Addenda & Corrigenda (in progress)

(X^T means X lines from the top; Y_B means Y lines from the bottom.)

Chapter 1

• Misprint p5, 5^T (thank you Rich Senko): ida should be idea.

• Misprint p24, in reference [6]: 243 should be 243-244.

• Add after Exercise 1.13 an exercise about showing that for any odd integer k, $\sqrt{k^2+1}$ is irrational [N. Lord: Extending the parity proof that $\sqrt{2}$ is irrational, *Math. Gazette* 99 (2015), p155].

• Perhaps of interest are 1. D.M. Bloom: A once-sentence proof that $\sqrt{2}$ is irrational, *Math. Mag* 68 (1995)), p286; and follow-up 2. J. Bergen: Is this the easiest proof that nth roots are always integers or irrational?, *Math. Mag.* 90 (2017), p225.

Chapter 2

• The paper [C. Mortici: A very elementary proof of Bernoulli's inequality, *College Math. J.* 46 (2015), 136-137] contains an excellent simple proof of Bernoulli's Inequality for $x > 0$, and extends it somewhat for $x > -1$. It is also noted there that Isaac Barrow (Newton's teacher) had a version of Bernoulli's Inequality 19 years before Bernoulli!

• Add after Exercise 2.3:
Show that for $x_j \geq 0$, $\prod_{j=1}^{n}(1+x_j) \ \geq \ 1 + \sum_{j=1}^{n} x_j$. (All x_j's equal is Bernoulli's Inequality.)

• In Exercise 2.24 (thank you Mehdi Hassani, Dragan Banjevic): $\frac{2+4+6+\cdots+(2n)}{1+3+5+\cdots++(2n-1)}$ should be $\left(\frac{2+4+6+\cdots+(2n)}{1+3+5+\cdots++(2n-1)}\right)^n$. Without the power n the limit is 1. With it, the limit is e.

• In Exercise 2.29, something subtle here needs consideration. See [A. Beardon: The area of a quadrilateral, *Math. Gazette* 101 (2017), 492-494].

• Add after Exercise 2.33 [Problem 11751 (C. Kempiak, A. Viejo, B. Suceavă & B. Karaivanov), *Amer. Math. Monthly* 122 (2015), 905-906]: In a triangle with angles of radian measure A, B, C, show that

$$\frac{\csc A + \csc B + \csc C}{2} \ \geq \ \frac{1}{\sin A + \sin B} + \frac{1}{\sin B + \sin C} + \frac{1}{\sin C + \sin A},$$

with equality occurring if and only if the triangle is equilateral.

• Add after Exercise 2.50 [D.M. Bătineţu-Giurgiu, N. Stanciu, and E. Lampakis: Problem 1050, *College Math. J.* 47 (2016), 143-144]:
We saw in Remark 2.14 that the isoperimetric inequality for an n-sided polygon with area T and perimeter P is

$$T \leq \frac{P^2}{4n\tan(\pi/n)}.$$

Suppose now that the polygon is convex with sides lengths a_k, and let $m \geq 0$. Show that

$$\left(\sum_{k=1}^{n} a_k^{2m+4}\right)\left(\sum_{k=1}^{n} \frac{1}{a_k^{2m}}\right) \geq 16T^2 \tan^2(\pi/n).$$

• Add after Exercise 2.56 [M. Can, G.E. Bilodeau, & M. Vowe: Problem 1059, *College Math. J.* 47 (2016), 304-305]: (If you did Exercise 2.54) Let A, B, C be a triangle. Use Chebyshev's Inequality (and Jensen's: $\cos(x/2)$ is concave on $(0, \pi)$) to show that

$$\frac{\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}}{\sin A + \sin B + \sin C} \geq \frac{1}{\sqrt{3}}.$$

Chapter 3

• In Exercise 3.37 (thank you Meng Lin Ma): Assume also that $f > 0$.

• Add a Remark at the end of Section 3.3: In [K. Razminia: A short proof of symmetric inequalities, *College. Math. J.* 46 (2015), 364-366], the Extreme Value Theorem is leaned upon heavily, in proofs of the AGM Inequality (and more) and the Cauchy-Schwarz inequality.

