Comprehensive School Mathematics Program

CSMP

PROBABILITY AND STATISTICS

a collection of papers on the teaching of probability and statistics in CSMP's elementary school curriculum

The material in this publication was prepared under a contract with the National Institute of Education, U.S. Department of Education. Its contents do not necessarily reflect the views or policy of the National Institute of Education or of any other agency of the United States Government.

Published October, 1982

Printed in the United States of America

CEMREL, Inc.

PROBABILITY AND STATISTICS

a collection of papers on the teaching
of probability and statistics in
CSMP's elementary school curriculum

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Preface

The Comprehensive School Mathematics Program (CSMP) is a complete mathematics curriculum for students of all ability levels, grades K-6. The program's goals of improving the effectiveness of mathematics instruction assume that students can learn and enjoy learning mathematics, not only standard arithmetic but also areas of mathematics not traditionally taught in the elementary school. To accomplish these goals, CSMP presents content as an extension of experiences that children have encountered in their development. Using a "pedagogy of situations," students are led through problem-solving experiences in an atmosphere of constant applications, for example, in stories or game-like settings. A feature unique to CSMP is the development of pictorial languages which foster student understanding of mathematical concepts and provide students the means to solve problems without burdensome terminology.

Topics in probability and statistics find a natural place in the CSMP curriculum. Students find the stories and games appealing and often relate them to everyday experiences. The development of innovative, pictorial techniques allows the analysis of probability and statistical situations to be a part of an elementary mathematics curriculum. The articles in this book summarize these activities and methods proven successful by the enthusiastic reception by CSMP students.

We extend our deepest gratitude to Frédérique Papy, former CSMP Associate Director for Research & Development, whose creativity and tireless efforts shaped the CSMP spirit and produced many of the ideas in this book. Our special thanks also go to Lennart Råde who brought his clever probability stories and innovative solution techniques from Sweden to St. Louis classrooms. We thank Burt Kaufman, former CSMP Director, and Clare Heidema, current CSMP Director, who suggested a need for this book and supported its development. Our thanks are especially due to the CSMP writers, typist Deborah Wriede, and artist Steven Sims, who survived the seemingly endless editorial changes.

We publish this book as a resource of ideas for classroom teachers and for educators responsible for mathematics teacher education. Our hope is that our experiences will enhance the role of probability and statistics in classrooms. We welcome hearing of your experiences.

August, 1982

Richard D. Armstrong
Pamela Pedersen

Introduction

Lennart Råde

A fundamental goal of education is to prepare children for life in a society, a society where mathematics is becoming increasingly important. Accordingly, one goal of mathematics teaching is to provide children the proper background for an understanding of the world around them. Both goals are strong reasons for including probability and statistics in a school's mathematics curriculum. These areas of knowledge are fundamental to the present-day modeling of our world in mathematical terms. Probabilistic and statistical methods are important tools in industry and in business, and such methods are essential in both physical and social sciences. It is also important for daily life in our society that people have some knowledge about the use and misuse of statistical reasoning. For instance, advertisements often use "statistical" reasoning in the form of graphs, tables, and verbal arguments in their attempts to influence consumers.

It is well documented that the study of combinatorics, probability, and statistics strongly motivates children by presenting the challenge and the intrinsic appeal of applications. Inclusion of these areas in the mathematics curriculum will further help to foster a positive attitude toward mathematics in elementary school children.

Probability theory is a very rich mathematical theory in close contact with many other parts of present-day mathematics. Also, probability theory employs many different mathematical tools. So with probability theory in the curriculum, students encounter and use a rich variety of mathematical tools and concepts. For example, already in elementary school they meet such basic mathematical concepts as sets, functions, and relations and use such basic mathematical tools as tables, graphs, codes, and abaci.

The Comprehensive School Mathematics Program (CSMP) has from its start been very interested in the possibilities of including probability and statistics in the elementary school mathematics curriculum. Lessons dealing with these areas appear in all parts of the CSMP curriculum and much effort has been used to investigate appropriate ways to introduce probability and statistics at the elementary school level. An internationally well-known indication of this interest is the book The Teaching of Probability and Statistics [1], which includes the proceedings of the CSMP international conference on teaching probability and statistics at the pre-college level. This conference was held in Carbondale, Illinois in March 1962. The participants of this conference adopted a number of recommendations of which the following may be quoted.

> The participants strongly endorse CSMP's efforts to introduce probability and statistics as subjects for study at elementary and secondary school levels. They believe that these subjects should be taught starting from a wealth of realistic examples. Some emphasis should be placed on their use as tools, both for the development of mathematical structures and in the building of applied models.
>
> In teaching probability, full advantage should be taken of practical experiments, and in particular of simulation methods. The knowledge acquired from such experiments should be directly reinforced by a theoretical framework; this should not be too rigid. In view of the different possible approaches to the subject, the formal concepts and theories presented should be eclectic. . . .
>
> Descriptive statistics of physical, biological, and social data are subjects of great importance to every citizen. They can be taught at almost every level. Material of this kind could serve as an introduction to a school course which might include further topics in statistical theory and inference. Such a course should be taught in careful coordination with probability theory and should make use of realistic data wherever possible. . . .

The CSMP work on curriculum development is based on some general pedagogical principles, which also have guided the work presented here. The following three tenets are basic to the CSMP view of mathematics teaching:

1) Mathematics should be taught as a unified whole.
2) Learning occurs best through interrelated experiences.
3) Children learn by reacting to problem-solving situations.

In the CSMP curriculum, the learning process is regarded as a spiral process where children learn by interacting with sequences of related situations.

The CSMP curriculum is published as a sequence of lessons in detailed Teacher's Guides, supplemented by colorful student workbooks and storybooks [2]. References to lessons described in this book are listed at the end of each article.

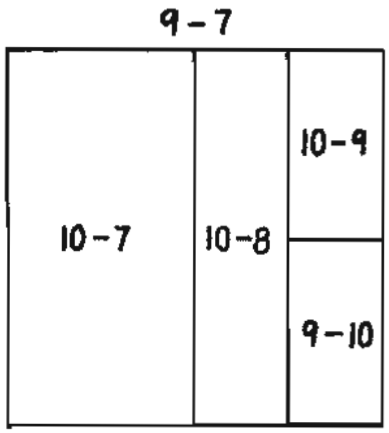
The papers in this book offer a selection of the ideas that CSMP has developed in its effort to effectively teach probability and statistics. The suggestions and the lessons are the results of many years of discussions and experimentation with various strategies. All of the ideas reflect classroom experiences.

In a mathematics curriculum, the goal of the earliest activities in probability and statistics should be to provide students with experiences involving fundamental concepts such as randomness, combinatorics, and the display of information. In their paper Probability and Statistics in Grades 1 to 3, Mark Driscoll and Richard Armstrong describe the stories and games in the CSMP curriculum that introduce these concepts. A key to maximizing the children's benefit from these experiences is to encourage student discussion about them. In these stories and games, teachers continually give students an opportunity to state their opinions, to consider the possibilities, to make predictions, and to discuss the results. Such interactive involvement prepares students for the probability and statistics situations encountered in the CSMP Intermediate Grades curriculum as described in other papers of this book.

