A Sampler of Graphs

By Robin Sanders

In this notebook. we look at the *shape* of some important graphs. These are graphs that you should immediately know what kind of graph you'll get from the formula AND also you should know what kind of formula is likely to be involved from looking at a picture. *This notebook is based on the material in Section 1.3 of the text.*

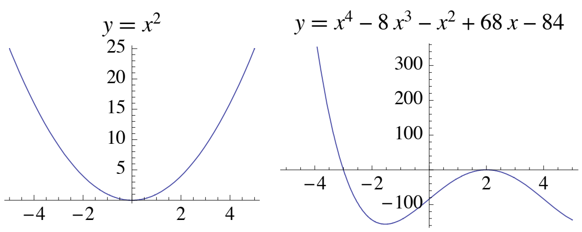
Polynomials

For a complete review of polynomial functions you should go through the material in Appendix D in the textbook. Polynomials come in two main flavors: Even degree polynomials and odd degree polynomials. If you zoom out *far enough* to see the *global behavior,* all even degree polynomials eventually have the same basic shape. Likewise, if you zoom out *far enough* to see the *global behavior,* all odd degree polynomials eventually have the same basic shape.

▪ Even degree polynomials

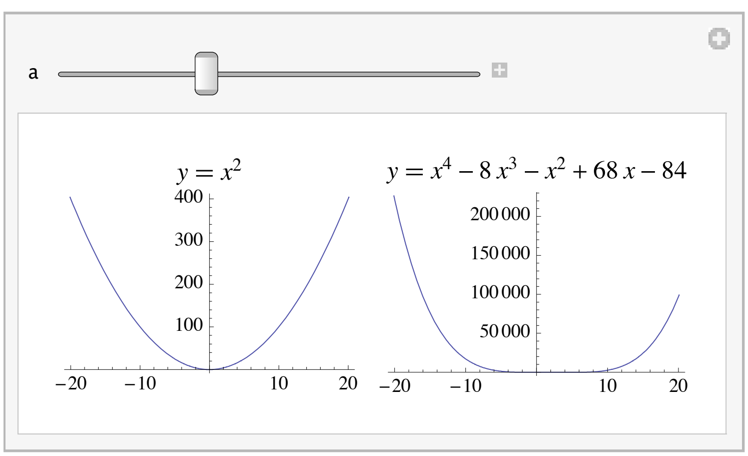
For example, let's compare the graph of $f(x) = x^2$ to the graph of $g(x) = x^4 - 8x^3 - x^2 + 68x - 84$. If we plot over a narrow range of x values the pictures are quite different:

```
f1[x_] := x^2;    g1[x_] := x^4 - 8 x^3 - x^2 + 68 x - 84;
TableForm[{{Plot[f1[x], {x, -5, 5}, PlotLabel → y == f1[x]],
            Plot[g1[x], {x, -5, 5}, PlotLabel → y == g1[x]]}}]
```



But watch what happens when we zoom out---i.e. when we *increase* the range of x-values in the plot:

```
Manipulate[TableForm[{{Plot[f1[x], {x, -a, a}, PlotLabel → y == f1[x]],
             Plot[g1[x], {x, -a, a}, PlotLabel → y == g1[x]]}}], {a, 5, 50}]
```



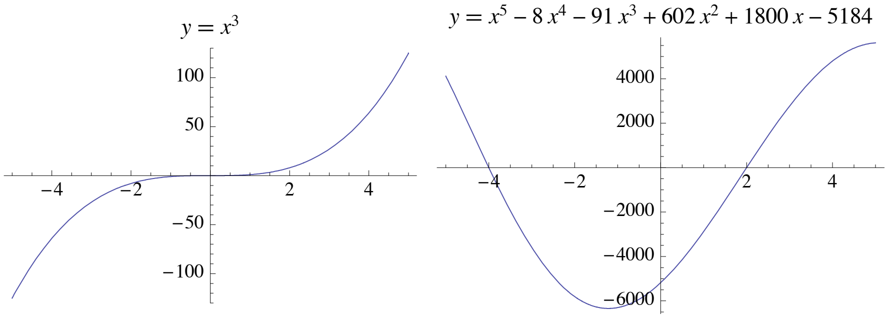
Except for the *size* of the y-values (and the flatness at the bottom of the U), this graph looks similar to the graph of $y = x^2$. In fact

ALL even degree polynomials with a POSITIVE leading coeffient eventually have a  ***shape when you zoom out far enough.***

