G15 – Developing Conic Sections with GSP

Conics using Locus Definitions

Exploration #1 Parabola as a locus

Objective: The definition of a parabola is the set of points that are equidistant from a line (directrix) and a point (focus). In the figure below, the line is the directrix and B is the focus. If point P is on the parabola, then \( AP = PB \).

To construct the locus of all points \( P \), we need to observe that \( \triangle APB \) would be isosceles and \( AP \perp \) to the directrix. The following steps will construct the locus of point \( P \).

a. Get a new sketch.
   Draw a horizontal line at the bottom of the screen. [Hold down the shift key while using the line tool to make a horizontal line.] Select the line and construct a point on the line (different from the two points that define the line). Label the point \( A \).

b. Place a point about an inch above the line. Label this point \( B \).
   Select the point \( A \) and the line and Construct a Perpendicular.
   The desired point \( P \) is located on this perpendicular. Since this point is also the vertex of an isosceles triangle, then it is located on the perpendicular bisector of the base, \( AB \).

c. Construct a Segment joining the focus and the point on the line. (\( AB \) in my sketch.) With the segment selected, Construct a Midpoint.

d. Select the midpoint and the line segment \( AB \), then Construct a Perpendicular. (See sketch at right.)

e. Select the perpendicular, then under Display choose Trace Perpendicular. If you drag \( A \) along the line you will see shadows left of the locus of the perpendicular. Under the Display menu choose Erase Traces.

f. We are now ready to animate this sketch.
   Select the point \( A \). Under Display menu choose Animate.
   To stop the animation click on the \( \square \) button on the Motion Controller.
   What does the locus of the perpendicular bisector outline?

This is called an “envelope.” You can achieve this same effect using a sheet of paper thin enough to see through, such as tracing paper, wax paper, or patty paper.

Draw a line and a focus point. Put several points on the line. Now fold the paper so that the focus point is matched with one of the points on the line; make a crease.
Repeat this several times to see the “envelope” of the parabola.
You can accomplish this same effect using a plain sheet of paper and a Mira. As before, draw a directrix line and a focus point. With the point toward you and the Mira between the focus and the directrix, adjust the Mira until you can see the focus reflected on the directrix. Draw a line using the edge of the Mira. After sufficient lines are drawn, you should see the parabola envelop.

g. Under the Display menu, choose Erase Traces. Construct the point P where the perpendiculars intersect. Select all of the following: the two perpendiculars, $\overline{AB}$, and the midpoint of $\overline{AB}$. Then under the Display menu choose Hide Objects. Choose point P, and then point A, and under Construct choose Locus.

Now you can see that as you move point A along the directrix, the locus of the point P will outline the parabola — the set of all points where $PA = PB$. Your screen should look like the figure above.

h. You can change the location of the focus in relation to the directrix to determine the effect this distance has on the shape of the parabola. As you move the focus B away from the directrix, how is the parabola changed?

As you move the focus B closer to the directrix, how is the parabola changed?
Exploration #2  Parabola Graph Paper

Objective: To create graph paper in order to plot parabolas using the definition.

a. Start with a new sketch. Put a grid on the screen. Select the x and y-axes and hide them. Select the points defining the origin and unit and hide them.

b. Turn on Snap Points. Draw a line on one of the horizontal grid lines near the bottom of the screen. This will be the directrix.

c. Place a point about 4 units above the directrix. This will be the focus.

d. With the focus as the center, construct concentric circles to fill the screen above the directrix. This can be done by placing points on the lattice points horizontally or vertically from the focus, then constructing Circle by Center+Point. Another method is to make the focal point a center, create one circle with radius = 1, then dilate this circle. (See sketch previous page.)

e. Plot the first point halfway between the focus and directrix.

f. Subsequent points are plotted by going one more circle out from F and one more line up from the directrix. These points can then be connected to form the parabola. (See sketch previous page.)
Exploration #3  Ellipses and hyperbolas — emulating paper folding:

