§1.1 Functions, Calculus Style

1. (a) Reading from the altitude graph, $A(1) \approx 5700$ feet.

   (b) Reading from the velocity graph, $V(1) \approx 650$ feet/minute.

3. (a) The balloon was ascending at time $t = 1$ and descending at time $t = 6$.

   (b) Near time $t = 1$, the altitude graph is increasing; values of $A(t)$ get larger as one moves to the right from $t = 1$. Near time $t = 6$, the altitude graph is decreasing; values of $A(t)$ get smaller as one moves to the right from $t = 1$.

   (c) At time $t = 1$, the value of the $V(t)$ is positive. At time $t = 6$, the value of the $V(t)$ is negative.

5. The approximate population function is $p(t) = 906e^{0.008\cdot(t-1800)}$. Recall that the function p is only an approximation—it can't give exact population figures. Here are the predictions in question:

   (a) For year 2010, we'd predict $p(2010) = 906e^{0.008\cdot(2010-1800)} \approx 4861.19$ millions, or about 4.861 billion people.

   (b) For the year 1000, we get $p(1000) = 906e^{0.008\cdot(1000-1800)} \approx 1.51$ million.

   (c) For the year 1000 B.C.E., use $t = -1000$. This gives the ridiculous prediction $p(-1000) = 906e^{0.008\cdot(-1000-1800)} \approx 0.00000017$ millions, or about 0.17 people!

   (d) None of these answers are completely convincing. This isn't surprising, since we're using a simple formula to predict a complex phenomenon. Inputs near $t = 1800$ produce the most believable outputs.

7. In each part, the question is *which* line of the multiline definition of m to use.

   (a) $m(-4) = -3 \times (-4)/2 - 9/2 = 3/2 = 1.5$

   (b) $m(0) = -\sqrt{4 - (0+1)^2} = -\sqrt{3}$

   (c) $m(2.3) = 2.3 - 1 = 1.3$

   (d) $m(\pi) = -\pi + 5 \approx 1.8584$

9. (a) The circle with radius r and center at the point (a, b) can be described by the equation $(x-a)^2 + (y-b)^2 = r^2$. Thus, the circle with radius $r = 2$ and center at the point $(-1, 0)$ is described by the equation $(x+1)^2 + y^2 = 4$.

   (b) The equation for the circle found in part (a) can be written in the form $y^2 = 4 - (x+1)^2$. Thus, the equation $y = -\sqrt{4 - (x+1)^2}$ describes a portion of this circle. This is the equation that appears in the defintion of m. Thus, the graph of m over the interval $[-3, 1]$ is an arc of the circle of radius 2 centered at $(-1, 0)$.

11. Call the function f. Since the pieces of f are lines, the "formula" for f is

$$f(x) = \begin{cases} 3 & \text{if } -5 \le x < -3 \\ -x - 2 & \text{if } -3 \le x \le 2 \\ 2x - 8 & \text{if } 2 < x \le 5 \end{cases}$$

13. (a) The variable x is the edge length of the removed corners. Negative edge length doesn't make physical sense in the context of this example.

    (b) Since two edge lengths are removed from each side of the piece of paper, $2x$ must be less than the length of the shortest side of the paper. Thus, $2x < 8$ must be true for physical reasons.

15. (a) The volume is the product of the three dimensions: $V(w) = w(24 - 2w)(32 - 2w)$.

(b) The problem is to find, from among all legal inputs $0 \le w \le 12$, the one that produces the largest output $V(w)$. A graph of V for $0 \le w \le 12$ shows that the volume takes its largest value—a bit less than 1600 cubic inches—at $w \approx 4.5$. [NOTE: With calculus methods we could improve these guesses slightly, to get $w = (28 - \sqrt{208})/3 \approx 4.526$; $V \approx 1552.539$.]

17. (a) Since the area of a rectangle is width times height, $A(t) = 3t$.

 (b) The graph is a line segment joining the points $(0, 0)$ and $(5, 15)$.

19. (a) For each t, the area of the trapezoid is

$$A(t) = \frac{1 + (2t + 1)}{2}(t) = \frac{t}{2}(2t + 2) = t^2 + t.$$

[NOTE: The area of a trapezoid is base $\times$ $(\text{ht}_1 + \text{ht}_2)/2$.]

21. (a) The domain of g is $[0, \infty)$. In other words, it is the set $\{ w \mid w \ge 0 \}$.

 (b) Yes — The range of g is $(-\infty, 12]$.

23. (a) 3 is in the domain of f.

 (b) 7 is in the range of f.

 (c) Yes — if $x = 3$, $2x + 1 = 7$.

 (d) Yes — if $t = 3$, $t^2 - 2 = 7$.

 (e) No — if $z = 3$, $z^3 - z^2 = 18 \ne 7$.

25. A function has a *unique* output for each input. Since, for example, the points $(x, y) = (0, -1)$ and $(x, y) = (0, 1)$ are on the graph — both with $x = 0$, the graph cannot be the graph of a function.

27. The domain of f is the set of inputs for which a value of the function is defined. Thus, the domain of f is the interval $[0, \infty)$. From the first two lines of the definition of f, it is apparent that the values 0 and -2 are in the range of f. From the third line of the definition, we can see that the interval $[2, \infty)$ is in the range of f. Thus, the range of f is $\{ 0, -2 \} \cup [2, \infty)$

29. Because the domain of the square root function is $[0, \infty)$, the function f is defined if and only if $x^2 - 4 \ge 0$. Therefore, the domain of f is $(-\infty, -2] \cup [2, \infty)$.

31. The rule defining r makes sense as long as $(x + 2)(1 - x) > 0$. Now, the expression $(x + 2)(1 - x)$ can changes sign the points $x = -2$ and $x = 1$. Checking nearby values of x shows that $(x + 2)(1 - x) > 0$ only if $-2 < x < 1$. Therefore the natural domain of r is the interval $(-2, 1)$.

33. (a) $g(0) = 0$, $g(-1) = \sqrt{2} \approx 1.4142$, $g(1) = \sqrt{2} \approx 1.4142$, $g(2) = \sqrt{20} \approx 4.4721$, and $g(500) = \sqrt{62500250000} \approx 250000.5$.

 (b) The formula for the distance between the points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. It follows that the formula for g is $g(x) = \sqrt{x^2 + f(x)^2} = \sqrt{x^2 + x^4} = |x|\sqrt{1 + x^2}$.

35. (a) Specific values of j can be found by computing slopes directly: $j(2) = 3$, $j(1.1) = 2.1$, $j(1.01) = 2.01$, $j(0.99) = 1.99$, $j(0.9) = 1.9$, and $j(0) = 1$.

 (b) $x = 1$ is not in the domain of j because then the definition doesn't make sense. (A single point doesn't determine a line.)

 (c) The following formula defining $j(x)$ holds for all $x \ne 1$:

$$j(x) = \frac{f(x) - 1}{x - 1} = \frac{x^2 - 1}{x - 1} = \frac{(x + 1)(x - 1)}{x - 1} = x + 1.$$

 (d) The graph of j is the line $y = x + 1$ with a “hole” at $x = 1$.

37. The rental company charges \$30 for the first 100 miles, and \$0.07 per mile for each mile over 100 miles.

39. (a) Volume = length × width × depth. Since the tank is a cube, $L = W = 10$ feet. Thus, the volume of the tank is $V = 100d$, where d is the depth of the water in the tank.

(b) The *domain* is the interval of sensible depth values, so $0 \le d \le 10$ is the domain of V. (Depth can't be negative; depth over 10 feet overflows the tank.)

(c) Plugging in $d = 0$ and $d = 10$ shows that the *range* of possible volumes (in cubic feet) is the interval $[0, 1000]$.

(d) $V = 100d \implies d = V/100$. The latter expresses d as a function of V.

41. (a) The absolute value function is defined for all real numbers x. Thus, the domain of f is the interval $(-\infty, \infty)$. The range of f is the set of nonnegative real numbers; the interval $[0, \infty)$.

(b) The slope of the f-graph is not defined at $x = 0$ because it has a sharp corner there. (Elsewhere, its slope is ± 1.)

(c) Since the f-graph has a slope for all $x \ne 0$, the domain of g is $(-\infty, 0) \cup (0, \infty)$. Since the slope of the f-graph, when it is defined, is always ± 1, the range of g is the set $\{-1, 1\}$.

(d) $g(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$

§1.2 Graphs

1. (a) The rule defining R makes sense if $t \neq 0$. Therefore, the domain of R is $(-\infty, 0) \cup (0, \infty)$.
   (b) The expression $1/t^2 > 0$ for all $t > 0$, so the range of R is $(0, \infty)$.
   (c) No, because 0 is not in the domain of R.
   (d) Yes; if $-5 < a < b < -1$, then $R(a) < R(b)$.
   (e) No; if $1 < a < b < 5$, then $R(b) < R(a)$ (i.e., R is decreasing on this interval).