In An Area Model for Solving Probability Problems, Richard Armstrong presents a very interesting method of solving probability problems. The method makes use of a graphical representation in which a square is divided into regions according to the probabilities present in the problem. This technique allows the solution of problems dealing with multi-stage random experiments in a very elegant and concrete way that avoids multiplication of fractions.

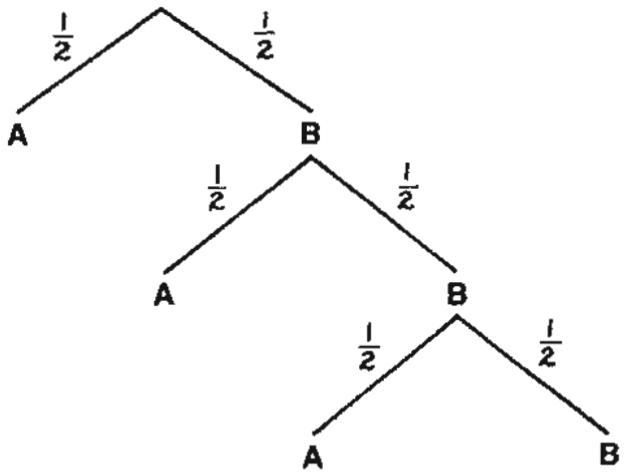
The paper includes solutions to some cases of the problem of points, a classical problem of probability theory that was in the focus of interest when the theory was developed by Pierre Fermat, Blaise Pascal, and other mathematicians during the 17th century. An example of this class of problems is to determine each

player's probability of winning a game to 10 points when player A has scored 9 and player B has scored 7. The following illustration shows the area method for attaining the solution.



It is seen from the graph that player A has probability $\frac{7}{8}$ of winning and that the corresponding probability for player B is $\frac{1}{8}$.

Usually this kind of problem is solved with the aid of tree diagrams, where probabilities are found by multiplying fractions along the branches. In this case, the following tree diagram would be used.

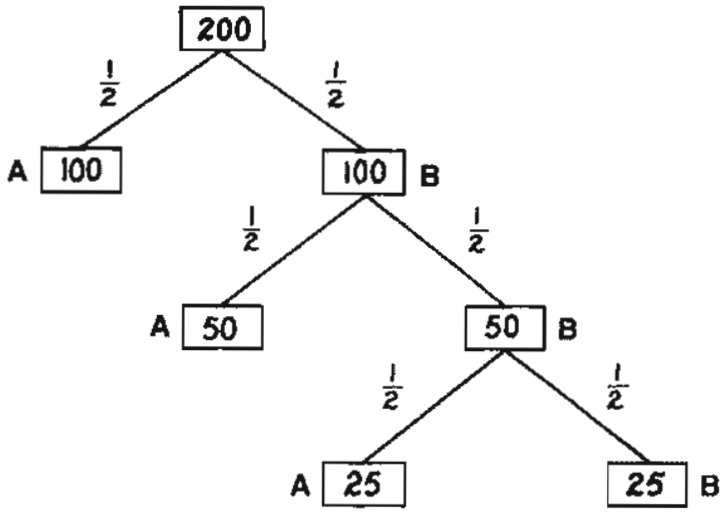


From the above diagram we calculate, with the aid of multiplication and addition rules, that player A wins with probability $\frac{7}{8}$.

$$\frac{1}{2} + \left(\frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) = \frac{7}{8}$$

This example clearly shows the merits of the area method compared to the tree diagram method. The latter method is, of course, very powerful and certainly should also be presented in introductory courses in probability. The paper Fair Games? in this book describes this method.

The study of different random games has always been important for the development and teaching of probability theory. The correspondence between Pierre Fermat and Blaise Pascal focused on problems concerning random games. In the paper Fair Games?, Jim Harpel discusses a sequence of such games. His paper describes how to use these games to introduce the method of using tree diagrams. Observe that such diagrams do not require the multiplication of fractions. For instance, in the problem of points discussed earlier, an alternative is to consider what is expected to occur in 200 trials. This approach leads to the following tree diagram, from which also it is seen that player A wins with probability $\frac{100 + 50 + 25}{200}$, or $\frac{7}{8}$.



The paper Whose Triangle Is It? by Richard Armstrong introduces a classic pattern of combinatorics, the Pascal Triangle, which incidentally is much older than Blaise Pascal (1623-1662), who used the triangle in connection with his treatment of the problems of points. Students are led to discover the pattern in connection with a challenging story, and then they use the Pascal Triangle to solve other probability problems.

In the paper Codes to Solve Problems, Pamela Pedersen presents three different situations from the CSMP curriculum. These situations lead to combinatoric and probabilistic problems that are solved with clever choices of appropriate codes or abaci, which are very efficient tools for these kind of problems. For students familiar with these tools from other parts of the CSMP curriculum, the codes and abaci allow them to solve quite complex combinatoric and probabilistic problems.

Probability theory has relations to many other fields of mathematics even, surprisingly, to geometry. As a matter of fact, random or stochastic geometry is an important area of present-day probability theory with numerous applications. A classic problem in this field is the problem of finding the probability that one may construct a triangle with the pieces formed when a stick is broken at random in three parts. In the paper Breaking a Stick: Probability Without Counting, Joel Schneider discusses how this problem is presented and solved in the CSMP curriculum. The treatment of this problem gives students a first contact with how probabilities may be calculated when the set of outcomes is a continuum.

In Shunda's Newsstand, Clare Heidema considers an advanced operations research problem. In the literature, the problem is usually called the "Newsboy's Problem" and is taken from inventory theory. Characteristic for this problem is that a decision about inventory is made once for the entire demand process. Every day Shunda has to decide how many newspapers to buy from her dealer. However, the demand is uncertain and the problem for Shunda is to order enough to realize full profit but not too much so as to avoid losses on the excess. Shunda uses a graph of daily demand to determine the most profitable inventory. This paper provides an excellent example on how a problem, which is usually discussed on a high mathematical level, can be made understandable already at the elementary school level. The paper shows how statistics as the art of making decisions when faced with uncertainty can be treated in a meaningful way at an elementary level.

One important field for applications of probabilistic and statistical methods is that of demography, that is, the statistical study of human populations especially with reference to size and distribution. CSMP has developed a sequence of

lessons dealing with this topic, which Tom Giambrone describes in the paper Population Growth. These lessons include work with population growth, the organization of population data in graphs and tables, and work with population pyramids. Also, students learn how to find the median age of a population. These lessons allow students to use hand-calculators in order to work with real population data.

There is a great growing interest today in the didactics of probability and statistics and especially so with regard to how these subjects should be introduced at the elementary school level. A sign of this interest is the 1982 NCTM Year Book [3], which is devoted to the teaching of probability and statistics at the school level. CSMP has pioneered work in this area. It is my hope that many will find the ideas and strategies described here useful and inspiring in future work in this important field.