■ Odd degree polynomials

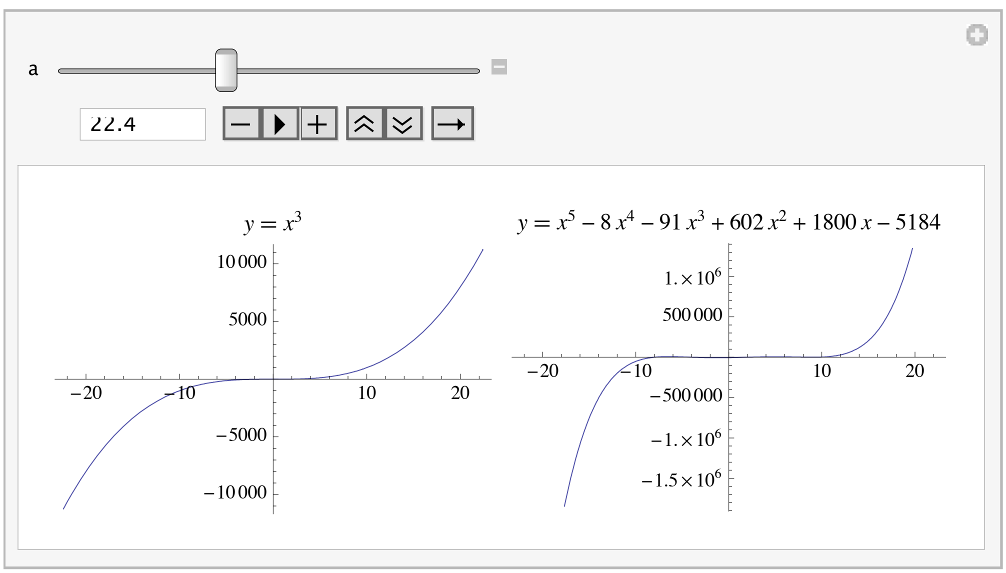
For example, let's compare the graph of $f(x) = x^3$ to the graph of $g(x) = x^5 - 8x^4 - 91x^3 + 602x^2 + 1800x - 5184$. If we plot over a narrow range of x values the pictures are quite different:

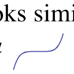
```
f2[x_] := x^3; g2[x_] := x^5 - 8 x^4 - 91 x^3 + 602 x^2 + 1800 x - 5184
TableForm[{{Plot[f2[x], {x, -5, 5}, PlotLabel → y == f2[x]],
            Plot[g2[x], {x, -5, 5}, PlotLabel → y == g2[x]]}}]
```



But watch what happens when we zoom out---i.e. when we *increase* the range of x-values in the plot:

```
Manipulate[TableForm[{{Plot[f2[x], {x, -a, a}, PlotLabel → y == f2[x]],
             Plot[g2[x], {x, -a, a}, PlotLabel → y == g2[x]]}}],
 {a, 5, 50}, TableSpacing → {0, 10}]
```



Except for the *size* of the y-values , this graph looks similar to the graph of $y = x^3$. In fact ***ALL ODD degree polynomials with a POSITIVE leading coefficient eventually have a***  ***shape if you zoom out far enough.***

Note about polynomials with NEGATIVE leading coefficients

If the leading coefficient of a polynomial is NEGATIVE, the global shaped is flipped over the *x*-axis. So we get TWO IMPORTANT FACTS:

If a polynomial has EVEN degree and a NEGATIVE leading coefficient, then it's global shape looks like 

If a polynomial has ODD degree and a NEGATIVE leading coefficient, then it's global shape looks like 

Rational functions

For a complete review of rational functions you should go through the material in Appendix D in the textbook.

A *rational function's symbolic formula* is the ratio of two polynomials. In other words, if $f(x)$is a rational function, then $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$and $q(x)$are polynomials.

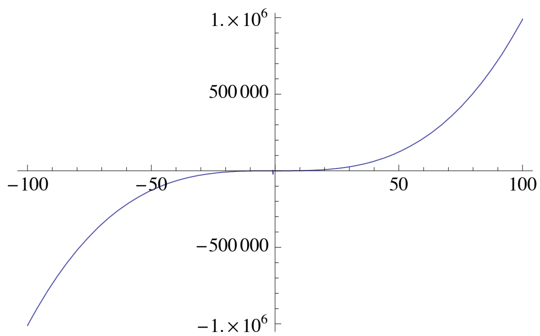
The *global shape* of any RATIONAL FUNCTION is determined by nothing more than the dominant terms of its numerator and denominator. Ignore all the lower order terms and just look at the ratio of the two dominant terms. The *local behavior* of the function depends on both the dominant terms and the lower order terms. As you saw in the first MAT163 lab, the roots of the rational function and the roots of the polynomial in the numerator are typically the same; vertical asymptotes occur where the denominator equals 0, and we get *holes* in the rational function at places where ***both the numerator and denominator equal 0***.

- **Example 1: Predict the global behavior of $f(x) = \frac{x^9 + 25x^3 - 17}{x^6 + x^5 + x^4 + x^3 + x^2 + x + 1.}$.**
  WRITE YOUR PREDICTION HERE:

  WRITE YOUR ANSWER BEFORE TURNING THE PAGE!