3. B has a root at x if $B(x) = 0$. Thus, since the graph of B crosses the x-axis at $x = -3$, B has a root at $x = -3$.

5. The number x is a root of a function f if and only if $f(x) = 0$. Since $u = \sqrt{2}$ is a root of the function Y, $Y(\sqrt{2}) = 0$.

7. The function g is neither even nor odd. To see this, note that $g(1) = 2$ and $g(-1) = 0$. Thus, there is an x in the domain of g for which $g(-x) \neq \pm g(x)$; this implies that g is neither even nor odd.

9. No; it is apparent from a graph that g is concave down over a portion of the itnerval $[-2, 2]$.

11. (a) $f(x) = (x-4)^3$ has the desired properties.
    (b) W has an inflection point at $x = 4$ because its graph changes its direction of concavity at this point.

13. The graph shows that I has a local maximum near $t = 12$ (i.e., on day 12). This means that the number of individuals with measles is largest at this time.

15. The inflection point in the graph of $S(t)$ near $t = 7$ means that the rate of change of the susceptible population is slowing down (because the concavity changes from concave down to concave up).

17. (a) The graph of j can be obtained from the graph of f by "stretching" the graph of f by a factor of 2.
    (b) The graph of k can be obtained from the graph of f by "stretching" the graph of f by a factor of $1/2$ (i.e., by compressing it).
    (c) The graph of m can be obtained from the graph of f by "stretching" the graph of f by a factor of 2, then reflecting it about the x-axis.

19. If x and $-x$ are both in the domain of f, then $f(x) = f(-x)$ since f is an even function. Thus, $f(-1) = f(1) = 2$.

21. Since T has period 7, $T(x+7) = T(x)$. Thus, $T(7) = T(0+7) = T(0) = 3$.

23. Since T has period 7, $T(x + k \cdot 7) = T(x)$ for any integer k. Thus, $T(-30) = T(-30 + 10 \cdot 7) = T(-30 + 70) = T(40)$.

25. (a) $U(1.4) = T(7) = T(0) = 3$, since T has period 7.
    (b) Suppose that $U(x) = U(x+P)$. Then, $U(x+P) = T\big(5(x+P)\big) = T(5x + 5P) = T(5x) = U(x)$ if $5P = 7$. Since $U(x + 7/5) = U(x)$, U is periodic with period $7/5 = 1.4$.

27. Lines with positive slope go "up and to the right". Here, lines A, B, and C have positive slopes; lines D and E have negative slope. The slope of E is most negative; the slope of A is most positive. Thus, $m_E < m_D < 0 < m_B < m_C < m_A$.

29. From the graph, it appears that the lowest point of f over the interval $[-3, 3]$ is $(-1.5, -1.8)$. Thus, the minimum value of f over the interval is -1.8; it is achieved at $x = -1.5$.

31. No, because the graph shows that f is decreasing over part of this interval.

33. Yes. The sine function is one possible example.

41. The information given implies that $f(-3) = -5$. Since f is an even function, it follows that $f(3) = -5$. Thus, the point $(3, -5)$ is also on the graph of f.

43. (a) $g(x) = x^3 + 2$

    (b) The graph of g can be obtained by shifting a graph of f up by 2 units.

45. (a) $g(x) = 2x^3$

    (b) The graph of g can be obtained by stretching a graph of f vertically by a factor of 2.

47. The function is odd. Its graph is symmetric with respect to the origin.

49. The function is neither even nor odd. It is not symmetric about the origin and it is not symmetric about the y-axis.

51. The line A has slope 2 and passes through the point $(2, 3)$. Therefore, it can be described by the equation $y = 2(x - 2) + 3 = 2x - 1$.

53. Line A is described by the equation $y = 2x - 1$ and line B is described by the equation $y = 5 - x$. At $x = 4$, line A has height 7 and line B has height 1. Since the graph $y = g(x)$ lies between the lines A and B at $x = 4$, $1 < g(4) < 7$. Thus, $L = 1$ and $U = 7$.

55. (a) When $D =$ water depth is 0, there is no water in the tank; hence $V(0) = 0$.

    (b) The tank is half of a sphere of radius 10 ft. Thus, when $D = 10$ ft, the tank is full of water. So, $V(10)$ is half the volume of a sphere of radius 10 ft. Thus, $V(10) = \frac{1}{2}\left(\frac{4}{3}\pi(10)^3\right)$ ft$^3 = \frac{2000\pi}{3}$ ft^3. [NOTE: The formula for the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$. (Be careful, r is not the water depth!)]

    (c) When $D = 0$, $V = 0$. Also, the shape of the tank means that the volume increases faster and faster with respect to depth. Only Graph C has these properties.

57. If $C(t)$ is the cost of health insurance at time t, then any correct graph of C should be *increasing and concave up*.

59. The point $(4, 5)$ is another local maximum on the graph of F. Since F is periodic with period 7, $F(x) = F(x + 7)$. Therefore, $5 = F(-3) = F(7 - 3) = F(4)$. Since $(-3, 5)$ is a local maximum of F, $(4, 5)$ is also a local maximum.

61. (a) $g(x) = 2x^2 - 4x + 5 = 2\left(x^2 - 2x + 1\right) + 3 = 2(x - 1)^2 + 3 = 2f(x - 1) + 3$

    (b) The graph of g can be obtained from the graph of f by "stretching" vertically by a factor of 2, translating to the right by 1 unit, and then translating upward by 3 units.

63. (a) Yes, a periodic function can be even. The cosine function is an example.

    (b) Yes, a periodic function can be odd. The sine function is an example.

    (c) Yes, a periodic function can be neither even nor odd. The function $f(x) = 1 + \sin x$ provides an example.

65. Since the graph of f is symmetric with respect to the origin, f is an odd function. Therefore, $f(-1) = -f(1) = -3$.

67. (a) $y = f(a) + \dfrac{f(b) - f(a)}{b - a}(x - a)$ is the equation of the secant line through the points $\big(a, f(a)\big)$ and $\big(b, f(b)\big)$. Therefore, every point on the line segment between $x = a$ and $x = b$ lies above the graph of f.

    (b) $f(a) + \dfrac{f(b) - f(a)}{b - a}(x - a) > f(x) \implies \dfrac{f(b) - f(a)}{b - a} > \dfrac{f(x) - f(a)}{x - a}$.

    (c) The line segment lies below the graph of g.

69. (a) Let ℓ be the linear function that passes through the points $(1, 4)$ and $(5, 2)$ on the graph of g; an equation for this line is $\ell(x) = (9 - x)/2$. Since g is concave down on the interval [0, 10], $g(3) > \ell(3) = 3$.

(b) Proceeding as in part (a) leads to the result $g(4) > 5/2$.

§1.3 A Field Guide to Elementary Functions

1. The expression involves a fractional power (i.e., $x^{1/2}$) rather than just positive integer powers.

3. The expression $(1+2x)/4x^3 = 0.25x^{-3} + 0.5x^{-2}$ involves negative powers of the variable.

5. Yes, because $(x^3+1)/(x+1) = x^2 - x + 1$ for all $x \neq -1$.

7. (a) The natural domain of any polynomial is $(-\infty, \infty)$.
   (b) Since Q is a 6^{th}-degree polynomial, it can have at most 6 roots.
   (c) Any function can have at most one y-intercept (i.e., if $x = 0$ is in the domain of the function, it must have a unique output value) . Since Q is a function that includes 0 in its domain, it has one y-intercept.
   (d) Since Q is an even-degree polynomial, it is possible that $Q(x) > 0$ for all x. (For example, $Q(x) = x^6 + 1$ has no real roots.)
   (e) Since Q is an even-degree polynomial, it is possible that $Q(x) < 11$ for all x. (For example, $Q(x) = -x^6$ has this property.)
   (f) No. Every polynomial p has the property that $|p(x)| \to \infty$ as $x \to \pm\infty$.

9. $m(x) = \dfrac{3x}{x^2+1} + \dfrac{4}{5x+6} = \dfrac{3x(5x+6) + 4(x^2+1)}{(x^2+1)(5x+6)} = \dfrac{19x^2+18x+4}{5x^3+6x^2+5x+6}$. Thus, $p(x) = 19x^2 + 18x + 4$ and $q(x) = 5x^3 + 6x^2 + 5x + 6$.

11. π^x involves a variable power of a fixed base, so it is an exponential function. x^π involves a fixed power of a variable base, so it is not an exponential function.