References

[1] L. Råde (editor), The Teaching of Probability and Statistics, Almqvist & Wiksell, Förlag AB, Stockholm, Sweden 1970 Wiley Interscience Division, John Wiley & Sons, New York, London, Sydney

[2] Comprehensive School Mathematics Program, CEMREL, Inc., St. Louis, Missouri

[3] A. P. Shulte, J. R. Smart (editors), Teaching Statistics and Probability, National Council of Teachers of Mathematics, Reston, Virginia 1981

Probability and Statistics in Grades 1 to 3

Mark Driscoll
Richard Armstrong

Young children first encounter the notion of randomness in their everyday experiences. Games often involve spinners or dice. Their parents warn them, "Stop that, you might hurt yourself," or "It will probably rain this afternoon." CSMP extends these experiences by presenting appealing stories and games for students to consider. The appeal arises not only from the settings but also from the challenge to the students' intuition and problem-solving skills. To prepare students for the probability and statistics concepts they will encounter in CSMP's Intermediate Grades Program, CSMP's Primary Grade activities focus on three fundamental notions: randomness, combinatorics, and the display of information.

A key to understanding the concept of randomness is the role of the "unknown." Some facts are unknown simply because sufficient information is not available. For example, only after several clues can students determine a secret number that their teacher has selected. Other events are unknown because they are, by nature, random. Thus no one can consistently predict the result of rolling two dice. A significant insight occurs when children realize that, despite the randomness, experience or analysis may reveal the likelihood of possible outcomes.

A discussion of the likelihood of various outcomes when dealing with random devices such as dice, coins, spinners, or marbles leads naturally to combinatorial questions such as "How many different outcomes are possible?" CSMP students' early experiences with arithmetic problems with multiple solutions, for example, "Find pairs of numbers whose sum is 10," provide their first exposure to combinatorics. Their initial tendency in tackling these problems is to list solutions as they find them, usually unsystematically. Only gradually do they recognize the need for organizing a problem's solutions to guarantee that all of the possibilities have been found. In addition to being a rich source of

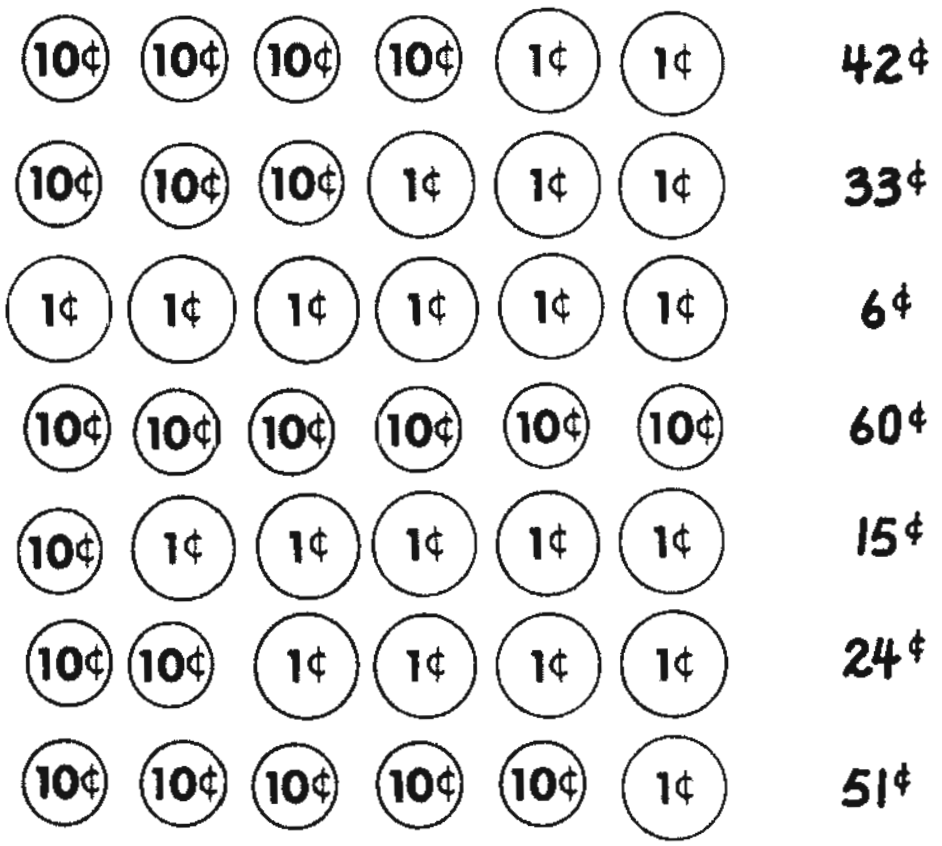
problems for investigation, such combinatorial situations prepare students to determine the probability of an event since often a first step in studying a probability problem is to consider all possible outcomes.

The systematic listing of solutions to combinatorial problems is an example of the third focus of the probability and statistics strand in CSMP, namely the development of efficient means of organizing information. In a variety of activities in the Primary Grades, students experience the value of using lists, tables, and graphs to record solutions, results of games, or data they have collected. They discover that organizing the information makes it easier to answer questions and to draw conclusions about the problem. These early experiences with tables and graphs prepare students to analyze many statistical and numerical problems in the Intermediate Grades.

The following three lessons provide a sampling of activities from the CSMP Primary Grades Program that illustrate the development of the three concepts outlined above.

A COIN PUZZLE - FIRST GRADE

Ms. Kavanaugh takes a paper cup from her desk drawer and shakes it. Her first-grade students hear coins jingling and try to guess the amount of money hidden in the cup. However, a few students remember similar activities they have done and say, "Don't tell us what is in the cup. Give us a clue." Ms. Kavanaugh carefully states her first clue, "I have exactly six coins. Each coin is either a dime or a penny." Martha suggests four dimes and two pennies and the whole class helps her count the amount of money that would be: "10¢, 20¢, 30¢, 40¢, 41¢, 42¢." On the board, Ms. Kavanaugh records the five combinations her students find. To maintain the pace of the lesson, she provides the other two possibilities herself.



Several students again prefer guessing the amount of money in the cup. Tommy seems convinced it's his suggestion, 6¢. Then a second clue is provided: "There are <u>at least</u> two pennies in the cup."

"It could still be 6¢."
"Cross out 51¢, there's only one penny."
Shortly the class agrees that only the combinations for 51¢ and 60¢ can be erased.

"Here is my last clue," continues Ms. Kavanaugh. "There are more dimes than pennies in this cup. If you know the amount of money in this cup, whisper it to me."

Most students whisper the correct answer, 42¢, and then Ms. Kavanaugh lets them see the four dimes and two pennies in the cup.

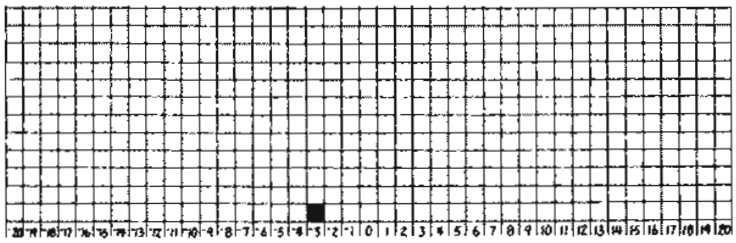
The discussion in this lesson highlights the distinction between the uncertainty of "It could be 6¢" and the deduction "It must be 42¢." This experience with an undetermined event is part of the students' preparation for encountering random events.