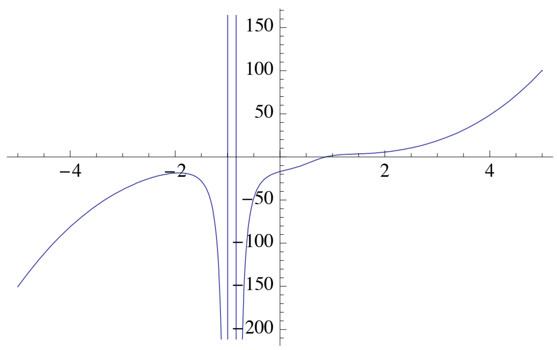
And now we check it out:

```
f3[x_] = (x^9 + 25 x^3 - 17) / (x^6 + x^5 + x^4 + x^3 + x + 1);
Plot[f3[x], {x, -100, 100}]
```



Note that if we zoom in near 0, we will see some of the local behavior:

```
Plot[f3[x], {x, -5, 5}]
```

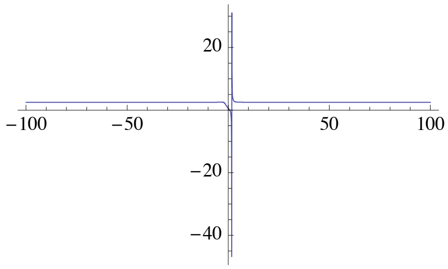


- **Example 2: Predict what happens to $f(x) = \frac{5x^5 + 24x - 19}{2x^5 - 32}$ as $x \to \pm\infty$.**
  WRITE YOUR PREDICTION HERE:

DO NOT TURN THE PAGE UNTIL YOU'VE WRITTEN DOWN YOUR PREDICTION ABOUT THE GLOBAL SHAPE

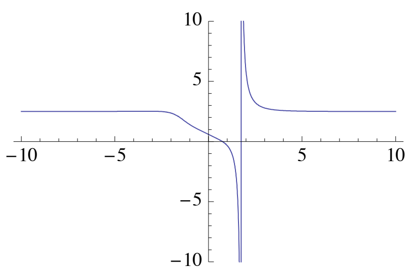
And now we check it out:

```
f4[x_] = (5 x^5 + 24 x - 19) / (2 x^5 - 32);
Plot[{f4[x]}, {x, -100, 100}, PlotRange → All]
```



Local behavior seems to start kicking in when $|x| \leq 10$. You can further explore this for homework if you want, but here's a starting plot similar to what **ZStandard** should show on your calculator:

```
Plot[f4[x], {x, -10, 10}, PlotRange → {-10, 10}]
```

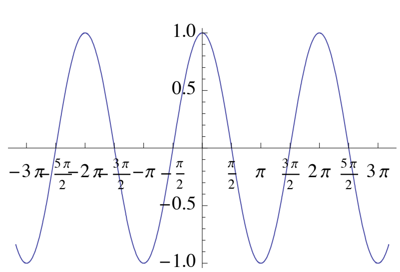


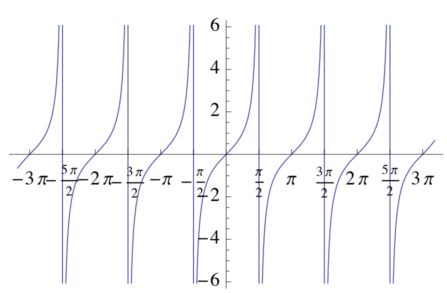
Trig functions

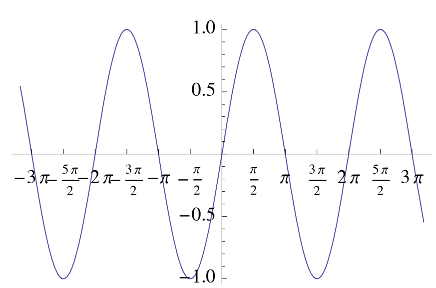
For additional information about these six trig functions you should consult Appendix F in the textbook.

■ Sin[x] , Cos[x], and Tan[x]

There are a total of *six* common trignometric functions used in calculus. Three of them (sin[x], cos[x], and tan[x]) are *very important for* you to know on sight. So let's play ***identify the trig graphs:*** Which is $y = \sin(x)$? which is $y = \cos(x)$? which is $y = \tan(x)$? Label the graphs directly on this sheet.

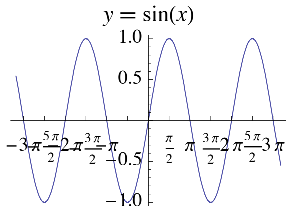






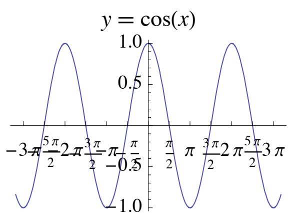
Now lets check our guesses:

```
TrigTicks = Table[(k Pi) / 2, {k, -100, 100}];
Plot[Sin[x], {x, -10, 10}, Ticks → {TrigTicks, Automatic}, PlotLabel → y == Sin[x]]
```

