§4.1 Slope Fields; More Differential Equation Models

1. (a) The solution curves are "parallel" to each other in the sense that they differ from each other only in their *horizontal* position. Thus, e.g., all the curves have the same slope where $y = 2$.

   (b) It *does* appear that each of the five "upper" curves has the same slope when $y = 3$. Carefully draw a tangent line to any one of the curves at the appropriate point; measure its slope. The result should be 3 (or very close to 3).

   The answer *could* have been predicted in advance. The fact that each curve is a solution to the DE $y' = y$ means precisely that when $y = 3$, $y' = 3$, too.

   (c) At the level $y = -4$, each curve has slope -4. Again, this is exactly what the DE predicts.

   (d) All curves appear to be very nearly *horizontal* near $y = 0$. The only solution curve that actually touches the line $y = 0$ is the solution curve $y = 0$ itself. Appropriately, this curve has slope 0 everywhere.

3. Since $f(1) = 2$, the point $(1, 2) = \big(t, y(t)\big)$ is on the solution curve. From the DE, we find that $y'(1) = 1^2 \cdot y(1) + 1 = 3$ so the tangent line has slope 3. Since the tangent line passes through the point $(1, 2)$, it is described by the equation $y = 3t - 1$.

5. The slope of the tangent line at $(-2, 1)$ is $y'(-2) = (-2)^2 \cdot y(-2) = 4$. Thus, $y = 4(t + 2) + 1 = 4t + 9$ is an equation of the tangent line.

7. This is the slope field of DE (vii), $y' = \cos y$. Observe that the slope is the same at each x-value — the slope at each grid point depends only on the value of y.

9. This is the slope field of DE (iii), $y' = ty$.

11. This is the slope field of DE (viii), $y' = \sin t$.

13. This is the slope field of DE (x), $y' = y(1 - y)$.

15. All ticks at the same vertical position are parallel.

17. (b) If $y(t) = \sqrt{t^2 + C}$, then $y'(t) = t/\sqrt{t^2 + C} = t/y$. Similarly, if $y(t) = -\sqrt{t^2 + C}$, then $y'(t) = -t/\sqrt{t^2 + C} = t/y$.

    (c) $y(t) = \sqrt{t^2 + 1}$

    (d) $y(t) = \pm\sqrt{t^2 - 1}$

19. (b) If $y(t) = e^{-t} + Ce^{-2t}$, then $y'(t) = -e^{-t} - 2Ce^{-2t} = e^{-t} - 2\left(e^{-t} + Ce^{-2t}\right) = e^{-t} - 2y$.

    (c) $y(t) = e^{-t} - e^{-2(t+1)}$

    (d) $y(t) = e^{-t} - e^{2-2t} - e^{1-2t}$

    (e) $y(t) = e^{-t} + e^{2-2t} - e^{1-2t}$

21. (a) $y(t) = t - 1 + 2e^{-t}$

    (b) $y(t) = t - 1 + 4e^{-t}$

    (c) $y(t) = t - 1$

    (d) If $y(t) = t - 1 + Ce^{-t}$, then $y'(t) = 1 - Ce^{-t} = t - (t - 1 + Ce^{-t}) = t - y(t)$.

    (e) $y' = t - y \implies y''(t) = 1 - y'(t) = 1 - \big(t - y(t)\big) = 1 - t + y(t)$. This implies that the solution curve in part (a) is concave up, since $y''(t) > 0$; the solution curve in part (b) is concave down since $y''(t) < 0$, and the solution curve in part (c) is linear since $y''(t) = 0$.

23. (a) Since the outside temperature is $-10°$ C and the coffee is initially $90°$ C, the temperature $y(t)$ of the coffee at time t is $y(t) = -10 + 100e^{kt}$. For the foam cup, $k = -0.05$, so the temperature reaches $70°$ C at time $t = -\ln(80/100)/0.05 \approx 4.46$ minutes after leaving the store. For the cardboard cup, $k = -0.08$, so the temperature reaches $70°$ C after $t = -\ln(80/100)/0.08 \approx 2.79$ minutes.

    After $t = 5$ minutes, the coffee in the foam cup is $-10 + 100e^{-0.25} \approx 68°$ C while the coffee in the paper cup is $-10 + 100e^{-0.4} \approx 57°$ C.

    (b) In the store, the temperature of the coffee at time t is $y(t) = 25 + 65e^{kt}$. Therefore, the coffee reaches $70°$ C after $t = \ln(45/65)/k$ minutes. This implies that Boris's coffee reaches $70°$ C in approximately 7.4 minutes and Natasha's coffee reaches this temperature in approximately 4.6 minutes.

    After $t = 5$ minutes, Boris's coffee is approximately $76°$ C and Natasha's coffee is approximately $69°$ C.

    (c) If the coffee is to be at least $70°$ C after 5 minutes outdoors, k must be chosen so that $70 \le -10 + 100e^{5k}$. This implies that $k \ge \ln(80/100)/5 \approx -0.0446$.

25. (a) $P(t)$ is an increasing function, so $P'(t)$ must never be negative. The factor $M - P(t)$ causes the rate of learning to decrease as the value of the performance function approaches the maximum. Thus, the rate of learning is large when $M - P(t)$ is large and approaches zero as $P(t)$ approaches M. The value of P never exceeds M because $P' = 0$ when $P = M$.

    (b) If $P(t) = M - Ae^{-kt}$, then $P'(t) = Ake^{-kt} = kAe^{-kt} = k(M - P)$, as desired.

    (c) From the previous part, solutions are of the form $P(t) = M - Ae^{-0.05t}$. Since $P(0) = 0.1M$, it follows that $A = 0.9M$. Therefore, $P(t) = M - 0.9Me^{-0.05t}$.

    We want t such that $P(t) = 0.9M$, so we solve the equation $M - 0.9Me^{-0.05t} = 0.9M$ for t. The solution is $t = 20\ln 9 \approx 44$ hours.

27. (a) The DE $v' = g - kv$ can be rewritten in the form $v' = (-k)\big(v - (-g/k)\big)$ which has the same form as the DE describing Newton's law of cooling. Thus, the solution of the DE is
    $v(t) = \dfrac{g}{k} - \left(\dfrac{g}{k} - v_0\right)e^{-kt}$

    (b) Since $y'(t) = v(t)$, $y(t) = \dfrac{g}{k}t + \left(\dfrac{g}{k^2} - \dfrac{v_0}{k}\right)e^{-kt} + \left(y_0 - \dfrac{g}{k^2} + \dfrac{v_0}{k}\right)$.

29. Observe that the graph of $y = f(x)$ is decreasing when $1 < x < 2$ and increasing when $2 < x < 5$. Thus, y' must be negative on $(1, 2)$ and positive on $(2, 5)$. Also, observe that $y > x$ on $(1, 2)$ and $y < x$ on $(2, 5)$. Furthermore, $y < x^2$ on $(2, 5)$.

    The differential equation (a) cannot be the correct answer since the expression $(y - x)/x$ is negative on the interval $(2, 5)$. Similarly, the differential equation (c) cannot be the correct answer because the expression $(x^2 - y)/x$ is not negative over the entire interval $(1, 2)$. On the other hand, the expression $(x - y)/y$ is negative on the interval $(1, 2)$ and positive on the interval $(2, 5)$. Therefore, the correct differential equation is (b).

31. For each value of y, the slopes do not depend on x (i.e., all ticks at the same vertical position are parallel).

33. (b) Since the DE does not depend explicitly on t, the solutions for different initial conditions can be obtained by horizontal shifts. Thus, since $y(t) = (t + 1)e^t$ is the solution of the DE with the initial condition $y(0) = 1$, $y(t) = (t + 2)e^{t+1}$ is the solution of the DE with the initial condition $y(0) = 2e$.

§4.2 More on Limits: Limits Involving Infinity and l'Hôpital's Rule

1. Note that $p(x) = (x+1)(2-x) = 2 + x - x^2 = x^2(2/x^2 + 1/x - 1)$.

   (a) Since $\lim_{x\to\infty}(2/x^2 + 1/x - 1) = -1$ and $\lim_{x\to\infty} x^2 = \infty$, $\lim_{x\to\infty} p(x) = -\infty$.

   (b) Since $\lim_{x\to-\infty}(2/x^2 + 1/x - 1) = -1$ and $\lim_{x\to-\infty} x^2 = \infty$, $\lim_{x\to-\infty} p(x) = -\infty$.