Objective: This method was originally done with wax paper, tracing paper, or patty paper. Draw a circle on the wax paper with the center indicated, then chose a point outside the circle. Place about 20 or more points on the circle. Match the point outside the circle with a point on the circle and crease the paper. By continuing in this fashion, the hyperbola envelope is formed. If the point is placed inside the circle and folding done as before, the ellipse envelope is formed.

a. Get a new sketch. Draw a circle in the middle of the screen. Select the circle and construct a point on the circle. (Do not use the point that defines the radius of the circle.)
b. Place a point outside the circle.
c. Construct a Segment joining the point on the circle and the point outside the circle. With the segment selected, Construct a Midpoint.
d. Select the midpoint and the line segment, then Construct a Perpendicular Line.
e. Select the perpendicular, then under Display choose Trace Perpendicular Line.
f. Since we want to reposition the point and animate this several times, we will create an “Animation button”. Select the point on the circle and the circle. Under Edit choose the Action Buttons submenu, then choose Animation. Click on OK. You will now have a button on the screen.
g. Click on the button to animate the sketch. What does the locus of the perpendicular bisector outline?
   To stop the animation, click the animation button.
h. Move the point inside the circle. Under Display, Erase Traces. Click the animation button. Now what does the locus of the perpendicular bisector outline?
i. Move point D on top of the center of the circle. Animate again. Now what do you have? Since the center of the circle is just a particular point inside the circle, then this shape is just a modification of the previous shape.

Exploration #4  Ellipses using the locus definition

Objective: An ellipse is defined to be the set of all points such that the sum of the distances
to two fixed points (foci) is constant. In the figure below, C and D are the two foci and AB is the constant sum of the distances. If point F is on the ellipse, then CF + FD = AB.

To construct the locus of all points F, we need to locate the set of all points that are a distance CF from C, which is a circle. Likewise, we need to locate the set of all points distance DF from D, which is a circle.

a. Get a new sketch. Draw segment and two points. Label the segment AB and the points C and D.

b. Select the segment and Construct a Point On Segment. Label the point E.

c. Select A and E, and then Construct a Segment.

d. While AE is selected, select C and Construct a Circle By Center+Radius.

e. Similarly Construct Segment EB and a circle with center at D and radius EB.

f. If the two circles do not intersect, adjust the location of point E on AB. Construct points at the intersections of the two circles. Verify that your construction now looks like the figure below.
g. Select the two intersection points, under the **Display** menu choose **Trace Intersections**.

h. Select the circles and under the **Display** menu choose **Hide Circles**.

Now as you move point E along segment AB, the locus of the intersection points will outline the ellipse — the set of all points where \( FD + FC = GD + GC = AB \).

i. Select F and E, then under the **Construct** menu choose **Locus**. Repeat for point G and E. The result should look like the figure at the right.

How does changing the distance between points C and D change the shape of the ellipse?

When points C and D are farther apart, what happens to the ellipse?

When does the ellipse ‘disappear’?

(Look at the length CD in relation to the length of line segment AB to determine how this affects the ellipse.)

Move point C close to D, so that they are nearly the same point. You should see the ellipse become nearly circular before it disappears. What is the relationship between an ellipse and a circle?

You can say that a circle is a special case of the ellipse. What are the foci of a circle?
Exploration #5  Hyperbola using the locus definition

Objective: A hyperbola is the set of all points such that the difference of the distances to two given points (foci) is a constant. In the figure below, C and D are the two foci and AE is the constant difference. If point G is on the hyperbola, then CG – DG = AE.

To construct the locus of all points G, we need to construct a point F on segment EB such that AF = CG and EF = GD. Then CG – DG = AF – EF = AE.

a. Draw segment and two focal points. Label the segment AB and the foci C and D.
b. Select the segment and Construct a Point On Segment. Label it E.
c. Construct segment BE by selecting point B and E and then go to the Construct menu and choose Segment. While BE is selected Construct a Point On Segment. Label this point F.
d. Construct segment AF. While AF is selected, select C and Construct a Circle By Center+Radius.
e. Similarly construct segment EF, and then construct a circle with center at D and radius EF.
f. Construct points at the intersections of the two circles. Hide the two circles.
g. Select one intersection point and F, and under Construct choose Locus. Repeat for the other intersection point and F.

