed for dominance; it was difficult, for example, to advocate both that students acquire fluency with algorithms for addition, subtraction, multiplication, and division in elementary school, and that they engage in serious work with statistics in high school, without being viewed askance by both camps. But CCSS incorporates both stances, viewing the two as integral parts of a coherent progression in skill and understanding.

What are the opportunities? First, the opportunity for curriculum developers to produce more focused and coherent materials, without having to attend to diverse demands for topic placement made by different state standards; second, the opportunity for teacher preparation and professional development to become less generic and more focused on the mathematics taught at a given grade level; and finally, the opportunity for teachers from across the country to share tools for implementation based on common standards.

What should standards look like?

In countries with a fully functioning education system, they can look like Figure 1. I was one of the lead writers of the Common Core; we sometimes dreamed of the ability to make simple bulleted lists like this. How does Singapore get away with this? we asked ourselves. The answer is that Singapore has a Ministry of Education that produces curriculum and exams; their standards document is a description of that system, not a prescription for it.

Figures 2 and 3 show two images of standards in the U.S. The NCTM standards have 14 “standards” (bulleted items) for Number and Operations, Grades 6–8, followed by 7 pages of narrative. To a certain extent the NCTM had an education system available as well, or rather two systems: the system of commercial and NSF-funded textbook projects for producing curriculum, and the system of 50 state departments of education for producing exams. The system was divergent, chaotic, and voluntary.

Secondary One	
Topic/Sub-topics	Content
Algebraic representation and formulae	Include: <ul style="list-style-type: none"> • using letters to represent numbers • interpreting notations: <ul style="list-style-type: none"> * ab as $a \times b$ * $\frac{a}{b}$ as $a \div b$ * a^2 as $a \times a$, a^3 as $a \times a \times a$, a^2b as $a \times a \times b$, ... * $3y$ as $y + y + y$ or $3 \times y$ * $\frac{3 \pm y}{5}$ as $(3 \pm y) \div 5$ or $\frac{1}{5} \times (3 \pm y)$ • evaluation of algebraic expressions and formulae • translation of simple real-world situations into algebraic expressions • recognising and representing number patterns (including finding an algebraic expression for the nth term)
Algebraic manipulation	Include:

Figure 1: A page from the Singapore secondary standards

Number and Operations

STANDARD

for Grades

6–8

*Instructional programs from
prekindergarten through grade 12
should enable all students to—*

Expectations

In grades 6–8 all students should—

Understand numbers, ways of representing numbers, relationships among numbers, and number systems

- work flexibly with fractions, decimals, and percents to solve problems;
- compare and order fractions, decimals, and percents efficiently and find their approximate locations on a number line;
- develop meaning for percents greater than 100 and less than 1;
- understand and use ratios and proportions to represent quantitative relationships;
- develop an understanding of large numbers and recognize and appropriately use exponential, scientific, and calculator notation;
- use factors, multiples, prime factorization, and relatively prime numbers to solve problems;
- develop meaning for integers and represent and compare quantities with them.

Understand meanings of operations and how they relate to one another

- understand the meaning and effects of arithmetic operations with fractions, decimals, and integers;
- use the associative and commutative properties of addition and multiplication and the distributive property of multiplication over addition to simplify computations with integers, fractions, and decimals;
- understand and use the inverse relationships of addition and subtraction, multiplication and division, and squaring and finding square roots to simplify computations and solve problems.

Compute fluently and make reasonable estimates

- select appropriate methods and tools for computing with fractions and decimals from among mental computation, estimation, calculators or computers, and paper and pencil, depending on the situation, and apply the selected methods;
- develop and analyze algorithms for computing with fractions, decimals, and integers and develop fluency in their use;
- develop and use strategies to estimate the results of rational-number computations and judge the reasonableness of the results;
- develop, analyze, and explain methods for solving problems involving proportions, such as scaling and finding equivalent ratios.