3. $\lim_{x\to 2}\dfrac{f(x)}{g(x)}$ does not exist because $\lim_{x\to 2^-}\dfrac{f(x)}{g(x)} = \infty$ but $\lim_{x\to 2^+}\dfrac{f(x)}{g(x)} = -\infty$.

5. Near $x = -2$, $f(x) = -x - 2 = -(x+2)$. Therefore, $\lim_{x\to-2}\dfrac{x+2}{f(x)} = \lim_{x\to-2}\dfrac{x+2}{-(x+2)} = -1$.
   [NOTE: l'Hôpital's rule could also be used to obtain this result.]

7. $\lim_{x\to 1}\dfrac{f(x)+2}{g(x)+1}$ does not exist because $\lim_{x\to 1^-}\dfrac{f(x)+2}{g(x)+1} = \lim_{x\to 1^-}\dfrac{-2x+2}{-x+1} = 2$ but
   $\lim_{x\to 1^+}\dfrac{f(x)+2}{g(x)+1} = \lim_{x\to 1^+}\dfrac{x-3}{x-2} = 1$.

9. $\lim_{x\to\infty}\dfrac{3}{x^2} = 0$

11. $\lim_{t\to\infty}\dfrac{2t+3}{5-4t} = -\dfrac{1}{2}$.

13. $\lim_{x\to\infty}\dfrac{x^2+1}{x} = \infty$

15. $\lim_{x\to\infty}\dfrac{\sin x}{x} = 0$

17. $\lim_{x\to\infty}\dfrac{2^x}{x^2} = \infty$

19. $\lim_{x\to\infty}\dfrac{\ln x}{x^{2/3}} = \lim_{x\to\infty}\dfrac{x^{-1}}{2x^{-1/3}/3} = \lim_{x\to\infty}\dfrac{3}{2x^{2/3}} = 0$.

21. $\lim_{x\to 0}\sin\big(\sin x\big) = 0$

23. $\lim_{x\to 0}\dfrac{\tan x}{x} = \lim_{x\to 0}\sec^2 x = 1$ so $\lim_{x\to 0}\cos\left(\dfrac{\tan x}{x}\right) = \cos 1$

25. $\lim_{x\to 1}\dfrac{x^3+x-2}{x^2-3x+2} = \lim_{x\to 1}\dfrac{3x^2+1}{2x-3} = -4$

27. $\lim_{x\to 0}\dfrac{1-\cos x}{\sin(2x)} = \lim_{x\to 0}\dfrac{\sin x}{2\cos(2x)} = 0$

29. $\lim_{x\to\infty}\dfrac{e^x}{x^2+x} = \lim_{x\to\infty}\dfrac{e^x}{2x+1} = \lim_{x\to\infty}\dfrac{e^x}{2} = \infty$

31. Since $\lim_{x\to 1} f(x) = \lim_{x\to 1} x^2 - 1 = 0$, l'Hôpital's rule can be used to evaluate the desired limit. Thus,
    $\lim_{x\to 1}\dfrac{f(x)}{x^2-1} = \lim_{x\to 1}\dfrac{f'(x)}{2x} = \dfrac{f'(1)}{2}$. It appears from the graph that $f'(1) \approx 3/2$, so $\lim_{x\to 1}\dfrac{f(x)}{x^2-1} \approx \dfrac{3}{4}$.

33. $\lim_{x\to 4}\dfrac{f(x)}{(x-4)^2} = \infty$ since $\lim_{x\to 4} f(x) \approx 3.6$ and since $\lim_{x\to 4}(x-4)^{-2} = \infty$.

35. Since $\lim_{x\to 1} f(x-3) = 0$ and $\lim_{x\to 1} f(x+3) = 3.6$, $\lim_{x\to 1} \dfrac{f(x-3)}{f(x+3)} = 0$.

37. No. Since $\lim_{x\to\pi/2^-} \tan x = \infty$ and $\lim_{x\to\pi/2^+} \tan x = -\infty$, $\lim_{x\to\pi/2} \tan x$ does not exist. Therefore, l'Hôpital's rule cannot be used. Moreover, the limits $\lim_{x\to\pi/2^\pm} \dfrac{\tan x}{x-\pi/2}$ have the form $\mp\infty/0$ so l'Hôpital's rule cannot be used to evaluate these limits either.
[NOTE: L'Hôpital's rule can be used only for limits of the form $0/0$ or ∞/∞.]

39. No. The limit $\lim_{x\to\pi/2^-} \dfrac{\tan x}{x-\pi/2}$ has the form $\infty/0$ so l'Hôpital's rule cannot be used to evaluate this limit.
[NOTE: L'Hôpital's rule can be used only for limits of the form $0/0$ or ∞/∞.]

41. Since f is a non-constant periodic function, the values of $f(x)$ do not approach a single number L as $x \to \infty$.

43. There are many ways to draw such a graph. Any suitable graph, though, should have a horizontal asymptote at $y = -3$ (to the right) and another horizontal asymptote at $y = 3$ (to the left). A simple way to satisfy condition (iii) is to have the graph pass through $(2, -2)$.

45. (a) Yes. The line $y = -1$ is a horizontal asymptote of h since $\lim_{x\to\infty} h(x) = -1$.

(b) No. If h were a rational function with a vertical asymptote at $x = 3$ and the property $\lim_{x\to\infty} h(x) = -1$, then h would have to be of the form

$$h(x) = \frac{-x^n + \cdots}{(x-3)(x^{n-1} + \cdots)}$$

for some positive integer n. But then $\lim_{x\to-\infty} h(x) = -1$ not ∞.

47. p and q can be any polynomials such that the coefficient of the highest power of x in each polynomial is positive, and the degree of p is greater than the degree of q. For example, the polynomials $p(x) = x^2 - 2x + 3$ and $q(x) = x + 1$ have the desired properities:
$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} q(x) = \lim_{x\to\infty} p(x)/q(x)$.

49. p and q can be any polynomials such that the degree of p is less than the degree of q. For example, the polynomials $p(x) = x + 1$ and $q(x) = x^2 + 2$ have the desired properities: $\lim_{x\to\infty} p(x) = \lim_{x\to\infty} q(x) = \infty$, but $\lim_{x\to\infty} p(x)/q(x) = 0$.

51. (a) If $a = -21$, then $\lim_{x\to 3} f(x) = \dfrac{13}{4}$.

(b) $\lim_{x\to\infty} f(x) = 2$ for every value of a.

53. g is **not** continuous at $x = 0$ because $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$.

55. $\lim_{x\to\infty} e^{-x} \ln x = \lim_{x\to\infty} \left(\dfrac{\ln x}{e^x}\right) = \lim_{x\to\infty} \left(\dfrac{\frac{1}{x}}{e^x}\right) = \lim_{x\to\infty} \left(\dfrac{1}{xe^x}\right) = 0$ [Denominator blows up!]

57. $\lim_{x\to 8} \left(\dfrac{x-8}{\sqrt[3]{x}-2}\right) = \lim_{x\to 8} \left(\dfrac{1}{\frac{1}{3}x^{-2/3}}\right) = \dfrac{1}{\frac{1}{3}\cdot 8^{-2/3}} = 3 \cdot 8^{2/3} = 3 \cdot 4 = 12$

59. $\lim_{x\to 0} \left(\dfrac{\sin x}{x - \sin x}\right) = \lim_{x\to 0} \left(\dfrac{\cos x}{1-\cos x}\right) = \infty$ [NOTE: numerator $\longrightarrow 1$ and denominator $\longrightarrow 0$ from *above*.]

61. $\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} \frac{e^x}{1} = 1$

63. $\lim_{x\to 0} \frac{e^x - e^{-x}}{x} = \lim_{x\to 0} \left(e^x + e^{-x}\right) = 2$

65. $\lim_{x\to 0} \frac{1-\cos^2 x}{x^2} = \lim_{x\to 0} \frac{2\cos x \sin x}{2x} = \lim_{x\to 0} \frac{2\cos^2 x - 2\sin^2 x}{2} = 1$

67. $\lim_{x\to 1} \frac{\ln x}{x^2 - x} = \lim_{x\to 1} \frac{1}{2x^2 - x} = 1$

69. $\lim_{x\to 1} \frac{\sin(\pi x)}{x^2 - 1} = \lim_{x\to 1} \frac{\pi \cos(\pi x)}{2x} = -\frac{\pi}{2}$

71. $\lim_{x\to 1} \frac{\cos^3(\pi x/2)}{\sin(\pi x)} = \lim_{x\to 1} \frac{-3\cos^2(\pi x/2)\sin(\pi x/2)}{2\cos(\pi x)} = 0$

73. $\lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-2}} = \lim_{x\to 0^+} -\frac{x^2}{2} = 0$

75. $\lim_{x\to 0} x^2 \ln(\cos x) = 0$ since $\lim_{x\to 0} x^2 = 0$ and $\lim_{x\to 0} \ln(\cos x) = 0$.