Chapter 4

• Section 4.2: A similar approach to our proof of the Product Rule (and perhaps even slicker!) can be found in P. Josevich: An alternative approach to the product rule, *Amer. Math. Monthly* 123 (2016), p470.

• Add after Exercise 4.11 [A. Soares & A.L. dos Santos: The straddle lemma in an introductory real analysis course, *Int. J. Math. & Math. Ed. Sci. Tech.* 48 (2016), 428-434]: Say that f is *strongly differentiable* at x_0 to mean

$$\lim_{(x,y)\to(x_0,x_0)} \frac{f(x) - f(y)}{x - y}$$

exists. Show that f is strongly differentiable at x_0 if and only if f' is continuous at x_0.

Chapter 5

• Add at the very end of section 5.1: See also [M.J. Poliferno: A natural auxiliary function for the mean value theorem, *Amer. Math. Monthly* 69 (1962), 45-47] and [A.P. de Camargo: The geometric mean value theorem, *Int. J. Math. & Math. Ed. Sci. Tech.* 49 (2018), 613-615.]

• Add after Exercise 5.36 [Problem 11788 (S. Andriopoulos & B. Karaivanov), *Amer. Math. Monthly* 123 (2016), p619] Let n be a positive integer and let $0 < y_j \leq x_j < 1$ for $1 \leq j \leq n$. Show that

$$\frac{\ln x_1 + \cdots + \ln x_n}{\ln y_1 + \cdots + \ln y_n} \leq \sqrt{\frac{1 - x_1}{1 - y_1} + \cdots + \frac{1 - x_n}{1 - y_n}}, \quad (\textit{and this is strict for } n > 1).$$

Hint (suggestion): Use induction; show first that $f(x) = \dfrac{\ln x}{\sqrt{1 - x}}$ is increasing on $(0, 1)$.

• In reference [54]: 1980 should be 1989.

Chapter 6

• p122, 6_B in Section 6.2: Perhaps also of interest is the paper [I. Patyi: On some elementary functions, *Math. Gazette* 99 (2015), 263-275].

• For some collateral reading, see also the excellent paper [N. Lord: The versatile exponential inequality $e^x \geq 1 + x$, *Math. Gazette* 101 (2017), 470-475].

• Add after Exercise 6.32 [S.P. Andriopoulos, K. Kaczkowski: Problem 11770, *Amer. Math. Monthly* 123 (2016), p300]: Let $a > b > 1$ and $x > y > 0$. Show that

$$\frac{a^x - b^y}{x - y} \;>\; a^{(x+y)/2}\ln(a) \;>\; \left(\frac{a+b}{2}\right)^{(x+y)/2} \ln\left(\frac{a+b}{2}\right).$$

• Misprint p157, in reference [7]: Should be p651 here, not p615.

• Oversight p157, in reference [29]: This paper is authored by Hansheng, Y. and Lu, B.

• Oversight p157, in reference [36] (thank you Eunjeong Yi): This paper is authored by Kan, C.X and Yi, E.

• Misprint p157, in [39] (thank you Fuad Kittaneh): This paper is authored by Kittaneh, F. & Hirzallah, O.

Chapter 7

• Add to general remarks at the end of Section 7.2 the recent paper:
[C. Tana & S. Lia: Some new mean value theorems of Flett type, *Int. J. Math. Ed. Sci. Tech.* 45 (2014), 1103-1107].

• Misprint in Exercise 7.14, p168: 5.39 should be 5.10.

• Add after Exercise 7.14 p169 [C. Mortici: Funny forms of the Mean value theorem, *Amer. Math. Monthly* 122 (2015), p780]: **(a)** Let $a < b$ with $a \neq -b$. Let f be continuous on $[a, b]$ and differentiable on (a, b), with $af(b) = bf(a)$. Show that there is $c \in (a, b)$ such that $f'(c) = \frac{f(a)+f(b)}{a+b}$.
(b) Let $a < b$ with $ab > 0$. Let f be continuous on $[a, b]$ and differentiable on (a, b), with $1/f(b) - 1/f(a) = 1/b - 1/a$. Show that there is $c \in (a, b)$ such that $f'(c) = \frac{f(af(b)}{ab}$.

