CHAPTER 1

THE PATTERN OF TWO LOCI

1.1. Geometric constructions

Describing or constructing figures with ruler and compasses has traditional place in the teaching of plane geometry. The simplest co structions of this kind are used by draftsmen, but otherwise the practi importance of geometric constructions is negligible and their theoreti importance not too great. Still, the place of such constructions in curriculum is well justified: they are most suitable for familiarizing beginner with geometric figures, and they are eminently appropriate acquainting him with the ideas of problem solving. It is for this lat reason that we are going to discuss geometric constructions.

As so many other traditions in the teaching of mathematics, geomet constructions go back to Euclid in whose system they play an import role. The very first problem in Euclid's Elements, Proposition One Book One, proposes "to describe an equilateral triangle on a given fi straight line." In Euclid's system there is a good reason for restrict the problem to the equilateral triangle but, in fact, the solution is just easy for the following more general problem: *Describe* (or construct *triangle being given its three sides.*

Let us devote a moment to analyzing this problem.

In any problem there must be an *unknown*—if everything is known, th is nothing to seek, nothing to do. In our problem the unknown (the th desired or required, the *quaesitum*) is a geometric figure, a triangle.

Yet in any problem something must be known or *given* (we call the gi things the *data*)—if nothing is given, there is nothing by which we co recognize the required thing: we would not know it if we saw it. In problem the data are three "finite straight lines" or line segments.

Finally, in any problem there must be a *condition* which specifies how the unknown is linked to the data. In our problem, the condition specifies that the three given segments must be the sides of the required triangle.

The condition is an essential part of the problem. Compare our problem with the following: "Describe a triangle being given its three altitudes." In both problems the data are the same (three line segments) and the unknown is a geometric figure of the same kind (a triangle). Yet the connection between the unknown and the data is different, the condition is different, and the problems are very different indeed (our problem is easier).

The reader is, of course, familiar with the solution of our problem. Let a, b, and c stand for the lengths of the three given segments. We lay down the segment a between the endpoints B and C (draw the figure yourself). We draw two circles, one with center C and radius b, the other with center B and radius c; let A be one of their two points of intersection. Then ABC is the desired triangle.

1.2. From example to pattern

Let us look back at the foregoing solution, and let us look for promising features which have some chance to be useful in solving similar problems.

By laying down the segment a, we have already located two vertices of the required triangle, B and C; just one more vertex remains to be found. In fact, by laying down that segment we have transformed the proposed problem into another problem equivalent to, but different from, the original problem. In this new problem

the unknown is a point (the third vertex of the required triangle);
the data are two points (B and C) and two lengths (b and c);
the condition requires that the desired point be at the distance b from the given point C and at the distance c from the given point B.

This condition consists of two parts, one concerned with b and C, the other with c and B. *Keep only one part of the condition, drop the other part; how far is the unknown then determined, how can it vary?* A point of the plane that has the given distance b from the given point C is neither completely determined nor completely free: it is restricted to a "locus"; it must belong to, but can move along, the periphery of the circle with center C and radius b. The unknown point must belong to two such loci and is found as their intersection.

We perceive here a pattern (the "pattern of two loci") which we can imitate with some chance of success in solving problems of geometric construction:

First, reduce the problem to the construction of ONE point.
Then, split the condition into TWO parts so that each part yields a locus for the unknown point; each locus must be either a straight line or a circle.

Examples are better than precepts—the mere statement of the pattern cannot do you much good. The pattern will grow in color and interest and value with each example to which you apply it successfully.

1.3. Examples

Almost all the constructions which traditionally belong to the high school curriculum are straightforward applications of the pattern of two loci.

(1) *Circumscribe a circle about a given triangle.* We reduce the problem to the construction of the center of the required circle. In the so reduced problem

the unknown is a point, say X;
the data are three points A, B, and C;
the condition consists in the equality of three distances:

$$XA = XB = XC$$

We split the condition into two parts:

First	$XA = XB$
Second	$XA = XC$

To each part of the condition corresponds a locus. The first locus is the perpendicular bisector of the segment AB, the second that of AC. The desired point X is the intersection of these two straight lines.

We could have split the condition differently: first, $XA = XB$, second, $XB = XC$. This yields a different construction. Yet can the result be different? Why not?

(2) *Inscribe a circle in a given triangle.* We reduce the problem to the construction of the center of the required circle. In the so reduced problem

the unknown is a point, say X;
the data are three (infinite) straight lines a, b, and c;
the condition is that the point X be at the same (perpendicular) distance from all three given lines.
We split the condition into two parts:
First, X is equidistant from a and b.
Second, X is equidistant from a and c.

The locus of the points satisfying the first part of the condition consists of *two* straight lines, perpendicular to each other: the bisectors of the angles included by a and b. The second locus is analogous. The two loci have four points of intersection: besides the center of the inscribed circle of the triangle we obtain also the centers of the three escribed circles.

Observe that this application calls for a slight modification of our formulation of the pattern at the end of sect. 1.2. What modification?

(3) *Given two parallel lines and a point between them. Draw a circle that is tangent to both given lines and passes through the given point.* If we visualize the required figure (it helps to have it on paper) we may observe that we can easily *solve a part of the problem*: the distance of the two given parallels is obviously the diameter of the required circle and half this distance is the radius.

We reduce the problem to finding the center X of the unknown circle. Knowing the radius, say r, we split the condition as follows:

First, X is at the distance r from the given point.
Second, X is at the distance r from both given lines.

The first part of the condition yields a circle, the second part a straight line midway between, and parallel to, the two given parallels.

Without knowing the radius of the desired circle, we could have split up the condition as follows:

First, X is at the same distance from the given point and the first given line.
Second, X is at the same distance from the given point and the second given line.

Splitting the condition into these two parts is logically unobjectionable but nevertheless useless: the corresponding loci are *parabolas*; we cannot draw them with ruler and compasses—it is an essential part of the scheme that the loci obtained should be circular or rectilinear.

This example may contribute to a better understanding of the pattern of two loci. This pattern helps in many cases, but not in all, as appropriate examples show.

1.4. Take the problem as solved

Wishful thinking is imagining good things you don't have. A hungry man who had nothing but a little piece of dry bread said to himself: "If I had some ham, I could make some ham-and-eggs if I had some eggs."

People tell you that wishful thinking is bad. Do not believe it, this is just one of those generally accepted errors. Wishful thinking may be bad as too much salt is bad in the soup and even a little garlic is bad in the chocolate pudding. I mean, wishful thinking may be bad if there is too much of it or in the wrong place, but it is good in itself and may be a great help in life and in problem solving. That poor guy may enjoy his dry bread more and digest it better with a little wishful thinking about eggs and ham. And we are going to consider the following problem (see Fig. 1.1).

Given three points A, B, and C. Draw a line intersecting AC in the point X and BC in the point Y so that

$$AX = XY = YB$$

Imagine that we knew the position of one of the two points X and Y (this is wishful thinking). Then we could easily find the other point (by drawing a perpendicular bisector). The trouble is that we know neither of the two—the problem does not look easy.

Let us indulge in a little more wishful thinking and *take the problem as solved.* That is, assume that Fig. 1.1 is drawn according to the condition laid down by our problem, so that the three segments of the broken line $AXYB$ are exactly equal. Doing so we imagine a good thing we have not got yet: we imagine that we have found the required location of the line XY; in fact, we imagine that we *have found the solution.*

Yet it is good to have Fig. 1.1 before us. It shows all the geometric elements we should examine, the elements we have and the elements we want, the data and the unknown, assembled as specified by the condition. With the figure before us, we can speculate as to which useful elements we could construct from the data, and which elements could be used in constructing the unknown. We can start from the data and work forward, or start from the unknown and work backward—even side trips could be instructive.

Could you put together at least a few pieces of the jigsaw puzzle? *Could you solve some part of the problem?* There is a triangle in Fig. 1.1, $\triangle XCY$. Can we construct it? We would need three data but, unfortunately, we have only one (the angle at C).

Use what you have, you cannot use what you have not. *Could you derive something useful from the data?* Well, it is easy to join the given points A and B, and the connecting line has some chance to be useful; let us draw it (Fig. 1.2). Yet it is not so easy to see *how* the line AB can be useful—should we rather drop it?

Figure 1.1 looks so empty. There is little doubt that more lines will be needed in the desired construction—what lines?

The lines AX, XY, and YB are equal (we regard them as equal—wishful thinking!). Yet they are in such an awkward relative position—equal lines can be arranged to form much nicer figures. Perhaps we should add more equal lines—or just one more equal line to begin with.

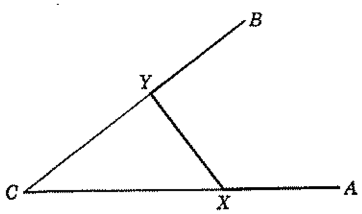

Fig. 1.1. Unknown, data, condition.

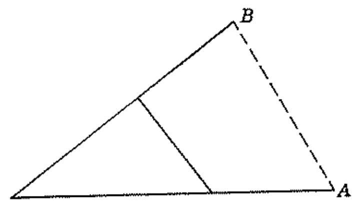

Fig. 1.2. Working forward (from the data).

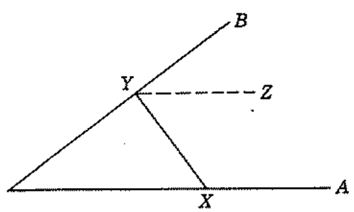

Fig. 1.3. Working backward (from the unknown).

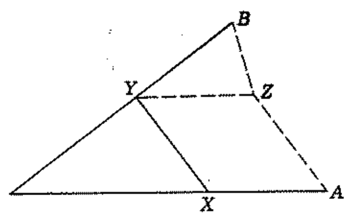

Fig. 1.4. Contacts with previous knowledge.

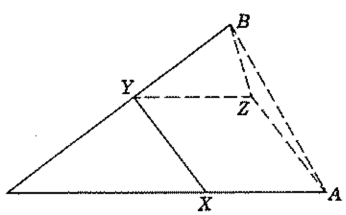

Fig. 1.5. Superposition.

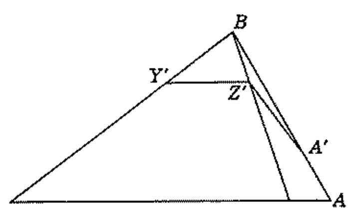

Fig. 1.6. Stepping stone.

Chance or inspiration may prompt us to introduce a line into the picture which, on the face of it, fits quite well into the intended connection: draw YZ parallel and equal to XA, see Fig. 1.3. (We are starting now from the desired unknown—wishful thinking—and trying to work backward toward the data.)

Introducing the line YZ was a trial. Yet the line does not look bad; it brings in familiar shapes. Join Z to A and B, see Fig. 1.4; we obtain the rhombus $XAZY$ and the isosceles triangle BYZ. *Could you solve some part of the problem?* Can we construct $\triangle BYZ$? We would need two data for an isosceles triangle but, unfortunately, we have only one (the angle at Y is equal to the given angle at C). Still, we have something here. Even if we do not know $\triangle BYZ$ completely, we know its shape; although we do not know its size, we could construct a triangle similar to it.