The situation in this lesson exemplifies the students' initial experiences with combinatorial problems in the first grade. The emphasis is on finding many solutions, not necessarily all solutions. The list simply records solutions; it need not be organized systematically to suggest missing solutions.

TEMPERATURE BAR GRAPHS - SECOND GRADE

On the first day of school after Christmas vacation, Mr. Warren shows his class a new Celsius thermometer. The students' curiosity and questions initiate a discussion about temperature, thermometers, above zero and below zero readings, and Celsius and Fahrenheit scales. Mr. Warren passes the thermometer around the class, and everyone confirms that the indoor temperature is 23°C. He then pins a duty roster and the following poster to the bulletin board.

TEMPERATURE CHART



°Celsius

"I'll hang the thermometer outside the classroom window," explains Mr. Warren. "Each school day morning, the assigned student will read the thermometer and record the reading on this bar graph. I've already shown that this morning's outside temperature was $^{-}3^{\circ}$ Celsius."

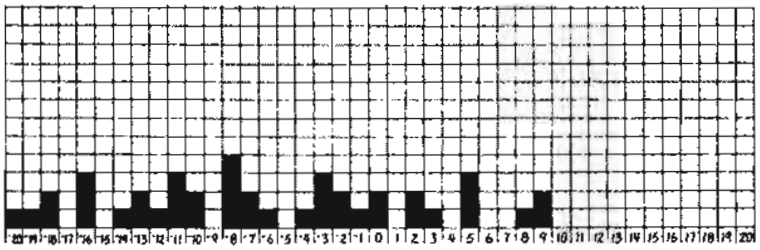
The students read and record the outside morning temperature every school day in January and February. Often Mr. Warren briefly asks questions about the day's temperature and the bar graph, for example:

- "Before Nguyen reads today's temperature, who wants to guess what it might be?"
- "What do you predict tomorrow's temperature will be?"
- "How much colder (or warmer) is today than yesterday?"
- "How often have we previously matched today's temperature?"
- "What has been our coldest temperature? our warmest temperature?"
- "Which is warmer, 2°C or ⁻10°C? How much warmer?"

After two months of recording temperatures, Mr. Warren suggests that they compare their data to the temperatures for St. Louis and New Orleans. Each child receives a listing of the two cities' temperatures for January and February and uses that data to draw a bar graph for each city:

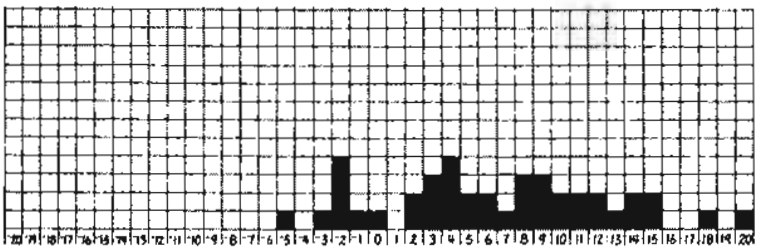
TEMPERATURE CHART

Our School



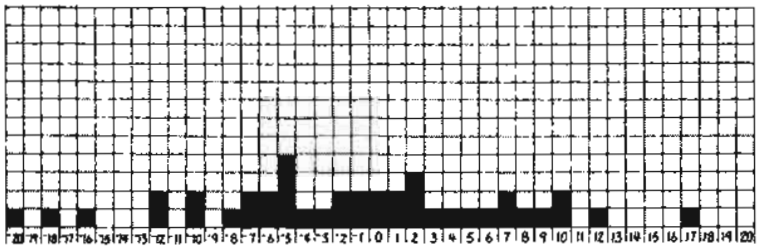
°Celsius

New Orleans



°Celsius

St. Louis



°Celsius

Referring to the three bar graphs, the students respond to Mr. Warren's questions:

- "Which city had the warmest weather in January and February? the coldest?"
- "What was the highest temperature for each city? the lowest temperature?"
- "Which city had the greatest variation in temperatures?"
- "What was the most common temperature in each city?"

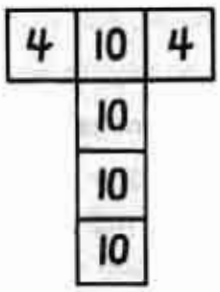
As the bell rings, the students insist on drawing a new graph for March and April and on continuing to record the morning temperature each day. They seem curious to observe the gradual warming as winter changes to spring.

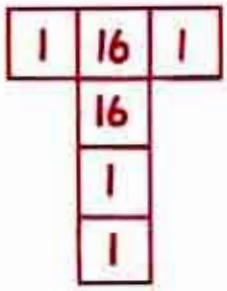
This two-month activity exposes second-grade students to several benefits of graphing data. First, the graph imposes an organized presentation of the data, ordering the temperatures from lowest to highest. Also, the bar graph provides a very strong visual overview of the data; for example, it is clear that New Orleans has a warmer winter than St. Louis. These two features — the orderliness and the visual impact — allow students to answer questions about the data and to draw conclusions much more readily than if the data were in a table or in a list.

Other second-grade lessons continue the development of the themes of randomness and of combinatorics. For example, two lessons concern the rolling of two dice. By rolling the dice many times and drawing a bar graph of the sum of the two dice, students conclude that some sums (e.g., 6, 7, and 8) occur more frequently than other sums (e.g., 2, 3, 11, and 12). Hence students experience that even though the outcome is random, certain events are more likely than other events.

Ms. Schneider shows her class a red cube and a blue cube that she has cut and folded from cardboard.

Each cube has a number on each of its six faces. Unfolding each cube, Ms. Schneider shows the shape that she made each cube from. She then draws the labeled shapes on the board.





"Suppose," says Ms. Schneider, "that we toss the blue cube three times and add the numbers that appear on top. What sum could we get?"

A student suggests 30. Several classmates agree, pointing out that 10 could be rolled three times. This combination is recorded on the board:

BLUE $10 + 10 + 10 = 30$

The students then offer the other possibilities for the blue cube.

$$10 + 10 + 4 = 24$$
$$10 + 4 + 4 = 18$$
$$4 + 4 + 4 = 12$$

Joanne confidently concludes, "That's all of the possibilities for the blue cube because with three rolls you will roll "4" no times, once, twice, or three times." Similarly, the students find the possibilities for the red cube.

RED $16 + 16 + 16 = 48$

$16 + 16 + 1 = 33$

$16 + 1 + 1 = 18$

$1 + 1 + 1 = 3$

Ms. Schneider then suggests playing a two-person game with the two cubes, "One player rolls the red cube three times; the other player rolls the blue cube three times. The player with the highest sum for the three rolls wins. Which cube would you rather play with?"

Some students prefer the red cube:

> "Two of the red cube's sums, 48 and 33, are higher than any of the blue cube's sums."
> "You can get 16's with the red cube."

Others prefer the blue cube:

> "The blue cube has four 10's while the red cube has four 1's."
> "You might get a sum of 3 on the red cube and lose for sure."
> "You won't roll many 16's with the red cube."
> "It's easier to get a "30" on the blue cube than it is to get a "48" on the red cube."