Chapter 8

• Add after Exercise 8.3 [J. Singh, Another proof of the Binomial theorem, *Amer. Math. Monthly* 124 (2017), p658]: Here's another proof of the Binomial theorem: **(a)** Show that $\dfrac{d}{dt}\dfrac{t^k}{(1+t)^n} \;=\; \dfrac{kt^{k-1} - (n-k)t^k}{(1+t)^{n+1}}$.
(b) Multiply through by $\frac{n!}{k!(n-k)!}$ and sum from 0 to n to show that

$$\frac{d}{dt}\sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\frac{t^k}{(1+t)^n} \;=\; \cdots\cdots \;=\; 0\,.$$

(c) Make a couple of observations which allow you to conclude that $\sum\limits_{k=0}^{n} \frac{n!}{k!(n-k)!} t^k \;=\; (1+t)^n$.

• Add after Exercise 8.39 [G. Apostolopoulos, the Hofstra Univ. Problem Solvers & Á. Plaza: Problem 1039, *College. Math. J.* 46 (2015), 143-144]: Let ABC be an acute triangle. Show that

$$\sum_{\text{A,B,C cyclic}} \frac{\sec A}{\sqrt{\cos A + \cos B}} \;\geq\; 6.$$

• Add after Exercise 8.41 [M. Bence & J.C. Smith: Problem 11843, *Amer. Math. Monthly* 124 (2017), p374]:
Show that $f(t) = \frac{1}{1+e^t}$ is convex on $[0, \infty)$. Show that

$$\sum_{j=1}^{M} \frac{1}{1+z_j} \;\geq\; \frac{M}{1+(z_1 z_2 \cdots z_M)^{1/M}}\,.$$

• Add after Exercise 8.48 [J.L. Diaz-Barrero & B. Bradie: Problem 1039, *College. Math. J.* 46 (2015), p373]: Let $x_1, x_2, \cdots, x_n > 0$ with $x_1 x_2 \cdots x_n = 1$ (and $n > 1$). Show that for each $m \geq 2$, we have

$$\frac{n-1}{n} \sum_{\text{cyclic}} \frac{x_1^m}{x_2 + x_3 + \cdots + x_n} \;\geq\; 1.$$

Hint: Cauchy-Schwarz Inequality, AGM Inequality, Power Mean Inequality (see exercise above).

Chapter 9

• p213: Computation of $\int_0^1 \frac{1}{1+x}\,dx = \ln(2)$, as in [F. Sánchez & J.M. Sanchis: Darboux sums and the alternating harmonic series, *Math. Mag.* 91 (2018), p96], could be added here as well (perhaps before Example 9.7).

• p215. At the end of Section 9.1, add a remark or two about the cool paper [I.C. Bivens & B.G. Klein: The median value of a continuous function, *Math. Mag.* 88 (2015), 39-51].

• p229, 8^T. After [33], add also: E. Omey: On Xiang's observations concerning the Cauchy-Schwarz Inequality, *Amer. Math. Monthly* 122 (2015), 696-698. — which extends [33] very nicely.

• Misprint in Exercise 9.7, p235 (thank you Felipe Filho): The exponent on the right-hand side of the inequality to be shown should be $-xf(x)$, not $-xg(x)$.

• Add after Exercise 9.22 [Problem 1064 (M. Merca & B. Dunn), *College. Math. J.* 48 (2017), 140-141]:
Let n be a positive integer. Show that

$$0 \;<\; \frac{1}{\pi}\frac{2^{2n}}{\binom{2n}{n}} \;-\; \sum_{k=1}^{n}\left(\cos\left(\frac{k\pi}{2n+1}\right)\right)^{2n+1} \;<\; 1\,.$$

• Add after Exercise 9.36 [Problem 1024 (O. Furdui & E. Herman), *College. Math. J.* 46 (2015), p146.]: Let f be continuous on $[a, b]$ with

$$\int_a^b f^2(x)\,dx \;=\; \left(\int_a^b f(x)\,dx\right)^2.$$

(a) Show that $b - a = 1$. (b) Show that f is constant.