This may bring us a little nearer to the solution, but we have not got it yet: we must try a few more things. Sooner or later we may remember a former trial, the Fig. 1.2. How about combining it with later remarks? By superposing Figs. 1.2 and 1.4 we obtain Fig. 1.5 in which there is a new triangle, $\triangle BZA$. Can we construct it? We could, if we knew $\triangle BYZ$; in that favorable case, we could muster three data: two sides, ZB and $ZA = ZY$, and the angle at B. Well, we do not know $\triangle BYZ$; at any rate, we do not know it completely, we know only its shape. Yet then, we can...

We can draw the quadrilateral $BY'Z'A'$, see Fig. 1.6, similar to the quadrilateral $BYZA$ in Fig. 1.5, which is an essential part of the desired configuration. This may be a stepping stone!

1.5. The pattern of similar figures

We carry out the construction, the discovery of which is told by the sequence of Figs. 1.1–1.6.

On the given line BC, see Fig. 1.6, we choose a point Y' at random (but not too far from B). We draw the line $Y'Z'$ parallel to CA so that

$$Y'Z' = Y'B$$

Then, we determine a point A' on AB so that

$$A'Z' = Y'Z'$$

Draw a parallel to $A'Z'$ through A and determine its intersection with the prolongation of the line BZ': this intersection is the desired point Z. The rest is easy.

The two quadrilaterals $AZYB$ and $A'Z'Y'B$ are not only similar but also "similarly located" (homothetic). The point B is their center of similarity. That is, any line connecting corresponding points of the two similar figures has to pass through B.

Here is a remark from which we can learn something about problem solving: Of the two similar figures, the one that came to our attention first, $AZYB$, was actually constructed later.[1]

The foregoing example suggests a general pattern: *If you cannot construct the required figure, think of the possibility of constructing a figure* SIMILAR *to the required figure.*

[1] In this "case history" which we have just finished (we started it in sect. 1.4) the most noteworthy step was to "take the problem as solved." For further remarks on this, cf. HSI, Figures 2, pp. 104–105, and Pappus, pp. 141–148, especially pp. 146–147.

There are examples at the end of this chapter which, if you work them through, may convince you of the usefulness of this pattern of "similar figures."

1.6. Examples

The following examples differ from each other in several respects; their differences may show up more clearly the common feature that we wish to disentangle.

(1) *Draw common tangents to two given circles.* Two circles are given in position (plotted on paper). We wish to draw straight lines touching both circles. If the given circles do not overlap they have four common tangents, two exterior and two interior tangents. Let us confine our attention to the exterior common tangents, see Fig. 1.7, which exist unless one of the two given circles lies completely within the other.

If you cannot solve the proposed problem, look around for an appropriate related problem. There is an obvious related problem (of which the reader is supposed to know the solution): to draw tangents to a given circle from an outside point. This problem is, in fact, a limiting case or *extreme case* of the proposed problem: one of the two given circles is shrunken into a point. We arrive at this extreme case in the most natural way by *variation of the data.* Now we can vary the data in many ways: decrease one radius and leave the other unchanged, or decrease one radius and increase the other, or decrease both. And so we may hit upon the idea of letting both radii decrease *at the same rate*, uniformly, so that both are diminished by the same length in the same time. Visualizing this change, we may observe that each common tangent is shifting, but remains parallel to itself while shifting, till ultimately Fig. 1.8 appears—and here is the solution: draw tangents from the center of the smaller given circle to a new circle which is concentric with the larger given circle and the radius of which is

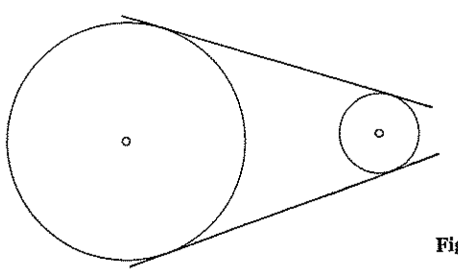

Fig. 1.7. Unknown, data, condition.

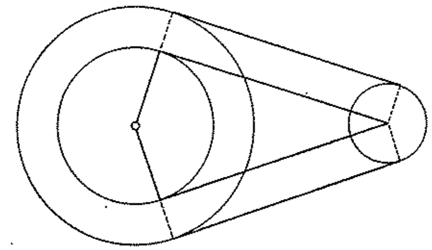

Fig. 1.8. Stepping stone.

the difference of the given radii. Use the figure so obtained as a stepping stone: the step from it to the desired figure is easy (there are just two rectangles to construct).

(2) *Construct a triangle being given the three medians.* We "take the problem as solved"; that is, we draw the (desired) triangle in which the three (given) medians are duly assembled; see Fig. 1.9. We should recollect that the three medians meet in one point (the point M in Fig. 1.9, the centroid of the triangle) which divides each median in the proportion 1:2. To visualize this essential fact, let us mark the midpoint D of the segment AM; the points D and M divide the median AE into three equal parts; see Fig. 1.10.

The desired triangle is divided into six small triangles. *Could you solve a part of the problem?* To construct one of those small triangles we need three data; in fact, we know two sides: one side is one third of a given median, another side is two thirds of another given median—but we do not see a third known piece. Could we introduce some other triangle with three known data? There is the point D in Fig. 1.10 which is obviously eager for more connections—if we join it to a neighboring point we may notice $\triangle MDG$ each side of which is one third of a median—and so we can construct it, from three known sides—here is a stepping stone! The rest is easy.

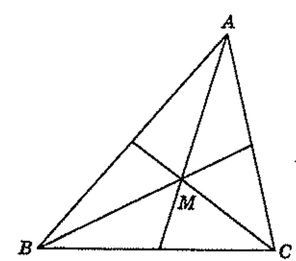

Fig. 1.9. Unknown, data, condition.

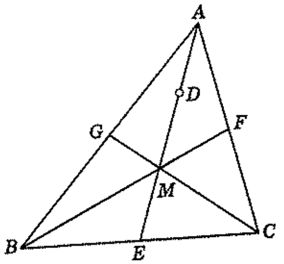

Fig. 1.10. A point eager for more connections.

(3) To each problem concerned with ordinary triangles there corresponds a problem concerned with spherical triangles or trihedral angles. (A trihedral angle is contained between three planes; a sphere described about its vertex as center intersects it in a spherical triangle.) These problems of solid geometry may be reduced to problems of plane geometry. Such reduction of problems about figures in space to drawings in a plane is, in fact, the object of *descriptive geometry*, which is an interesting branch of geometry indispensable to engineers and architects for the accurate drafting of machinery, vessels, buildings, and so on.

The reader needs no knowledge of descriptive geometry, just a little solid geometry and some common sense, to solve the following problem: *Being given the three face angles of a trihedral angle, construct its dihedral angles.*

Let a, b, and c denote the face angles of the trihedral angle (the sides of the corresponding spherical triangle) and α the dihedral angle opposite to the face a (α is an angle of the spherical triangle). Being given a, b, c, construct α. (The same method can serve to construct all three dihedral angles, and so we restrict ourselves to one of them, to α.)

To visualize the *data*, we juxtapose the three angles b, a, and c in a plane; see Fig. 1.11. To visualize the *unknown*, we should see the configuration in space. (Reproduce Fig. 1.11 on cardboard, crease the line between a and b and also that between a and c, and then fold the cardboard to form the trihedral angle.) In Fig. 1.12, the trihedral angle is seen in perspective; A is a point chosen at random on the edge opposite the face a; two perpendiculars to this edge starting from A, one drawn in the face b, the other drawn in the face c, include the angle α that we are required to construct.

Look at the unknown!—It is an angle, the angle α in Fig. 1.12.

What can you do to get this kind of unknown?—We often determine an angle from a triangle.

Is there a triangle in the figure?—No, but we can introduce one.

In fact, there is an obvious way to introduce a triangle: the plane that contains the angle α intersects the trihedral angle in a triangle; see Fig. 1.13. This triangle is a promising auxiliary figure, a likely stepping stone.

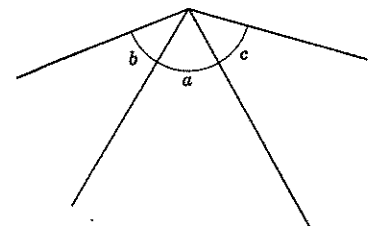

Fig. 1.11. The data.

In fact, the solution is not far. Return to the figure in the plane, to Fig. 1.11, where the data, the angles a, b, and c, appear in true magnitude. (Unfold the cardboard model we have folded together in passing from Fig. 1.11 to Fig. 1.12.) The point A appears twice, as A_1 and A_2 (by unfolding, we have separated the two faces b and c which are adjacent in space). These points A_1 and A_2 are at the same distance from the vertex V. A perpendicular to A_1V through A_1 meets the other side of the angle b in C, and B is analogously obtained; see Fig. 1.14. Now we know A_2B, BC, and CA_1, the three sides of the auxiliary triangle introduced in Fig. 1.13, and so we can readily construct it (in dotted lines in Fig. 1.14): it contains the desired angle α.

The problem just discussed is analogous to, and uses the construction of, the simplest problem about ordinary triangles which we discussed in sect. 1.1. We can see herein a sort of justice and a hint about the use of analogy.

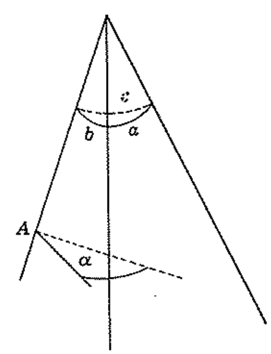

Fig. 1.12. The unknown.

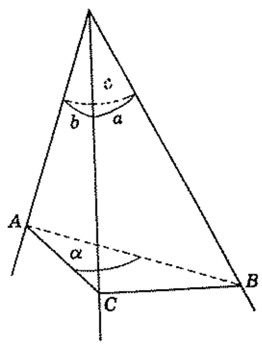

Fig. 1.13. A likely stepping stone.

1.7. The pattern of auxiliary figures

Let us look back at the problems discussed in the foregoing sect. 1.6. They were quite different, and their solutions were quite different too, except that in each case the key to the solution was an *auxiliary figure*: a circle with two tangents from an outside point in (1), a smaller triangle carved out from the desired triangle in (2), another triangle in (3). In each case we could easily construct the auxiliary figure from the data and, once in possession of the auxiliary figure, we could easily construct the originally required figure by using the auxiliary figure. And so we attained our goal in two steps; the auxiliary figure served as a kind of stepping stone; its discovery was the decisive performance, the culminating point of our work. There is a pattern here, the *pattern of auxiliary*

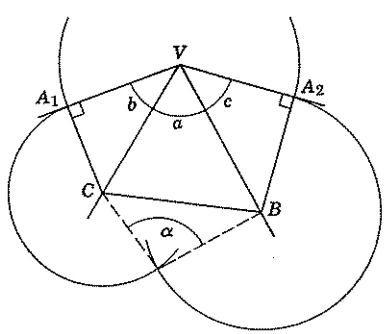

Fig. 1.14. The solution.

figures, which has some promise and which we can describe as follows: *Try to discover some part of the figure or some closely* RELATED FIGURE *which you can construct and which you can use as a stepping stone in constructing the original figure.*

This pattern is very general. In fact, the pattern of similar figures formulated in sect. 1.5 is just a particular case: a figure similar to the required figure is related to it in a particular manner and can serve as a particularly handy auxiliary figure.