After the lengthy discussion, no consensus is reached and Ms. Schneider selects two students to play the game. The student with the red cube wins; the score is 33 to 24. Other pairs of students play the game, one at a time. The

players with the blue cube win 3 out of 5 games. As the students continue to play the game, Ms. Schneider often interrupts a game to ask some questions about the situation, for example,

- "The first two rolls of the blue cube are 10 and 4. What could this player's score be after three rolls?" (Answer: 18 or 24)
- "The player with the red cube has a total score of 3. Can he win?" (Answer: No)
- "The player with the red cube scores 18. The blue player's first roll is a 10. Can she still win? lose? tie?" (Answer: She can't lose. She will either tie or win.)
- "The score of the player with the blue cube is 30. The first two rolls of the red cube are 16 and 1. Which player is more likely to win?" (Answer: The player with the blue cube. The other player needs a 16, but there are more 1's than 16's on the red cube.)

At the end of the lesson, the players with the red cube have won 17 out of the 30 games. This evidence convinces many students to prefer the red cube; a few students remain undecided or still prefer the blue cube. Ms. Schneider, realizing that a deeper analysis of probability is more appropriate for a later lesson, brings the discussion to an end.

This lesson illustrates an application of combinatorics and provides a setting for an intuitive discussion of probabilistic questions. Third-grade activities place an emphasis on finding all of the possibilities in combinatorial situations. As demonstrated in this lesson, an organized list aids in reaching the conclusion that no combinations have been missed.

The teacher's questions about what could happen motivate the discussion of possible outcomes versus impossible outcomes and of likely events versus unlikely events. Through these discussions, students learn that random events are not entirely chaotic. An analysis of the random device, the cube in this lesson, yields information on the likelihood of certain events.

SUMMARY

A primary goal of the probability and statistics activities in Grades 1-3 is to provide a variety of experiences involving randomness, combinatorics, and organizing information. By developing these three topics as described in the above lessons, CSMP prepares its students for the more sophisticated problems in their Intermediate Grades Program described in the other articles of this book.

* * *

In the CSMP curriculum, the activities described in this paper appear in the following lessons:

CSMP Mathematics for the First Grade, Part II, Lesson S73.2

CSMP Mathematics for the Upper Primary Grades, Part I, Lesson L15

CSMP Mathematics for the Upper Primary Grades, Part II, Lessons L6, L12, L14

CSMP Mathematics for the Upper Primary Grades, Part III, Lesson L12

Fair Games?

Jim Harpel

Many mathematical problems either do not interest children or cannot be presented in ways that are accurate and yet accessible for elementary-school students. Fortunately, probability provides exceptions to these limitations. Paralleling the historical role of games in the development of the theory of probability, the lessons summarized in this article focus on games involving coins and marbles. The games are not only enjoyable, but also easy to understand.

Based on their experiences and due to the apparent simplicity of the games, students have considerable trust in their intuition as they consider the fairness of the games. They feel that they understand the situations and therefore confidently make predictions about the expected outcomes. Yet in probabilistic situations, the intuition can often be fooled. Paradoxes abound in probability. A key question in curriculum development is to determine an appropriate role for paradoxes. Handled carelessly, paradoxes can destroy the students' trust in intuition and convince them that probability is inscrutable. Rather, the pedagogical goals of using paradoxes should be to intrigue students with situations having surprising results and to refine each student's intuition to encompass these results.

To achieve these goals, CSMP employs a three-step procedure for presenting paradoxical situations: prediction, experimentation, and analysis. Once a game is explained, the prediction step allows students to express their opinions based on their intuition. The predictions force discussion and clearly stated commitments which set the stage for revealing the paradox.

In the second step, experimentation, students use dice, coins, spinners, or other random devices to test their predictions by actually playing the game many times. The conflict between the predictions and the experimental results serves

to dramatically pose the paradox. A strongly felt need has been created within the students — the discrepancy between predictions and results cries for an explanation. This need motivates the third step in the process: mathematical modeling and analysis of the situation.

The need to analyze a probability problem often becomes a roadblock which ultimately precludes the study of probability in the early grades. Admittedly, most traditional analyses of probability problems are too complex for elementary-school students. The papers in this book illustrate several techniques used in the CSMP curriculum that are appropriate for these students. In particular, the activities in this article illustrate the use of pictorial methods and probability trees to accurately model the problems and to appeal to students. The analyses tend to confirm the experimental results and often reveal the source of any discrepancy between those results and the students' predictions. The paradox within the game situation has motivated the students to proceed through steps of prediction, experimentation, and analysis. The active personal involvement with the story provides a basis for refining the student's intuition with regard to probability situations.

To captivate the students' interest, the paradoxes occur in stories about the protagonist Bruce, a boy who invents games to play with his friends. The games appear fair but usually favor Bruce.

SAME OR DIFFERENT?

"Two children, Alice and Bruce, are responsible for washing the dinner dishes. In order that they both not have to wash and dry each night, they decide that some method be used to select randomly who will wash and dry the dinner dishes. Bruce suggests that 2 black marbles and 1 white marble be used. Alice will mix the marbles in her hands behind her back and draw two of the marbles without looking. What could Alice draw?"

"She could draw two black marbles, or she could draw a black and a white marble."

"Yes, Alice could draw marbles of the same color or of different colors. Bruce will try to predict what Alice has drawn. If Bruce correctly predicts what Alice has drawn, Alice must wash the dishes. If he is wrong, he must wash the dishes. Is this a fair way to decide who washes the dishes?"

With two black marbles and one white marble, the students sense that the game is unfair but they don't reach a concensus on who is favored. The students discuss this issue for a few minutes. They insist on playing the game. Two volunteers play the game 10 times and record the outcomes on the board.

Different: 𝍸 |

Same: ||||

"Can we tell from these 10 trials if this game is fair?"

This result convinces some students that "different" is favored. Other students are uncertain as to the fairness of the game since the results are so close to 5-5. The teacher suggests that the students pair off and that each pair plays the game 10 times and records the outcomes. In this classroom, there are 15 pairs of students.

"150 games will be played. If this is a fair game, in about how many of those games do you think you will choose marbles of different colors?"

"If the game is fair, 'same' and 'different' will each come up about one half of the time — about 75 times apiece."

As each pair of students completes 10 games, the results are recorded and totaled. The grand total is:

Different: 103

Same: 47

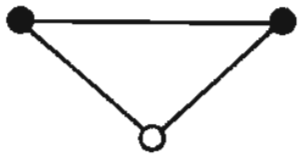
"Do you think this is a fair game? Should Alice play this game with Bruce to decide who will wash the dishes?"

"No! The game appears to favor Bruce. He could always guess 'different' and usually win."

"Let's find out if that's really so; here are the three marbles."

"What pairs of marbles could Alice choose?"

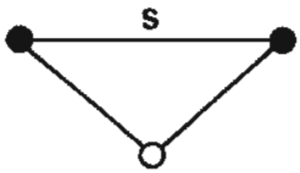
Students draw cords to indicate the pairs of marbles that Alice could select.