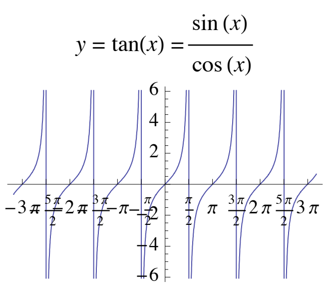


And now check your graph for Cos[x]:

```
Plot[Cos[x], {x, -10, 10}, Ticks → {TrigTicks, Automatic}, PlotLabel → y == Cos[x]]
```




```
Plot[Tan[x], {x, -10, 10}, Ticks → {TrigTicks, Automatic}, PlotLabel → "y = tan(x) = sin (x) / cos (x)"]
```

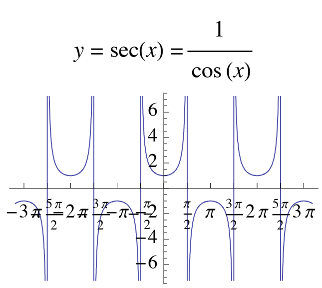


■ The other three trig functions

The other three trig functions are the reciprocals of the first (important) three trig functions.

Secant is the reciprocal of the cosine. Here's a plot of $y = \sec(x) = \frac{1}{\cos(x)}$.

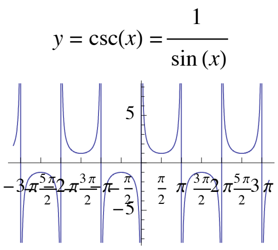
```
Plot[Sec[x], {x, -10, 10}, Ticks → {TrigTicks, Automatic}, PlotLabel → "y = sec(x) = 1 / cos (x)"]
```



The vertical asymptotes for $\sec(x)$ occur when $\cos(x) = 0$; the tops & bottoms of the U's have $y = \pm 1$and occur where $\cos(x) = \pm 1$.

Cosecant is the reciprocal of the sine. Here's a plot of $y = \csc(x) = \frac{1}{\sin(x)}$.

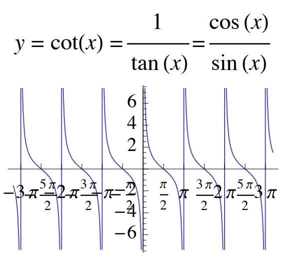
```
Plot[Csc[x], {x, -10, 10}, Ticks → {TrigTicks, Automatic}, PlotLabel → "y = csc(x) = 1/sin(x)"]
```



The vertical asymptotes for csc(x) occur when $\sin(x) = 0$; the tops & bottoms of the U's have $y = \pm 1$and occur where $\sin(x) = \pm 1$.

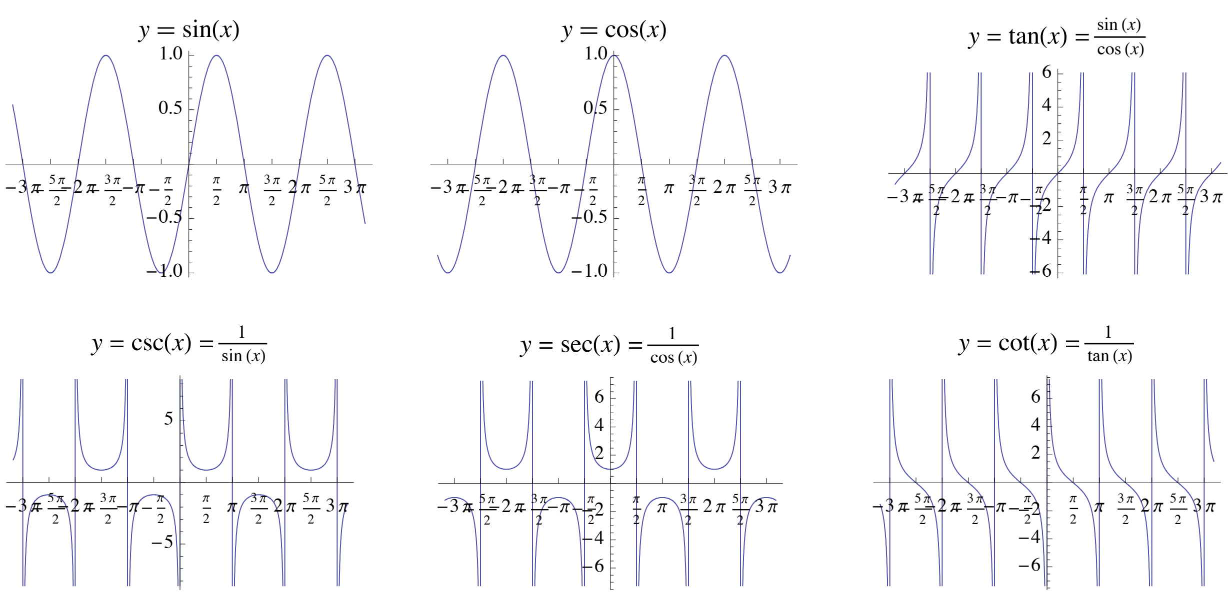
Cotangent is the reciprocal of the tangent. Here's a plot of $y = \cot(x) = \frac{1}{\tan(x)} = \frac{\cos(x)}{\sin(x)}$.

```
Plot[Cot[x], {x, -10, 10}, Ticks → {TrigTicks, Automatic},
 PlotLabel → "y = cot(x) = 1/tan(x) = cos(x)/sin(x)"]
```



Note the vertical asymptotes for $y = \cot(x)$ occur where $\sin(x) = 0$ and $y = \cot(x)$ crosses the x-axis when $\cos(x) = 0$.