13. If $b > 0$, $b^0 = 1$. Therefore, the point $(0, 1)$ is on the graph of $y = b^x$ for every positive number b.

15. (a) All real numbers are in the natural domain of f. That is, the domain of f is the interval $(-\infty, \infty)$.
    (b) The range of f is all positive real numbers. That is, the range of f is the interval $(0, \infty)$.
    (c) f is increasing everywhere on its domain: if $a < b$, then $f(a) < f(b)$.
    (d) Since f is increasing everywhere on its domain, it does not have any local maximum or local minimum points.
    (e) f is concave up everywhere on its domain.
    (f) Since f is concave up everywhere, there are no points at which its concavity changes (i.e., f has no inflection points).
    (g) Since $f(x) > 0$ for all x, f has no roots.

17. (a) The domain of the cosine function includes all real numbers, so the domain of f is $(-\infty, \infty)$.
    (b) The range of the cosine function is the interval $[-1, 1]$, so the range of f is $[-1, 1]$.
    (c) Yes, f is an even function because the cosine function is an even function: $f(-x) = \cos(-x/2) = \cos(x/2) = f(x)$.
    (d) Yes, f is periodic with period 4π: $f(x+4\pi) = \cos\big((x+4\pi)/2\big) = \cos(x/2 + 2\pi) = \cos(x/2)$.
    (e) f is decreasing throughout the interval $(0, 2\pi)$. Therefore, f is increasing nowhere in the interval $(0, 2\pi)$.
    (f) Since f is decreasing throughout the interval $(0, 2\pi)$, f has no local maximum or local minimum points in this interval.
    (g) By examining a graph of f, it can be seen that f is concave up on the interval $(\pi, 2\pi)$. [NOTE: This is because the cosine function is concave up on the interval $(\pi/2, \pi)$.]
    (h) f has an inflection point at $x = \pi/2$ because the concavity of f changes from down to up at this point.
    (i) Since $f(\pi) = \cos(\pi/2) = 0$, f has a root at $x = \pi$. This is the only root of f in the interval $(0, 2\pi)$.

19. (a) The tangent function is not defined at x if $\cos x = 0$. Thus, the domain of f is $(-\infty, \infty)$ except $x = \pi/2 + k\pi$, where $k = 0, \pm 1, \pm 2, \ldots$.

 (b) The range of the tangent function includes all real numbers; it is the interval $(-\infty, \infty)$.

 (c) $\tan(-x) = \dfrac{\sin(-x)}{\cos(-x)} = \dfrac{-\sin x}{\cos x} = -\tan x$. Thus, $f(x) = \tan x$ is an odd function.

 (d) Yes, f is periodic and has period π because $\tan(x+\pi) = \dfrac{\sin(x+\pi)}{\cos(x+\pi)} = \dfrac{-\sin x}{-\cos x} = \dfrac{\sin x}{\cos x} = \tan x$.

 (e) f is increasing everywhere it is defined. Thus, f is increasing on the intervals $(0, \pi/2)$, $(\pi/2, 3\pi/2)$, and $(3\pi/2, 2\pi)$.

 (f) Since f is increasing everywhere it is defined, it has no local maximum or local minimum points.

 (g) f is concave up on the intervals $(0, \pi/2)$ and $(\pi, 3\pi/2)$.

 (h) f changes from concave down to concave up at $x = \pi/2$, $x = \pi$, and at $x = 3\pi/2$. Thus, these are inflection points of f.

 (i) Since $\tan(\pi) = 0$, f has a root at $x = \pi$.

 (j) f has vertical asymptotes at $x = \pi/2$ and at $x = 3\pi/2$.

21. (a) If $b > 0$, the exponential function b^x is defined for all real numbers x. Thus, the natural domain of h is the interval $(-\infty, \infty)$.

 (b) If $b > 0$, then $0 < b^x$. Thus, the range of h is the interval $(0, \infty)$.

 (c) Since $0 < b < 1$, the exponential function b^x is decreasing everywhere. Thus, there is no interval on which h is increasing.

 (d) Since h is decreasing throughout its domain, h has no local maximum or local minimum points.

 (e) For any $b > 0$, the exponential function b^x is concave up everywhere. Therefore, h is concave up on the interval $(-\infty, \infty)$.

 (f) Since h is always concave up, it has no inflection points.

 (g) Since $h(x) > 0$ for all x, h has no roots.

23. (a) For any $b > 0$, the logarithm function $\log_b x$ is defined for all $x > 0$. Thus, the natural domain of H is the interval $(0, \infty)$.

 (b) For any $b > 0$, the range of the logarithm function $\log_b x$ is the interval $(-\infty, \infty)$. Thus, the range of H is the interval $(-\infty, \infty)$.

 (c) If $0 < b < 1$, the logarithm function $\log_b x$ is decreasing throughout its domain. Therefore, H is not increasing anywhere.

 (d) Since H is decreasing throughout its domain, H does not have any local maximum or local minimum points.

 (e) If $0 < b < 1$, the logarithm function $\log_b x$ is concave up throughout its domain. Therefore, H is concave up on the interval $(0, \infty)$.

 (f) Since H is concave up throughout its domain, it has no inflection points.

 (g) For any $b > 0$, $\log_b 1 = 0$. Thus, H has exactly one root, at $x = 1$.

25. The domain of the cosine function is all real numbers; its range is the interval $[-1, 1]$. Therefore, the domain of f is the interval $(-\infty, \infty)$ and its range is $[-1, 1]$. [NOTE: The graph of f can be obtained from a graph of the cosine function by horizontal shifting and horizontal compressing.]

27. The domain of the exponential function e^u is $(-\infty, \infty)$. Thus, the domain of f is also $(-\infty, \infty)$. Since $0 \le x^2$ for all x, $1 \le e^{x^2}$ for all x. Therefore, the range of f is the interval $(-\infty, -1)$.

29. Both the sine and the natural exponential functions include all real numbers in their domain. Therefore, the domain of f is the interval $(-\infty, \infty)$. Since the range of the sine function is the interval $[-1, 1]$, $e^{-1} \leq e^{\sin x} \leq e^{1}$ for all x. This implies that the range of f is the interval $[e^{-1}, e]$.

31. The domain of the natural logarithm function is all positive real numbers, so x is in the domain of f if $x^3 + 1 > 0$. The latter condition will be true if $x > -1$, so the domain of f is the interval $(-1, \infty)$. Since the expression $x^3 + 1$ takes on all real values greater than 0 as x ranges through the interval $(-1, \infty)$, the range of f is the same as the range of the natural logarithm function. That is, the range of f is the interval $(-\infty, \infty)$.

33. Since the sine function is defined for all x, and since $2 + \sin x > 0$ for all x, the domain of f is the interval $(-\infty, \infty)$. Since the range of $2 + \sin x$ is the interval $[1, 3]$, the range of f is the interval $[3^{-1/3}, 1^{-1/3}] = [3^{-1/3}, 1^{1/3}]$.

35. The graph of f is the graph of the sine function shifted horizontally to the right by one unit. Since the sine function has period 2π, f also has period 2π.

37. The graph of f can be obtained by horizontally stretching the graph of the sine function by a factor of 2. Since the sine function has period 2π, f has period 4π.

39. The graph of f can be obtained by horizontally stretching the graph of the sine function by a factor of π. Since the cosine function has period 2π, f has period $2\pi^2$.

    Here's a symbolic proof of this result:
    $f(x + 2\pi^2) = \cos\big((x + 2\pi^2)/\pi\big) = \cos(x/\pi + 2\pi) = \cos(x/\pi) = f(x)$.

41. (a) For any $b > 0$, the natural domain of the logarithm function $\log_b x$ is the interval $(0, \infty)$. Thus, this interval is the domain of g.

    (b) Since $\log_{10} x = \dfrac{\ln x}{\ln 10}$, $g(x) = f(x)/\ln 10$ (i.e., $k = 1/\ln 10$).

    (c) The graph of g can be obtained by vertically stretching the graph of f by a factor of $1/\ln 10 \approx 0.43429$ (i.e., by compressing the graph of f vertically).

43. $e^B = e^{2\ln A} = \left(e^{\ln A}\right)^2 = A^2$

45. If the point (A, B) is on the graph of the exponential function b^x, then $B = b^A$. This, in turn, implies that $\log_b B = A$. In other words, the point (B, A) is on the graph of the logarithm function $\log_b x$.

47. (a) The graph $y = g(x)$ is the graph $y = f(x)$ shifted vertically upwards by 3 units. Since the line $y = 2$ is a horizontal asymptote of f, this means that the line $y = 5$ is a horizontal asymptote of g.

    (b) The line $y = 2$ is a horizontal asymptote of g. Since the graph of g is a horizontal translation of the graph of f, both graphs have the same horizontal asymptotes.

    (c) The line $y = 7$ is a horizontal asymptote of g. Here's why: The function $f(x + 4)$ has the line $y = 2$ as a horizontal asymptote since f does. This implies that the function $3f(x + 4)$ has the line $y = 3 \cdot 2 = 6$ as a horizontal asymptote. Therefore, the function $3f(x + 4) + 1$ has the line $y = 6 + 1 = 7$ as a horizontal asymptote.