77. $\lim_{w\to 0^+} w(\ln w)^2 = \lim_{w\to 0^+} \frac{(\ln w)^2}{1/w} = \lim_{w\to 0^+} -\frac{2\ln w}{1/w} = \lim_{w\to 0^+} 2w = 0.$

79. $\lim_{x\to\infty} x\left(\frac{\pi}{2} - \arctan x\right) = \lim_{x\to\infty} \frac{\pi/2 - \arctan x}{x^{-1}} = \lim_{x\to\infty} \frac{x^2}{1+x^2} = 1$

81. $\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0} \frac{x - \sin x}{x\sin x} = \lim_{x\to 0} \frac{1-\cos x}{\sin x + x\cos x} = \lim_{x\to 0} \frac{\sin x}{2\cos x - x\sin x} = 0$

83. Using l'Hôpital's rule, $\lim_{x\to 1} \frac{\left(f(x)\right)^2 - 4}{x^2 - 1} = \lim_{x\to 1} \frac{f(x)f'(x)}{x} = 6.$

85. $\lim_{x\to 0} \frac{xf(x)}{(e^x - 1)\,g(x)} = \lim_{x\to 0} \frac{f(x) + xf'(x)}{e^x g(x) + (e^x - 1)\,g'(x)} = \frac{f(0)}{g(0)}$

87. $\lim_{x\to 0^+} 2x\ln x = 0$ so $\lim_{x\to 0^+} x^{2x} = e^0 = 1$

89. $\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{1}{1+x} = 1$ so $\lim_{x\to\infty} (1+x)^{1/x} = e^1 = e.$

91. $\lim_{x\to 1} \left(\ln x\right)^{\sin x} = 0$ since $\lim_{x\to 1} \ln x = 0$ and $\lim_{x\to 1} \sin x = \sin 1 \approx 0.84147.$

93. $\lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}$. The limit on the right is the definition of $f'(0)$.

95. $f'(1)$ does not exist because f is not continuous at $x = 1$. Here are the details:
$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$. However, the two-sided limit in the definition of $f'(1)$ does not exist because $\lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^+} \frac{(1+h)^2 + 3 - 5}{h} = \lim_{h\to 0^+} \frac{h^2 + 2h - 1}{h} = -\infty$ and
$\lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^-} \frac{2(1+h) + 2 - 5}{h} = \lim_{h\to 0^-} \frac{2h - 1}{h} = \infty.$

§4.3 Optimization

1. A function achieves its minimum value over a closed interval at an endpoint of the interval or at a stationary point within the interval. Since $f'(1) = 0$, $f(x)$ could attain its minimum value at $x = -5$, $x = 1$, or $x = 5$.

3. Yes — the function h fails to be differentiable at $x = 1$ so it has a critical point there.

5. (a) We can write $g(x) = 5x^2 - 8x + 4$, so $g'(x) = 10x - 8$. The only stationary point occurs where $g'(x) = 10x - 8 = 0$, i.e., at $x = 4/5$. Also, $g(4/5) = 4/5$, so $4/5$ is also the minimum value of g.

   (b) The previous part shows $g(x)$ is least if $x = 4/5$. This means that the square of the distance (and hence the distance itself) is least if $x = 4/5$, just as we found in Example 3. This minimum distance, moreover, is $\sqrt{4/5} = 2/\sqrt{5} \approx 0.8944$.

7. (a) The sketch should suggest that P has coordinates somewhere near $(0.7, 0.5)$.

   (b) The function $f(x) = x^2 + (1 - x^2)^2$ gives the square of the distance from the origin to the point $(x, 1 - x^2)$ on the parabola. Minimizing f in the usual way shows that the minimum distance occurs if $x = \sqrt{2}/2$ and $y = 1/2$—i.e., P is the point $(\sqrt{2}/2, 1/2)$.

   (c) The line from the origin to $P(\sqrt{2}/2, 1/2)$ has slope $1/\sqrt{2}$. At $x = \sqrt{2}/2$, the parabola $y = 1 - x^2$ has slope $-\sqrt{2}$. This means that the two lines are perpendicular, as claimed.

9. The equations $P = xy$ and $x + y = 10$ imply that $P(x) = x(10 - x) = 10x - x^2$; the exercise is meaningful for $0 \le x \le 10$. Now $P'(x) = 10 - 2x = 0$ if $x = 5$, and $P(5) = 25$.

11. The constraint $x + y = 4$ implies that $y = 4 - x$. Thus, we wish to find the minimum value of $f(x) = x^3 + (4 - x)$ if $x \ge 0$. Since $f'(x) = 3x^2 - 1$, the only critical point of f in the interval $[0, \infty)$ is at $x_* = 1/\sqrt{3} = \sqrt{3}/3$. Since $f''(x_*) = 6x_* = 2\sqrt{3} > 0$, this value of x is a local minimum point of f. It follows that the minimum value of f over the interval $[0, \infty)$ is $f(x_*) = x_*^3 + 4 - x_* = (\sqrt{3}/3)^3 + 4 - \sqrt{3}/3 = 4 - 2\sqrt{3}/9 \approx 3.6151$.

13. If $H(w) = 4w^5 - 5w^6$, $H'(w) = 20w^4 - 30w^5 = 10w^4(2 - 3w)$. Thus, $H'(w) > 0$ on the interval $(-\infty, 2/3)$ and $H'(w) < 0$ on the interval $(2/3, \infty)$. This implies that H is increasing everywhere to the left of $w = 2/3$ and decreasing everywhere to the right of $w = 2/3$. Therefore, $w = 2/3$ is the global maximum point of $H(w)$.

15. If the base of the rectangle is on the x-axis and has corners at $(-a, 0)$ and $(a, 0)$, the area of the inscribed rectangle is $A(a) = 2a \cdot \sqrt{3}(1 - a) = 2\sqrt{3}a(1 - a)$. Now, $A'(a) = 2\sqrt{3}(1 - 2a) = 0$ if $a = 1/2$. Since $A(0) = A(1) = 0$ and $A(1/2) = \sqrt{3}/2$, $a = 1/2$ corresponds to the rectange with largest area. This rectangle has corners at the points $(\pm 1/2, 0)$, $(\pm 1/2, \sqrt{3}/2)$; it occupies 50% of the area of the triangle.

17. Suppose that the side of the triangle in the first quadrant is tangent to the parabola at the point $x = a > 0$. Then the tangent line is described by the equation $y = -2a(x - a) + (1 - a^2)$. The x-intercept of this tangent line is the point $\big((1 + a^2)/2a, 0\big)$; its y-intercept is the point $(0, 1 + a^2)$. By symmetry, the area of the triangle corresponding to this tangent line is $A(a) = \dfrac{1}{2} \cdot \dfrac{1 + a^2}{a} \cdot (1 + a^2) = \dfrac{\big(1 + a^2\big)^2}{2a}$.

    Since we wish to find the value of a that minimizes $A(a)$, we solve for the stationary points of A': $A'(a) = 0 \implies a = \pm\sqrt{3}/3$. Since $A'(x) < 0$ if $0 < x < \sqrt{3}/3$ and $A'(x) > 0$ if $\sqrt{3}/3 < x < 1$, $a = \sqrt{3}/3$ corresponds to the triangle with smallest possible area. Its area is $A(\sqrt{3}/3) = 8\sqrt{3}/9$ square units.

19. (a) The trajectory formula gives $y = x - 9.8x^2/900$. The peak occurs where $dy/dx = 1 - 9.8x/450 = 0$, i.e., at $x \approx 45.92$ meters. If $x \approx 45.92$, then $y \approx 22.96$ meters. The ball lands where $y = 0$, i.e., at $x \approx 91.84$ meters.

    (b) The trajectory is a parabola, intersecting the x-axis at $x = 0$ and $x = 91.84$; the peak occurs at $(45.92, 22.96)$.

21. (a) Set the trajectory equation to zero and solve for x. There are two solutions—$x = 0$ and $x = R$, where R is the range.