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Principles and Standards for School Mathematics

Figure 2: A page from the 2000 NCTM Principles and Standards for School Mathematics

MATHEMATICS STANDARD ARTICULATED BY GRADE LEVEL

GRADE 6

Strand 1: Number Sense and Operations

Every student should understand and use all concepts and skills from the previous grade levels. The standards are designed so that new learning builds on preceding skills and are needed to learn new skills. Communication, Problem-solving, Reasoning & Proof, Connections, and Representation are the process standards that are embedded throughout the teaching and learning of mathematical strands.

Concept 1: Number Sense
Understand and apply numbers, ways of representing numbers, the relationships among numbers and different number systems.
PO 1. Express fractions as ratios, comparing two whole numbers (e.g., $\frac{3}{4}$ is equivalent to 3:4 and 3 to 4).
PO 2. Compare two proper fractions, improper fractions, or mixed numbers.
PO 3. Order three or more proper fractions, improper fractions, or mixed numbers.
PO 4. Determine the equivalency between and among fractions, decimals, and percents in contextual situations.
PO 5. Identify the greatest common factor for two whole numbers.
PO 6. Determine the least common multiple for two whole numbers.
PO 7. Express a whole number as a product of its prime factors, using exponents when appropriate.

Figure 3: A page from a typical set of state standards, 2008

The document illustrated in Figure 3 (which is representative of state standards at the time) has 82 standards for Number and Operations in Grade 6 alone, and no pages of narrative. This is much more detailed and performance-based than the NCTM standards. Unlike the NCTM standards, state standards have direct policy and legal consequences, and are used a basis for writing assessments. They are flat lists of performance objectives of even grain size, designed to be delivered into the hands of assessment writers without the need for too much discussion or interpretation.

It was against this background that the Common Core State Standards were written. On the one hand they were commissioned by the states and therefore had to be the type of document states were used to: detailed bulleted lists describing what we want students to know and be able to do. On the other hand, we were being asked to do something new, to break out of the system that produced the mile wide inch deep curriculum.

Design of the Standards

The fundamental design principles for the Standards are focus, coherence, and rigor.

Focus means attending to fewer topics in greater depth at any given grade level, giving teachers and students time to complete that grade’s learning.

Coherence means attending to the structure of mathematics and the natural pathways through that structure, where “natural” means taking into account both the imperatives of logic and the imperatives of cognitive development in designing the sequence of ideas. Since these two imperatives are sometimes in conflict, attaining coherence is a complex exercise in judgement, requiring a certain amount of professional craft and wisdom of practice not easily obtained from any one source.

Rigor means balancing conceptual understanding, procedural fluency, and meaningful applications of mathematics. Here the word rigor is used not in the way that mathematicians use it, to indicate a correct and complete chain of logical reasoning, but in the sense of a rigorous preparation for a sport or profession: one that exercises all the necessary proficiencies in a balanced way.

Organization of the standards

The Standards are divided into Standards for Mathematical Content and Standards for Mathematical Practice. The content standards are further subdivided into K–8 standards and high school standards. The K–8 standards are specified by grade level and organized into *domains*, topics which follow a coherent progression over a certain grade span (see Figure 4).

K	1	2	3	4	5	6	7	8
Geometry								
Measurement and Data					Statistics and Probability			
Number and Operations in Base Ten					The Number System			
Operations and Algebraic Thinking					Expressions and Equations			
Counting and Cardinality		Number and Operations—Fractions			Ratios and Proportional Relationships		Functions	

Figure 4: Domains in the Common Core, Grades K–8

The organization by domains is different in an important way from the organization by strands typical of previous state standards. Under the latter scheme, four or five strands (e.g., Number and Operations, Algebra and Functions, Data and Measurement, Geometry) would extend from Kindergarten to

Grade 12. The homogeneity of this scheme with respect to time is at odds with the progressive nature of mathematics, and resulted in a tendency to fill in every cell of the grade-by-strand matrix, one of the causes of the mile wide inch deep curriculum.