49. (a) If $x \neq \pm 3$, $g(x) = \dfrac{1}{x+3} = \dfrac{1}{x+3} \cdot \dfrac{x-3}{x-3} = \dfrac{x-3}{x^2-9} = f(x)$.

    (b) Since f is a rational function, it is defined for all values of x where its denominator is nonzero. Thus, the domain of f is $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$. The range of f is the set of values $f(x)$ if x is in the domain interval. Thus, the range of f is $(-\infty, 0) \cup (0, 1/6) \cup (1/6, \infty)$. [NOTE: The values 0 and 1/6 are not in the range of f because $x = -3$ and $x = 3$ are not in the domain of f.]

    (c) Since g is a rational function, its domain includes all values of x for which its denominator is nonzero (i.e., $x \neq -3$). Therefore, the domain of g is $(\infty, -3) \cup (-3, \infty)$. The range of g is the set of values that $g(x)$ assumes as x ranges through the domain of g. Thus, the range of g is $(-\infty, 0) \cup (0, \infty)$.

(d) They are the same except at $x = 3$. f is undefined at $x = 3$, but $g(3) = 1/6$.

(e) Yes. The line $y = 0$ is a horizontal asymptote of f.

(f) Yes. The line $x = -3$ is a vertical asymptote of f.

§1.4 Amount functions and rate functions: the idea of the derivative

1. $P(2) = 100$ means that the car is 100 miles east of the reference point (the State Capitol on-ramp in Bismarck, North Dakota) at 2:00 p.m. on October 1, 2000.

3. $V(2) = 70$ means that at 2:00 p.m. on October 1, 2000, the car is traveling east at 70 miles per hour.

5. V is the rate function associated with P. That is, $V(t)$ is the rate of change of P at time t.

7. If P is measured in kilometers and time is measured in hours, then the rate of change of P is expressed in kilometers per hour.

9. The line tangent to the graph of $P(t)$ at $t = 2$ has slope $P'(2) = V(2) = 70$.

11. $B'(t)$ represents the rate at which the bank balance is changing at time t (in dollars per year).

13. (a) $\Delta g = g(x + \Delta x) - g(x) = 3(x + \Delta x) - 1 - (3x - 1) = 3\Delta x$
    (b) Since g is a linear function, $g'(x)$ is the slope of the line defined by g. Thus, $g'(x) = 3$.

15. $f'(-2) \approx 10$

17. (a) Since the rate of change of v is a constant (-9.8), v must be a linear function with slope -9.8. Thus, $v(t) = v_0 - 9.8t$, where v_0 is a constant.
    (b) The constant v_0 is the velocity of the object at time $t = 0$ (i.e., the object's initial velocity).

19. Just before the instant of time at which the height is greatest, the velocity is positive. Just after the instant of time at which the height is greatest, the velocity is negative. Since velocity changes continuously, its value must be zero at the instant of time at which the height is greatest.

21. The height of the ball at time t is $h(t) = h_0 + v_0 t - 16t^2$, where h_0 is the height of the ball at time $t = 0$ and v_0 is the ball's velocity at time $t = 0$. Since the ball was thrown from a height of 5 feet with an initial vertical velocity of 30 ft/sec, $h(t) = 5 + 30t - 16t^2$.

    The velocity of the ball at time t is $v(t) = v_0 - 32t$, where v_0 is the ball's initial vertical velocity. Since the initial vertical veclocity of the ball is 30 ft/sec, its velocity at time t is $v(t) = 30 - 32t$ ft/sec.

23. A car with a top speed of 10 miles per hour can't travel more than 10 miles in one hour. Thus, after 1 hour, the car can't have traveled more than 10 miles (i.e., it can't be more than 10 miles from its starting position).

25. (a) Since $h'(x) \le -2$ for all $x \ge 0$, $h(5) - h(0) \le (-2)(5 - 0) = -10$. Since $h(0) = 0$, this implies that $h(5) \le -10$.
    (b) Since $h'(x) \le -2$ for all $x \ge 0$, $h(3) - h(0) \le (-2)(3 - 0) = -6$. Since $h(0) = 0$, this implies that $h(3) \le -6 < 5$.
    (c) Let $h'(x)$ be the eastward velocity of a car at time x. The inequality $h'(x) \le -2$ for all $x \ge 0$ means that the car's eastward velocity is no greater than -2 (i.e., the car is always moving west at a speed of at least 2 miles per hour). Thus, after five hours, the car must be at least 10 miles west of its position at time $t = 0$. Similarly, after three hours, the car must be at least 6 miles west of its initial position and, therefore, it must be west of a position 5 miles east of its starting position.

27. Since $f'(x) \le 4$, the speed limit principle implies that $f(x) - f(0) \le 4x$ if $x \ge 0$, so $f(x) \le 4x + 2$ if $x \ge 0$. Thus, $f(1) \le 6$. The speed limit principle also implies that $f(x) \ge 4x + 2$ if $x \le 0$, so $f(-3) \ge -10$.

29. (a) The speed limit principle implies that $g(x) - g(0) \le 3x$ if $x \ge 0$, so $g(x) \le 1 + 3x$ if $x \ge 0$. The speed limit principle also implies that $g(x) - g(0) \ge 3x$ if $x \le 0$, so $g(x) \ge 1 + 3x$ if $x \le 0$.
    (b) The curve $y = g(x)$ lies above the line $y = 1 + 3x$ to the left of $x = 0$ and below this line to the right of $x = 0$.

41. (a) 100 days after January 1 (i.e., on April 10), the length of a day is increasing at a rate of 0.05 hours per day.

    (b) In the northern hemisphere, days get shorter between about June 21 and December 21. Thus, $H'(t) < 0$ if $171 \leq t \leq 354$.

43. Yes. $f'(-1)$ is the slope of the line tangent to f at $x = -1$. Comparing the graph of f near $x = -1$ with the line through the points $(-2, -3)$ and $(0, -1)$ makes it clear that that $f'(-1) > 1$.

45. (a) f is a constant function, so $f'(x) = 0$ for any x. Thus, $f'(11) = 0$.

    (b) $f'(x) = 0$

47. Since $f'(1) = 2$, the tangent line has slope 2. Since $f(1) = 5$, the tangent line passes through the point $(1, 5)$. Thus, an equation of the tangent line is $y = 2(x - 1) + 5$.

49. (a) $g'(4)$ is the slope of the curve $y = g(x)$ at $x = 4$. Thus, $g'(4) = -2$.

    (b) The slope of the tangent line is $g'(4) = -2$. Therefore, since the tangent line passes through the point $(4, 3)$, the tangent line is described by the equation $y = -2(x - 4) + 3 = -2x + 11$.

51. Let $W(t)$ be the size of the work force as a function of time t. Then, $W'(t)$ is the rate at which the size of the workforce is changing. Since the workforce is growing more slowly now than it was five years ago, $W'(t_{now}) < W'(t_{now} - 5)$.

53. (b) Since the units associated with T are $^\circ$ C and the units associated with t are minutes, the units associated with $T'(t)$ are $^\circ$ C/minute.

    (c) Yes. $T(5)$ is the temperature of the coffee 5 minutes after it was poured into the cup. Since the temperature of the room is 25° C, $T(5) \geq 25 > 0$.

    (d) No. $T'(5)$ is the rate of change of the temperature of the coffee in the cup 5 minues after it was poured. Since the initial temperature of the coffee is greater than the temperature of the room, the coffee is cooling. Thus, $T'(5) < 0$.

    (e) $T(300)$ is the temperature of the coffee 300 minutes (i.e., 5 hours) after it was poured into the cup. Since the temperature of the room is 25° C, this will be (approximately) the temperature of the coffee in the cup.

    (f) $T'(300)$ is the rate of change of the temperature of the coffee in the cup 5 hours after it was poured into the cup. After this amount of time, the temperature of the coffee will be constant (i.e., it will remain at room temperature).

55. $f'(x) = 1$ implies that f is a linear function of the form $f(x) = x + C$, where C is a constant. Since $f(0) = 2$, $f(x) = x + 2$.

57. $f'(x) = -3$ implies that f is a linear function of the form $f(x) = -3x + C$, where C is a constant. Since $f(0) = -2$, $f(x) = -3x - 2$.

59. Since the line is tangent to the graph of f at $(5, 2)$, this point must be on the graph of f. Hence, $f(5) = 2$. The slope of the line tangent to the curve $y = f(x)$ at $x = 5$ is $f'(5)$. Since the line passing through the points $(5, 2)$ and $(0, 1)$ has slope $(2 - 1)/(5 - 0) = 1/5$, $f'(5) = 1/5$.