(b) Let $R(m) = \dfrac{2v_0^2}{g}\dfrac{m}{1+m^2}$; let's maximize this for $m > 0$. Well, $R'(m) = \dfrac{2v_0^2}{g}\dfrac{1-m^2}{(1+m^2)^2}$; thus $R'(m) = 0$ for $m = 1$. Thus the range is maximum if the initial slope is 1, or, equivalently, the initial angle is $\pi/4$.

23. (a) Maximizing $A = xy$ subject to $2hx + 2vy = b$ leads to $hx = vy = b/4$, so $x = b/(4h)$, $y = b/(4v)$, and the maximum possible area $xy = b^2/(16hv)$ square feet.

(b) Minimizing $C = 2hx + 2vy$ subject to $xy = a$ leads to $hx = vy$. Combining this with $a = xy$ gives $x = \sqrt{av/h}$ and $y = \sqrt{ah/v}$. Thus the minimum possible cost is $2hx + 2vy = 4\sqrt{avh}$.

(c) The two parts are consistent—both say that the best scheme is to spend half the money on east-west fence and half on north-south fence.

25. The volume of the can is $V = \pi r^2 h$ and the surface area is $A = 2\pi rh + 2\pi r^2$. Since the can must hold 168 cm^3, $168 = \pi r^2 h$ or $h = 168/\pi r^2$. This allows us to express A as a function of r:

$$A(r) = 2\pi r\left(\frac{168}{\pi r^2}\right) + 2\pi r^2 = \frac{336}{r} + 2\pi r^2.$$

Therefore,

$$\frac{dA}{dr} = -\frac{336}{r^2} + 4\pi r = 0$$

if $r = \sqrt[3]{84/\pi} \approx 2.99$ cm. Since $(d^2A/dr^2 > 0$ if r has this value, it corresponds to a local minimum of the function $A(r)$.) It follows that $h = 2\sqrt[3]{84/\pi} \approx 5.98$ cm.

27. Let x be the length of a side of the base and h be the height of the box. Then $100 = 8x + 4h$, so $h = 25 - 2x$.

(a) The volume of the box is $x^2h = x^2(25-2x) = V(x)$. Since $V'(x) = 50x - 6x^2$ and $V''(x) = 50 - 12x$, the volume is maximized if $x = 25/3$; $V(25/3) = 15625/27 \approx 578.7$ cm^3.

(b) The surface area of the box is $2x^2 + 4xh = 2x^2 + 4x(25-2x) = 100x - 6x^2 = A(x)$. Since $A'(x) = 100 - 12x$ and $A''(x) = -12$, the surface area is maximized if $x = 25/3$; $A(25/3) = 1250/3$ cm^2.

29. (a) Both conditions are satisfied if $x \in [0, 6]$.

(b) The results in part (a) imply that the dam can be built at most 6 miles downstream. If the dam were constructed at this point, it would be $W(6) = 100$ feet wide and $D(6) = 130$ feet high.

(c) $W(x)$ achieves its maximum value on the interval $[0, 6]$ at $x = 0$; $W(0) = 220$.

(d) $W(x)$ achieves its minimum value on the interval $[0, 6]$ at $x = 4$; $W(4) = 60$. [$W'(x) = 20(x-4)$ and $W''(x) = 20$.]

(e) The cost of building the dam x miles downstream is $C(x) = kW(x)D(x) = 100k\left(2x^3 - 15x^2 + 36x + 22\right)$, where k is a positive constant. Now, $C'(x) = 600k\left(x^2 - 5x + 6\right)$ so $x = 2$ and $x = 3$ are stationary points of C. Finally, since $C(0) < C(3) < C(2) < C(6)$, the dam should be built $x = 0$ miles downstream.

31. If x is the length of an edge of the square and r is the radius of the circle, then $L = 4x + 2\pi r$. We wish to maximize $S = x^2 + \pi r^2$. Since $r = (L-4x)/2\pi$, the equation for S can be written in the form $S(x) = x^2 + (L-4x)^2/4\pi$, where $0 \le x \le L/4$. Now, $S'(x) = 2x - 2(L-4x)/\pi = 0$ if $x = L/(4+\pi)$. Since $S''(x) = 2 + 8/\pi > 0$, this value of x corresponds to a local minimum of S! Thus, the maximum value of S must occur when $x = 0$ or when $x = L/4$. Since $S(0) = L^2/4\pi$ and $S(L/4) = L^2/16$, it follows that the sum of the areas is maximized when all of the wire is used to form a circle.

§4.4 Parametric Equations, Parametric Curves

1. The curve is the upper unit semi-circle plotted from $(-1, 0)$ to $(0, 1)$ to $(1, 0)$.

3. The curve is the right unit semi-circle plotted from $(0, -1)$ to $(1, 0)$ to $(0, 1)$.

5. The curve is the unit circle plotted clockwise from $(0, -1)$ to $(0, 1)$ to $(0, -1)$.

7. In each case the idea is to calculate $\sqrt{f'(t)^2 + g'(t)^2}$; if the result is constant, then the curve has constant speed. Among the given choices only the last—$x = \sin(\pi t)$, $y = \cos(\pi t)$—has constant speed.

9. (a) The spacing of bullets suggests that P moves quickly at $t = 3$, $t = 4$, $t = 9$, and $t = 10$, and slowly at $t = 0$, $t = 1$, $t = 6$, and $t = 7$.

   (b) The distance along the curve from $t = 2.5$ to $t = 3.5$ seems to be about 3 units. Thus P appears to travel about 3 units per second at $t = 3$.

   (c) Use the curve to estimate the speed of P at $t = 6$. The distance along the curve from $t = 5.5$ to $t = 6.5$ seems to be about 1 unit. Thus P appears to travel about 1 unit per second at $t = 6$.

11. (a) The result is the circle of radius 2, centered at $(2, 1)$.

   (b) Here's the calculation: Since $x = a + r\cos t$ and $y = b + r\sin t$,

$$(x - a)^2 + (y - b)^2 = r^2(\cos t)^2 + r^2(\sin t)^2 = r^2.$$

   (c) Setting $x = 2 + \sqrt{13}\cos t$, $y = 3 + \sqrt{13}\sin t$, and $0 \le t \le 2\pi$, gives the circle of radius $\sqrt{13}$, centered at $(2, 3)$.

   (d) No proper "curve" results: for all t, (x, y) stays put at $(2, 3)$.

13. (a) The origin corresponds to $t = 0$; $P(0.1) \approx (0.48, 0.56)$; $P(\pi/2) = (1, 0)$. Thus P starts at the origin and starts off in a northeasterly direction.

   (b) Both x and y are 0 if and only if both $5t$ and $6t$ are integer multiples of π. This occurs only for $t = 0$, $t = \pi$, and $t = 2\pi$.

   (c) Using the t-interval $0 \le t \le 4\pi$ would produce exactly the same curve, but it would be traversed twice.

15. (a) The curve starts at $(at_0 + b, ct_0 + d)$ and ends at $(at_1 + b, ct_1 + d)$.

   (b) $y = \dfrac{c}{a}(x - b) + d$

   (c) $x = \dfrac{a}{c}(y - d) + b$

   (d) If $a = c = 0$, the parametric curve is just the point (b, d).

17. (a) The model would be more realistic if it took wind resistance into account. To do so, one would need some mathematical information about wind resistance.

   (b) Imitate the argument given for $f(t)$. Notice, too, that if $g(t) = 7 - 16t^2$, then $g'' = -32$, $g(0) = 7$, and $g'(0) = 0$, just as claimed.

   (c) By definition, $s(t) = \sqrt{f'(t)^2 + g'(t)^2} = \sqrt{150^2 + (-32t)^2} = \sqrt{22500 + 1024t^2}$. Plotting this function over the interval $0 \le t \le 0.661$ (when the ball hits the ground) gives almost a horizontal line—the velocity changes very little over the short time interval.

19. (a) If $x = f(t) = s_0 t$ and $y = g(t) = 7 - 16t^2$ it's easy to check directly that $f''(t) = 0$, $f'(0) = s_0$, $f(0) = 0$, $g''(t) = -32$, $g'(0) = 0$, and $g(0) = 7$. These are the necessary conditions.

   (b) The ball reaches home plate when $f(t) = s_0 t = 60.5$, i.e., at $t = 60.5/s_0$ seconds.

(c) The trajectory is parabolic for any $s_0 > 0$. (If $s_0 = 0$, the ball drops straight down.) This can be seen by eliminating t. Since $x = s_0 t$, $t = x/s_0$, so $y = 7 - 16t^2 = 7 - 16x^2/s_0^2$. This is the equation of a parabola in the xy-plane.