- **Putting all the Trig functions together: Here's are all six trig functions shown together.**



Exponential functions

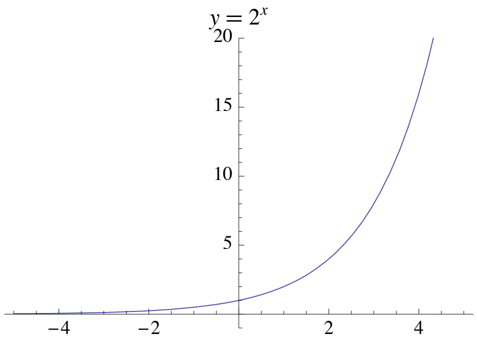
■ The graph of $f(x) = b^x$

The standard formula for a basic exponential function is $f(x) = b^x$. What's it look like if $b > 1$? **Draw your guess here:**

DO NOT TURN THE PAGE UNTIL YOU'VE DRAWN YOUR GUESS!

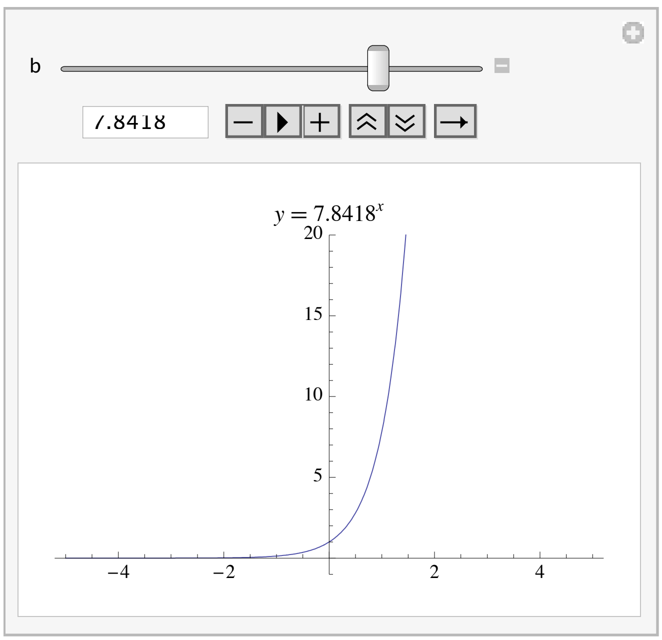
And let's now check if you're correct:

```
b = 2; Plot[b^x, {x, -5, 5}, PlotRange → {-1, 20}, PlotLabel → y == b^x]
```



Now let's play with what happens to the graph as we change the value of *b*:

```
Manipulate[
 Plot[b^x, {x, -5, 5}, PlotRange → {-1, 20}, PlotLabel → y == b^x], {{b, 2}, 1 / 10, 10}]
```



Your observations about all exponential graphs: (FILL IN THE BLANKS BEFORE YOU TURN THE PAGE)

1) All the $y = b^x$ graphs go through the point ______________________________

2) When $b > 1$, the graph of $y = b^x$ is ______________________________

and we have an ____________ asymptote at ______________________________

3) When $b = 1$, the graph of $y = b^x$ is ______________________________

4) When $0 < b < 1$, the graph of $y = b^x$ is ______________________________

and we have an ____________ asymptote at ______________________________

Our class observations about all exponential graphs:

(These are the answers to the FILL IN THE BLANKS on the previous page.)

1) All the $y = b^x$ graphs go through the point **(1, 0) because** $b^0 = 1$ **for all values of** b.

2) When $b > 1$, the graph of $y = b^x$ is ***increasing and concave up. Also notice that as x → -∞, $b^x \to 0$. So we also have a one-sided horizontal asymptote at x = 0.***

3) When $b = 1$, the graph of $y = b^x$ is ***the horizontal line y = 1.***

4) When $0 < b < 1$, the graph of $y = b^x$ is ***decreasing and concave up. Also notice that as x → +∞, $b^x \to 0$. So we also have a one-sided horizontal asymptote at x = 0.***