61. No. For example, if $f(x) = x$ and $g(x) = x + 2$, then $f'(x) = g'(x) = 1$ but $f(x) \neq g(x)$ for all x.

63. Since $f'(x) \leq 0$ if $2 < x < 7$, the speed limit principle implies that $f(6) - f(3) \leq 0 \cdot (6 - 3) = 0 \implies f(6) \leq f(3) \implies f(3) \geq f(6)$.

§1.5 Estimating Derivatives: A Closer Look

1. $f'(1) \approx 2$

3. Since $f'(1) = 2$, an equation of the line tangent to f at $x = 1$ is $y = 2(x-1) + 1 = 2x - 1$.

5. $f'(-1) = -2$ means that that at $x = -1$, y is decreasing twice as fast as x is increasing.

7. The line ℓ is tangent to the graph of f of $x = 1$. Thus, near $x = 1$, $\ell(x)$ and $f(x)$ are visually indistinguishable.

9. (a)

x	0.7	0.8	0.9	1.0	1.1	1.2	1.3
$f(x)$	0.49	**0.64**	**0.81**	1.00	**1.21**	1.44	**1.69**

(b) $f'(1) \approx \dfrac{f(1.1) - f(1.0)}{1.1 - 1.0} = \dfrac{1.21 - 1.00}{0.1} = 2.1$

11. $f'(-1) \approx \dfrac{f(-1.001) - f(-1)}{-0.001} = \dfrac{1.002001 - 1.00}{-0.001} = -2.001$. Alternatively, $f'(-1) \approx \dfrac{f(-0.999) - f(-1)}{0.001} = \dfrac{0.998001 - 1.00}{0.001} = -1.999$. Thus, it seems reasonable to guess that $f'(-1) \approx -2$.

13. $f'(1.4) \approx \dfrac{f(1.42) - f(1.38)}{1.42 - 1.38} \approx \dfrac{0.98865 - 0.98185}{0.04} = 0.17$

15. (a)

x	1.37	1.38	1.39	1.40	1.41	1.42	1.43
$f(x)$	0.97991	**0.98185**	**0.98370**	0.98545	**0.98710**	0.98865	**0.99010**

(b) $f'(1.4) \approx \dfrac{f(1.41) - f(1.40)}{0.01} \approx \dfrac{0.98710 - 0.98545}{0.01} = 0.165$. Alternatively, $f'(1.4) \approx \dfrac{f(1.39) - f(1.40)}{-0.01} \approx \dfrac{0.98370 - 0.98545}{0.01} = 0.175$. Thus, it is reasonable to guess that $f'(1.4) \approx 0.17$.

17. (a) For all $x > 0$, $f(x) = x$ so $f'(2) = 1$.
(b) For all $x < 0$, $f(x) = -x$ so $f'(-3) = -1$.
(c) $f'(0)$ does not exist because f is not locally linear at $x = 0$. The graph of f has a sharp corner at $x = 0$ so the graph of f does not have a well-defined slope there.

19. The graph of f is symmetric about the y-axis (i.e., f is an even function). The line tangent to f at the point $(1, 1)$ is described by the equation $y = 2x - 1$. The reflection of this line across the y-axis is the line described by the equation $y = -2x - 1$. By symmetry, this line is tangent to the graph of f at the point $(-1, 1)$. Since the slope of this line is -2, $f'(-1) = -2$.

21. The graph of f is symmetric about the origin (i.e., f is an odd function). The line tangent of f at the point $(-1, -1)$ is described by the equation $y = 3x + 2$. The reflection of this line about the origin is the line described by the equation $y = 3x - 2$. By symmetry, this line is tangent to the graph of f at the point $(1, 1)$. Since the slope of this line is 3, $f'(1) = 3$.

23. (a) The formula $f'(x) = 2x$ predicts that $f'(3) = 6$.
(b) $f'(3) \approx \dfrac{f(3.001) - f(3)}{0.001} = \dfrac{9.006001 - 9}{0.001} = 6.001$. Alternatively, $f'(3) \approx \dfrac{f(2.999) - f(3)}{-0.001} = \dfrac{8.994001 - 9}{-0.001} = 5.999$. Both results are consistent with the prediction in part (a) that $f'(3) = 6$.

25. (a) $f'(0.5) \approx \dfrac{f(0.501) - f(0.5)}{0.001} \approx 1.65$

(b) Yes — $f(0.5) = e^{0.5} \approx 1.65 \approx f'(0.5)$.

27. (a)

x	-2	-1	0	1
$f'(x)$	6.0	2.0	-2.0	-2.0

x	2	2.5	3	4
$f'(x)$	0.0	1.0	2.0	3.5

29. (a) $f'(1/4) \approx 1.0$, $f'(1) \approx 0.5$, $f'(9/4) \approx 0.33$, $f'(4) \approx 0.25$, $f'(25/4) \approx 0.20$, and $f'(9) \approx 0.17$.

(b) $f'(x) = 1/2\sqrt{x}$

31. (a) $f'(-1) \approx -2.72$, $f'(0) \approx 1.00$, $f'(1) \approx -0.368$, $f'(1.5) \approx -0.223$, and $f'(2) \approx -0.135$.

(b) $f'(x) = -e^{-x}$

33. Yes; for every increase of 0.1 in x, y increases by 1.4. Thus, the points lie on the line $y = 14x - 45$.

35. (a) $f'(0) = 0$, $f'(1/2) = 3/4$, $f'(1) = 3$, $f'(3/2) = 27/4$, $f'(2) = 12$.

(b) Since f is an odd function, f' will be an even function. (Reflecting a tangent line through the origin does not change its slope.)

(c) $f'(x) = 3x^2$

37. (a) $f'(1/4) = -16$, $f'(1/2) = -4$, $f'(1) = -1$, $f'(2) = -1/4$, $f'(3) = -1/9$.

(b) f is an odd function, so f' is an even function. (Reflecting a tangent line about the y-axis produces a line whose slope is -1 times the slope of the original line.)

(c) $f'(x) = -1/x^2$

39. (a) $f'(0) \approx 0$, $f'(\pi/6) \approx -0.5$, $f'(\pi/4) \approx -0.71$, $f'(\pi/3) \approx -0.87$, $f'(\pi/2) \approx -1$, and $f'(\pi) \approx 0$.

(b) $f'(x) = -\sin x$

41. (a) This line has slope $m_a \approx 1$.

(b) This line has slope $m_b \approx 0.62$.

(c) This line has slope $m_c \approx 0.43$.

(d) $\ln e = 1$, $\ln 5 \approx 1.61$, and $\ln 10 \approx 2.3$. $m_a \approx 1/\ln e$, $m_b \approx 1/\ln 5$, and $m_c \approx 1/\ln 10$.

43. (a) The slope of the line tangent to the graph of $y = 2^x$ at $x = 0$ is ≈ 0.693; at $x = 1$, the slope is ≈ 1.386; and, at $x = 2$, the slope is ≈ 2.773. Thus, $y'(0)/y(0) \approx 0.693$; $y'(1)/y(1) \approx 0.693$; and, $y'(2)/y(2) \approx 0.693$. Thus, the constant of proportionality is, approximately, 0.693.

(b) $\ln 2 \approx 0.693$

45. (a) The slope of the line tangent to the graph of $y = 10^x$ at $x = 0$ is ≈ 2.30; at $x = 1$, the slope is ≈ 23.03; and, at $x = 2$, the slope is ≈ 230.3. Thus, $y'(0)/y(0) \approx 2.30$; $y'(1)/y(1) \approx 2.30$; and, $y'(2)/y(2) \approx 2.30$. Thus, the constant of proportionality is, approximately, 2.30.

(b) $\ln 10 \approx 2.30$

47. (a) The slope of the line tangent to the graph of $y = \log_2 x$ at $x = 1/2$ is ≈ 2.89; at $x = 2$, the slope is ≈ 0.721; and, at $x = 10$, the slope is ≈ 0.144. Thus, $y'(1/2)/(1/2)^{-1} \approx 1.44$; $y'(2)/2^{-1} \approx 1.44$; and, $y'(10)/10^{-1} \approx 1.44$. Thus, the constant of proportionality is, approximately, 1.44.

(b) $1/\ln 2 \approx 1.44$

49. (a) The slope of the line tangent to the graph of $y = \log_{10} x$ at $x = 1/2$ is ≈ 0.869; at $x = 2$, the slope is ≈ 0.217; and, at $x = 10$, the slope is ≈ 0.0434. Thus, $y'(1/2)/(1/2)^{-1} \approx 0.434$; $y'(2)/2^{-1} \approx 0.434$; and, $y'(10)/10^{-1} \approx 0.434$. Thus, the constant of proportionality is, approximately, 0.434.