"Altogether there are three possible ways to draw two marbles. In how many ways could we get a pair of marbles of the same color?"

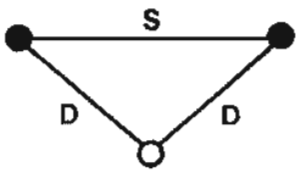
"Only by drawing the two black marbles."

"Therefore we have only one chance out of three of getting marbles of the same color."

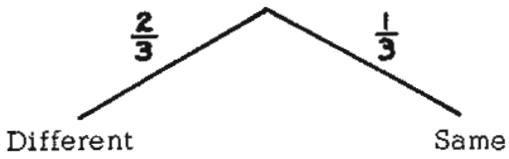


"What about marbles of different colors?"

"There are two ways to get marbles of different colors. So there are two chances out of three of drawing marbles of different colors."



The teacher draws a probability tree to summarize the information.



"Do you see what this means? When we play Bruce's game, we are more likely to get marbles of different colors than we are to get marbles of the same color. If we play the game many times, we can expect that about two thirds of the time we will get marbles of different colors and about one third of the time we will get marbles of the same color."

"So what result could we have expected in the 150 games we just played? About how many times could we have expected to get marbles of the same color?"

"We should have gotten marbles of the same color about 50 times, because $\frac{1}{3} \times 150 = 50$, and marbles of different colors about 100 times, because $\frac{2}{3} \times 150 = 100$."

"How does that compare with what actually happened?"

"103 to 47 is close to 100 to 50."

"Is Bruce's game fair?"

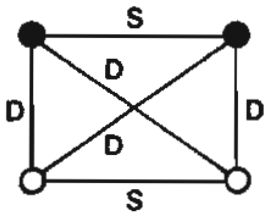
"No, he's very likely to win."

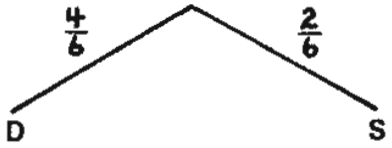
The students express little surprise that Bruce's game is unfair. They strongly doubted that a game with two black marbles and one white marble would be fair. Now the stage is properly set for a paradox.

"If Alice discovers Bruce's game to be unfair, she could refuse to play with Bruce or she could suggest altering the game to make it fair. What changes could we make so that the game is fair?"

Nearly all of the students suggest adding another white marble so that there are two white marbles and two black marbles. A few students express the opinion that any game with equal numbers of white marbles and black marbles should be fair.

"Let's look at the game with two white marbles and two black marbles. Rather than play this new game 150 times, we'll analyze it."



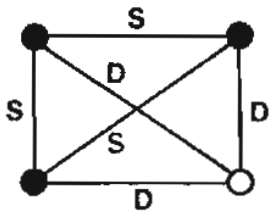


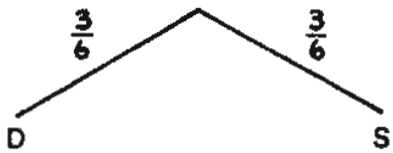
Much to their surprise, the students notice that this game also is not fair. In fact, it has the same probabilities as the original game Bruce proposed.

"Neither of these games are fair, but there <u>are</u> fair games with the same rules but with different numbers of white marbles and black marbles. Try to find a fair game."

Individually, students test various combinations of marbles. They find several games that are almost fair, and a few students find a fair game.

"Use one white marble and three black marbles! There are three out of six chances to select 'same' and three out of six chances to select 'different.'"





The students play this game 150 times and record that they draw marbles of the same color 71 times and of different colors 79 times. These results tend to confirm the analysis; certainly the game seems much more fair than the original game. There are other fair games with two colors of marbles, but the number of

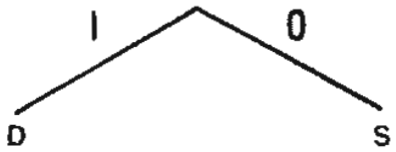
marbles involved increases quickly.†

Other variations of Bruce's game can be analyzed. What happens if more black marbles are added? If more white marbles are added? If a third color marble is introduced?

The following question motivates another version of the game.

"What happens to Bruce's game if only one white marble and one black marble are used?"

"You will only get 'different' every time you select a pair of marbles. You have one chance out of one of drawing two marbles of different colors and no chance of drawing marbles of the same color."

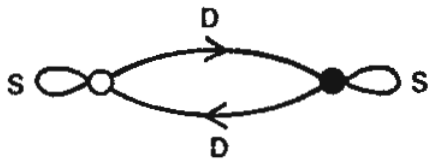


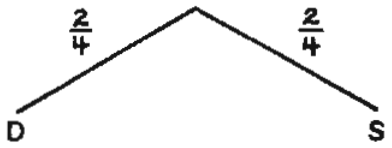
"To make a more interesting game, let's change a rule. What if we keep one white marble and one black marble but we draw one marble then replace it and draw again?"

With this replacement rule, the order in which the marbles are drawn is important. The method of analysis must be adapted to take into account the outcomes white-white and black-black and the order of the draw. The drawing of loops provides for the white-white and black-black outcomes.

† To find additional fair games is an excellent, challenging project for students. The increased complexity of the cord diagrams requires the development of new techniques for counting occurrences of "same" and "different." Fortunately the diagrams themselves suggest the needed rules. The "next" fair game involves 3 white marbles and 6 black marbles. An algebraic analysis reveals that the game is fair if and only if the number of white marbles and the number of black marbles are two consecutive triangular numbers.

But now the cord represents two distinct outcomes: "white then black" and "black then white." To represent this, replace the cord by two arrows because an arrow indicates the order of the draw.



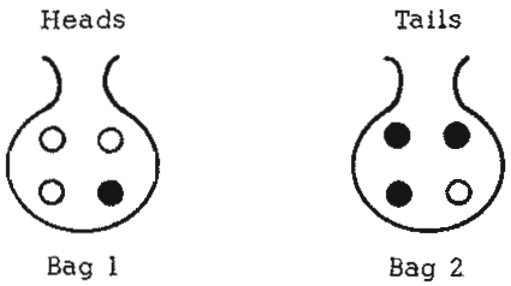


The shift from drawing without replacement to drawing with replacement yields a fair game. In fact, any "same-different" game with replacement and with equal numbers of white marbles and black marbles is fair. This result partially justifies any intuitive feelings based on symmetry that students might have had originally about the situation.

TWO-STAGE PROBABILITY GAMES

The "Same or Different?" lessons and the use of trees to solve combinatorics problems prepares students to consider multi-stage probability situations. Once again, Bruce provides the intriguing games.

"Abby and Charles are neighborhood friends of Bruce. One day, Bruce puts three white marbles and one black marble in a bag. In a second bag, he puts three black marbles and one white marble. Bruce's game is to flip a coin. If 'heads' comes up, Abby picks two marbles from the first bag. If 'tails' comes up, she picks two marbles from the second bag."