Unavoidably, its greater generality renders the pattern of auxiliary figures less concrete, less tangible: it gives no specific advice about what kind of figure we should seek. Experience, of course, can give us some directives (although no hard and fast rules): we should look for figures which are easy to "carve out" from the desired figure, for "simple" figures (as triangles), for "extreme cases," and so on. We may learn procedures, such as the variation of the data, or the use of analogy, which, in certain cases, may indicate an appropriate auxiliary figure.

We have now isolated three different patterns which we may use in dealing with problems of geometric construction. The pattern of *auxiliary figures* leaves us more choice, but offers a less definite target, than the pattern of *similar figures*. The pattern of *two loci* is the simplest—you may try it first just because, in most cases, it is best to try the simplest thing first. Yet do not commit yourself, keep an open mind: take the problem as solved, draw a figure in which the *unknown* and the *data* are appropriately assembled, each element at its right place, all elements connected by the right relations, as required by the *condition*. Study this figure, try to recognize in it some familiar configuration, try to recall any relevant knowledge you may have (related problems, applicable theorems), look out for an opening (a more accessible part of the figure, for instance). You may be lucky: a bright idea may emerge from the figure and suggest an appropriate auxiliary line, the suitable pattern, or some other useful step.

Examples and Comments on Chapter 1

1.1. What is the locus of a variable point that has a given distance from a given point?

1.2. What is the locus of a variable point that has a given distance from a given straight line?

1.3. A variable point remains equidistant from two given points; what is its locus?

1.4. A variable point remains equidistant from two given parallel straight lines; what is its locus?

1.5. A variable point remains equidistant from two given intersecting straight lines; what is its locus?

1.6. Of a triangle, given two vertices, A and B, and the angle γ, opposite to the side AB; the triangle is not determined, its third vertex (that of γ) can vary. What is the locus of this third vertex?

1.7. *Notation.* In dealing with a triangle, it is convenient to use the following notation:

$A,\ B,\ C$	vertices
$a,\ b,\ c$	sides
$\alpha,\ \beta,\ \gamma$	angles
$h_a,\ h_b,\ h_c$	altitudes ("heights")
$m_a,\ m_b,\ m_c$	medians
$d_\alpha,\ d_\beta,\ d_\gamma$	bisectors of the angles ("disectors"?!)
R	radius of circumscribed circle
r	radius of inscribed circle

It is understood that the side a is opposite the angle α, the vertex of which is the point A which is the common endpoint of the three lines h_a, m_a, and d_α. According to common usage, a stands both for the side (a line segment) and for the length of the side; the reader has to find out from the context which meaning is intended. The same ambiguity is inherent in the symbols $b, c, h_a, \ldots d_\gamma, R, r$. We follow this usage although it is objectionable.

The problem "Triangle from a, b, c" means, of course, "construct a triangle being given a, b, and c." Observe that there may be no solution (the figure satisfying the proposed condition may not exist) if the data are adversely chosen; for example, there is no triangle with given sides a, b, and c, if $a > b + c$. Experiment first with data for which the required figure is likely to exist.

1.8. Triangle from a, b, m_a.

1.9. Triangle from a, h_a, m_a.

1.10. Triangle from a, h_a, α.

1.11. Triangle from a, m_a, α.

1.12. Given three (infinite) straight lines. Construct a circle that touches the first two lines and has its center on the third line.

1.13. Given two intersecting infinite straight lines and a line segment of length r. Construct a circle with radius r that touches the two given lines.

1.14. Construct a circle, being given one point on it, one straight line tangent to it, and its radius.

1.15. *Three lighthouses* are visible from a ship; their positions on the map are known, and the angles between the rays of light coming from them have been measured. Plot the position of the ship on the map.

1.16. Within a given circle, describe three equal circles so that each shall touch the other two and also the given circle. (This figure can sometimes be seen in Gothic tracery where analogous figures, with four or six inner circles, are more frequent.)

1.17. Inside a given triangle find a point from which all three sides are seen under the same angle.

1.18. Trisect the area of a given triangle.

That is, you should locate a point X inside the given $\triangle ABC$ so that $\triangle XBC$, $\triangle XCA$, and $\triangle XAB$ are equal in area.

[*Keep only a part of the condition, drop the other part:* if only the two triangles $\triangle XCA$ and $\triangle XCB$ are supposed to be equal, what is the locus of X? The answer to this question may show you a way to the solution, but there are also other approaches.]

1.19. Triangle from a, α, r.

[*Keep only a part of the condition, drop the other part:* disregard r, but keep a and α; what is the locus of the center of the inscribed circle?]

1.20. Triangle from a, h_b, c.

1.21. Triangle from a, h_b, d_γ.

1.22. Triangle from a, h_b, h_c.

1.23. Triangle from h_a, h_b, β.

1.24. Triangle from h_a, β, γ.

1.25. Triangle from h_a, d_α, α.

1.26. Construct a parallelogram, being given one side and two diagonals.

1.27. Construct a trapezoid being given its four sides a, b, c, and d; a and c should be parallel.

1.28. Construct a quadrilateral being given a, b, c, and d, its four sides, and the angle ϵ, included by the opposite sides a and c produced.

1.29. Triangle from a, $b + c$, α.

[Do not fail to introduce *all the data* into the figure. Where is the "right place" for $b + c$?]

1.30. Triangle from a, $b + c$, $\beta - \gamma$.

1.31. Triangle from $a + b + c$, h_a, α.

[Symmetry: b and c (not given) play interchangeable roles.]

1.32. Given two circles exterior to each other, draw their interior common tangents. (The two circles are situated in the same halfplane with respect to an exterior common tangent, in different halfplanes with respect to an interior common tangent.)

1.33. Given three equal circles, construct a circle containing, and tangent to, all three given circles.

1.34. Triangle from α, β, d_γ.

1.35. Inscribe a square in a given right triangle. One corner of the square is required to coincide with the right-angle corner of the given triangle, the opposite vertex of the square should lie on the hypotenuse, the two other vertices on the legs of the right triangle, one on each.

1.36. Inscribe a square in a given triangle ABC. Two vertices of the square are required to lie on AB, one on AC, and one on BC.

1.37. Inscribe a square in a given sector of a circle. Two vertices of the square are required to lie on the arc, one vertex on each of the two sides of the central angle of the sector.

1.38. Construct a circle being given two points on it and one straight line tangent to it.

1.39. Construct a circle, being given one point on it and two straight lines tangent to it.

1.40. Construct a pentagon circumscribable about a circle being given its five angles α, β, γ, δ, and ϵ (subject, of course, to the condition $\alpha + \beta + \gamma + \delta + \epsilon = 540°$) and the length of its perimeter l.

1.41. Triangle from h_a, h_b, h_c.

1.42. *A flaw.* It may happen that a problem of geometric construction has no solution: there may be no figure satisfying the proposed condition with the proposed data. For instance, there exists no triangle the sides of which have the given lengths a, b, and c if $a > b + c$. A perfect method of solution will either obtain a figure satisfying the proposed condition or show in failing that there exists no such figure.

There can arise, however, the following situation: the proposed problem itself does possess a solution, yet an auxiliary problem does not—an auxiliary figure, which our scheme would need for the construction of the originally required figure, is impossible to construct. This is, of course, a flaw in our scheme.

Is your method for solving ex. 1.41 perfect in this respect? (The triangle with sides 65, 156, 169 is a right triangle—the sides are proportional to 5, 12, 13—with heights 156, 65, 60.) If the answer is No, can you improve your method?

1.43. Triangle from a, α, R.

1.44. *Looking back* at the solution of ex. 1.43, you may ask some instructive questions and propose some related problems.

(*a*) An analogous problem?
(*b*) A more general problem?
(*c*) Triangle from a, β, R.
(*d*) Triangle from a, r, R.

1.45. *Three listening posts.* The time at which the sound of an enemy gun is heard has been exactly observed at three listening posts A, B, and C. On the basis of these data, plot on the map the position X of that enemy gun.

Regard the velocity of sound as known. Explain the analogy with, and the difference from, the problem of the three lighthouses, ex. 1.15.

1.46. *On the pattern of two loci.* Are the loci with which exs. 1.2, 1.5, and 1.6 are concerned usable in connection with the pattern of two loci? Cf. the statement at the end of sect. 1.2.

1.47. *The pattern of three loci.* A concept of plane geometry may have various analogues in solid geometry. For instance, in sect. 1.6(3) we regarded a spherical triangle or a trihedral angle as analogous to an ordinary plane triangle. Yet we could also regard a tetrahedron as analogous to an ordinary triangle; seen from this viewpoint, the following problem appears as analogous to the problem of sect. 1.3(1).

Circumscribe a sphere about a given tetrahedron.

Let us work out the analogy in some detail. We reduce the problem to obtaining the center of the required sphere. In the so reduced problem

the unknown is a point, say X;
the data are four points (the vertices of the given tetrahedron), say A, B, C, and D;
the condition consists in the equality of four distances

$$XA = XB = XC = XD$$

We may split this condition into three parts:

First	$XA = XB$
Second	$XA = XC$
Third	$XA = XD$

To each part of the condition corresponds a locus. If the point X satisfies the first part of the condition, its locus is (it can vary on) a plane, the perpendicular bisector of the segment AB; to each other part of the condition there corresponds an analogous plane. Finally, the desired center of the sphere is obtained as the intersection of three planes.

Let us assume that we have instruments with which we can determine the points of intersection of three given surfaces when each of these surfaces is either a plane or a sphere. (In fact, we have made this assumption implicitly in the foregoing. By the way, ruler and compasses are such instruments—we can determine with them those points of intersection if we know enough descriptive geometry.) Then we can propose and solve problems of geometric construction in space. The foregoing problem is an example and its solution sets an example: with the help of analogy, we can disentangle from it a pattern for solving problems of construction in space, the *pattern of three loci.*

1.48. In the foregoing ex. 1.47, as in the example of sect. 1.3(1), we could have split the condition differently and so obtain another (although a pretty similar) construction. Yet can the result be different? Why not?

1.49. *On geometric constructions.* There are many problems of geometric construction where the required figure obviously "exists" but cannot be constructed with ruler and compasses (it could be constructed with other—equally idealized—instruments). A famous problem of this kind is the trisection of the angle: a *general* angle cannot be divided into three equal parts by ruler and compasses; see Courant and Robbins, pp. 137–138.

A perfect method for geometric constructions should *either* lead us to a construction of the required figure by ruler and compasses *or* show that such a construction is impossible. Our patterns (two loci, similar figures, auxiliary figures) are not useless (as, I hope, the reader has had opportunity to convince himself) but they yield no perfect method; they frequently suggest a construction but, when they do not suggest one, we are left in the dark about the alternative with which we are most concerned: is the construction impossible in itself, or is it possible and just our effort insufficient?

