The ArcTangent Function

You have been given a transparency with the graph of $y = \tan(x)$ on it. Use this transparency to help answer the questions on this sheet.

1. *Start of development of the arctangent.* Look at the sketch of $y = \tan(x)$ and answer the following questions.
   A. Explain why $\tan(x)$ does not have an inverse on its natural domain.

   B. What is the *largest* interval containing $x = 0$ where $\tan(x)$ is indeed a one-to-one function? [In other words, what's the largest part of the graph that passes *both* the horizontal and vertical line tests?]

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2. *Getting the graph of the arctangent function.* Now use a RED or BLUE marker to highlight the part of $y = \tan(x)$ that gives the graph of $g(x) = \tan(x)$ on the restricted domain $[-\pi/2, \pi/2]$. Use the highlighted graph to answer these questions:
   A. Does $g(x)$ have an inverse function? Why or why not?

   B. Physically flip your transparency over the line $y = x$. You should now be looking at the BACK side of your transparency. Note that the "old" x-axis is now the "new" y-axis and the "old" y-axis is now the "new" x-axis. Label these NEW axes on the BACK side of your transparency with a RED marker. Is the highlighted curve that you see (when you're looking at the NEW axes) a function? Why or why not?

Part 2 Continued

C. This new function (the one you're looking at on the BACK side of the transparency) is the inverse of the original function. Draw a copy of this inverse function here:

We call this function $\arctan(x)$ or $\tan^{-1}(x)$. We read $\arctan(x)$ as "the arctangent of x" and we read $\tan^{-1}(x)$ as "the tangent-inverse of x." Label the curve (on the BACK side of the transparency) with the label $y = \arctan(x)$ or $y = \tan^{-1}(x)$.

WARNING: Do NOT interpret the notation $\tan^{-1}(x)$ as the *multiplicative inverse* of $\tan(x)$. Remember that $1/\tan(x) = \cot(x)$, which is NOT the same function as the inverse tangent function. Fill in the blanks below:

$\tan^{-1}(x) =$

$(\tan(x))^{-1} = \dfrac{1}{\tan(x)} =$

and $\arctan(x) \neq \cot(x)$.

D. Look at the graph of $\arctan(x)$. Find the *domain* and the *range* of the arctangent function and write them here:

The domain of $\arctan(x)$ is

The range of $\arctan(x)$ is

Have me or the t.a. check your answers before you go any further. You may also want to add this information to the BACK SIDE of the overhead with the picture of $y = \arctan(x)$.

Part 2 Continued

E. Explain why the lines $y = \pi/2$ and $y = -\pi/2$ are both horizontal asymptotes of $y = \arctan(x)$.

Re-write this information in terms of limits by filling in the blanks:

$$\lim_{x\to\infty} \arctan(x) = \underline{\qquad\quad}$$

$$\lim_{x\to-\infty} \arctan(x) = \underline{\qquad\quad}$$

You may want to add this information to the BACK SIDE of the overhead with the picture of $y = \arctan(x)$.

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3. *Practice with the arctangent function.* It is important to realize what we mean by the inverse relation of $y = \tan(x)$ and $y = \arctan(x)$. Since the graphs are reflections over the line $y = x$, we know that

$$\begin{aligned} \tan(\arctan(x)) = x &\quad \text{for all real } x \\ \arctan(\tan(x)) = x &\quad \text{for all } x \text{ in the RESTRICTED domain of } (-\pi/2, \pi/2) \end{aligned}$$

Another way of stating this relationship is to say:

$$\theta = \arctan(x) \iff [\tan(\theta) = x \text{ AND } -\pi/2 \le \theta \le \pi/2]$$

In English, this says that $\arctan(x)$ is the ANGLE between $= \pi/2$ and $\pi/2$ with tangent equal to x. We use this inverse relationship to find the derivative of the arctangent function in the next part of this handout.

Make sure your calculator is set to RADIANS before continuing with this problem.

Part 3 Continued

A. Explain why $(1/\sqrt{3}, \pi/6)$ and $(-1, -\pi/4)$ are on the graph of $y = \arctan(x)$.

B. Use your calculator to find the following values:

$\arctan(\tan(0)) =$

$\arctan(\tan(0.5)) =$

$\arctan(\tan(1)) =$

$\arctan(\tan(1.5)) =$

$\arctan(\tan(2)) =$

Are you surprised with any of these values? Why or why not?

C. Note that in one problem in part B we did NOT get the original x value back when we evaluated $\arctan(\tan(x))$. Carefully look at the information you know about the arctangent function to explain why $\arctan(\tan(2)) \neq 2$.

Part 3 Continued

D. Use your unit circle handout to find the following values by hand.

$\arctan(1) =$

$\arctan(-\sqrt{3}) =$

$\cos(\arctan(0)) =$

$\cos(\arctan(1/\sqrt{3})) =$

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4. *The derivative of the arctangent function.*

   A. Now make a hand-drawn sketch of $y = \arctan(x)$ on some paper. Use your hand-drawn graph to answer the following questions. [Your graph should be easier to look at than the graph on the back side of the transparency.]

Part 4 Continued

B. By looking at the graph of $\arctan(x)$, what can you say about the derivative of the $\arctan(x)$? Note that you are NOT asked to find a formula for the derivative of arctangent function yet, but we do want you to give some precise information: Can you tell whether the derivative has a constant sign? Why or why not? Can you say anything intelligent about what happens to the derivative as $x \to \infty$? Can you say anything intelligent about what happens to the derivative as $x \to -\infty$? Why or why not?

C. Recall that we know $\tan(\arctan(x)) = x$. We can differentiate both sides of this equation with respect to x and algebraically solve for $\frac{d}{dx}\arctan(x)$. I've provided the first two (trivial) steps. You need to use the chain rule for the derivative of the left-hand side and then solve for $\frac{d}{dx}\arctan(x)$. Have me or the t.a. check your work before you go any further. Note that you will not yet have an aesthetically pleasing (or meaningful) answer yet.

$$\begin{aligned} & \tan(\arctan(x)) = x \\ \Rightarrow \quad & \frac{d}{dx}\tan(\arctan(x)) = \frac{d}{dx}\,x \end{aligned}$$

D. Let $\theta = \arctan(x)$. [So you know $\tan(\theta) = x$.] Notice that the right triangle below has one angel labeled θ. Use right triangle trig definitions to label *all three* sides of the triangle in terms of x and the number 1. Then use some more right triangle trig to re-write your answer from part A in a much simpler and more elegant way that involves the variable x, but NO TRIG FUNCTIONS! Have me or the t.a. check your work on this problem before you continue.

E. Now write the companion antiderivative formula and add it to your list of known antiderivatives.