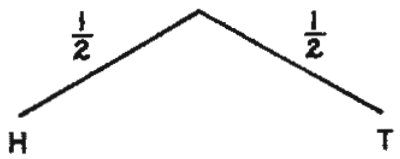
"If 0 black marbles are drawn, Abby wins.
If 1 black marble is drawn, Bruce wins.
If 2 black marbles are drawn, Charles wins."

"Abby and Charles are always suspicious of their friend's games, so they wonder whether or not it is a fair game. Do you think Bruce has invented a fair game?"

The students spend several minutes discussing the game. Some students suggest that the game is fair because there are three possible outcomes and each child has one chance to win. Others are suspicious of the game because it is possible to draw two black marbles from only one of the bags while one black marble may be drawn from either bag. The disagreement provides a need to analyze this game.

"What is the first step or stage of this game?"

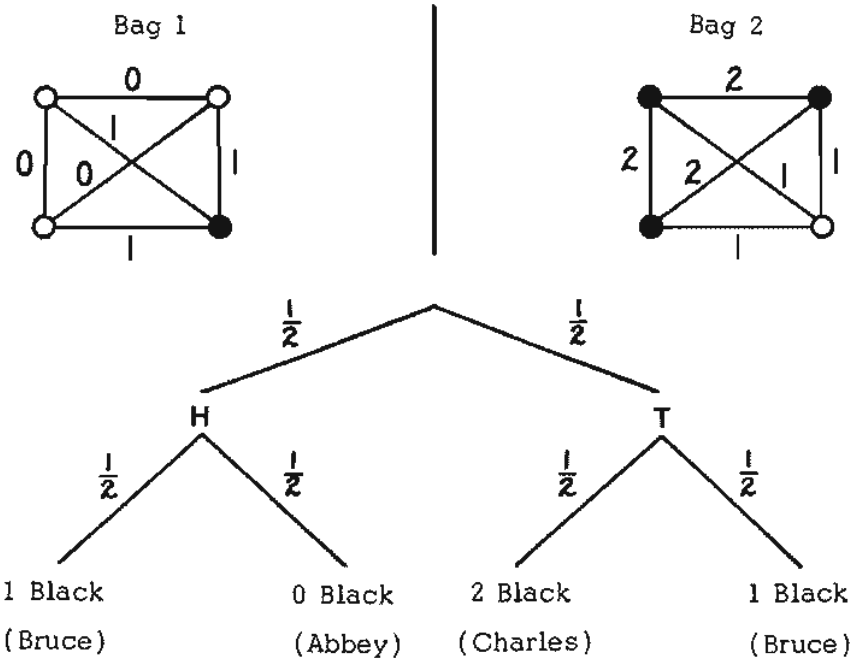
"Flipping a coin — you get either 'heads' or 'tails'."



"Yes. What happens next?"

"Marbles are drawn from either Bag 1 or Bag 2."

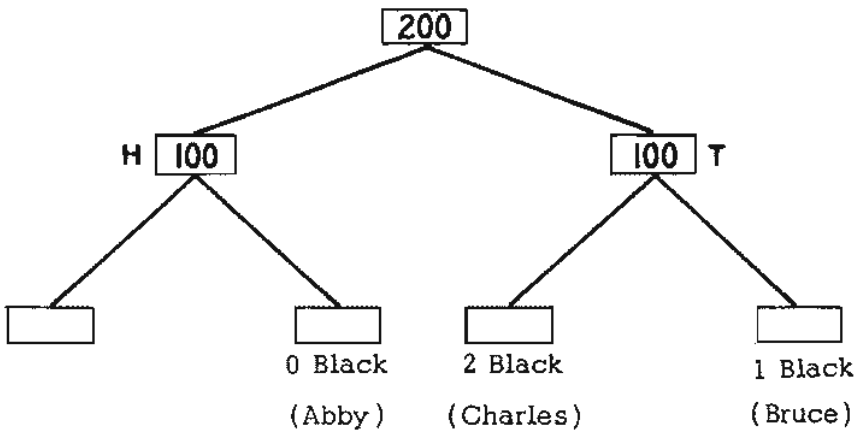
Several students recognize the similarity of this stage with previous work and suggest using cord pictures to analyze the results. The labels on the cords indicate the number of black marbles chosen.



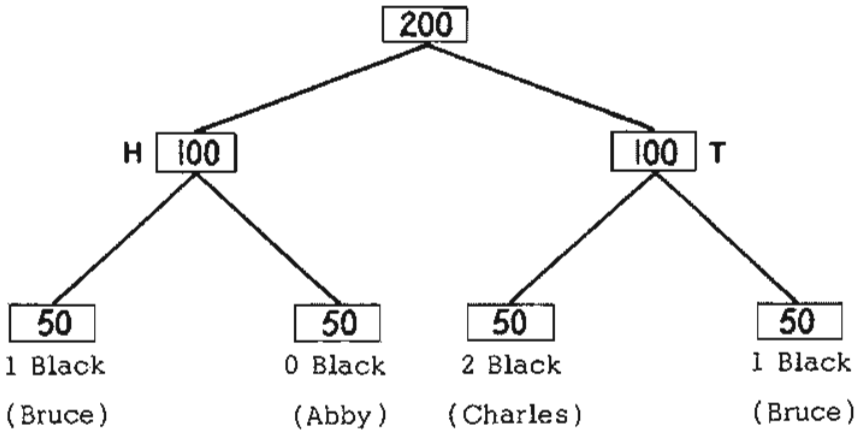
The tree representation suggests that Bruce is favored as only he can win in two ways. Since the product rule has not yet been introduced, other methods must be used to quantify the situation.

"Suppose that the three children play the game 200 times. What would we expect to happen? About how many times do we expect to get 'heads'? About how many times do we expect to get 'tails'?

"About 100 times each because $\frac{1}{2} \times 200 = 100$."



"Since $\frac{3}{6} = \frac{1}{2}$ and $\frac{1}{2} \times 100 = 50$, each outcome should occur about 50 times."



"Now we can decide how heavily Bruce's game favors himself. About how many games out of 200 would we expect Abby to win?"

"Abby should win about 50 games."

"What is Abby's probability of winning?"

"$\frac{1}{4}$; her chances are 50 out of 200."

Similar questioning determines that Bruce's probability of winning is $\frac{1}{2}$ and Charles' probability of winning is $\frac{1}{4}$. The intuitive approach of "let's pretend to play 200 games" allows students to calculate these probabilities without recourse to the multiplication and addition of fractions.

"The game is not fair. Bruce has the best chance of winning."

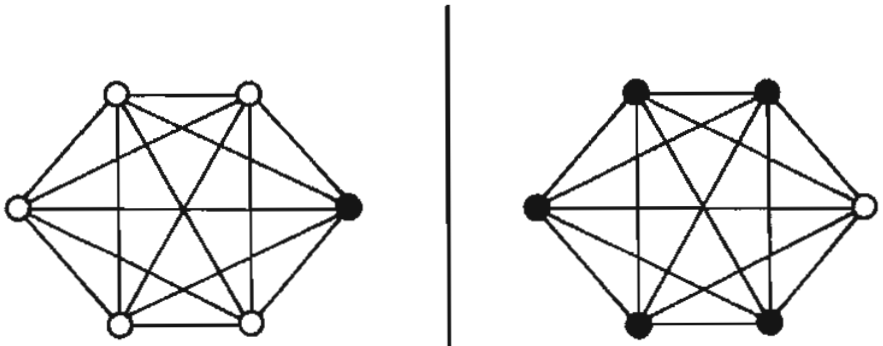
"Yes; that is what Abby and Charles concluded too, and they were not very happy with Bruce's game. Could we modify this game so that it would be a fair game?"