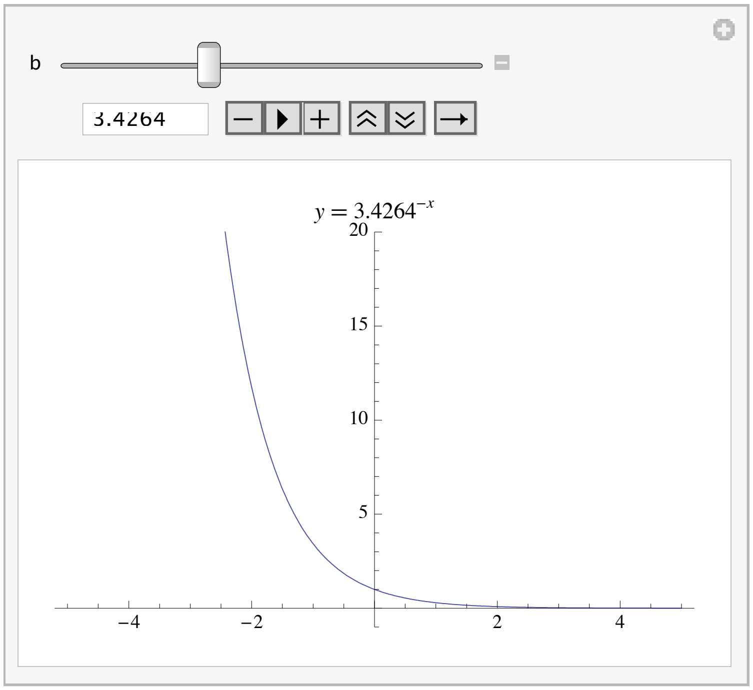
▪ The graph of $f(x) = b^{-x}$

An important variation on the theme of exponential functions are exponential functions with ***negative*** exponents. What's the shape of $y = b^{-x}$ look like? **Draw what you think $y = e^{-x}$ looks like here:**

DO NOT TURN THE PAGE UNTIL YOU'VE DRAWN YOUR GUESS!

Now let's check:

```
Manipulate[
 Plot[b^(-x), {x, -5, 5}, PlotRange → {-1, 20}, PlotLabel → y == b^(-x)], {{b, 2}, 1/10, 10}]
```



Your observations about all exponential graphs with negative exponents:
(FILL IN THE BLANKS BEFORE YOU TURN THE PAGE)

1) All the $y = b^x$ graphs go through the point ____

2) When $b > 1$, the graph of $y = b^x$ is ____

and we have an ______________________ asymptote at __________________________________

3) When $b = 1$, the graph of $y = b^x$ is ____

4) When $0 < b < 1$, the graph of $y = b^x$ is ____

and we have an ______________________ asymptote at __________________________________

Our class observations about all exponential graphs with negative exponents:
(These are the answers to the FILL IN THE BLANKS on the previous page.

1) All the $y = b^{-x}$ graphs go through the point **(1, 0) because** $b^0 = 1$ **for all values of** b.

2) When $b > 1$, the graph of $y = b^{-x}$ is ***decreasing and concave up. Also notice that as x → ∞, $b^x \to 0$. So we also have a one-sided horizontal asymptote at x = 0.***

3) When $b = 1$, the graph of $y = b^{-x}$ is ***the horizontal line y = 1.***

4) When $0 < b < 1$, the graph of $y = b^x$ is ***increasing and concave up. Also notice that as x → -∞, $b^x \to 0$. So we also have a one-sided horizontal asymptote at x = 0.***

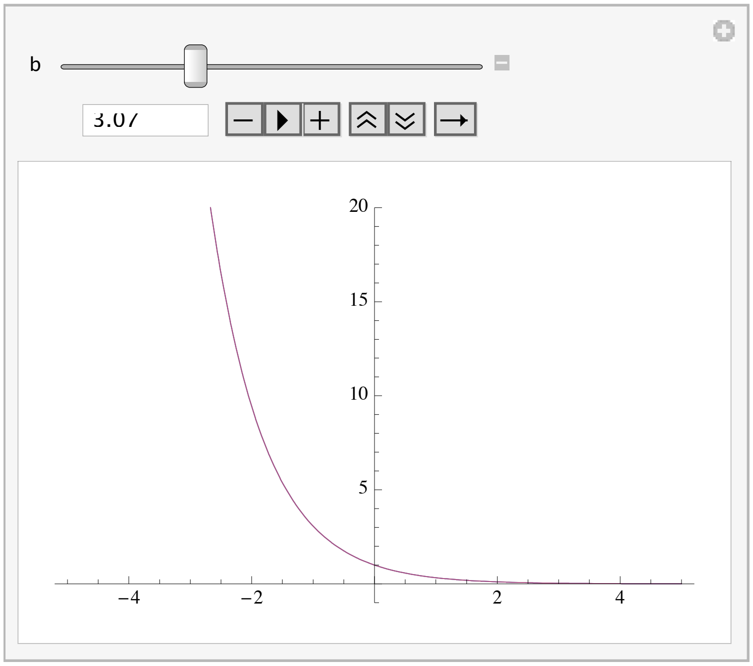
■ TWO important connections concerning exponential graphs

■ *CONNCTION 1: What's the connection between the graph of $y = b^{-x}$ and the graph $y = (1/b)^x$?* Fill in your prediction here*:*

DO NOT TURN THE PAGE UNTIL YOU'VE MADE YOUR PREDICTION!