This would give us the locus of points where the longer distance is from C and the shorter from D. The opposite condition may also be true.

h. Select segment AF again. (If the wrong segment is highlighted, then click again.
i. While AF is selected, select D and Construct a Circle By Center+Radius. Similarly select segment EF again, and then construct a circle with center at C and radius EF.
j. Construct points at the intersections of the two circles. Your construction should now look like the figure on the next page.
k. Select the circles and under the Display menu choose **Hide Circles**.
l. Select one of the intersection points and F, and under Construct choose **Locus**. Repeat for the other intersection point and F.

This would give us the locus of points where the longer distance is from D and the shorter from C.

Now as you move point F along segment EB, the locus of the intersection points will outline the hyperbola — the set of all points where the difference of the distances = AF – EF = AE. (See sketch below.)

m. You can change the distance between points C and D and the length of line segment AE. How do these distances affect the shape of the hyperbola? Determine what happens if AE > CD. What conditions make the hyperbola narrower? Flatter?
n. Rearrange points C and D so that segment CD would be vertical. What happens to the hyperbola? Does the definition still hold?
   In either position, the ‘center’ of the hyperbola is the midpoint of CD.

Exploration #6  Conic Graphing Paper

Objective: Included in the University of Chicago School Mathematics Project supplementary material is conic graphing paper. This idea really stems from the 1960s at the University of Illinois (according to Zal Usiskin). This graph paper can be constructed using Geometer’s Sketchpad. It consists of two focal points, each surrounded by concentric circles (see sample below). Students can then graph ellipses and hyperbolas by repeatedly applying the rules for the locus definitions.
1. Graph an ellipse defined as the set of all points such that the sum of the distances to the foci is 16.
   a. Since there are 10 units between the foci (1 unit = distance between circles), then start by placing one point 3 units to the right of the right focus point. This point would be 13 units from the left focus point and $3 + 13 = 16$.
   b. Continue to place points by moving in one unit toward the left focus and out one unit away from the right focus until you reach the point 3 units to the left of the left focus.
   c. Continue around, now moving one unit in toward the right focus and one unit away from the left focus until you get back to the point 3 units to the right of the right focus.
   d. Select the circle tool, now under the Display menu choose Hide Circles. The result should look something like the picture below.

   e. Connecting the dots gives the image of the ellipse.
2. Similarly, graph a hyperbola where the
difference of the distances from each focus is 4.
   a. Since there are 10 units between the foci, start
      by placing one point 3 units to the right of the left
      focus point. This point would be 7 units from the
      right focus point and $7 - 3 = 4$.
   b. Continue to place points by moving upward one
      unit away from the left focus and out one unit away
      from the right focus until you reach a point at the top
      of the paper. $(8 - 4 = 9 - 5 = \ldots = 4)$ Go back to your
      initial point and move downward one unit away from
      the left focus and one unit away from the right focus
      until you reach a point at the bottom of the paper.
   c. Place a second vertex point 3 units to the left of
      the right focus point. This point would be 7 units
      from the left focus point and $7 - 3 = 4$.
   d. Continue to place points by moving upward one
      unit away from the left focus and one unit away from
      the right focus until you reach a point at the top of
      the paper. Go back to your second vertex point and
      move downward one unit away from the left focus
      and one unit away from the right focus until you
      reach a point at the bottom of the paper.
   e. Select the circle tool, now under the Display
      menu choose Hide Circles. The result should look
      something like the picture above on the left.
   f. Connecting the dots gives the image of the hyperbola.

3. To make your own conic graph paper, simply choose two points and a unit radius.
   Then simply construct concentric circles with radii one unit longer than the previous. I
   used the axes from the Graph menu to help me. (See below.)