• Add after Exercise 9.36 [Problem 11780 (C. Lupu, T. Lupu, & R. Tauraso), *Amer. Math. Monthly* 123 (2016), 614-615]:
Let f be positive and concave on $[0,1]$. Show that

$$\frac{3}{4}\Big(\int_0^1 f(x)\,dx\Big)^2 \;\le\; \frac{1}{8} \;+\; \int_0^1 \left(f(x)\right)^3\,dx\,.$$

• Add after Exercise 9.37 [C. Lupu & R. Boukharfane: Problem 11819, *Amer. Math. Monthly* 123 (2016), p1054]:
Let f be continuous and nonnegative on $[0,1]$. Show that

$$\int_0^1 f(x)^3\,dx \;\ge\; 4\int_0^1 x^2 f(x)\,dx \int_0^1 x f(x)^2\,dx.$$

• Add after Exercise 9.40 [Problem 1069 (Á. Plaza & M. Andreoli) *College Math. J.* 48 (2017), 60-61]:
Let $f_1,\ f_2, \cdots, f_n$ be positive and continuous on $[0,1]$. Use Exercise 9.40 to show that

$$\int_0^1 \frac{f_1(x)}{f_2(1-x)}\,dx \cdot \int_0^1 \frac{f_2(x)}{f_3(1-x)}\,dx \cdots \int_0^1 \frac{f_n(x)}{f_1(1-x)}\,dx \;\ge\; 1\,.$$

• Add after Exercise 9.51(c): Show that $\left(\int_1^{e^2} \left(\frac{\ln x}{x}\right)^n dx\right)^{1/n} \;\to\; 1/e$ as $n \to \infty$.

[G. Apostolopoulos, Missouri State Univ. Problem Solving Proup: Problem 1954, *Math. Mag.* 88 (2015), 381-382.]

• Misprint p247, in reference [1]: Wilkins, E.J. Jr. should be Wilkins, J. E., Jr.

• Misprint p248, in reference [28]: Sieffert should be Seiffert.

Chapter 10

• Add at the end of Section 10.1: Perhaps also of interest is the reference: [J.R. Nurcombe: Rearranging the signs of the alternating harmonic series, *Math. Gazette* 98 (2014), 321-324].

• Add to Exercise 10.21:
Show that $\int \sec(x)\,dx \;=\; \frac{1}{2}\ln|\frac{1+\sin(x)}{1-\sin(x)}| \,+\, C$, and that $\int \csc(x)\,dx \;=\; \frac{1}{2}\ln|\frac{1-\cos(x)}{1+\cos(x)}| \,+\, C$.

• Example 10.12, p258:
This curious fact was apparently first observed by D.P. Dalzell (On 22/7, *J. London Math. Soc.* 19 (1944), 133-134.)

• Add after Exercise 10.12 [J. Sandor: A note on the Logarithmic mean: *Amer. Math. Monthly* 123 (2016) p112]:
(a) Verify that for $t > 1$, we have

$$\frac{4}{(t+1)^2} \;<\; \frac{1}{t} \;<\; \frac{1}{2\sqrt{t}} \;+\; \frac{1}{2t\sqrt{t}}\,.$$

(b) For $0 < a < b$, integrate over $[1, b/a]$ and do some tidying, to obtain (again!) $G \,<\, L \,<\, A\,.$

• Exercise 10.15b (thank you Dragan Banjevic) is clearly nonsense. (My word, not Dragan's!) Omit it post-haste. Also, in Exercise 10.15c, the $a > 1$ there must be an integer.

• Add after Exercise 10.48
[C. Lupu & NY Math Circle: Problem 11814, *Amer. Math. Monthly* 123 (2016), 1051-1052]:
Let ϕ, defined on $[0,1]$, have a continuous nonzero derivative, with $\phi(0) = 0$ and $\phi(1) = 1$. Let f, continuous on $[0,1]$, satisfy $\int_0^1 f(x)\,dx = \int_0^1 \phi(x)f(x)\,dx$. Show that there is $t \in (0,1)$ such that $\int_0^t \phi(x)f(x)\,dx = 0$.

• In the references, p280, item 39: The volume number 20 there should be volume 29.