Here are the plots:

```
Manipulate[Plot[{b^(-x), (1/b)^x}, {x, -5, 5}, PlotRange → {-1, 20}], {{b, 2}, 1/10, 10}]
```



We're only getting one graph here! Can you explain why? Use the space below.
Hint: Use exponent rules in your explanation.

DO NOT TURN THE PAGE UNTIL YOU'VE TRIED TO ANSWER THE QUESTION!

Recall that b^{-x} means to take the reciprocal of b^x. In other words,

$$y = b^{-x} = 1/(b^x) = (1/b)^x$$

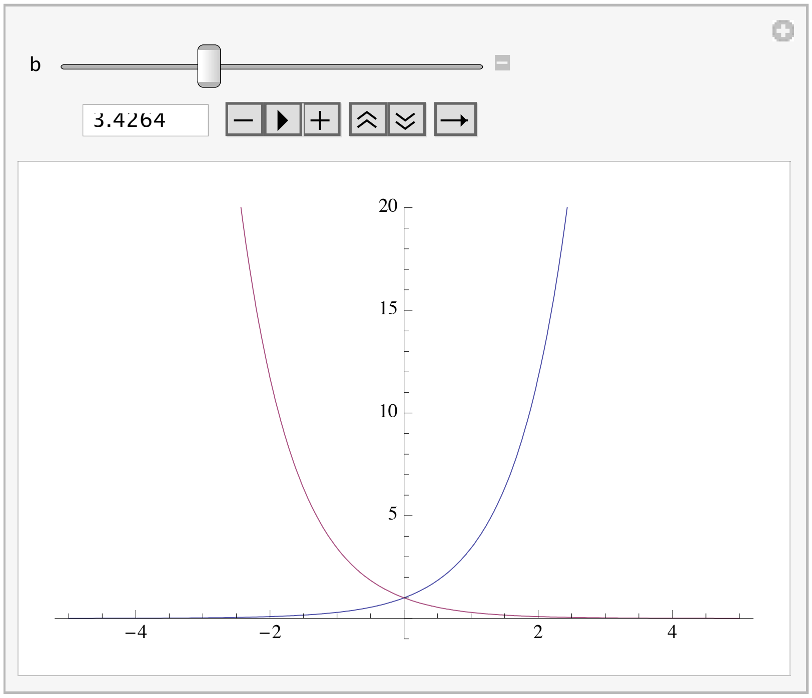
by standard exponent rules. So of course $y = b^{-x}$ and $y = (1/b)^x$ give us the same graph.

- *CONNCETION 2: What's the connection between the graph of $y = b^{-x}$ and the graph $y = b^x$?* Fill in your prediction here:

DO NOT TURN THE PAGE UNTIL YOU'VE MADE YOUR PREDICTION!

Here are the plots:

```
Manipulate[Plot[{b^x, b^(-x)}, {x, -5, 5}, PlotRange → {-1, 20}], {{b, 2}, 1/10, 10}]
```



The graphs are *mirror images across the y-axis!* Can you explain why? Use the space below.
HINT: Do you remember any connection between the graph of $y = f(x)$ and the graph of $y = f(-x)$?

DO NOT TURN THE PAGE UNTIL YOU'VE TRIED TO ANSWER THE QUESTION!

SHORT EXPLAINATION: Note that if we let $f(x) = b^x$ then $f(-x) = b^x$. The graphs of $y = f(x)$ and $y = f(-x)$ are **always** reflections across the y-axis. You might rember learning this fact back in pre-calculus.

LONGER EXPLAINATION: This full explanation was probably covered in your pre-calculus course:
We start by letting $(a,\ b)$ denote a point on the curve $y = f(x)$. That means that we know $f(a) = b$. The reflection of $(a,\ b)$ across the y-axis is the point $(-a,\ b)$. And we now examine why $(-a,\ b)$ must lie on the curve $y = f(-x)$. First recall that any point on $y = f(-x)$ has coordinates of the form: $(x,\ f(-x))$. So $(-a,\ b)$ lies on $y = f(-x)$ if and only if b equals the value of $f(-(-a))$. But we already know $f(-(-a)) = f(a) = b$ since $(a,\ b)$ lies on $y = f(x)$.

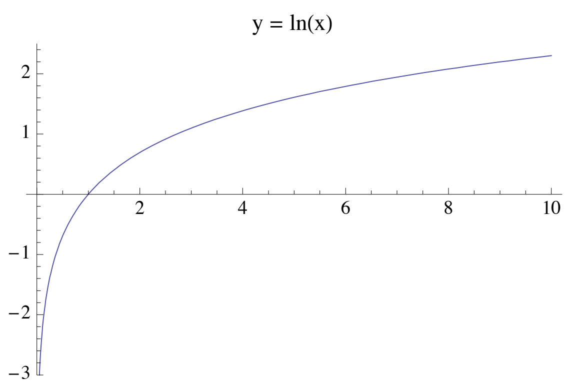
Logarithmic functions

▪ The natural logarithm: $y = \ln(x)$ (done)