By contrast, domains operate at a finer level (there are 12 domains in the K–8 standards), and have a beginning and an end, each preparing for and eventually giving way to higher domains that both build on and encapsulate previous work. Domains allow for convergence and consolidation of ideas, as when the K–5 number work in the domains Operations and Algebraic Thinking, Number and Operations in Base Ten, and Fractions, is consolidated into a unified understanding of The Number System in Grades 6–8. The abbreviated life time of a domain also allows for the delineation of foundational domains that support more than one future domain: the work on Fractions in Grades 3–5 is a basis for The Number System, but also for the work on Ratios and Proportional Relationships in Grades 6–7, leading to Functions in Grade 8.

The high school standards are arranged into the broad *conceptual categories* shown in Table 5, which are further divided into domains as in the K–8 standards. However, the high school standards are not arranged into grade levels, and so the domains do not always exhibit a temporal progression. Some of the domains are conventional topics (e.g. Congruence is a domain in Geometry); others describe ways of thinking that help students bind their mathematical knowledge into coherent packages rather than trying to remember innumerable different formulas and techniques. For example, the Algebra category has a domain Seeing Structure in Expressions, which undergirds a student’s work during the entire high school experience, from linear functions to logarithms.

Number and Quantity
Algebra
Functions
Modeling
Geometry
Statistics and Probability

Figure 5: High school conceptual categories in the Common Core

Just as the content standards attempt to describe the complex structure of mathematical knowledge, the Standards for Mathematical Practice (see Figure 6) describe the contours of mathematical practice; the various ways in which proficient practitioners of mathematics carry out their work. These are not intended to free floating proficiencies, unconnected with content, nor are they uniformly applied over all the work that students do. Just as a rock climber’s various skills are called on differently during different parts of a climb, so specific aspects of practice become salient in specific pieces of mathematical work. For example, students learning how to complete the square in a quadratic expression benefit from consciously looking for structure and seeing the regularity in reasoning with a sequence of well-chosen examples (SMP 7 and 8); students

constructing geometric proofs will learn to critique arguments and use precise language (SMP 3 and 6); students designing a study to see if there is a connection between athletic and academic proficiency will construct a statistical model and choose appropriate methods and technologies (SMP 4 and 5).

- | |
|---|
| SMP.1 Make sense of problems and persevere in solving them |
| SMP.2 Reason abstractly and quantitatively |
| SMP.3 Construct viable arguments and critique the reasoning of others |
| SMP.4 Model with mathematics |
| SMP.5 Use appropriate tools strategically |
| SMP.6 Attend to precision |
| SMP.7 Look for and make use of structure |
| SMP.8 Look for and express regularity in repeated reasoning |

Figure 6: Standards for Mathematical Practice

Taking focus seriously

Four out of the six domains in K–5 deal with number and operations (see Figure 4): Counting and Cardinality (Kindergarten), Number and Operations in Base Ten (K–5), Operations and Algebraic Thinking (K–5), and Fractions (3–5).

The focus on number and operations in elementary school is even stronger than this count would suggest, because many standards in the other domains are designed to support the focus on number and operations. For example, the following data standard in Grade 2 supports the principle work with addition and subtraction of whole numbers:

2.MD.10. Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put-together, take-apart, and compare problems using information presented in a bar graph.

As another example, many geometry standards in elementary school deal with composing and decomposing figures, and dealing with composite figures, supporting the unit fraction approach to fractions starting in Grade 3.

In order to make room for the focus on number and operations, some topics are given much less time in elementary school than was the case with previous state standards. This was a necessary step to make good on the promise of repairing the mile wide inch deep curriculum. For example, standards on data, patterns, and symmetry are reduced to a trickle in elementary school. This

was one of the more controversial shifts in the Common Core, and it is worth looking at in a little more detail. Debate about curriculum in the United States has suffered from an all-or-nothing quality, and nowhere is this seen more clearly than in the debate about data and statistics in elementary school: it has seemed that the only choices were embracing a rich stream on data work in elementary school, as advocated by the GAISE report[4], or drying it up to nothing. In contrast, the Common Core is based on progressions that start with a trickle before they grow into the full flow of a domain. Thus the data standards in elementary school are neither to be ignored nor to be given undue prominence. In due time, in high school, statistics and probability becomes a major topic.