21. Now, $x = 200\ln(3t/4 + 1)$.

(a) $x(t) = 60.5$ at time $t = 4\left(e^{121/400} - 1\right)/3 \approx 0.47098$. Thus, the air-dragged ball takes about 0.0677 seconds longer to reach the plate.

(b) $y(t) \approx 3.4508$ feet at the time when $x(t) = 60.5$

(c) When $x = 60.5$, the ball's speed is approximately 111.87 ft/sec.

(d) When $y = 0$, $x \approx 80.569$ feet.

§4.5 Related Rates

1. (a) $x + 2y = 3 \implies x'(t) + 2y'(t) = 0 \implies x'(t) = -2y'(t)$.
   (b) Using part (a), $1 = -2y'$ so $y' = -1/2$.
   (c) The line $x + 2y = 3$ has slope $-1/2$. Thus, a change of Δx in x leads to a change $\Delta y = -\Delta x/2$. In other words, the rate of change of y is $-1/2$ the rate of change of x for all x.

3. (a) Using implicit differentiation,
   $x^2(t) + y^2(t) = 1 \implies 2x(t)\,x'(t) + 2y(t)\,y'(t) = 0 \implies x(t)\,x'(t) + y(t)\,y'(t) = 0$ for all t.
   (b) If $x(0) = 1$ and $y'(0) = 1$, then $y(0) = 0$ and $x'(0) = 0$. This means that the moving point is at $(1, 0)$ at this time. Since $x'(0) = 0$ and $y'(0) = 1$, the point is moving vertically upwards.
   (c) If $x(t_0) = 1/2$, the equation $x^2 + y^2 = 1$ implies that $y(t_0) = \pm\sqrt{3}/2$. If $x(t_0) = 1/2$ and $x'(t_0) = 1$, the equation in part (a) implies that $1/2 \pm \dfrac{\sqrt{3}}{2}y'(t_0) = 0$. Thus, if $y(t_0) = \sqrt{3}/2$, $y'(t_0) = -\sqrt{3}/3$; if $y(t_0) = -\sqrt{3}/2$, $y'(t_0) = \sqrt{3}/3$.
   (d) At the time t_0, the moving point is at $(1/2, \sqrt{3}/2)$ or at $(1/2, -\sqrt{3}/2)$. If it is at the first point, then it is moving to the right and downward. If it is at the second point, then it is moving to the right and upward.

5. Using similar triangles,

$$\frac{12}{x+s} = \frac{6}{s} \implies 12s = 6x + 6s \implies 6s = 6x \implies s = x$$

for every time t. Thus, if Hal is 30 feet from the lamppost (i.e., $x = 30$), the length of his shadow is $s = 30$ feet. Furthermore, since $s'(t) = x'(t)$ and $x'(t) = 7$ feet per second, his shadow is lengthening at the rate of 7 feet per second.

7. If $L(t)$ and $W(t)$ are the length and width of the rectangle at time t, then the area of the rectangle at time t is $A(t) = L(t) \cdot W(t)$ and $A'(t) = L'(t)W(t) + L(t)W'(t)$ is the rate of change of the rectangle's area. Plugging in the values given in the problem, we find that the area of the rectangle is increasing at a rate of 22 cm^2/sec.

9. Let $E(t)$ be the distance from the intersection of the bicyclist traveling west and $S(t)$ be the distance from the intersection of the bicyclist traveling south. From the information given in the problem, we have $E = 4$ miles, $E' = -9$ miles/hour, $S = 3$ miles, and $S' = 10$ miles per hour. The distance between the two bicyclists $D(t)$ at any time can be determined from the equation $D^2 = E^2 + S^2$. Differentiating both sides of this equation with respect to time (using the product rule), we find that

$$2D \cdot D' = 2E \cdot E' + 2S \cdot S'$$

Now, at the time of interest $D = 5$ miles, so we may use the previously given values of E, E', S, and S', so $5 \cdot D' = 4 \cdot (-9) + 3 \cdot 10$. This implies that $D' = -\frac{6}{5}$ miles per hour. Therefore, the distance between the bicyclists is *decreasing* at a rate of 1.2 miles per hour.

11. Let $x(t)$ be the distance from the runner to first base at time t. Then the distance from the runner to second base is $D(t) = \sqrt{90^2 + x(t)^2}$ and $D'(t) = x(t)x'(t)/D(t)$. When the runner is halfway to first base $x(t) = 45$ and $x'(t) = -20$ ft/sec, so $D'(t) = -20/\sqrt{5} = -4\sqrt{5}$ ft/sec.

13. The area of the ring between the two circles is increasing.

Let $r(t)$ be the radius of the inner circle at time t and $R(t)$ be the radius of the outer circle at time t. Then, the area of the ring between the two circles is $A(t) = \pi\left(\left(R(t)\right)^2 - \left(r(t)\right)^2\right)$. Therefore, $A'(t) = 2\pi\left(R(t)R'(t) - r(t)r'(t)\right)$. At the time when $R = 10$, $R' = 2$, $r = 3$, and $r' = 5$, $A' = 10\pi$. Since this value (the rate of change of the area of the ring between the two circles) is positive, the area is increasing.

15. Since both flights are at the same elevation, we may describe the positions of the planes in terms of just their x- and y-coordinates. The coordinates of the Pachyderm plane t hours after observation are $(0, -36+410t)$ and the coordinates of the Peterpan plane are $(41-455t, 0)$. Thus, the distance between the two planes at time t is $D(t) = \sqrt{(-36+410t)^2 + (41-455t)^2}$.

    (a) At the time of closest approach the planes are $\sqrt{7396/15005} \approx 0.702$ nautical miles apart.

    (b) At time $t = 6683/75025 \approx 0.0891$ hours, $D'(t) = 0$ and $D''(t) > 0$. Thus, the controllers have approximately 5.345 minutes before the time of closest approach.

17. Since the slick has the form of a circular cylinder, the volume of oil in the slick is $V = Ah$ where A is the area of the slick and h is its depth. At the moment of time described in the problem, the slick has area $A = \pi r^2 = \pi 500^2$ square feet, $h = 0.01$ feet, and $h' = -0.001$ feet/hour. Since $V' = A'h + Ah' = -5$ cubic feet per hour, the surface area is increasing at a rate of $A' = (-5 + 250\pi)/0.01 \approx 78{,}040$ square feet per hour.

19. (a) The volume of the ice is $V = 4\pi\left(3R^2T + 3RT^2 + T^3\right)/3$ where $R = 4$ inches is the radius of the iron ball and $T = 2$ inches is the thickness of the ice. Now, the rate of change of the volume of the ice can be related to the rate of change in the thickness of the ice by differentiating: $V' = 4\pi\left(2RR'T + R^2T' + 2RTT' + R'T^2 + T^2T'\right)$. Since $R' = 0$ and $V' = -10$ in^3/min, we may solve for $T' = -10/144\pi$ in/min.

    (b) The surface area of the ice is $S = 4\pi\,(R+T)^2$. Thus, the rate of change of the surface area is $S' = 8\pi\,(R+T)\left(R' + T'\right) = 8\pi \cdot 6 \cdot T' = -10/3$ in^2/min.

21. (a) Home plate, the position of the ball, and first base can be considered to be the vertices of a right triangle. Let S be the length of the hypotenuse of this right triangle and F and T be the lengths of the other two sides. At the instant of time when the ball is halfway to third base $F = 90$ feet and $T = 45$ feet. Since $S^2 = F^2 + T^2$, $S = \sqrt{10125}$ feet. Differentiating both sides of the equation relating S to F and T, we obtain $2SS' = 2FF' + 2TT'$, or $SS' = FF' + TT'$. Now, $T' = 100$ ft/sec and $F' = 0$, so $S' = (45 \cdot 100)/\sqrt{10125} \approx 44.721$ ft/sec.

    (b) This is similar to part (a) except that F is now the distance between home plate and the runner. The ball reaches the point halfway to third base in 0.45 seconds, so $F = 25$ ft/sec $\cdot$ 0.45 sec $= 11.25$ feet. Thus,
    $S' = (11.25 \cdot 25 + 45 \cdot 100)/\sqrt{11.25^2 + 45^2} \approx 103.08$ ft/sec.