There is a well known more perfect method for geometric constructions (reduction to algebra—but we need not enter upon details now). Yet for another kind of problem which we may face another day there may be no perfect method known at that time—and still we have to try. And so the patterns considered may contribute to the education of the problem solver just by their inherent imperfection.

1.50. *More problems.* Devise some problems similar to, but different from, the problems proposed in this chapter—especially such problems as you can solve.

1.51. *Sets.* We cannot define the concept of a *set* in terms of more fundamental concepts, because there are no more fundamental concepts. Yet, in fact, everybody is familiar with this concept, even if he does not use the word "set" for it. "Set of elements" means essentially the same as "class of objects" or "collection of things" or "aggregate of individuals." "Those students who will make an A in this course" form a set even if, at this moment, you could not tell all their names. "Those points in space that are equidistant from two given points" form a very clearly defined set of points, a plane. "Those straight lines in a given plane that have a given distance from a given point" form an interesting set consisting of all the tangents of a certain circle. If a, b, and c are any three distinct objects, the set to which just these three objects belong as elements is clearly defined.

Two sets are *equal* if every object that belongs to one of them belongs also to the other. If any element that belongs to the set A belongs also to the set B, we say that A is *contained* in B; there are many ways to say the same thing: B contains A, B includes A, A is a *subset* of B, and so on.

It is often convenient to consider the *empty set*, that is, the set to which no element belongs. For example, the "set of those students who will make A in this course" could well turn out to be the empty set, if no student makes a better grade than B, or if the course should be discontinued without a final examination. The empty set is a useful set as 0 is a useful number. Now, 0 is less than any positive integer; similarly, the empty set is considered as a subset of any set.

The greatest common subset of several sets is termed their *intersection.* That is, the intersection of the sets $A, B, C, \ldots,$ and L consists of those, and only those, elements that belong simultaneously to each of the sets $A, B, C, \ldots,$ and L.

For example, let A and B denote two planes, each considered as a set of points; if they are different and nonparallel, their intersection is a straight line; if they are different but parallel, their intersection is the empty set; if they are identical, their "intersection" is identical with any of them. If A, B, and C are three planes and there is no straight line parallel to all three of them, their intersection is a set containing just one element, a point.

The term "locus" means essentially the same as the term "set": the set (or locus) of those points of a plane that have a given distance from a given point is a circle.

In this example, we define the set (or locus) by stating a *condition* that its elements (points) must satisfy, or a *property* that these elements must possess: the points of a circle satisfy the condition, or have the property, that they are all contained in the same plane and all have the same distance from a given point.

The concepts of "condition" and "property" are indissolubly linked with the concept of a set. In many mathematical examples we can clearly and simply state the condition or property that characterizes the elements of a set. Yet, if a more informative description is lacking we can always say: the elements of the set S have the property of belonging to S, and satisfy the condition that they belong to S.

The consideration of the pattern of three loci (after that of two loci; see ex. 1.47) may have given us already a hint of a wider generalization. The consideration of sets and their intersections intensifies the suggestion. We now leave this suggestion to mature in the mind of the reader and we shall return to it in a later chapter.

(The least extensive set of which each one of several given sets is a subset is called the *union* of those given sets. That is, the union of the sets A, B, $\ldots$, and L contains all the elements of A, all the elements of $B, \ldots,$ and all the elements of L, and any element that the union contains, must belong to at least one of the sets $A, B, \ldots,$ and L (it may belong to several of them).

Intersection and union of sets are closely allied concepts (they are "complementary" concepts in a sense which we cannot but hint), and we could not very well discuss one without mentioning the other. In fact, we shall have more opportunity to consider the intersection of given sets than their union. The reader should familiarize himself from some other book with the first notions of the theory of sets which may be introduced into the high schools in the near future.)

CHAPTER 2

THE CARTESIAN PATTERN

2.1. Descartes and the idea of a universal method

René Descartes (1596–1650) was one of the very great. He is regarded by many as the founder of modern philosophy, his work changed the face of mathematics, and he also has a place in the history of physics. We are here mainly concerned with one of his works, the *Rules for the Direction of the Mind* (cf. ex. 2.72).

In his "Rules," Descartes planned to present a universal method for the solution of problems. Here is a rough outline of the scheme that Descartes expected to be applicable to all types of problems:

First, reduce any kind of problem to a mathematical problem.

Second, reduce any kind of mathematical problem to a problem of algebra.

Third, reduce any problem of algebra to the solution of a single equation.

The more you know, the more gaps you can see in this project. Descartes himself must have noticed after a while that there are cases in which his scheme is impracticable; at any rate, he left unfinished his "Rules" and presented only fragments of his project in his later (and better known) work *Discours de la Méthode*.

There seems to be something profoundly right in the intention that underlies the Cartesian scheme. Yet it is more difficult to carry this intention into effect, there are more obstacles and more intricate details than Descartes imagined in his first enthusiasm. Descartes' project failed, but it was a great project and even in its failure it influenced science much more than a thousand and one little projects which happened to succeed.

Although Descartes' scheme does not work in all cases, it does work in

an inexhaustible variety of cases, among which there is an inexhaustible variety of *important* cases. When a high school boy solves a "word problem" by "setting up equations," he follows Descartes' scheme and in doing so he prepares himself for serious applications of the underlying idea.

And so it may be worthwhile to have a look at some high school work.

2.2. A little problem

Here is a brain teaser which may amuse intelligent youngsters today as it probably amused others through several centuries.

A farmer has hens and rabbits. These animals have 50 *heads and* 140 *feet. How many hens and how many rabbits has the farmer?*

We consider several approaches.

(1) *Groping*. There are 50 animals altogether. They cannot all be hens, because then they would have only 100 feet. They cannot all be rabbits, because they would then have 200 feet. Yet there should be just 140 feet. If just one half of the animals were hens and the other half rabbits, they would then have.... Let us survey all these cases in a table:

Hens	*Rabbits*	*Feet*
50	0	100
0	50	200
25	25	150

If we take a smaller number of hens, we have to take a larger number of rabbits and this leads to more feet. On the contrary, if we take a larger number of hens.... Yes, there must be more than 25 hens—let us try 30:

Hens	*Rabbits*	*Feet*
30	20	140

I have got it! Here is the solution!

Yes, indeed, we have got the solution, because the given numbers, 50 and 140, are relatively small and simple. Yet if the problem, proposed with the same wording, had larger or more complicated numbers, we would need more trials or more luck to solve it in this manner, by merely muddling through.

(2) *Bright idea*. Of course, our little problem can be solved less "empirically" and more "deductively"—I mean with fewer trials, less guesswork, and more reasoning. Here is another solution.

The farmer surprises his animals in an extraordinary performance: each hen is standing on one leg and each rabbit is standing on its hind legs. In this remarkable situation just one half of the legs are used, that is, 70 legs. In this number 70 the head of a hen is counted just *once* but the head of a rabbit is counted *twice*. Take away from 70 the number of all heads, which is 50; there remains the number of the rabbit heads—there are

$$70 - 50 = 20$$

rabbits! And, of course, 30 hens.

This solution would work just as well if the numbers in our little problem (50 and 140) were replaced by less simple numbers. This solution (which can be presented less whimsically) is ingenious: it needs a clear intuitive grasp of the situation, a little bit of a bright idea—my congratulations to a youngster of fourteen who discovers it by himself. Yet bright ideas are rare—we need a lot of luck to conceive one.

(3) *By algebra.* We can solve our little problem without relying on chance, with less luck and more system, if we know a little algebra.

Algebra is a language which does not consist of words but of symbols. If we are familiar with it we can translate into it appropriate sentences of everyday language. Well, let us try to translate into it the proposed problem. In doing so, we follow a precept of the Cartesian scheme: "reduce any kind of problem to a problem of algebra." In our case the translation is easy.

State the problem

in English	*in algebraic language*
A farmer has	
a certain number of hens	x
and a certain number of rabbits	y
These animals have fifty heads	$x + y = 50$
and one hundred forty feet	$2x + 4y = 140$

We have translated the proposed question into a system of two equations with two unknowns, x and y. Very little knowledge of algebra is needed to solve this system: we rewrite it in the form

$$x + 2y = 70$$
$$x + y = 50$$

and subtracting the second equation from the first we obtain

$$y = 20$$

Using this we find, from the second equation of the system, that

$$x = 30$$

This solution works just as well for large given numbers as for small ones, works for an inexhaustible variety of problems, and needs no rare bright idea, just a little facility in the use of the algebraic language.

(4) *Generalization.* We have repeatedly considered the possibility of substituting other, especially larger, numbers for the given numbers of our problem, and this consideration was instructive. It is even more instructive to substitute *letters for the given numbers.*

Substitute h for 50 and f for 140 in our problem. That is, let h stand for the number of heads, and f for the number of feet, of the farmer's animals. By this substitution, our problem acquires a new look; let us consider also the translation into algebraic language.

A farmer has	
a certain number of hens	x
and a certain number of rabbits.	y
These animals have h heads	$x + y = h$
and f feet.	$2x + 4y = f$

The system of two equations that we have obtained can be rewritten in the form

$$x + 2y = \frac{f}{2}$$
$$x + y = h$$

and yields, by subtraction,

$$y = \frac{f}{2} - h$$

Let us retranslate this formula into ordinary language: the number of rabbits equals one half of the number of feet, less the number of heads: this is the result of the imaginative solution (2).

Yet here we did not need any extraordinary stroke of luck or whimsical imagination; we attained the result by a straightforward routine procedure after a simple initial step which consisted in replacing the given numbers by letters. This step is certainly simple, but it is an important step of generalization.[1]

(5) *Comparison.* It may be instructive to compare different approaches to the same problem. Looking back at the four preceding approaches, we may observe that each of them, even the very first, has some merit, some specific interest.

[1] Cf. HSI, Generalization 3, pp. 109–110; Variation of the problem 4, pp. 210–211; Can you check the result? 2, p. 60.

The first procedure which we have characterized as "groping" and "muddling through" is usually described as a solution by *trial and error*. In fact, it consists of a series of trials, each of which attempts to correct the error committed by the preceding and, on the whole, the errors diminish as we proceed and the successive trials come closer and closer to the desired final result. Looking at this aspect of the procedure, we may wish a better characterization than "trial and error"; we may speak of "successive trials" or "successive corrections" or "successive approximations." The last expression may appear, for various reasons, to be the most suitable. The term *method of successive approximations* naturally applies to a vast variety of procedures on all levels. You use successive approximations when, in looking for a word in the dictionary, you turn the leaves and proceed forward or backward according as a word you notice precedes or follows in alphabetical order the word you are looking for. A mathematician may apply the term successive approximations to a highly sophisticated procedure with which he tries to treat some very advanced problem of great practical importance that he cannot treat otherwise. The term even applies to science as a whole; the scientific theories which succeed each other, each claiming a better explanation of phenomena than the foregoing, may appear as successive approximations to the truth.

Therefore, the teacher should not discourage his students from using "trial and error"—on the contrary, he should encourage the intelligent use of the fundamental method of successive approximations. Yet he should convincingly show that for such simple problems as that of the hens and rabbits, and in many more (and more important) situations, straightforward algebra is more efficient than successive approximations.