Chapter 11

• Add after Exercise 11.30 [M.W. Botsko and K.W. Lau: Problem 1945, *Math. Mag.* 88 (2015), p241]:
(a) Let f be continuous on $[0,1]$ with $\int\limits_0^1 f(x)\,dx = 0$. Let ϕ be differentiable on $[0,1]$ with $\phi(0) = 0$ and $\phi'(x) > 0$ for $x \in (0,1)$. Show that there exists $x_0 \in (0,1)$ such that

$$\int_0^{x_0} f(x)\phi(x)\,dx = 0\,.$$

• Misprint p310, in reference [35]: Sieffert should be Seiffert.

Chapter 12

• At beginning of Section 12.4, p318, the modifications of [18] which are given in Daners' paper [5] are rediscovered in [D.J. Velleman: Monthly Gems, *Amer. Math. Monthly* 123 (2016), p77].

• Daners' ideas in [5], which we follow in Section 12.4, are extended nicely in [B.D. Sittinger: Computing $\zeta(2m)$ by using telescoping sums, *Amer. Math. Monthly* 123 (2016), 710-715].

• A wonderful approach to Theorem 12.7 in Section 12.4 appears in [S.G. Moreno: A short and elementary proof of the Basel problem, *College Math. J.* 47 (2016), 134-135]. This approach is arguably simpler than the one in the book!

• Another excellent and very elementary approach to Theorem 12.7 in Section 12.4 appears in [N. Lord: The most elementary proof that $\sum\limits_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$?, *Math. Gazette* 100 (2016), 429-434]. This paper is definitely worth a good look.

• As regards Exercise 12.4 about Vieta's formula, see also the cool paper [C. Boucher: Viète's formula and an error bound without Taylor's formula, *Int. J. Math. & Math. Ed. Sci. Tech.* 49 (2018), 455-458].

• Add after Exercise 12.8 [G. Stoica & E.A. Herman, Problem 1991, *Math. Mag.* 90 (2017), 232-233]:
Show that

$$\lim_{n\to\infty} \sum_{k=1}^{n-1} \frac{n^2}{k^2(n-k)^2} = \frac{\pi^2}{3}\,.$$

• Add after Exercise 12.9 [J. Gräter and K.J. Withs: On elementary bounds for $\sum_{k=n}^{\infty} k^{-s}$, *Amer. Math. Monthly* 122 (2015), 155-158]:

(a) Show that for $s > 1$, $h_s(x) = (1-x)^{1-s} - (1+x)^{1-s} - 2x(s-1)$ is strictly positive on $(0,1)$.

(b) Use (a) to show that

$$\left(1 - \frac{1}{2k}\right)^{1-s} - \left(1 + \frac{1}{2k}\right)^{1-s} > \frac{s-1}{k}.$$

(c) Use (b) to show that

$$k^{-s} < \frac{\left(k - \frac{1}{2}\right)^{1-s}}{s-1} - \frac{\left(k + \frac{1}{2}\right)^{1-s}}{s-1}.$$

(d) Conclude that

$$\sum_{k=n}^{\infty} k^{-s} < \frac{\left(n - \frac{1}{2}\right)^{1-s}}{s-1}.$$

• Oversight p329, in reference [26]: This paper is authored by Sondow, J. and Yi, H.

Chapter 13

• Misprint p338, line 13^T: The 0 there should be a 1. That is, $e^{\frac{1}{4n}} \to 1$ as $n \to +\infty$. (Blushes.)

• Add in (or after) Example 13.10 p337: In [M. Shauo: Bounding the Euler-Mascheroni constant, *College. Math. J.* 46 (2015), p347], the Midpoint Rule is used to obtain also $\gamma < 2(1 - \ln(2)) \cong 0.6137$. I should have thought of this!

Chapter 14

Appendix

• p 405, 8^T: To references [2,7,8], add the interesting paper [R. Kantrowitz & M. Neumann: Another face of the Archimedean property, *College. Math. J.* 46 (2015), 139-141], in which the Archimedean Property (in *any* ordered field) is shown to be equivalent with conditions involving various simple geometric sequences, including the geometric series test (cf. Example 2.3 in Chapter 2).