23. The volume of a cone with radius r and height h is $V = \pi r^2 h/3$. If R is the ratio h/r, then $V = \pi h^3/3R^2$. Therefore, $V'(t) = \pi h(t)^2 h'(t)/R^2$. Since $V'(t) = -10$ cm^3/min when $h'(t) = -2$ cm/min and $h(t) = 8$ cm, we find that $R^2 = 8^2\pi/5$ so $R = 8\sqrt{\pi/5}$.

25. If t is measured in hours and $t = 0$ corresponds to 12:00, the coordinates of the tip of the minute hand are $(x_m(t), y_m(t))$ where $x_m(t) = 7\cos(2\pi t - \pi/2) = 7\sin(2\pi t)$ and $y_m(t) = 7\cos(2\pi t)$. Similarly, the coordinates of the tip of the hour hand are $(x_h(t), y_h(t))$ where $x_h(t) = 5\cos(\pi t/6)$ and $y_h(t) = 5\sin(\pi t/6)$. Thus, the distance between the tips of the hands at time t is

$$\begin{aligned} D(t) &= \sqrt{\Big(x_m(t) - x_h(t)\Big)^2 + \Big(y_m(t) - y_h(t)\Big)^2} \\ &= \sqrt{74 - 70\Big(\sin(2\pi t)\cos(\pi t/6) + \cos(2\pi t)\sin(\pi t/6)\Big)} \\ &= \sqrt{74 - 70\sin(13\pi t/6)} \end{aligned}$$

so the distance between the hands at time t is changing at the rate

$$D'(t) = \frac{770\pi}{12 \cdot D(t)} \sin\left(\frac{13\pi}{6}t\right)$$

Thus, at time $t = 9$ the distance between the hands is increasing at the rate of $D'(9) = 770\pi/12\sqrt{74} \approx 23.434$ feet/hour or approximately 4.6868 inches/minute or 0.3906 feet/minute.

27. The elevation of the rocket at time t is $y = 100\tan\theta$ where θ is the angle of elevation at time t. Therefore, since both y and θ are functions of time, the speed of the rocket at time t is $y'(t) = 100 \cdot \theta'(t) \cdot \sec^2\theta(t)$ where θ' is rate of change in the angle of elevation at time t. From the problem statement, we are interested in the value of y' at the time when $\theta = \pi/3$ radians and $\theta' = \pi/15$ radians/sec. Thus, at this time, $y' = 100 \cdot 4 \cdot \pi/15 = 80\pi/3 \approx 83.776$ m/sec.

29. When the water in the tank is h feet deep, the volume of water in the tank is $V = \frac{1}{2}\big(3 + (3+h)\big)h \cdot 10 = 30h + 5h^2$ cubic feet. (The volume is the cross-sectional area times the length of the tank. When the water has height h, the upper base of the trapezoidal cross-section has length $3 + h$.) Thus, $V' = 10(3+h)h'$ cubic feet/minute. When $h = 1$ foot and $h' = 1/48$ feet/minute, $V' = 5/6$ cubic feet/minute.

§4.6 Newton's Method: Finding Roots

1. The first three answers, written as fractions, are $x_1 = 9/4$, $x_2 = 161/72$, and $x_3 = 51841/23184$. (The last answer, by the way, is correct to 9 decimal places!)

3. Newton's method will converge to the leftmost root (i.e., the root near -1.88) because the x-intercept of the line tangent to the graph at $x = 0.95$ is to the left of the local maximum point at $x = -1$. (In fact, the x-intercept of this tangent line is approximately -2.44.) It follows that subsequent iterations of Newton's method will converge to the root near -1.88 (all tangent lines based at points to the left of $x = -1$ have x-intercepts to the left of $x = -1$).

5. (a) Newton's method with $x_0 = 0.5$ gives $x_1 = 0.724638$, $x_2 = 0.7063515$, and $x_3 = 0.706115$. Thus, to three-decimal-place accuracy, the solution is $x = 0.706$.

   (b) The function f has no other roots, because $f'(x) = 5x^4 + 4 > 0$ for all x. (This means that f is increasing everywhere.)

7. (a) $N(x) = x - \dfrac{x^2 - a}{2x} = \dfrac{x}{2} + \dfrac{a}{2x}$.

   (b) If $x > \sqrt{a}$, then $a/x < a/\sqrt{a} = \sqrt{a}$. If $x < \sqrt{a}$, then $a/x > a/\sqrt{a} = \sqrt{a}$.

   (c) The result follows from simple algebra.

   (d) The estimates are 1, $3/2$, $17/12$, $577/408$, $665857/470832$.

9. (a) The first few Newton estimates are 1.250000000, 1.0250000001.000304878, 1.000000046. They are accurate to 0, 1, 3, and 7 decimal places respectively.

   (b) Newton's method finds $x = 1$ if $x_0 > 0$; it finds $x = -1$ if $x_0 < 0$. It fails if $x_0 = 0$.

11. x_{n+1} is the x-intercept of the line tangent to the graph of f that passes through $\big(x_n,\ f(x_n)\big)$; this line has slope $f'(x_n)$. Therefore, equation **??** implies that $x_{n+1} = x_n - f(x_n)/f'(x_n)$.

13. (a) Suppose that $0 < x < \sqrt{a}$. Then $\sqrt{a}x < \sqrt{a}\sqrt{a} = a \implies \sqrt{a} < a/x$. Alternatively, suppose that $x > \sqrt{a} > 0$. Then $x > \sqrt{a} \implies \sqrt{a}x > a \implies \sqrt{a} > a/x$.

    (b) $N(x) = x - \dfrac{x^2 - a}{2x} = x - x/2 + a/2x = (x + a/x)/2$.

    (c) If $x = \sqrt{a}$, then $N(x) = x$. In other words, Newton's method "stops"—as it should—when it finds the *exact* root $\sqrt{a}$.

15. (a) The approximate roots are -0.244817, 3.80675, and 6.43807.

    (b) Newton's method jumps back and forth between the estimates $x = 2$ and $x = 5$. Since $f(2)/f'(2) = -3$, and $f(5)/f'(5) = 3$, applying Newton's method to 2 gives 5; applying it to 5 gives 2. (Draw the graph to see the situation more clearly.)

    (c) $f'(1.39)$ is a small number because $x = 1.39$ is near a critical point of f (i.e., the tangent line is nearly horizontal). This means that the x-intercept of the tangent line may be far from the current estimate of the root. This causes Newton's method to converge slowly.

17. To find the minimum value of $g(x)$, consider $g'(x) = -20x^{-3} + 6x + 1$. For $x > 0$, $g''(x) = 60x^{-4} + 6 > 0$, so g is concave up for x in $[1, 10]$. Thus g has at most one local minimum on $[1, 10]$; it must occur at the one place where where $g'(x) = 0$. (Since $g''(x) > 0$, $g'(x)$ is always increasing, so $g'(x)$ can equal 0 for at most one value of $x > 0$.)

    Applying Newton's method to $g'(x)$, starting from $x_0 = 2$, locates the root $x \approx 1.3114$. Thus $g(1.3114) \approx 8.285$ is the minimum value of g.

19. From a graph, it appears that f achieves its maximum value between $x = 2.5$ and $x = 3$. To find the maximum value of f, therefore, we need to identify the corresponding root of f'. To do this, we use the Newton iteration formula $x_{n+1} = x_n - \dfrac{f'(x_n)}{f''(x_n)}$, where $f'(x) = 2x\sin(x^2) + 2x^3\cos(x^2)$ and $f''(x) = 2\sin(x^2) + 4x^2\cos(x^2) + 6x^2\cos(x^2) - 4x^4\sin(x^4)$. Using $x_0 = 3$, we obtain $x_1 = 2.78236$, $x_2 = 2.82791$, $x_3 = 2.82467$, $x_4 = 2.82465$, and $x_5 = 2.82465$. Thus, the maximum value of f is $f(2.82465) = 7.91673$.

21. The Newton's estimates "blow up." (Draw the graph to see why.) The underlying reason is that if $f(x) = x^{1/3}$, then (as algebra shows), $x - f(x)/f'(x) = -2x$.

§4.7 Building Polynomials to Order; Taylor Polynomials

1. The value and derivatives are (in order), 1, 1, 2, 6, 24, 120, 0, 0. Note that all derivatives beyond the fifth are zero.

3. (a) $P_2(x) = 1 + 2(x-1) + (x-1)^2$.
   (b) Multiplying out $P_2(x)$ gives x^2. This happens because the quadratic approximation to a quadratic function f is f itself.