2.3. Setting up equations

In the foregoing, cf. sect. 2.2(3), we have translated a proposed problem from the ordinary language of words into the algebraic language of symbols. In our example, the translation was obvious; there are cases, however, where the translation of the problem into a system of equations demands more experience, or more ingenuity, or more work.[2]

What is the nature of this work? Descartes intended to answer this question in the second part of his "Rules" which, however, he left unfinished. I wish to extract from his text and present in contemporary language such parts of his considerations as are the most relevant at this stage of our study. I shall leave aside many things that Descartes did say, and I shall make explicit a few things that he did not quite say, but I still think that I shall not distort his intentions.

[2] Cf. HSI, Setting up equations, pp. 174–177.

I wish to follow Descartes' manner of exposition: I shall begin each explanation by a concise "advice" (in fact, it is rather a summary) and then expand that advice (summary) by adding comments.

(1) *First, having well understood the problem, reduce it to the determination of certain unknown quantities* (Rules XIII–XVI).

To spend time on a problem that we do not understand would be foolish. Therefore, our first and most obvious duty is to understand the problem, its meaning, its purpose.

Having understood the problem as a whole, we turn our attention to its principal parts. We should see very clearly

what kind of thing we have to find (the UNKNOWN or unknowns)
what is given or known (the DATA)
how, by what relations, the unknowns and the data are connected with each other (the CONDITION).

(In the problem of sect. 2.2(4) the unknowns are x and y, and the data h and f, the numbers of hens and rabbits, heads and feet, respectively. The condition is expressed first in words, then in equations.)

Following Descartes, we now confine ourselves to problems in which the unknowns are quantities (that is, numbers but not necessarily integers). Problems of other kinds, such as geometrical or physical problems, may be reduced sometimes to problems of this purely quantitative type, as we shall illustrate later by examples; cf. sects. 2.5 and 2.6.

(2) *Survey the problem in the most natural way, taking it as solved and visualizing in suitable order all the relations that must hold between the unknowns and the data according to the condition* (Rule XVII).

We imagine that the unknown quantities have values fully satisfying the condition of the problem: this is meant essentially by "taking the problem as solved"; cf. sect. 1.4. Accordingly, we treat unknown and given quantities equally in some respects; we visualize them connected by relations as the condition requires. We should survey and study these relations in the spirit in which we survey and study the figure when planning a geometric construction; see the end of sect. 1.7. The aim is to find some indication about our next task.

(3) *Detach a part of the condition according to which you can express the same quantity in two different ways and so obtain an equation between the unknowns. Eventually you should split the condition into as many parts, and so obtain a system of as many equations, as there are unknowns* (Rule XIX).

The foregoing is a free rendering, or *paraphrase*, of the statement of Descartes' Rule XIX. After this statement there is a great gap in

Descartes' manuscript: the explanation which should have followed the statement of the Rule is missing (it was probably never written). Therefore, we have to make up our own comments.

The aim is stated clearly enough: we should obtain a system of n equations with n unknowns. It is understood that the computation of these unknowns should solve the proposed problem. Therefore, the system of equations should be equivalent to the proposed condition. If the whole system expresses the whole condition, each single equation of the system should express some part of the condition. Hence, in order to set up the n equations we should split the condition into n parts. But how?

The foregoing considerations under (1) and (2) (which outline very sketchily Descartes' Rules XIII–XVII) give some indications, but no definite instructions. Certainly, we have to understand the problem, we have to see the unknowns, the data, and the condition very, very clearly. We may profit from surveying the various clauses of the condition and from visualizing the relations between the unknowns and the data. All these activities give us a chance to obtain the desired system of equations, but no certainty.

The advice that we are considering (the paraphrase of Rule XIX) stresses an additional point: in order to obtain an equation we have to *express the same quantity in two different ways*. (In the example of sect. 2.2(3) an equation expresses the *number of feet* in two different ways.) This remark, properly digested, often helps to discover an equation between the unknowns—it can always help to explain the equation after it has been discovered.

In short, there are some good suggestions, but there is no foolproof precept for setting up equations. Yet, where no precept helps, practice may help.

(4) *Reduce the system of equations to one equation* (Rule XXI).

The statement of Descartes' Rule XXI which is here paraphrased is not followed by an explanation (in fact, it is the last sentence in Descartes' manuscript). We shall not examine here under which conditions a system of algebraic equations can be reduced to a single equation or how such reduction can be performed; these questions belong to a purely mathematical theory which is more intricate than Descartes' short advice may lead us to suppose, but is pretty well explored nowadays and no concern of ours at this point. Very little algebra will be sufficient to perform the reduction in those simple cases in which we shall need it.

There are other questions which remain unexplored although we should concern ourselves with them. Yet we may take them up more profitably after some examples.

2.4. Classroom examples

The "word problems" of the high school are trivial for mathematicians, but not so trivial for high school boys or girls or teachers. I think, however, that a teacher who makes an earnest effort to bring Descartes' advice, presented in the foregoing, down to classroom level and to put it into practice will avoid many of the usual pitfalls and difficulties.

First of all, the student should not start doing a problem before he has understood it. It can be checked to a certain extent whether the student has really understood the problem: he should be able to repeat the statement of the problem, point out the unknowns and the data, and explain the condition in his own words. If he can do all this reasonably well, he may proceed to the main business.

An equation expresses a *part of the condition*. The student should be able to tell which part of the condition is expressed by an equation that he brings forward—and which part is not yet expressed.

An equation expresses the *same quantity in two different ways*. The student should be able to tell which quantity is so expressed.

Of course, the student should possess the *relevant knowledge* without which he could not understand the problem. Many of the usual high school problems are "rate problems" (see the next three examples). Before he is called upon to do such a problem, the student should acquire in some form the idea of "rate," proportionality, uniform change.

(1) *One pipe can fill a tank in* 15 *minutes, another pipe can fill it in* 20 *minutes, a third pipe in* 30 *minutes. With all three pipes open, how long will it take to fill the empty tank?*

Let us assume that the tank contains g gallons of water when it is full. Then the rate of flow through the first pipe is

$$\frac{g}{15}$$

gallons per minute. Since

$$\text{amount} = \text{rate} \times \text{time}$$

the amount of water flowing through the first pipe in t minutes is

$$\frac{g}{15}t$$

If the three pipes together fill the empty tank in t minutes, the *amount of water in the full tank* can be expressed in two ways:

$$\frac{g}{15}t + \frac{g}{20}t + \frac{g}{30}t = g$$

The left-hand side shows the contribution of each pipe separately, the right-hand side the joint result of these three contributions.

Division by g yields the equation for the required time t:

$$\frac{t}{15} + \frac{t}{20} + \frac{t}{30} = 1$$

Of course, the derivation of the equation could be presented differently and the problem itself could be generalized and modified in various ways.

(2) *Tom can do a job in* 3 *hours, Dick in* 4 *hours, and Harry in* 6 *hours. If they do it together* (and do not delay each other), *how long does the job take?*

Tom can do $\frac{1}{3}$ of the whole job in one hour; we can also say that Tom is working at the rate of $\frac{1}{3}$ of the job per hour. Therefore, in t hours Tom does $t/3$ of the job. If the three boys work together and finish the work in t hours (and if they do not delay each other—a very iffy condition), the *full amount of work* can be expressed in two ways:

$$\frac{t}{3} + \frac{t}{4} + \frac{t}{6} = 1$$

in fact, the 1 on the right-hand side stands for "one full job."

This problem is almost identical with the foregoing (1), even numerically since

$$15:20:30 = 3:4:6$$

It is instructive to formulate a common generalization of both (using letters). It is also instructive to compare the solutions and weigh the advantage and disadvantage of introducing the quantity g into the solution (1).

(3) *A patrol plane flies* 220 *miles per hour in still air. It carries fuel for* 4 *hours of safe flying. If it takes off on patrol against a wind of* 20 *miles per hour, how far can it fly and return safely?*

It is understood that the wind is supposed to blow with unchanged intensity during the whole flight, that the plane travels in a straight line, that the time needed for changing direction at the furthest point is negligible, and so on. All word problems contain such unstated simplifying assumptions and demand from the problem solver some preliminary work of *interpretation* and *abstraction*. This is an essential feature of the word problems which is not always trivial and should be brought into the open, at least now and then.

The problem becomes more instructive if for the numbers

$$220 \qquad 20 \qquad 4$$

we substitute general quantities

$$v \qquad w \qquad T$$

which denote the velocity of the plane in still air, the velocity of the wind, and the total flying time, respectively; these three quantities are the *data*. Let x stand for the distance flown in one direction, t_1 for the duration of the outgoing flight, t_2 for the duration of the homecoming flight; these three quantities are *unknowns*. It is useful to display some of these quantities in a neat arrangement:

	Going	Returning
Distance	x	x
Time	t_1	t_2
Velocity	$v - w$	$v + w$

(To fill out the last line we need, in fact, some "unsophisticated" knowledge of kinematics.) Now, as we should know,

$$\text{distance} = \text{velocity} \times \text{time}$$

We express each of the following three quantities in two ways:

$$x = (v - w)t_1$$
$$x = (v + w)t_2$$
$$t_1 + t_2 = T$$

We have here a system of three equations for the three unknowns x, t_1, and t_2. In fact, only x was required by the proposed problem; t_1 and t_2 are *auxiliary unknowns* which we have introduced in order to express neatly the whole condition. Eliminating t_1 and t_2, we find

$$\frac{x}{v - w} + \frac{x}{v + w} = T$$

and hence

$$x = \frac{(v^2 - w^2)T}{2v}$$

There is no difficulty in substituting numerical values for the data v, w, and T. It is more interesting to examine the result, and to check it by the *variation of the data*.

If $w = 0$, then $2x = vT$. This is right, obviously: the whole flight is now supposed to take place in still air.

If $w = v$, then $x = 0$. Again obvious: against a headwind with speed v, the plane cannot start at all.

If w increases from the value $w = 0$ to the value $w = v$, the distance x decreases steadily, according to the formula. And so, again, the formula agrees with what we can foresee without any algebra, just by visualizing the situation.

Working with numerical data instead of general data (letters) we would have missed this instructive discussion of the formula and the valuable checks of our result. By the way, there are still other interesting checks.

(4) *A dealer has two kinds of nuts; one costs* 90 *cents a pound, the other* 60 *cents a pound. He wishes to make* 50 *pounds of a mixture that will cost* 72 *cents a pound. How many pounds of each kind should he use?*

This is a typical, rather simple "mixture problem." Let us say that the dealer uses x pounds of nuts of the first kind, and y pounds of the second kind; x and y are the unknowns. We can conveniently survey the unknowns and the data in the array:

	First kind	Second kind	Mixture
Price per pound	90	60	72
Weight	x	y	50

Express in two ways the *total weight of the mixture*:

$$x + y = 50$$

Then express in two ways the *total price of the mixture*:

$$90x + 60y = 72 \cdot 50$$

We have here a system of two equations for the two unknowns x and y. We leave the solution to the reader, who should have no trouble in finding the values

$$x = 20, \qquad y = 30$$

In passing from "numbers" to "letters" the reader obtains a problem which, as it will turn out later, has still other (and more interesting) interpretations.