    (b) $1/\ln 10 \approx 0.434$

51. The rate of growth of h at x is $h'(x)$. Using numerical zooming, $h'(-3) \approx 0.141$, $h'(0) \approx 0$, $h'(1) \approx -0.841$, and, $h'(2) \approx -0.909$.

53. Since $h'(2) = -3$, the slope of the line tangent to h at $x = 2$ is -3. Thus, $\ell(x) = -3(x-2) + h(2)$ is an equation of this tangent line. If $x \to x + \Delta x$, $\ell(x + \Delta x) - \ell(x) = -3\Delta x$. Since $\ell(x)$ is a good approximation to h near $x = 2$, we conclude that h decreases by approximately $-3\Delta x$.

55. Since $f'(x) = 2x$ and $g'(x) = -2$, $f'(x) = g'(x)$ at $x = -1$.

57. By graphing f and g on the same set of axes, it appears that f and g have the same slope at $x = 1$. Numerical zooming confirms that $f'(1) \approx -2 = g'(1)$.

59. If f and g are plotted using the viewing window $[3, 3.2] \times [-0.1, 0.1]$, they are visually indistinguishable. This suggests that $f'(\pi) = g'(\pi) = -1$.

61. (a) Since g is 2π-periodic, $g'(9\pi/4) = g'(\pi/4 + 2\pi) = g'(\pi/4) = -\sqrt{2}/2$.

    (b) Since $g'(\pi/4) = -\sqrt{2}/2$, the slope of the line tangent to g at $x = \pi/4$ is $-\sqrt{2}/2$. Now, the cosine function is symmetric about the y-axis (i.e., it is an even function), so the line tangent to the cosine function at $x = -\pi/4$ is the reflection across the y-axis of the line tangent to g at $x = \pi/4$. This (reflected) line has slope $\sqrt{2}/2$. Thus, $g'(-\pi/4) = \sqrt{2}/2$.

    (c) By symmetry, $g(\pi/2 + x) = -g(\pi/2 - x)$ so $g'(\pi/2 + x) = -g'(\pi/2 - x)$. Substituting $x = \pi/4$ leads to $g'(3\pi/4) = -g'(\pi/2 - \pi/4) = -g'(\pi/4) = -\sqrt{2}/2$.

§1.6 The Geometry of Derivatives

1. (a) The tangent line is horizontal, so its slope is 0.
   (b) Yes. The point A' is on the x-axis, so it corresponds to an f' value of 0.

3. The function f achieves larger values elsewhere. For example, $f(3.2) \approx 5$ which is larger than the value of f at A.

5. If g has a local minimum point at 2, $g(2) \leq g(x)$ for all values of x in an interval containing $x = 2$.

7. If x_0 is a global maximum point of a function f, $f(x_0) \geq f(x)$ for every x in the domain of f. If x_0 is a local maximum point of a function f, the statement $f(x_0) \geq f(x)$ may not be true for every x in the domain of f (it only needs to be true for every x in an open interval containing x_0).

9. At a stationary point, the graph of a function has a flat spot (i.e., it has a horizontal tangent line).

11. Yes. f is decreasing to the left of $x = 3$ and increasing to the right of $x = 3$. Therefore, f has a local minimum point at $x = 3$.

13. An inflection point is a point at which a graph's direction of concavity changes.

17. f has a stationary point (i.e., a horizontal tangent line) between A and B, and also between C and D.

19. f' is inreasing between the points C and D because the graph of f is concave up between these two points.

21. (a) f' is negative where f is decreasing. Therefore, $f'(x)$ is negative on the intervals $(-5, -2)$ and $(2, 5)$.
    (b) f' is decreasing where f is concave down. Therefore, f' is decreasing on the interval $(0, 5)$

23. f' achieves its maximum value at the point where f changes its concavity from up to down (i.e., at an inflection point). This occurs at $x = 0$; $f'(0) \approx 1.4$.

25. g is increasing on those intervals where $g' \geq 0$. Therefore, g is increasing on the intervals $(0, 5.5)$ and $(9, 10)$.

27. g has stationary points where $g'(x) = 0$. Therefore, g has stationary points at $x = 1$, $x = 5.5$, and $x = 9$.

29. g has a local minimum where g' changes sign from negative to positive. Thus, $x = 9$ is a local minimum point for g.

31. No. Since g' is decreasing at $x = 6$, g is concave down at $x = 6$.

35. Since $g'(9) = 0$, the line tangent to g at $x = 9$ is a constant linear function. Since $g(9) = -1$, the tangent line is $y = -1$.

37. f is increasing on those intervals where $f' \geq 0$. Thus, f is increasing on the interval $(-3, 4)$.

39. f has a stationary point at x if $f'(x) = 0$. Thus, f has stationary points at $x = -3$ and at $x = 3$.

41. f has a local minimum where the sign of f' changes from negative to positive. Thus, f has a local minimum point at $x = -3$.

43. f has a point of inflection where f' has a local maximum or minimum. Thus, f has points of inflection at $x = -1$ and $x = 3$.

45. Since $f'(1) = 2$, the slope of the tangent line is 2. Since $f(1) = 3$, the tangent line passes through the point $(1, 3)$. Therefore, $y = 2(x - 1) + 3 = 2x + 1$ is an equation for the line tangent to f at $x = 1$.

47. (a) Roots of f' correspond to stationary points of f (i.e., places where the tangent line is horizontal). Since f has only one stationary point, at $x = 0$, this point is the only root of f'.

(b) Since f is increasing over the interval $(-\infty, 0)$, $f' > 0$ on this interval.

(c) Yes. The graph of f is concave up at $x = 2$, so f' is increasing at $x = 2$.

(d) $f'(x) = -2x/\left(1 + x^2\right)^2$

49. (a) Roots of f' correspond to stationary points of f (i.e., places where the tangent line is horizontal). Since f has no stationary points, f' has no roots.

(b) The graph of f is increasing on the interval $(0, \infty)$, so f' is positive on this interval.

(c) The graph of f is concave down on the interval $(0, \infty)$, so $f'' < 0$ on this interval. Therefore, f' is decreasing over the interval $(0, \infty)$.

(d) $f'(x) = 1/x$

51. (a) The graph of f is increasing on those intervals where $f' \geq 0$. Therefore, f is increasing on the intervals $(-\infty, -4)$ and $(2, \infty)$.

(b) x is a stationary point of f if $f(x) = 0$. Therefore, $x = -4$ and $x = 2$ are stationary points of f.

(c) $x = a$ is a local minimum point of f if the sign of f' changes from negative to positive at $x = a$. Since $f'(x)$ changes sign from negative to positive at $x = 2$, f has a local minimum point there.

(d) The graph of f is concave down on intervals where f' is decreasing. Thus, f is concave down on the interval $(-\infty, -1)$.

(e) $f(x) = x(x^2 + 3x - 24)/3$ is one possibility.

53. (a) $f'(x) > 0$ for all x, so f is increasing on the interval $(-\infty, \infty)$.

(b) The equation $f'(x) = 0$ has no solutions, so f has no stationary points.

(c) Since $f'(x) > 0$ for all x, there is no point at which the sign of f' changes from negative to positive. Therefore, f has no local minimum points.

(d) f' is decreasing on the interval $(0, \infty)$ so f is concave down on this interval.

55. $y = 0$

57. $y = -1/x$

59. $y = -2$

61. $g(x) = e^{-x}$

63. $k(x) = -e^{x}$

§1.7 The Geometry of Higher-Order Derivatives

1. (a) The graph of f is concave up at $x = -1$.
   (b) The graph of f' is increasing at $x = -1$.

3. (a) $h''(2) = 0$ and h'' changes sign at $x = 2$
   (b) The graph of h' has a local extreme point at $x = 2$.

5. Yes — the second derivative test implies that f has a local minimum at $x = -3$.

7. Stationary points of a function f are, by definition, roots of f'. Since f' is a quadratic function, it has only two roots so f has only two stationary points.

9. $f''(1) < f''(3) < f''(-2) < f''(-3)$ because f'' is the derivative of f'.

11. (b) Since g is concave down over the interval $[1, 5]$, the graph of f lies below the line tangent to g at $x = 2$ throughout this interval. In other words, $g(x) < \ell(x)$ if $1 < x < 2$ or $2 < x < 5$. Thus, in particular, $g(3) < \ell(3)$.
    (c) No. The argument in part (a) shows that $g(1.5) < \ell(1.5)$.

13. (a) f'' is positive where the graph of f is concave upward. Therefore, $f''(x)$ is positive if $-10 < x < 1$.
    (b) f'' is negative where the graph of f is concave downward. Therefore, $f''(x)$ is negative if $1 < x < 10$.
    (c) f'' is zero where the graph of f changes concavity. Therefore, $f''(x) = 0$ only for $x = 1$.

15. g is concave up on intervals where $g'' \geq 0$. Therefore, g is concave up on the interval $(0, 3)$.

