Zooming in on Differences

The derivative $f'(a)$ is the slope of the tangent line to the graph of $f$ at the point $(a, f(a))$. This tangent line has two main properties: (i) it passes through the point $(a, f(a))$ “fits” the $f$-graph as closely as possible at $(a, f(a))$. The first property is no big deal; it’s easy to find lines through a given point. The second property is trickier; in this interlude we explore further how to recognize the line that best “fits” a (curved) graph at a point.

In Example 1, page 47, we estimated that the tangent line to the graph of $y = f(x)$ at the point $(1, 1)$ has slope 2, that is, that $f'(1) \approx 2$. This estimate looks reasonable, graphs alone only yield estimates. Do we really prefer $f'(1) \approx 2$ to, say, $f'(1) \approx 2.1$?

These three guesses correspond, respectively, to three linear functions. equations

$$
\ell_2(x) = 2(x - 1) + 1, \quad \ell_{2.1}(x) = 2.1(x - 1) + 1, \quad \text{and} \quad \ell_{1.9}(x) = 1.9(x - 1) + 1
$$

(The letter $\ell$ stands for “linear”; the subscripts correspond to slopes.) Which guess is best?

To decide, we might plot all three linear functions and $f$ together; the linear graph of $f$ is “on the money.” Figure 1 shows two pictures in different windows:

![Graphs](image)

**Figure 1**
Tangent line candidates: graphs of $f$, $\ell_2$, $\ell_{2.1}$, and $\ell_{1.9}$

Alas, neither picture helps much—all three lines look like reasonable tangent candidates.

**A better way: look at differences** A better way to select the best candidate and to the duds is to compare the errors each candidate commits. To this end, we consider error functions

$$
f(x) - \ell_2(x), \quad f(x) - \ell_{2.1}(x), \quad \text{and} \quad f(x) - \ell_{1.9}(x),
$$

which measure the difference between the candidate functions and the “target” function ($f(x)$). This difference is the “error” committed by each tangent-line candidate in approximating the target function. The best candidate is the linear function whose error function is closest to zero near $x = 1$. Figure 2 shows graphs (in two different windows) of our error functions; the dashed curve is the function $f(x) - \ell_2(x)$ (which corresponds to $f'(1) \approx 2$).
All three graphs pass through the point \((1, 0)\)—not surprisingly, since all three tangent line candidates were chosen to agree with \(f\) at \(x = 1\). What we really care about is how the curves behave near \textit{but not at} \(x = 1\). Observe:

- The dashed curve “wins”; it’s \textit{much} closer than the others to the line \(y = 0\) for values of \(x\) near 1. (This is most obvious in the second picture; zooming in yet again would make the difference even more striking.)

- The two solid curves \textit{cross} the \(x\)-axis at \(x = 1\), so the two error functions \(f(x) - \ell_{2,1}(x)\) and \(f(x) - \ell_{1,9}(x)\) \textit{change sign} at \(x = 1\). This implies, in turn, that the lines \(\ell_{1,9}\) and \(\ell_{2,1}\) cross the curve \(y = x^2\) at \(x = 1\). (The upward concavity of the parabola \(y = x^2\) suggests that the “true” tangent line should \textit{not} cross the curve at \(x = 1\).) In the second picture, the two solid curves resemble \textit{straight lines} that cross the \(x\)-axis at the point \(x = 1\).

- The dashed curve looks like a \textit{parabola} with its vertex on the \(x\)-axis at \(x = 1\). (In the second picture, the choice of vertical units “flattens”—but doesn’t destroy—the parabolic shape.)

**Numerical differences** Zooming in numerically on the error functions can also help reveal the best linear approximation. Here are some sample values:

<table>
<thead>
<tr>
<th>(x)</th>
<th>1.000</th>
<th>1.001</th>
<th>1.002</th>
<th>1.003</th>
<th>1.004</th>
<th>1.005</th>
<th>1.006</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x) - \ell_2(x))</td>
<td>0</td>
<td>0.000001</td>
<td>0.000004</td>
<td>0.000009</td>
<td>0.000016</td>
<td>0.000025</td>
<td>0.000036</td>
</tr>
<tr>
<td>(f(x) - \ell_{2,1}(x))</td>
<td>0</td>
<td>-0.000099</td>
<td>-0.000196</td>
<td>-0.000291</td>
<td>-0.000384</td>
<td>-0.000475</td>
<td>-0.000564</td>
</tr>
<tr>
<td>(f(x) - \ell_{1,9}(x))</td>
<td>0</td>
<td>0.000101</td>
<td>0.000204</td>
<td>0.000309</td>
<td>0.000416</td>
<td>0.000525</td>
<td>0.000636</td>
</tr>
</tbody>
</table>

Notice the big difference between the best linear approximation \(\ell_2\) and the two “also-rans.” Values of \(f - \ell_2\) are less than those of the other two error functions throughout the table; even more striking is the relative advantage of \(f - \ell_2\) when \(x\) is near \(x = 1\). At \(x = 1.001\), for instance, \(\ell_2\) commits only about 1/100 the error committed by \(\ell_{1,9}\) and \(\ell_{2,1}\). The key advantage of \(\ell_2\) can be seen in the numbers: the error function \(f - \ell_2\) appears to be \textit{quadratic}, with vertex at \(x = 1\), while \(f - \ell_{1,9}\) and \(f - \ell_{2,1}\) behave like \textit{linear} functions.
A zooming strategy Here’s an informal strategy that often works: ✗

Let \( f \) be a well-behaved function and \((a, f(a))\) a point on its graph. Let \( \ell \) be a linear function whose graph passes through \((a, f(a))\). To decide whether \( \ell \) is the “right” tangent line function for \( f \) at \( x = a \), define the error function \( e(x) = f(x) - \ell(x) \) and zoom in (graphically or numerically) on \( e(x) \) near \( x = a \). If the result looks like a line that crosses the \( x \)-axis at \( x = a \), then \( \ell \) is probably wrong. If the result resembles a parabola with vertex \((a, 0)\), then \( \ell(x) \) is probably right.

Further Exercises

1. Let \( f(x) = \sqrt{x} \). The estimates \( f'(1) \approx 0.5, f'(1) \approx 0.4, \) and \( f'(1) \approx 0.6 \) are all reasonable, but only one is correct.
   (a) Write equations for the three linear functions and the three error function associated with the estimates above.
   (b) Use graphical zooming (as shown above) to identify the “true” value of \( f' \).
   (c) Tabulate some numerical values of the three error functions near \( x = 1 \). Which numbers support your answer in the preceding part?

2. Repeat the first problem, using the function \( f(x) = x^2 \) and the estimates \( f'(3) \approx 6.0, \) and \( f'(3) \approx 6.1. \)

3. Repeat the first problem, using the function \( f(x) = e^x \) and the estimates \( f'(0) \approx 1.00, \) and \( f'(0) \approx 1.01. \)

4. The strategy outlined above doesn’t always work the same way. To see this, apply the zooming strategy to the function \( f(x) = \sin x \), the estimates \( f'(0) \approx 0.9 \) and \( f'(0) \approx 1.1, \) and the exact answer, \( f'(0) = 1.0. \) What do you see? What accounts for the difference? (Hint: The function \( f(x) = \sin x \) has an inflection point at \( x = 0 \).)

5. Apply the strategy outlined above to the function \( f(x) = x^3 \) at \( x = 0 \) (which happens to be an inflection point). Use the estimates \( f'(0) = 0, f'(0) = 0.01, \) and \( f'(0) = -0.01 \). What do you see?