The ***only*** logarithm function that will be *critically important* to us is the natural log, and we'll study the formal (symbolic) properties of the natural logarithm function in some depth later in the semester. Here's a very good picture of $y = \ln(x)$:

```
In[30]:= Plot[Log[x], {x, 0, 10}, PlotRange → {-3, 2.5}, PlotLabel → "y = ln(x) "]
```

Out[30]=



***Mathematica* TIP**: *Mathematica* uses the syntax Log[x] to denote $\ln(x)$. (It's base is $e \approx 2.71828$.)

Your observations about $\ln(x)$***: (FILL IN THE BLANKS BEFORE YOU TURN THE PAGE)***

1) The DOMAIN of $\ln(x)$ is ______________________________

2) As $x \to 0$ from the RIGHT (positive) side, we see $\ln(x) \to$ ______________________________

And there is a ____________ **asymptote for $y = \ln(x)$** at ______________

3) As $x \to +\infty$, we see $\ln(x) \to$ ______________________________

4) The graph $y = \ln(x)$ passes through the point ______________________________

5) The graph y = ln(x) looks like ______________________________

Class Observations about $y = \ln(x)$***:***

(These are the answers to the fill in the blank questions on the previous page.)

1) The DOMAIN of ln(x) is **the set of POSITIVE NUMBERS. So ln(-5) is NOT defined.**

2) As $x \to 0$ from the RIGHT (positive) side, we see $\ln(x) \to -\infty$ **very, very quickly.**
[So there is a VERTICAL asymptote for $y = \ln(x)$ at $x = 0$.]

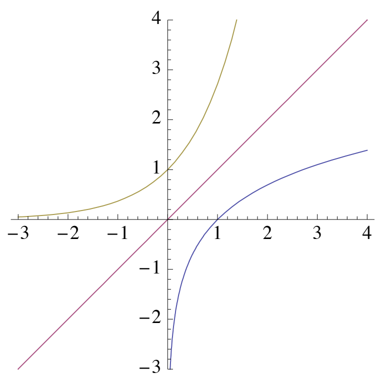
3) As $x \to +\infty$, we see $\ln(x) \to +\infty$ **very, very slowly.**
[So there is not a horizontal asymptote for $y = \ln(x)$.]

4) The graph $y = \ln(x)$ passes through the point **(1, 0)**

5) The graph y = ln(x) looks like ***an exponential function on its side!***

In fact, as the next true scale plot shows, the graphs of $y = \ln(x)$ and $y = e^x$ are reflections of each other across the line $y = x$:

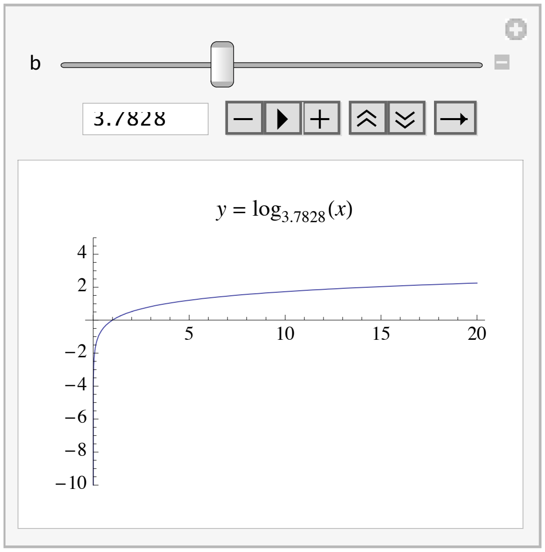
```
Plot[{Log[x], x, E^x}, {x, -3, 4}, PlotRange → {-3, 4}, AspectRatio → Automatic]
```



■ Other logarithms

You may recall from pre-calculus that there is a whole family of logarithm functions where we use $\log_b(x)$to denote the ``log-base-b'' function. If we have time, we may study the formal properties of these functions in a bit more detail later in the course, For now it' s enough to realize that **logs are exponential functions on their sides!** Hence they all have the same basic shape:

```
Manipulate[Plot[Log[b, x], {x, 0, 20}, PlotRange → {-10, 5},
  PlotLabel → TableForm[{{"y = log"_b, "(x)"}}, TableSpacing → {0, 0}]], {{b, E}, 1 / 10, 10}]
```



Observations:

1) All logs with $b > 1$ have the same basic shape as $\ln(x)$. As $b \to +\infty$, the graph of $y = \log_b(x)$ becomes even flatter--- i.e. it takes even longer for the value of $\log_b(x)$ to become large.

2) Logs with $0 < b < 1$ are reflections (across the x-axis) of log curves with base $(1/b) > 1$.

3) ALL LOGS pass through the point (1, 0).

***Mathematica* TIP:** *Mathematica* uses the syntax Log[b, x] to denote $\log_b(x)$, the so-called ``log-base-b'' function
(So Log[b, x] = $\log_b(x)$ in regular notation.)

Mathematics TIP: The *base* of a log function ***cannot equal 1.***
In other words, the symbol string $\log_1(x)$ is MEANINGLESS in mathematics