17. No. g does not have a point of inflection at $x = 1$ because $g''(x)$ does not change sign at $x = 1$.

19. Since $g''(x) \geq 0$ throughout the interval $[0, 3]$, g' is increasing on this interval. This implies that $g'(0) < g'(1) < g'(2) < g'(3)$.

21. Yes. $g''(x) < 1$ if $1 \leq x \leq 3$. Therefore, the speed limit principle implies that $g'(3) - g'(1) \leq 1 \cdot (3 - 1) = 2$ so $g'(3) < -2$. Since $g'(3)$ is negative, g is decreasing at $x = 3$.

23. (a) Values of g'' can be estimated by measuring slopes of lines tangent to g'. The results are $g''(0.5) = 0.125$, $g''(1) = -1/2$, $g''(1.5) = 1.125$, and $g''(2) = 2$.

25. (a) This function is concave upward if $x < 0$, concave downward if $x > 0$, and its second derivative doesn't exist at $x = 0$. Thus, graph (iv) must be the second derivative function.
    (b) This function is concave downward everywhere, so its second derivative must be negative everywhere. Thus, graph (iii) is the second derivative function.
    (c) This function is concave down except on an interval centered at the origin where it is concave up. Thus, the second derivative is negative except on an interval centered at the origin where it is positive. It follows that graph (i) is the second derivative function.
    (d) Since the function is a straight line, its first derivative is a horizontal line so its second derivative is zero everywhere. Thus, graph (ii) is the second derivative function.

27. **B** is the graph of f, **A** is the graph of f', and **C** is the graph of f''.

    Looking at the behavior of the three graphs near $x = 0$, one sees that **A** must be the derivative of **B** (**B** is increasing until its local maximum point; **A** is positive on the interval where **B** is increasing; etc.) and that **C** must be the derivative of **A** (**C** has a root at each local extreme point of **A**).

29. (a) $V''(t)$ is the rate at which V' is changing at time t. Since water is flowing into the tank at a *constant* rate, $V''(t)$ is **zero**.

(b) $H''(t)$ is the rate at which H' is changing at time t. As the tank fills, it takes more water (i.e., more time) to raise the height of the water in the tank by a fixed amount. Thus, H' is "large" when the tank is nearly empty and H' is "small" when the tank is nearly full. These considerations imply that $H''(t^*) < 0$.

31. Let $T(t)$ be the child's temperature at time t. $T'(t) > 0$ (that is, $T(t)$ is *increasing*), but $T''(t) < 0$ (that is, $T(t)$ is increasing at a *decreasing* rate.)

33. The function $f(x) = e^{-x}$ has the desired properties.

35. If f is an even function, then $g = f'$ is an odd function. This implies that $g' = f''$ is an even function.

37. (a) Since f is concave up everywhere, f' is increasing everywhere. This implies that f can have at most one root and the root, if it exists, must be located in the interval $(-\infty, 2)$.

    In fact, one can say more about the possible location of a root of f. Since f is concave up on the interval $(-\infty, 2)$, the curve $y = f(x)$ must always lie *above* the line tangent to f at $x = 2$. Since this line crosses the x-axis at $x = -1/2$, the root, if it exists, must be located in the interval $(-\infty, -1/2)$.

    (b) Since f is concave up on the interval $[0, 2]$, the line tangent to the graph of f at $x = 2$ lies *below* the curve $y = f(x)$. Thus, the y-intercept of the tangent line is a lower bound on the value of $f(0)$. Since the tangent line is $y = 2(x - 2) + 5 = 2x + 1$, we have that $f(0) \geq 1$.

    (c) Since f is concave up on the interval $[-2, 2]$, the line passing through the points $\big(-2, f(-2)\big) = (-2, -1)$ and $\big(2, f(2)\big) = (2, 5)$ must lie above the curve $y = f(x)$ over the interval $(-2, 2)$. The equation of this line is $y = \frac{3}{2}(x + 2) - 1 = \frac{3}{2}x + 2$. At $x = 0$ this line passes through the point $(0, 2)$, so $f(0) < 2$ must be true. This implies that $f(0) < 3$.

39. (a) $f''(-1) = 0$, $f''(0) = 2$, and $f''(2) = 4$.

    (b) f'' has a root at each stationary point of f'. Since f' has only one stationary point (a local minimum point at $x = -1$), f'' has only one root.

    (c) $f''(x) > 0$ on those intervals where f' is increasing. Thus, $f''(x) > 0$ on the interval $(-1, \infty)$.

    (d) $y = 2x + 2$

41. (a) $f''(-1) = -1/2$, $f''(0)$ does not exist, and $f''(2) = 1/3$.

    (b) f'' has a root at each stationary point of f'. Since f' has no stationary points, f'' has no roots.

    (c) $f''(x) > 0$ on those intervals where f' is increasing. Thus, $f''(x) > 0$ on the interval $(0, \infty)$.

    (d) $y = x/(|x| + x^2)$

43. (a) f is concave up on the intervals where $f''(x) > 0$. Thus, f is concave up on the interval $(2, \infty)$.

    (b) f' is decreasing on the the intervals where $f''(x) < 0$. Thus, f' is decreasing on the interval $(-\infty, 2)$.

    (c) f' has a stationary point at each root of f''. Thus, f' has two stationary points: at $x = -1$ and at $x = 2$.

    (d) f has an inflection point where the sign of f'' changes. Thus, f has an inflection point at $x = 2$.

    (e) $y = \left(x^4 - 6x^2 - 8x\right)/4$

45. (a) f is concave up on the intervals where $f''(x) > 0$. Thus, f is concave up on the interval $(-1, \infty)$.

    (b) f' is decreasing on the the intervals where $f''(x) < 0$. Thus, f' is decreasing on the interval $(-\infty, -1)$.

    (c) f' has a stationary point at each root of f''. Thus, f' has only one stationary point, at $x = -1$.

    (d) f has an inflection point where the sign of f'' changes. Thus, f has an inflection point at $x = -1$.

    (e) $y = \sqrt{1 + x^2} + \ln\left|x + \sqrt{1 + x^2}\right|$

47. (a) Yes. Since $f'(1) = 0$ and $f''(1) = 5 > 0$, f has a local minimum point at $x = 1$.
    (b) No. $f''(x) > 0$ if $-1 < x < 1$, so f is concave up over this interval. This implies that the curve $y = f(x)$ lies above the line tangent to f at $x = -1$ over the interval $-1 < x < 1$.

49. (a) No. Since $f'(1) = 0$ and $f''(1) = -1/6 < 0$, f has a local maximum point at $x = 1$.
    (b) Yes. $f''(x) < 0$ if $-1 < x < 1$, so f is concave down over this interval. This implies that the curve $y = f(x)$ lies below the line tangent to f at $x = -1$ over the interval $-1 < x < 1$.

51. $f''(x) = 3 + 2x$

§1.8 Chapter summary

1. In general, if f' is a constant function, then f is a linear function. Since $f'(x) = 3$, $f(x) = 3x + C$, where C is a constant.

3. (a) The graph of g is a vertical translate of the graph of f.
   (b) The graphs of g' and f' are identical (i.e., $g'(x) = f'(x)$ for every x in the domain of f'.
   (c) $g'(1) = f'(1) = 5$

5. Since h is a vertical translate of f, $h'(2) = f'(2) = 3$. Thus, the line $y = (3x - 4) + 1 = 3x - 3$ is an equation of the line tangent to h at the point $\big(2, h(2)\big) = (2, 3)$.

7. Since $g(x) = f(x - 2)$, the graph of g is a horizontal translate of the graph of f. This implies that $g'(x) = f'(x - 2)$. Thus, $g'(0) = f'(-2) = 1/7$.

9. The shape of the f-graph repeats every 5 units. Since f' is the slope function, it must also be periodic with period 5.

11. $f'(x)$ is the slope (or rate of growth) of f at x. From a graph of f it is apparent that $f'(-1) < f'(0) < f'(2) < f'(10)$.

13. For any function f, $f'(x)$ is the slope of f at x; it is also the rate of growth of f at that point. A close look at graphs of f and g near each of the listed points leads to the following conclusions: $f'(1/10) < g'(1/10)$, $f'(1/2) < g'(1/2)$, $g'(2) < f'(2)$, and $g'(7) < f'(7)$. Thus, g is growing faster at the first two points and f is growing faster at the other two points.

15. Neither graph **A** nor **B** can be f since then $f'(x) < 0$ would be true on the interval $[0, 1]$ and graph **C** does not have this property. Thus, graph **C** must be the graph of f. Estimating $f'(x)$ at some point (e.g., at $x = 2$, $f'(2) \approx -0.4$) allows us to determine that **A** is the graph of f'.

17. At $x = 3$ and at $x = 6$.

19. f is concave down wherever the slope of f' is negative. This occurs on the intervals $(1, 2)$ and $(4.5, 7)$.