5. $f'(x) = \frac{1}{3}x^{-2/3}$ and $f''(x) = -\frac{2}{9}x^{-5/3}$. Therefore, since, $f(8) = 2$, $f'(8) = 1/12$, and $f''(8) = -1/144$,
$$P_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 = \frac{10}{9} + \frac{5}{36}x + \frac{1}{288}x^2.$$

7. Theorem **??** says $p_2(x) = 2 + b_2(x-1)^2 + b_3(x-1)^3$. The conditions $p_2(2) = 1$ and $p_2'(2) = 0$ imply together that $b_2 = -3$ and $b_3 = 2$.

9. If $f(x) = \dfrac{1}{1-x}$, $n = 3$, and $x_0 = 0$, then $P_3(x) = 1 + x + x^2 + x^3$.

11. If $f(x) = \ln x$, $n = 3$, $x_0 = 1$, then $P_3(x) = (x-1) - \dfrac{(x-1)^2}{2} + \dfrac{(x-1)^3}{3}$.

13. If $f(x) = \sqrt{x}$, $n = 3$, $x_0 = 4$, then $P_3(x) = 2 + \dfrac{x-4}{4} - \dfrac{(x-4)^2}{64} + \dfrac{(x-4)^3}{512}$.

15. $\ell(x) = 1$; $q(x) = 1 - x^2/2$

17. $f(x) = e^x$ has $\ell(x) = 1 + x$, $q(x) = 1 + x + x^2/2$.

19. $f(x) = \arcsin x$ has $\ell(x) = x$, $q(x) = x$.

21. (a) The graph of an odd function is symmetric about the origin. All graphs shown have that property.
    (b) The *even-order* Maclaurin polynomials P_2, P_4, P_6, and P_8 are the same as the odd-order Maclaurin polynomials P_1, P_3, P_5, and P_7. (This happens because the sine function is odd. So, therefore, are all of its Maclaurin polynomials.)

23. (a) We'll use the linear approximation $l(x)$ at $x = 1$. Since $f(1) = 0$ and $f'(1) = \sin 1 \approx 0.84147$, $l(x) = 0 + \sin 1(x-1)$; $l(0.5) = 0 + \sin 1(-0.5) \approx -0.42074$.
    (b) Whether the estimate above is too big or too small depends on the concavity of f between $x = 0.5$ and $x = 1$. Notice that $f''(x) = 2x\cos(x^2)$; thus $f''(x) > 0$ for x in $[-0.5, 1]$; so f is concave *up*, and so the linear approximation *underestimates* f.
    (c) $f''(1) = 2\cos 1 \approx 1.08060$; therefore the quadratic approximation at $x = 1$ has the form $q(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 = 0 + \sin 1(x-1) + \cos 1(x-1)^2$. Therefore $q(0.5) \approx -0.28566$.

25. Let $f(x) = \sqrt{x}$ and $x_0 = 100$. Then $f'(x) = 1/2\sqrt{x}$, $f(x_0) = 10$, and $f'(x_0) = 1/20$. Therefore, $f(103) = f(x_0 + 3) \approx f(x_0) + 3f'(x_0) = 10 + 3/20 = 10.15$. A calculator gives $\sqrt{103} \approx 10.14889$; the difference is $10.15 - 10.14889 = 0.00111$.

27. Let $f(x) = \sin x$ and $x_0 = \pi/3$. Then $f'(x) = \cos x$, $f(x_0) = \sqrt{3}/2$ and $f'(x_0) = 1/2$. Therefore, $\sin 58^\circ = \sin(\pi/3 - \pi/90) \approx f(x_0) - (\pi/90)f'(x_0) = \sqrt{3}/2 - \pi/180 \approx 0.84857$. A calculator gives $\sin 58^\circ \approx 0.84805$; the difference is $0.84857 - 0.84805 = 0.00052$.

29. (a) $\ell_p(t) = 25t$; $\ell_p(1) = 25$; $\ell_p(-1) = -25$
    (b) $q_p(t) = 25t + t^2$; $q_p(1) = 26$; $q_p(-1) = -24$

(c) $\ell_v(t) = 25 + 2t$; $\ell_v(1) = 27$

31. (a) $\ell(t) = 100$ meters; $\ell(1) = 100$ meters
 (b) $q(t) = 100 - 4.9t^2$ meters; $q(1) = 95.1$ meters

33. (a) $f(1) = 1,\ f'(1) = 0$
 (b) no, $q(1) = 3 \neq 1$
 (c) no, $r'(1) = -4 \neq 0$
 (d) no, $s''(1) = 4 > 0$ but $f''(1) < 0$

35. No. Since f has a local maximum at $x = 1$, $f'(1) = 0$. Since $p'(1) \neq 0$, p cannot be a Taylor polynomial for f.

37. The polynomial $p(x) = x - x^3/6$ is the fourth-order Taylor polynomial approximation to $f(x) = \sin x$ based at $x_0 = 0$. Therefore, $p(0) = f(0)$, $p'(0) = f'(0)$, $p''(0) = f''(0)$, $p'''(0) = f'''(0)$, and $p^{(4)}(0) = f^{(4)}(0)$.

39. Let $f(x) = \sin x$ and $g(x) = x(\cos x)^{1/3}$. The graphs of f and g are almost indistinguishable in a viewing window centered at the origin because $f(0) = g(0)$, $f'(0) = g'(0)$, $f''(0) = g''(0)$, $f'''(0) = g'''(0)$, and $f^{(4)}(0) = g^{(4)}(0)$. [NOTE: $f^{(5)}(0) \neq g^{(5)}(0)$.]

41. Let $f(x) = e^x$ and $g(x) = 5/2 - \frac{5}{3}\cos x + \frac{1}{6}\cos(2x) + \frac{5}{3}\sin x - \frac{1}{3}\sin(2x)$. The graphs are almost indistinguishable in a viewing window centered at the point $(0, 1)$ because $f(0) = g(0)$, $f'(0) = g'(0)$, $f''(0) = g''(0)$, $f'''(0) = g'''(0)$, and $f^{(4)}(0) = g^{(4)}(0)$. [NOTE: $f^{(5)}(0) \neq g^{(5)}(0)$.]

43. (a) $q_{100}(x) = 1$
 (b) $q_{010}(x) = x$
 (c) $q_{001}(x) = x^2/2$
 (d) The graphs are two lines and a parabola.
 (e) If $q = aq_{100}(x) + bq_{010}(x) + cq_{001}(x)$, then $q(0) = aq_{100}(0) + bq_{010}(0) + cq_{001}(0) = a \cdot 1 + 0 + 0 = a$; the other parts are similar.

45. (a) Yes, they are inflection points, since $p_1''(1/2) = 0$ and $p_2''(3/2) = 0$.
 (b) Theorem **??** says that $p_3(x) = 2 + 0(x-1) - 6(x-1)^2 + a_3(x-1)^3$. To find a_3 we use $p_3(2) = 1$. But $p_3(2) = 2 - 6 + a_3 = 1 \implies a_3 = 5$, so $p_3(x) = 2 - 6(x-1)^2 + 5(x-1)^3$.
 (c) The inflection point on p_3 is at $x = 7/5$.

§4.8 Why Continuity Matters

1. The graph of f has points of discontinuity at -3, -1, 1, and 2. Thus, f is continuous on the intervals $[-4, -3)$, $(-3, -1)$, $(-1, 1)$, $(1, 2)$, and $(2, 4)$.

3. The function $h(x) = f(x) + g(x)$ will be continuous wherever *both* f and g are continuous. Thus, points of discontinuity of either f or g are possible points of discontinuity of h. By examining values of h near the points -3, -1, 1, 2, and 3, we find that h is continuous on the intervals $[-4, -3)$, $(-3, -1)$, $(-1, 1)$, $(1, 2)$, $(2, 3)$, and $(3, 4)$.

5. The function $h(x) = f(x)/g(x)$ will be continuous wherever $g(x) \neq 0$ and both f and g are continuous. Thus, the points -3, -1, $-1/3$, 1, 2, and 3 are possible points of discontinuity. By examining values of h near these points, we find that h is continuous on the intervals $(-4, -3)$, $(-3, -1)$, $(-1, -1/3)$, $(-1/3, 1)$, $(1, 2)$, $(2, 3)$, and $(3, 4)$.

7. Yes, there is an input to f that produces every output value between $f(-4) = -2$ and $f(4) = 1$.

9. Yes, $-2 = f(-4) \le f(x) \le f(0) = 2$ if $-4 \le x \le 4$. That is, f assumes both a maximum value and a minimum value over the interval $[-4, 4]$.