2.5. Examples from geometry

We shall discuss just two examples.

(1) *A problem of geometric construction.* It is possible to reduce any problem of geometric construction to a problem of algebra. We cannot treat here the general theory of such reduction,[3] but here is an example.

A triangular area is enclosed by a straight line AB and two circular arcs, AC and BC. The center of one circle is A, that of the other is B, and each circle passes through the center of the other. Inscribe into this triangular area a circle touching all three boundary lines.

The desired configuration, Fig. 2.1, is sometimes seen in Gothic tracery.

Obviously, we can reduce the problem to the construction of one point: the center of the required circle. One locus for this point is also obvious:

[3] See Courant-Robbins, pp. 117–140.

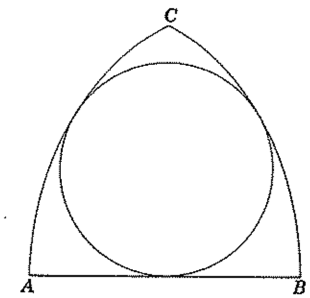

Fig. 2.1. From a Gothic window.

the perpendicular bisector of the segment AB which is a line of symmetry for the given triangular area. And so there remains to find another locus.

Keep only a part of the condition, drop the other part. We consider a (variable) circle touching not three, but only two boundary lines: the straight line AB and the circular arc BC; see Fig. 2.2. In order to find the locus of the center of this variable circle, we use analytic geometry. We let the origin of our rectangular coordinate system coincide with the point A, and let the x axis pass through the point B; see Fig. 2.2. Let x and y denote the coordinates of the center of the variable circle. Join this center to the two essential points of contact, one with the straight line AB, the other with the circular arc BC; see Fig. 2.2. The two radii have the same length which, therefore, can be expressed in two different manners (set $AB = a$):

$$y = a - \sqrt{x^2 + y^2}$$

By getting rid of the square root, we transform this equation into

$$x^2 = a^2 - 2ay$$

And so the locus of the center of the variable circle turns out to be a parabola—a locus of no immediate use in geometric constructions.

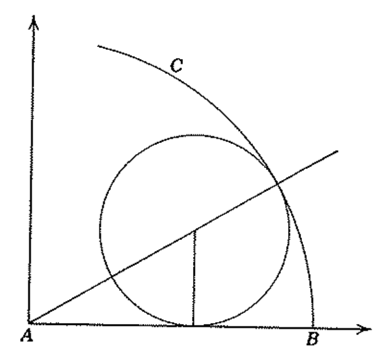

Fig. 2.2. We have dropped a part of the condition.

Yet the obvious locus mentioned at the beginning, the perpendicular bisector of AB, has the equation

$$x = \frac{a}{2}$$

which, combined with the equation of the parabola, yields the ordinate of the desired center of the circle.

$$y = \frac{3a}{8}$$

and this ordinate is easy to construct from the given length $a = AB$.

(2) *The analogue of Pythagoras' theorem in solid geometry.* Analogy is not unambiguous. There are various facts of solid geometry which can be quite properly regarded as analogous to the Pythagorean proposition. We arrive at such a fact if we regard a cube as analogous to a square, and a tetrahedron that we obtain by cutting off a corner of the cube by an oblique plane as analogous to a right triangle (which we obtain by cutting off a corner of a square by an oblique straight line). To the rectangular vertex of the right triangle there corresponds a vertex of the tetrahedron which we shall call a *trirectangular* vertex. In fact, the three edges of the tetrahedron starting from this vertex are perpendicular to each other, forming three right angles.

Pythagoras' theorem solves the following problem: In a triangle that possesses a rectangular vertex O, there are given the lengths a and b of the two sides meeting in O. Find the length c of the side opposite O.

We put the analogous problem: *In a tetrahedron that possesses a trirectangular vertex O, there are given the areas A, B, and C of the three faces meeting in O. Find the area D of the face opposite O.*

We are required to express D in terms of A, B, and C. It is natural to expect a formula analogous to Pythagoras' theorem

$$c^2 = a^2 + b^2$$

which solves the corresponding problem of plane geometry. A high school boy guessed

$$D^3 = A^3 + B^3 + C^3$$

This is a clever guess; the change in the exponent corresponds neatly to the transition from 2 to 3 dimensions.

(3) *What is the unknown?*—The area of a triangle, D.

How can you find such an unknown? *How can you get this kind of thing?*—The area of a triangle can be computed if the three sides are known, by Heron's formula. The area of our triangle is D. Let a, b,

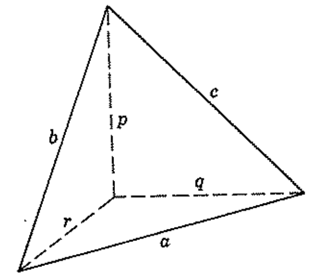

Fig. 2.3. Pythagoras in space.

and c denote the lengths of the sides, and set $s = (a + b + c)/2$; then

$$D^2 = s(s - a)(s - b)(s - c)$$

(This is a form of Heron's formula.) Let us label the sides of D in the figure; see Fig. 2.3.

Fine! But are the sides a, b, and c known?—No, but they are in right triangles; if the legs in these right triangles (labeled p, q, r in Fig. 2.3) were known, we could express a, b, and c:

$$a^2 = q^2 + r^2, \qquad b^2 = r^2 + p^2, \qquad c^2 = p^2 + q^2$$

That is good; but are p, q, and r themselves known?—No, but they are connected with the data, the areas A, B, and C:

$$\tfrac{1}{2}qr = A, \qquad \tfrac{1}{2}rp = B, \qquad \tfrac{1}{2}pq = C$$

That is right; but did you achieve anything useful?—I think I did. I now have 7 unknowns

$$\begin{matrix} & D & \\ a, & b, & c \\ p, & q, & r \end{matrix}$$

but also a system of 7 equations to determine them.

(4) There is nothing wrong with our foregoing reasoning, under (3). We have attained the goal set by Descartes' Rule (freely rendered in sect. 2.3(3)): we have obtained a system with as many equations as there are unknowns. There is just one thing: the number 7 may seem too high, to solve 7 equations with 7 unknowns may appear as too much trouble. And Heron's formula may not look too inviting.

If we feel so, we may prefer a new start.

What is the unknown?—The area of a triangle, D.

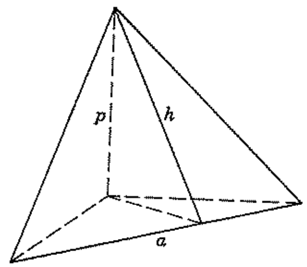

Fig. 2.4. A new departure.

How can you find such an unknown? *How can you get this kind of thing?*—The most familiar way to compute the area of a triangle is

$$D = \frac{ah}{2}$$

where a is the base, and h the altitude, of the triangle with area D; let us introduce h into the figure. See Fig. 2.4.

Yes, we have seen a before; but what about h?—The height h of the triangle with area D can be computed from a suitable triangle, I hope. In fact, intersect the tetrahedron with a plane passing through h and the trirectangular vertex. The intersection is a right triangle, its hypotenuse is h, one of its legs is p which we have seen before, and the other leg, say k, is the altitude perpendicular to the side a in the triangle with area A. Therefore,

$$h^2 = k^2 + p^2$$

Very good! But what about k?—We can get it somehow. In fact, express the area of the triangle in which, as I have just said, k is an altitude in two different manners:

$$\tfrac{1}{2}ak = A$$

Have you as many equations as you have unknowns?—I also have the former equations, and I have no time to count. I now see my way, I think. Let me just combine what is before me:

$$\begin{aligned}4D^2 &= a^2h^2\\ &= a^2(k^2 + p^2)\\ &= 4A^2 + a^2p^2\\ &= 4A^2 + (r^2 + q^2)p^2\\ &= 4A^2 + (rp)^2 + (pq)^2\\ &= 4A^2 + 4B^2 + 4C^2 .\end{aligned}$$

Let me bring together the beginning and the end and get rid of that superfluous factor 4. Here it is:

$$D^2 = A^2 + B^2 + C^2$$

This result is, in fact, closely analogous to Pythagoras' theorem. That guess with the exponent 3 was clever—it turned out wrong, but this is not surprising. What is surprising is that the guess came so close to the truth.

It may be quite instructive to compare the two foregoing approaches to the same problem; they differ in various respects.

And could you imagine a different analogue to Pythagoras' theorem?

2.6. An example from physics

We start from the following question.

An iron sphere is floating in mercury. Water is poured over the mercury and covers the sphere. Will the sphere sink, rise, or remain at the same depth?

We have to compare two situations. In both cases the lower part of the iron sphere is immersed in (is under the level of) mercury. The upper part of the sphere is surrounded by air (or vacuum) in the first situation, and by water in the second situation. In which situation is the upper part (the one over the level of the mercury) a greater fraction of the whole volume?

This is a purely qualitative question. Yet we can give it a quantitative twist which renders it more precise (and accessible to algebra): *Compute the fraction of the volume of the sphere that is over the level of the mercury for both situations.*

(1) We can give a plausible answer to the qualitative question by purely intuitive reasoning, just by visualizing a *continuous transition* from one proposed situation to the other. Let us imagine that the fluid poured over the mercury and surrounding the upper part of the iron sphere *changes its density continuously.* To begin with, this imaginary fluid has density zero (we have just vacuum). Then the density increases; it soon attains the density of the air, and after a while the density of water. If you do not see yet how this change affects the floating sphere, let the *density increase still further.* When the density of that imaginary fluid attains the density of iron, the sphere must rise clear out of the mercury. In fact, if the density increased further ever so little, the sphere should pop up and emerge somewhat from that imaginary fluid.

It is natural to suppose that the position of the floating sphere, as the density of the imaginary fluid covering it, changes all the time in the *same direction.* Then we are driven to the conclusion that, in the transition from covering vacuum or air to covering water, the sphere will *rise.*

(2) In order to answer the quantitative question, we need the numerical values of the three specific gravities involved which are

	1.00	13.60	7.84
for	water	mercury	iron

respectively. Yet it is more instructive to substitute letters for these numerical data. Let

$$a \qquad b \qquad c$$

denote the specific gravity of the

upper fluid lower fluid floating solid

respectively. Let v denote the (given) total volume of the floating solid, x the fraction of v that is over the level separating the two fluids, and y the fraction under that level; see Fig. 2.5. Our data are a, b, c, and v, our unknowns x and y. It is understood that

$$a < c < b$$

We may express the total volume of the floating body in two different ways:

$$x + y = v$$

Now, we cannot proceed beyond this point unless we know the pertinent physical facts. The *relevant knowledge* that we should possess is the law of Archimedes which is usually expressed as follows: the floating body is buoyed up by a vertical force equal in magnitude to the weight of the displaced fluid. The sphere that we are considering displaces fluid in two layers. The weights of the displaced quantities are

	ax	and	by
in the	upper layer	and	lower layer

respectively. These two upward vertical forces must jointly balance the

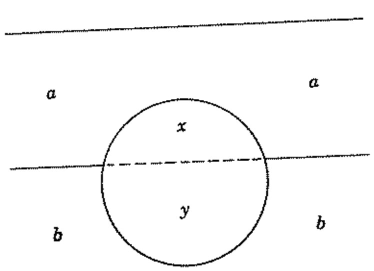

Fig. 2.5. Floating in two fluids.

weight of the floating sphere which we can, therefore, express in two different ways:

$$ax + by = cv$$

Now, we have obtained a system of two equations for our two unknowns x and y. Solving this system, we obtain

$$x = \frac{b - c}{b - a} v, \qquad y = \frac{c - a}{b - a} v$$

(3) Let us return to the original statement of the problem. In the first situation, if there is vacuum over the mercury

$$a = 0, \qquad b = 13.60, \qquad c = 7.84$$

which yield for the fraction of the iron sphere's volume over the level of mercury

$$x = 0.423v$$

In the second situation, when there is water over the mercury,

$$a = 1.00, \qquad b = 13.60, \qquad c = 7.84$$

which yield

$$x = 0.457v$$

and the latter fraction is larger, which agrees with the conclusion of our intuitive reasoning.