The function concept is another topic that is delayed compared to previous state standards. There is a trickle of pattern standards in elementary school, carefully worded to support the emergence of an incipient notion of function:

3.OA.9. Identify arithmetic patterns (including patterns in the addition table or multiplication table), and explain them using properties of operations.

4.OA.5. Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself.

5.OA.3. Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns, and graph the ordered pairs on a coordinate plane.

Figure 7: The trickle of pattern standards in elementary school

In middle school, further preparation for functions is provided in the domains Ratios and Proportional Relationships and Expressions and Equations. The function concept finally makes its appearance in its own domain in Grade 8, and becomes a major conceptual category in high school.

As the examples of statistics and functions illustrate, taking focus seriously means delaying favored topics until their time, which will be a difficult shift for the educational system in the U.S.

The payoff for this approach occurs in high school, where the subject matter focus broadens as the foundations developed in K–8 allow for a variety of work in number and quantity, algebra, functions, modeling, geometry, statistics, and probability. Focus in high school means not so much a small number of topics as a concentration of skills and practice into a small number of underlying principles.

Preserving coherence

The act of writing standards for a subject is inherently in conflict with the goal of showing the structure of the subject. In [3] this is likened to shattering

an intricately decorated Grecian urn into pieces and expecting the shape and decorative details to be visible in the pieces:

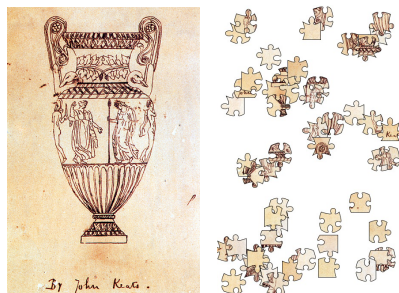


Figure 8: Standards for a Grecian Urn

In order to avoid this problem and preserve a coherent view of the subject, both in the broad contours and in the small details, the Common Core breaks with a long-standing tradition that each individual standard should have the same “grain-size”. Mathematics itself does not come in pieces of equal grain-size, and neither should a description of it. Consider, for example the Grade 2 cluster of standards shown in Figure 9.

Understand place value

2.NBT.1. Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases:

- a 100 can be thought of as a bundle of ten tens—called a “hundred.”
- b The numbers 100, 200, 300, 400, 500, 600, 700, 800, 900 refer to one, two, three, four, five, six, seven, eight, or nine hundreds (and 0 tens and 0 ones).

2.NBT.2. Count within 1000; skip-count by 5s, 10s, and 100s.

2.NBT.3. Read and write numbers to 1000 using base-ten numerals, number names, and expanded form.

2.NBT.4. Compare two three-digit numbers based on meanings of the hundreds, tens, and ones digits, using $>$, $=$, and $<$ symbols to record the results of comparisons.

Figure 9: Grade 2 cluster of standards on place value

The first standard is large and fundamental, figures in much of the work of

elementary school, and will show up again and again in a curricular implementation of the standards, reinforced and deepened by work in later grades. The second standard is a discrete performance objective that, once secured, recedes from importance.

This example illustrates another feature of the standards designed to provide coherence: the clusters and cluster headings. In the Common Core, the individual statements of what students are expected to understand and be able to do (the “standards”) are embedded within cluster headings, which are in turn embedded in domains. “The Standards” refers to all elements of the document’s design, including the wording of domain headings, cluster headings, and individual statements. In this case, the cluster heading “Understand place value” says clearly the fundamental purpose of this cluster, in a way that is not completely captured by any individual standard within the cluster.