11. Since f is continuous on the interval $[0.1, 1]$, the EVT says that f assumes both a minimum value and a maximum value over this interval.

13. The interval $(0, 1]$ is not a closed interval. Thus, since the hypotheses of the EVT are not satisfied, f need not assume a maximumm and minimum value on this interval.

15. The function f is not continuous on the interval $[-1, 1]$, so the hypotheses of the theorem are not satisfied.

17. Since f is a polynomial, it is continuous everywhere. Moreover, $f(0) = 2$ and $f(1) = -1$. Therefore, since $-1 < 0 < 2$, the IVT implies that f has a root somewhere in the interval $(0, 1)$.

19. Note that $1^2 < 3 < 2^2$, so $\sqrt{3}$ lies in the interval $[1, 2]$; the midpoint of this interval is $m_1 = 3/2$. Since $(m_1)^2 = 9/4 < 3$, $\sqrt{3}$ lies in the interval $[3/2, 2]$; the midpoint of this interval is $m_2 = 7/4$. Since $(m_2)^2 = 49/16 > 3$, $\sqrt{3}$ lies in the interval $[3/2, 7/4]$; the midpoint of this interval is $m_3 = 13/8$. Since $(m_3)^2 = 169/64 < 3$, $\sqrt{3}$ lies in the interval $[13/8, 7/4]$; the midpoint of this interval is $m_4 = 27/16$.

21. (a) Each iteration of the bisection method reduces the width of the interval in which the root is guaranteed to be located by a factor of 2. Thus, after 1 iteration, the interval has length $1/2^1 = 1/2$; after 2 iterations, the interval has length $1/2^2 = 1/4$; and so forth. The smallest integer n for which $1/2^n < 10^{-2}$ is $n = 7$. Therefore, 7 iterations of the bisection method are necessary to guarantee an estimate of the root within 10^{-2}.
    (b) The smallest integer n for which $1/2^n < 10^{-3}$ is $n = 10$. Therefore, 10 iterations of the bisection method are necessary to guarantee an estimate of the root within 10^{-3}.
    (c) The smallest integer n for which $1/2^n < 10^{-5}$ is $n = 17$. Therefore, 17 iterations of the bisection method are necessary to guarantee an estimate of the root within 10^{-5}.
    (d) The smallest integer n for which $1/2^n < 10^{-10}$ is $n = 34$. Therefore, 34 iterations of the bisection method are necessary to guarantee an estimate of the root within 10^{-10}.

23. For each distance between 37 and 12 miles, there was some time at which I was that far from home. (Distance is a continuous function of time.)

25. The IVT says nothing—the amount of money in my pocket is not a continuous function of time, since only 2-decimal place numbers are possible.

27. Let $g(x) = f(x) - x$. Then, g is a continuous function, $g(0) > 0$, and $g(1) < 0$. The IVT implies that g must have a root in the interval $[0, 1]$, so $f(x) - x = 0 \implies f(x) = x$ for some x in the interval $[0, 1]$.

29. Since $f(0) \cdot f(1) < 0$, the two factors must have opposite signs. Therefore, the IVT guarantees that $f(x) = 0$ for some x in the interval $(0, 1)$.

31. The statement **cannot** be true since $f(x) \leq 5$ for all x.

33. The statement **must** be true because, by hypothesis, there is a value of x for which $f(x) = -7 \implies |f(x)| = 7$.

35. The statement **might** be true. If f is continuous on $[1, 5]$, then the IVT guarantees that the statement is true. However, if f isn't continuous, the statement might be true, but it need not be.

37. The statement **might** be true.

    Notice first that
    $$f(1) \cdot f(5) < 0 \iff f(1) \text{ and } f(5) \text{ have opposite signs.}$$
    Thus the question is whether having a root in $(1, 5)$ means, necessarily, that f changes sign from $x = 1$ to $x = 5$. A little thought shows that f may or may not do so. E.g., $f(x) = x - 3$ *does* change sign from $x = 1$ to $x = 5$, but $f(x) = (x - 3)^2$ *doesn't*.

39. Since f is a polynomial, it is continuous everywhere. Also, since $f(0) = 2$ and $f(-1) = -5$, the IVT guarantees that a root of f lies in the interval $(-1, 0)$.

41. Consider the function $g(x) = f(x) - x$. Finding a *fixed point* of f is the same thing as finding a *root* of g, since $g(x) = 0 \iff f(x) = x$. So let's show that g has a root.

    By hypothesis, $g(0) = f(0) \geq 0$, and $g(1) = f(1) - 1 \leq 1 - 1 \leq 0$. Thus $g(0) \geq 0 \geq g(1)$.

    If either $g(0) = 0$ or $g(1) = 0$ we've *found* our root, and we're done. The only alternative is that $g(0) > 0 > g(1)$. In this case, the IVT guarantees that g has a root somewhere in $(0, 1)$, so we're done.

§4.9 Why Differentiability Matters; The Mean Value Theorem

1. The converse of the given statement is "If it's cloudy, then it's raining." The given statement is true, but its converse is false.

3. The converse of the given statement is "If $x > 1$, then $x > 2$." The given statement is true, but its converse is false.

5. The converse of the given statement is "If $f(x) = x^2$, then $f'(x) = 2x$." The given statement is false, but its converse is true.

7. If $m = (a+b)/2$, then $\big(q(b) - q(a)\big)/(b-a) = q(m)$.

9. No such function exists. Suppose that x_1, x_2, and x_3 are roots of f. Rolle's theorem implies that f must have a stationary point between x_1 and x_2 and also between x_2 and x_3 since $f(x_1) = f(x_2) = f(x_3) = 0$. Thus, f must have at least 2 stationary points.

11. The sine function is an example of a function with infinitely many roots and infinitely many stationary points.

13. Suppose that $f'(x) > 2$ for all x. Then the speed limit law implies that $f(1) - f(0) > 2$. This contradicts the fact that $f(1) - f(0) = 1$. Therefore, it is impossible to find a differentiable function f with the desired properties.

15. No, f is not differentiable at $x = 0$.

17. Yes, f is continuous on the closed interval $[1, 2]$ and differentiable on the open interval $(1, 2)$.

19. Take a look at the graph of $f'(x)$ on the interval $[-1, 1]$. You will see that it has a value of ≈ 8 at $x = -1$, and a value of 0 at $x = 1$. Since $f'(x)$ is a polynomial and therefore continuous, by the IVT there must exist a c on $[-1, 1]$ where $f'(c) = 2$. Since $f'(x)$ must equal 2 at some c, then $f(x)$ must have slope 2 at that same c.

21. The MVT says that $f'(c) = \big(f(2) - f(1)\big)/(2-1) = 3$ for some c in the interval $(1, 2)$. Since $f'(x) = 2x$, the only suitable value of c is $c = 1.5$.

23. Since g is not differentiable at 0, Rolle's theorem doesn't apply.

25. If f' is a continous function, f is differentiable on $(1, 4)$. Since $f(1) = f(4) = 0$, Rolle's theorem implies that there must be a c between 1 and 4 for which $f'(c) = 0$.

27. If f' is not continuous, f' can "jump" from a positive to a negative value or *vice versa*.

29. (a) Suppose that $f(2) = 3$. Then, according to the MVT, there would be a number c such that $0 \le c \le 2$ and $f'(c) = \big(f(2) - f(0)\big)/(2-0) = 3/2$. However, this contradicts the hypothesis that $f'(x) < 1$. Thus, $f(2) = 3$ is not possible.
    (b) $3/2 \le f(3) \le 3$
    (c) $-1 \le f(-1) \le -1/2$

31. (a) $f'(1) = 3$, and $f'(3) = -1$.
    (b) $f(1) = 2$. The equation of the line tangent to the graph of f at $x = 3$ is $y = 5 - x$. Thus, $f(3) = 2$.
    (c) Since the function is differentiable everywhere, $f'(1) > 0$, and $f'(3) < 0$, there must, by Rolle's Theorem, be some point where $f'(x) = 0$.
    (d) To get more than one maxima on that interval, the second derivative would have to have a sign change. Since it doesn't, that means $f'(x)$ must decrease all the time, and that means it can only cross the y axis once, which in turn means that $f(x)$ can have only one critical point, which must be a maximum, due to the negative second derivative.

33. (a) Yes. The trucker traveled 100 miles in 1.25 hours. The MVT asserts that the trucker's speed must have been 80 mph at some time during the trip.

(b) The trucker's fine will be at least $\$125 = \$50 + \$5 \cdot 15$.