The general formula (in letters) is, however, more interesting than any particular numerical result that we can derive from it. Especially, it fully substantiates our intuitive reasoning. In fact, keep b, c, and v constant and let a (the density of the upper layer) increase from

$$a = 0 \qquad \text{to} \qquad a = c$$

Then the denominator $b - a$ of x decreases steadily and so x, the fraction of v over the level of the mercury, increases steadily from

$$x = \frac{b - c}{b} v \qquad \text{to} \qquad x = v$$

2.7. An example from a puzzle

How can you make two squares from five? Fig. 2.6 shows a sheet of paper that has the shape of a cross; it is made up of five equal squares. Cut this sheet along a straight line in two pieces, then cut one of the pieces along another straight line again in two, so that the resulting three pieces, suitably fitted together, form two juxtaposed squares.

The cross in Fig. 2.6 is highly symmetric (it has a center of symmetry and four lines of symmetry). The two juxtaposed squares form a rectangle the length of which is twice the width. It is understood that the

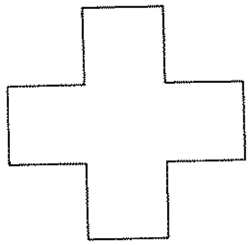

Fig. 2.6. Two from five?

three pieces into which the cross will be divided should fill up this rectangle without overlapping.

Could you solve a part of the problem? Obviously, the area of the desired rectangle is equal to the area of the given cross, and so it equals $5a^2$ if a denotes the side of one of the five squares forming the cross. Yet, having obtained its area, we can also find the sides of the rectangle. Let x denote the length of the rectangle; then its width is $x/2$. Express the area of the rectangle in two different ways; we obtain

$$x \cdot \frac{x}{2} = 5a^2$$

or

$$x^2 = 10a^2$$

from which we can find both sides of the rectangle.

We now have sufficient information about the rectangle, its shape and size, but the proposed puzzle is not yet solved: we still have to locate the two cuts in the cross. Yet the expression for x obtained above may yield a hint, especially if we write it in this form:

$$x^2 = 9a^2 + a^2$$

With this indication, I leave the solution to the reader.

We can derive some useful suggestions from the foregoing treatment of the puzzle.

First, it shows that algebra can be useful even when it cannot solve the problem completely: it can solve a part of the problem and the solution of that part can facilitate the remaining work.

Second, the procedure that we have employed may impress us with its peculiar *expanding* pattern. First, we have obtained only a small part of the solution: the area of the desired rectangle. We have used, however, this small part to obtain a bigger part: the sides of the rectangle, and hence complete information about the rectangle. Now, we are trying to use this bigger part to obtain a still bigger part which we may use afterwards, we hope, to obtain the full solution.

2.8. Puzzling examples

The problems that we have considered so far in this chapter are "reasonable." We are inclined to regard a problem as reasonable if its solution is uniquely determined. If we are seriously concerned with our problem, we wish to know (or guess) as early as possible whether it is reasonable or not. And so, from the outset, we may ask ourselves: *Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or inconsistent?*

These questions are important.[4] We postpone a general discussion of their role, but it will be appropriate to look here at a couple of examples.

(1) *A man walked five hours, first along a level road, then up a hill, then he turned round and walked back to his starting point along the same route. He walks* 4 *miles per hour on the level,* 3 *uphill, and* 6 *downhill. Find the distance walked.*[5]

Is this a reasonable problem? Are the data *sufficient to determine the unknown? Or are they insufficient? Or redundant?*

The data seem to be *insufficient*: some information about the extent of the nonlevel part of the route seems to be lacking. If we knew how much time the man spent walking uphill, or downhill, there would be no difficulty. Yet without such information the problem appears indeterminate.

Still, let us try. Let

x stand for the total distance walked,
y for the length of the uphill walk.

The walk had four different phases:

level, uphill, downhill, level.

Now we can easily express the total time spent in walking in two different ways:

$$\frac{x/2 - y}{4} + \frac{y}{3} + \frac{y}{6} + \frac{x/2 - y}{4} = 5$$

Just one equation between two unknowns—it is insufficient. Yet, when we collect the terms, the coefficient of y turns out to be 0, and there remains

$$\frac{x}{4} = 5$$

$$x = 20$$

And so the data are sufficient to determine x, the only unknown required

[4] Cf. HSI, p. 122: Is it possible to satisfy the condition?
[5] Cf. "Knot I" of "A Tangled Tale" by Lewis Carroll.

by the statement of the problem. Hence, after all, the problem is not indeterminate: we were wrong.

(2) We were wrong, there is no denying, but we suspect that the author of the problem took pains to mislead us by a tricky choice of those numbers 3, 6, and 4. To get to the bottom of his trick, let us substitute for the numbers

	3,	6,	4
the letters	u,	v,	w

which stand for the pace of the walk

uphill, downhill, on the level,

respectively. We should reread the problem, with the letters just introduced substituted for the original numbers, and then express the total time spent in walking in two different ways, using the appropriate letters:

$$\frac{x/2 - y}{w} + \frac{y}{u} + \frac{y}{v} + \frac{x/2 - y}{w} = 5$$

or

$$\frac{x}{w} + \left(\frac{1}{u} + \frac{1}{v} - \frac{2}{w}\right) y = 5$$

We cannot determine x from this equation, unless the coefficient of y vanishes. And so the problem *is* indeterminate, unless

$$\frac{1}{w} = \frac{1}{2}\left(\frac{1}{u} + \frac{1}{v}\right)$$

If, however, the three rates of walking are chosen at random, they do not satisfy this relation, and so the problem *is* indeterminate. We were put in the wrong by a vicious trick!

(We can express the critical relation also by the formula

$$w = \frac{2uv}{u + v}$$

or by saying that the pace on the level is the harmonic mean of the uphill pace and the downhill pace.)

(3) *Two smaller circles are outside each other, but inside a third, larger circle. Each of these three circles is tangent to the two others and their centers are along the same straight line. Given r, the radius of the larger circle, and t, that piece of the tangent to the two smaller circles in their common point that lies within the larger circle. Find the area that is inside the larger circle but outside the two smaller circles.* See Fig. 2.7.

Is this a reasonable problem? Are the data *sufficient to determine the unknown? Or are they insufficient? Or redundant?*

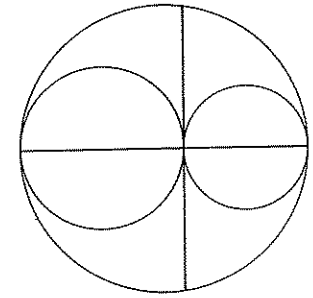

Fig. 2.7. Two data.

The problem seems perfectly reasonable. To determine the configuration of the three circles, it is both necessary and sufficient to know the radii of the two smaller circles, and any two independent data will be just as good. Now, the given r and t are obviously independent: we can vary one without changing the other (except for the inequality $t \leqq 2r$ which we may take for granted). Yes, the two data r and t seem to be just sufficient, neither insufficient nor redundant.

Therefore, let us settle down to work. Let A stand for the required area, x and y for the radii of the two smaller circles. Obviously

$$A = \pi r^2 - \pi x^2 - \pi y^2$$
$$2r = 2x + 2y$$

We have here two equations for our three unknowns, A, x, and y. In order to obtain a third equation, consider the right triangle inscribed in the larger circle, the base of which passes through the three centers and the opposite vertex of which is one of the endpoints of the segment t. The altitude in this triangle, drawn from the vertex of the right angle, is $t/2$; this altitude is a mean proportional (Euclid VI 13):

$$\left(\frac{t}{2}\right)^2 = 2x \cdot 2y$$

Now we have three equations. We rewrite the last two:

$$(x + y)^2 = r^2$$

$$2xy = \frac{t^2}{8}$$

Subtraction yields $x^2 + y^2$, and substitution into the first equation yields

$$A = \frac{\pi t^2}{8}$$

The data turned out to be *redundant*: of the two data, t and r, only the first is really needed, not the second. We were wrong again.

The curious relation underlying the example just discussed was observed by Archimedes; see his *Works* edited by T. L. Heath, pp. 304–305.

Examples and Comments on Chapter 2

First Part

2.1. Bob has three dollars and one half in nickels and dimes, fifty coins altogether. How many nickels has Bob, and how many dimes? (Have you seen the same problem in a slightly different form?)

2.2. Generalize the problem of sect. 2.4(1) by passing from "numbers" to "letters" and considering several filling and emptying pipes.

2.3. Devise some other interpretation for the equation set up in sect. 2.4(2).

2.4. Find further checks for the solution of the flight problem of sect. 2.4(3).

2.5. In the "mixture problem" of sect. 2.4(4) substitute the letters

$$a \quad b \quad c \quad v$$

for the numerical data

$$90 \quad 60 \quad 72. \quad 50$$

respectively. Read the problem after this substitution and set up the equations. Do you recognize them?

2.6. Fig. 2.8 (which is different from, but related to, Fig. 2.1) shows another configuration frequently seen in Gothic tracery.

Find the center of the circle that touches four circular arcs forming a "curvilinear quadrilateral."

Two arcs have the radius AB, the center of one is A, that of the other B.

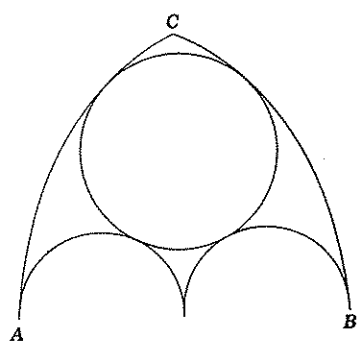

Fig. 2.8. From a Gothic window.

Two semicircles have the radius $AB/4$, the center of each lies on the line AB, one starts from the point A, the other from the point B, both end in the midpoint of the line AB where they are tangent to each other.

2.7. Carry through the plan devised in sect. 2.5(3); it should lead you to the same simple expression for D^2 in terms of A, B, and C that we have obtained in sect. 2.5(4) by other means.

2.8. Compare the approaches of sect. 2.5(3) and 2.5(4). (Emphasize general viewpoints.)

2.9. Find the volume V of a tetrahedron that has a trirectangular vertex O, being given the areas A, B, and C of the three faces meeting in O.