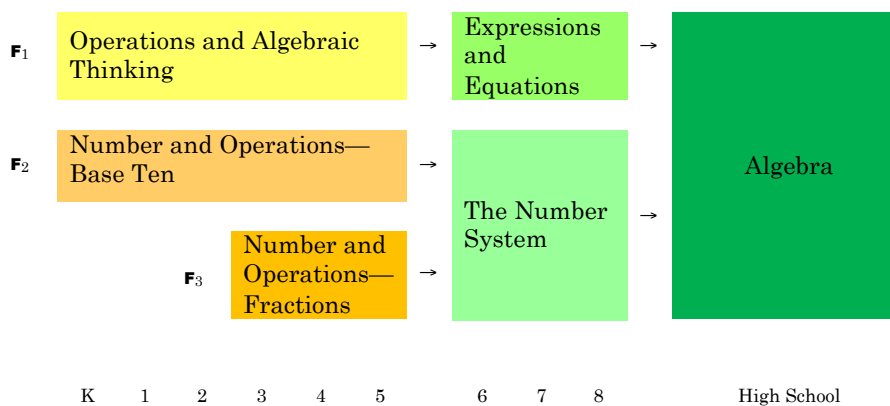


Figure 10: Flows leading to algebra

Another aspect of coherence is the flow of domains across grade levels, described on page 7. As a further example of this, Figure 10 shows the flow of domains to high school algebra. Building a viable ramp to algebra was a design requirement implied by the mandate to write standards that prepared students for college and career.

Balancing understanding, fluency, and applications

State standards before the Common Core were often formulated in terms of concrete observable performances, following a hierarchy of verbs, in which some verbs describe higher levels of performance than others (e.g., memorize, interpret, formulate, analyze). One verb in particular was often avoided, however, the verb “understand.” Many times during the writing of the standards we were told we could not use this verb because standards had to be measurable, and understanding was ill-defined and either impossible or very difficult to measure.

Nonetheless, if the goal of standards is to express our desires for our children’s achievements, it is hard to argue that understanding is not among them. The Common Core calls explicitly for understanding in a number of standards and cluster headings (see Figure 11).

- Understand and apply properties of operations and the relationship between addition and subtraction (Grade 2)
- Understand concepts of area and relate area to multiplication and to addition (Grade 3)
- Understand ratio concepts and use ratio reasoning to solve problems (Grade 6)
- Understand congruence and similarity using physical models, transparencies, or geometry software (Grade 8)
- Understand solving equations as a process of reasoning and explain the reasoning (High School)
- Understand and evaluate random processes underlying statistical experiments (High School)

Figure 11: Selected cluster headings using the word “understand”

Other standards explicitly call for fluency with addition and multiplication facts and with standard algorithms for addition, subtraction, multiplication, and division. These are capstone standards, occurring after adequate groundwork in earlier grades on strategies and algorithms based on place value and the properties of operations.

Yet other standards call for students to apply the mathematics they have learned. Modeling with mathematics is one of the Standards for Mathematical Practice (see Figure 6), and many of the high school standards are flagged as particularly important venues for modeling with a special symbol. The elementary and middle school standards build towards this with a progression of standards from simple word problems involving addition of whole numbers in Grade 2 to the following culminating standard in Grade 7:

7.EE.3. Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies.

Concluding thoughts

Let me conclude by mentioning two projects designed to round out the description of the standards:

- The Progressions Project, ime.math.arizona.edu/progressions/
- The Illustrative Mathematics Project, illustrativemathematics.org

The standards were founded on progressions, narrative descriptions of domains provided by experts on the working team. The original progressions documents did not keep up with the rapid revision process, and the Progressions Project aims to produce final versions of them. The progressions provide another view of the standards, useful for curriculum designers, teacher educators, and could support research aimed at making recommendations for revisions to the standards.

The Illustrative Mathematics project is collecting sample tasks to illustrate the standards. It uses a community based approach, in which tasks are submitted, reviewed, edited and finally published by a growing community of experts who gain discernment and craft by participating in the process. It aims to become a permanent virtual destination that is both a repository of materials and a place where an expert community works.

I have tried to give some idea of what the standards look like, but ultimately a close reading of the standards is necessary to gain a complete picture. The standards are not designed to be easy reading, but they are designed to be read. The promise of the Common Core is having a shared text that, whatever its virtues and flaws, provides the basis of disciplined innovation in curriculum and shared tools for teaching.

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