21. Since $f'(x) < 0$ on the interval $[0, 2]$, f is decreasing there. Thus, f achieves its maximum value at $x = 0$ and its minimum value at $x = 2$.

23. Notice that f' is negative on the interval $(1, 4]$ and positive on the interval $[-2, 1)$. Therefore:
   (a) f is increasing on $(-2, 1)$
   (b) Since $f'(1) = 0$, $x = 1$ is a stationary point of f. It is a local maximum point since f' changes from positive to negative there (i.e., f changes from increasing to decreasing at $x = 1$).
   (c) The graph has more or less the shape of an inverted "vee," with vertex at $x = 1$ (e.g., something similar to the graph of $y = e^{-x^2/40}$).

25. (a) Since $f'(1) = 0$ and $f'(x)$ changes sign at $x = 1$, f has a local extreme point there.
   (b) No, $x = 1$ is a local minimum point of f since $f'(1) = 0$ and the sign of $f'(x)$ changes from negative to positive at $x = 1$.
   (c) No, $f'(-2) = 0$ but $f'(x)$ doesn't change sign at $x = -2$ so $x = -2$ is not a local extreme point of f.
   (d) f is increasing where $f'(x) \geq 0$. Thus, f is increasing on the interval $(1, \infty)$.
   (e) f is concave down where f' is decreasing. Thus, f is concave down on the interval $(-2, 0)$.
   (f) f has inflection points at each point where f' has a local extreme point. Thus, f has inflection points at $x = -2$ and $x = 0$.

27. (a) Since $f(2) = 4$ and $f'(2) = \sqrt{8+1} = 3$, the line tangent to f at $x = 2$ passes through the point $(2, 4)$ with slope 3. An equation of this tangent line $y = 3(x-2) + 4 = 3x - 2$.

    (b) Since f' is an increasing function on the interval $[-1, \infty)$, f is concave up over this interval. This implies that the curve $y = f(x)$ lies above any tangent line in this interval. Thus, $\ell(0)$ underestimates $f(0)$.

    (c) Using the tangent line found in part (a), $f(2.1) \approx 4.3$. From part (b), we may conclude that $f(2.1) \geq 4.3$ (i.e., the tangent line estimate is too small).

29. Using the speed limit principle, $f(5) - f(1) \leq 3(5-1) \implies f(5) \leq f(1) + 12 = 10$.

31. Yes. Using the speed limit principle, $f(-5) - f(1) \geq 3\big((-5) - 1\big) \implies f(-5) \geq f(1) - 18 = -20 > -23$.

33. Using the speed limit principle, the condition $f'(x) \leq 3$ if $-10 \leq x \leq 10$ implies that $f(8) - f(1) \leq 3(8-1) \implies f(8) \leq f(1) + 21 = 19$. Thus it is possible that $f(8) = -25$.

35. **A** is the graph of f'', **B** is the graph of g, **C** is the graph of f', and **D** is the graph of f.

37. The function $f(x) = -e^{-x}$ has the desired properties.

39. No such function exists. The conditions given imply that f lies below the x-axis and that it is concave up everywhere (i.e., its slope is always increasing). Such a function must cross the x-axis if its domain is all real numbers.

41. Yes, it is easy to sketch a function with the desired properties.

43. No; since $x = 2$ is a local maximum point of g, $g''(2) \leq 0$ must be true.

45. Yes, the slope of g at $x = 3$ could be 1.

47. No — $f'(x) \geq 0$ throughout the interval $[-2, 3]$, so $f(3) > f(-2) = -3$.

49. Since $f'(x) \geq 2$ if $-2 \leq 0 \leq 0$, the speed limit principle implies that $f(0) - f(-2) \geq 2 \cdot \big(0 - (-2)\big) = 4 \implies f(0) \geq 1$ since $f(-2) = -3$.

51. f achieves its smallest value at $x = -2$. If $-2 \leq x \leq 2$, then $f'(x) > 0$ and the speed limit principle implies that $f(x) - f(-2) \geq 0 \implies f(x) \geq f(-2)$.

53. (a) Yes. The particle is moving up at $t = 1$ because $y(t)$ is increasing at $t = 1$.

    (b) Yes. $y'(1)$ is the slope of the curve $y(t) = \sin t$ at $t = 1$. Since the y-graph is increasing at $t = 1$, $y'(1) > 0$.

55. $f'(-2)$ is the slope of the tangent line, so $f'(-2) = 4$. Since the tangent line and the function f "touch" at $x = -2$, $f(-2) = 4 \cdot -2 + 3 = -5$.

57. No. The graph shows that $g'(x) > g'(2)$ if $2 < x \leq 5$. The tangent line has a constant growth rate, $g'(2)$. However, throughout the interval $(2, 5]$, the growth rate of g is always greater than $g'(2)$. Thus, the tangent line at $x = 2$ lies below the graph of g over the interval $(2, 5]$.

59. Yes. Since $g'(x) > 0$ if $3 \leq x \leq 5$, g is increasing over this interval. Thus, $g(3) < g(5)$.

61. In 1810, the population of the United States was growing at a rate of approximately 2.3 million people per year.

63. No — $g'(3) \approx -0.2781 < 0$ implies that g is *de*creasing at $r = 3$.

65. No — g' is a decreasing function at $r = 2$ so g is concave down there.

67. g has one local minimum point in the interval $[0, 5]$, at $r = \sqrt{10\pi/3} \approx 3.326$. This is the only point in the interval where the sign of g' changes from negative to positive.

69. Local extrema of g' correspond to inflection points of g. Since g' has a local minimum at $r = \sqrt{2\pi} \approx 2.5066$, g has an inflection point there.

71. The graph of g' is decreasing throughout the interval $[1, 1.1]$, so the graph of g is concave down on this interval. It follows that the line tangent to the graph of g at $r = 1$ lies above the graph of g over the interval $[1, 1.1]$.

73. (a) $h'(z) = 3 \implies h'(1) = 3$. (Since h is a linear function, h' is a constant.)
    (b) Since h' is a constant, $h''(z) = 0$ for any z. Therefore, $h''(4) = 0$.

75. A "qualitatively correct" graph of G must be (i) increasing on the interval $[1, 3]$, (ii) concave up on the interval $[1, 2]$, (iii) concave down on the interval $[2, 3]$, and (iv) have an x-intercept at $x = 1$. The curve $y = 6x^2 - x^3 - 5$ has the right shape.

77. $f'(x)$ is the slope of f at x. Using this idea, it is clear from the graph that $-2 < f'(1.5) < f'(3) < 0 < 0.5 < f'(4.5)$.

79. No. The maximum value of f must occur at a stationary point (i.e., at a root of f') or at an endpoint of I. Thus, the maximum value of f on I *cannot* occur at 4.

83. (a) No — $h'(1) = -2 < 0$ so h is decreasing at $w = 1$.
    (b) Yes — h is concave up at $w = 4$ because $h'(w)$ is increasing at $w = 4$.
    (c) The point $w = 9$ is a local minimum point because the sign of h' changes from negative to positive there.

85. $g(4) = -7$. Since g' is a constant function, g must be a linear function. From the information given, it follows that $g(t) = -3(t-1) + 2 = 5 - 3t$.

87. The statement *must* be true because $G''(w)$ changes sign at $w = -2$.

89. The statement *might* be true. G is concave up on the interval $[0, 2]$ (since $G''(w) > 0$ if $0 \leq w \leq 2]$), but G need not be decreasing over this interval.

91. (a) At the time of the election, 2 million people were unemployed.
    (b) Two months after the election, 3 million people were unemployed.
    (c) Twenty months after the election, unemployment was increasing at a rate of 10,000 people per month.
    (d) Three years (thirty-six months) after the election, the number of unemployed was at a local minimum.

93. (a) g has an inflection point at $w = \sqrt{\pi}$ because this point corresponds to a local maximum of $g'(w)$.
    (b) Yes, g also has an inflection point at $w = \sqrt{2\pi}$ because this point corresponds to a local minimum of $g'(w)$.

95. g is concave down on the interval $(\sqrt{\pi}, \sqrt{2\pi})$ because the graph of g' is decreasing on this interval.

97. g has a local maximum point where the sign of g' changes from positive to negative. This occurs once in the interval $[1, 3]$, at $w = \sqrt{15\pi}/3$.

99. (d) When the car has positive acceleration, the graph in part (a) is concave up, the graph in part (b) is increasing, and the graph in part (c) is positive.
    When the car has zero acceleration, the graph in part (a) is linear, the graph in part (b) is horizontal, and the graph in part (c) is zero.
    When the car has negative acceleration (i.e., it is decelerating), the graph in part (a) is concave down, the graph in part (b) is decreasing, and the graph in part (c) is negative.