2.10. *An analogue to Heron's theorem.* Find the volume V of a tetrahedron that has a trirectangular vertex, being given the lengths a, b, and c of the sides of the face opposite the trirectangular vertex.

(If we introduce the quantity

$$S^2 = \frac{a^2 + b^2 + c^2}{2}$$

into the expression of V in an appropriate, symmetric way, the result assumes a form somewhat similar to Heron's formula.)

2.11. *Another analogue to Pythagoras' theorem.* Find the length d of the diagonal of a box (a rectangular parallelepiped) being given p, q, and r, the length, the width, and the height of the box.

2.12. *Still another analogue to Pythagoras' theorem.* Find the length d of the diagonal of a box, being given a, b, and c, the lengths of the diagonals of three faces having a corner in common.

2.13. *Another analogue to Heron's theorem.* Let V denote the volume of a tetrahedron, a, b, and c the lengths of the three sides of one of its faces, and assume that each edge of the tetrahedron is equal in length to the opposite edge. Express V in terms of a, b, and c.

2.14. Check the result of ex. 2.10 and that of ex. 2.13 by examining the degenerate case in which V vanishes.

2.15. Solve the puzzle proposed in sect. 2.7. (The sides x and $x/2$ should result from the cuts—but how can you fit a segment of length x into the cross?)

2.16. Fig. 2.9 shows a sheet of paper of peculiar shape: it is a rectangle with a rectangular hole. The sides of the outer rectangular boundary measure 9 and 12, those of the inner 1 and 8 units, respectively. Both rectangular boundaries have the same center and their corresponding sides are parallel. Cut this sheet along just two lines in two pieces that, fitted together, form a complete square.

(*a*) *Could you solve a part of the problem?* How long is the side of the desired square?

(*b*) *Take the problem as solved.* Imagine that the sheet is already cut into

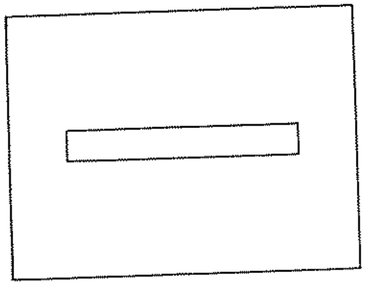

Fig. 2.9. By two cuts a square.

two pieces, the "left piece" and the "right piece." You keep the left piece where it is, and move the right piece into the desired position (where, with the other, it forms a complete square). Knowing the answer to (*a*), what kind of motion do you expect?

(*c*) *Guess a part of the solution.* The given sheet is symmetric with respect to its center and also with respect to two axes perpendicular to each other. Which kind of symmetry do you expect it to retain when cut along the two required lines?

Second Part

Some of the following examples are grouped according to subject matter which is hinted by an indication in front of the first example of the group (*Miscellaneous, Plane Geometry, Solid Geometry,* etc.) Some examples are followed by the name of Newton or Euler in parentheses; these are taken from the following sources, respectively:

Universal Arithmetick: or, a Treatise of Arithmetical Composition and Resolution. Written in Latin by Sir Isaac Newton. Translated by the late Mr. Ralphson. London, 1769. (Examples marked "After Newton" are derived from the same source, but some change is introduced into the formulation or into the numerical data.)

Elements of Algebra. By Leonard Euler. Translated from the French. London, 1797. (In fact, Euler's Algebra was originally written in German.)

Isaac Newton (1643–1727) is regarded by many as the greatest man of science who ever lived. His work encompasses the principles of mechanics, the theory of universal gravitation, the differential and integral calculus, theoretical and experimental optics, and several minor items each of which would be sufficient to secure him a place in the history of science. Leonard Euler (1707–1783) is also one of the very great; he left his traces on almost every branch of mathematics and on several branches of physics; he contributed more than anybody else to the development of the calculus discovered by Newton and Leibnitz. Observe that such great men did not find it beneath their dignity to explain and illustrate at length the application of equations to the solution of "word problems."

2.17. *Miscellaneous.* A mule and an ass were carrying burdens amounting to some hundred weight. The ass complained of his, and said to the mule: "I need only one hundred weight of your load, to make mine twice as heavy as yours." The mule answered: "Yes, but if you gave me a hundred weight of yours, I should be loaded three times as much as you would be."

How many hundred weight did each carry? (Euler)

2.18. When Mr. and Mrs. Smith took the airplane, they had together 94 pounds of baggage. He paid \$1.50 and she paid \$2 for excess weight. If Mr. Smith made the trip by himself with the combined baggage of both of them, he would have to pay \$13.50. How many pounds of baggage can one person take along without charge?

2.19. A father who has three sons leaves them 1600 crowns. The will precises, that the eldest shall have 200 crowns more than the second, and the second shall have 100 crowns more than the youngest. Required the share of each. (Euler)

2.20. A father leaves four sons, who share his property in the following manner:

The first takes the half of the fortune, minus 3000 livres.
The second takes the third, minus 1000 livres.
The third takes exactly the fourth of the property.
The fourth takes 600 livres and the fifth part of the property.
What was the whole fortune, and how much did each son receive? (Euler)

2.21. A father leaves at his death several children, who share his property in the following manner:

The first receives a hundred crowns and the tenth part of what remains.
The second receives two hundred crowns and the tenth part of what remains.
The third takes three hundred crowns and the tenth part of what remains.
The fourth takes four hundred crowns and the tenth part of what remains, and so on.

Now it is found at the end that the property has been divided equally among all the children. Required, how much it was, how many children there were, and how much each received. (Euler)

2.22. Three persons play together; in the first game, the first loses to each of the other two as much money as each of them has. In the next, the second person loses to each of the other two as much money as they have already. Lastly, in the third game, the first and second person gain each from the third as much money as they had before. They then leave off and find that they have all an equal sum, namely, 24 louis each. Required, with how much money each sat down to play. (Euler)

2.23. Three Workmen can do a Piece of Work in certain Times, viz. *A* once in 3 Weeks, *B* thrice in 8 Weeks, and *C* five Times in 12 Weeks. It is desired to know in what Time they can finish it jointly. (Newton)

2.24. The Forces of several Agents being given, to determine the Time wherein they will jointly perform a given Effect. (Newton)

2.25. One bought 40 Bushels of Wheat, 24 Bushels of Barley, and 20 Bushels of Oats together for 15 Pounds 12 Shillings.

Again, he bought of the same Grain 26 Bushels of Wheat, 30 Bushels of Barley, and 50 Bushels of Oats together for 16 Pounds.

And thirdly, he bought the like Kind of Grain, 24 Bushels of Wheat, 120 Bushels of Barley, and 100 Bushels of Oats together for 34 Pounds.

It is demanded at what Rate a Bushel of each of the Grains ought to be valued. (Newton)

2.26. (Continued) Generalize.

2.27. If 12 Oxen eat up $3\frac{1}{3}$ Acres of Pasture in 4 Weeks, and 21 Oxen eat up 10 Acres of like Pasture in 9 Weeks; to find how many Oxen will eat up 24 Acres in 18 Weeks. (Newton)

2.28. *An Egyptian problem.* We take a problem from the Rhind Papyrus which is the principal source of our knowledge of ancient Egyptian mathematics. In the original text, the problem is about hundred loaves of bread which should be divided between five people, but a major part of the condition is not expressed (or not clearly expressed); the solution is attained by "groping": by a guess, and a correction of the first guess.[6]

Here follows the Egyptian problem reduced to abstract form and modern terminology; the reader should go one step further and reduce it to equations: An arithmetic progression has five terms. The sum of all five terms equals 100, the sum of the three largest terms is seven times the sum of the two smallest terms. Find the progression.

2.29. A geometric progression has three terms. The sum of these terms is 19 and the sum of their squares is 133. Find the terms. (After Newton.)

2.30. A geometric progression has four terms. The sum of the two extreme terms is 13, the sum of the two middle terms is 4. Find the terms. (After Newton.)

2.31. Some merchants have a common stock of 8240 crowns; each contributes to it forty times as many crowns as there are partners; they gain with the whole sum as much per cent as there are partners; dividing the profit, it is found that, after each has received ten times as many crowns as there are partners, there remain 224 crowns. Required the number of partners. (Euler)

2.32. *Plane geometry.* Inside a square with side a there are five nonoverlapping circles with the same radius r. One circle is concentric with the square and touches the four other circles each of which touches two sides of the square (is pushed into a corner). Express r in terms of a.

2.33. *Newton on setting up equations in geometric problems.* If the Question be of an Isosceles Triangle inscribed in a Circle, whose Sides and Base are to be compared with the Diameter of the Circle, this may either be proposed of

[6] Cf. J. R. Newman, *The World of Mathematics*, vol. 1, pp. 173–174.

the Investigation of the Diameter from the given Sides and Base, or of the Investigation of the Basis from the given Sides and Diameter; or lastly, of the Investigation of the Sides from the given Base and Diameter; but however it be proposed, it will be reduced to an Equation by the same...Analysis. (Newton)

Let d, s, and b stand for the length of the diameter, that of the side, and that of the base, respectively (so that the three sides of the triangle are of length s, s, and b, respectively), and find an equation connecting d, s, and b which solves all three problems: the one in which d, the other in which b, and the third in which s is the unknown. (There are always two data.)

2.34. (Continued) Examine critically the equation obtained as answer to ex. 2.33. (*a*) Are all three problems equally easy? (*b*) The equation obtained yields positive values in the three cases mentioned (for d, b, and s, respectively) only under certain conditions: do these conditions correctly correspond to the geometric situation?

2.35. The four points G, H, V, and U are (in this order) the four corners of a quadrilateral. A surveyor wants to find the length $UV=x$. He knows the length $GH=l$ and measures the four angles

$$\angle GUH = \alpha, \qquad \angle HUV = \beta, \qquad \angle UVG = \gamma, \qquad \angle GVH = \delta.$$

Express x in terms of α, β, γ, δ, and l.

(Remember ex. 2.33 and follow Newton's advice: choose those "Data and Quaesita by which you think it is most easy for you to make out your Equation.")

2.36. The Area and the Perimeter of a right-angled Triangle being given, to find the Hypothenuse. (Newton)

2.37. Having given the Altitude, Base and Sum of the Sides, to find the Triangle. (Newton)

2.38. Having given the Sides of any Parallelogram and one of the Diagonals, to find the other Diagonal. (Newton)

2.39. The triangle with the sides a, a, and b is isosceles. Cut off from it two triangles, symmetric to each other with respect to the altitude perpendicular to the base b, so that the remaining symmetric pentagon is *equilateral*. Express the side x of the pentagon in terms of a and b.

(This problem was discussed by Leonardo of Pisa, called Fibonacci, with the numerical values $a = 10$ and $b = 12$.)

2.40. A hexagon is equilateral, its sides are all of the same length a. Three of its angles are right angles; they alternate with three obtuse angles. (If the hexagon is $ABCDEF$, the angles at the vertices A, C, and E are right angles, those at the vertices B, D, and F obtuse.) Find the area of the hexagon.

2.41. An equilateral triangle is inscribed in a larger equilateral triangle so that corresponding sides of the two triangles are perpendicular to each other.