Some students suggest that the composition of marbles in the bags does not need to be changed to get a fair game.

"Whenever 'tails' followed by a draw of one black marble occurs, we just start the game over again. If this game were played 200 times, Abby would win about

50 times, Bruce would win about 50 times, Charles would win about 50 times, and the game would have to be started over about 50 times."

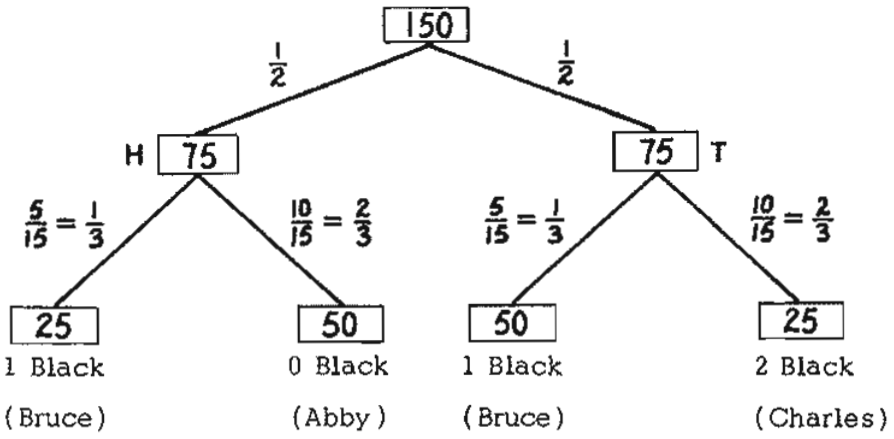
Other students discover that by adding two white marbles to the first bag and two black marbles to the second bag, a fair game results. Analysis of this situation verifies that the new game is fair.



0 Black: 10 cords
1 Black: 5 cords

1 Black: 5 cords
2 Black: 10 cords

Again, only Bruce can win regardless of which bag is chosen. Thus some students still suspect Bruce is favored. Only by constructing a probability tree and considering play of 150 games are the "hold-outs" persuaded.



Even though only Bruce has two ways to win, the game is fair:

Abby: 50 games; $\frac{50}{150} = \frac{1}{3}$

Bruce: 25 + 25 games; $\frac{50}{150} = \frac{1}{3}$

Charles: 50 games; $\frac{50}{150} = \frac{1}{3}$

SUMMARY

The immediate goals of these activities are to provide students with appealing probability problems that they are eager to understand and to develop the tools needed to analyze the problems. The paradoxes in Bruce's games usually lead the students to disagreeing predictions and experimental data. These discrepancies intrigue the students and thereby create a need for a deeper understanding of the problem. The analyses, based on dot and cord pictures and tree diagrams, provide visual means for explaining the paradoxes. After revealing the source of a paradox, the challenge is to use the same analytical tools together with trial and error to find modifications that produce a fair game.

The mathematical goal of these activities is to introduce tree diagrams as a means for determining probabilities. Tree diagrams are a powerful tool for analyzing probability problems because they explicitly present all of the random events within a situation in their logical order, and they offer strong visual support for the appropriate multiplication and addition of probabilities. Experiences with tree diagrams lead directly to the basic algebraic rules for combining probabilities.

This paper demonstrates a way to introduce tree diagrams to elementary school students; a key is to avoid any need to add or multiply fractions. Instead, students consider what "should" happen if a situation, for example, a game, is repeated a large number of times. Running, for example, 200 games through a probability tree determines each player's expected number of wins and thus his/her probability of winning. This technique, along with prediction and experimentation, serves to develop intuition with regard to probabilistic situations.

* * *

In the CSMP curriculum, the activities in this paper appear in the fourth-grade lessons from the Probability and Statistics strand.

Codes to Solve Problems

Pamela Pedersen

Situations involving equally likely outcomes provide a good place to begin studying probability, a place accessible to students at the intermediate grade levels. In problems involving a finite number of equally likely outcomes, the measure of the probability that a particular event will occur is simply the ratio of the number of favorable outcomes to the number of possible outcomes. To measure the probability of a particular event occurring in such situations, one needs to count:

a) the elements in the outcome set, and

b) the elements in a particular subset (event) of the outcome set.

In these situations probability questions reduce quickly to combinatorics questions, probability providing an appealing context in which to develop combinatoric techniques.

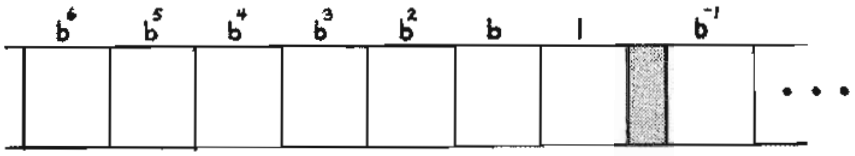
This paper describes three probability situations from the Comprehensive School Mathematics Program (CSMP), each situation involving a set of equally likely outcomes. To solve the problems posed, counting techniques are used that fit the interests and experiences of students in the intermediate grades. Each of the solutions involves a mathematical model of the situation in which the counting of outcomes is readily achieved, the necessary correspondence between the situation and the model being accomplished by a code.

In two of the three situations, the code sets up a one-to-one correspondence between the possible outcomes and configurations on base abaci — "pencil and paper" schematics upon which convenient number base systems are imposed. To understand these solutions, students need to have many prior experiences with various base abaci, gradually building confidence that for every number there is

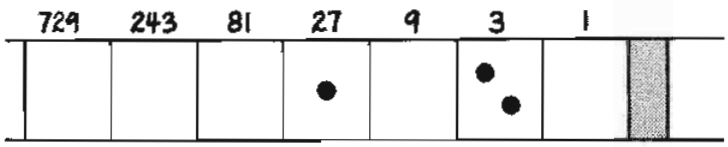
exactly one standard configuration on any given abacus, and that every configuration on an abacus represents exactly one number.

The third situation employs a rectilinear grid system as a coding device. The grid system provides a strong visual aid that makes clear how to apply the standard product rule for combinatorics in the context of this problem.

Before considering the three situations, let us look briefly at base abaci.



A checker on an abacus assumes the value of the board on which it is placed. The number represented by a configuration of checkers on an abacus is the sum of the values of the checkers. For example,

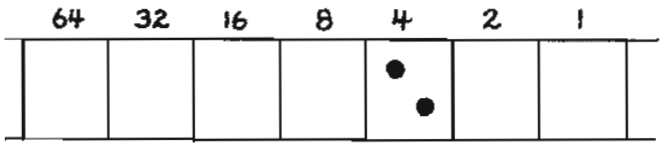


represents the decimal number 33 (27 + 3 + 3) on a Base Three abacus.

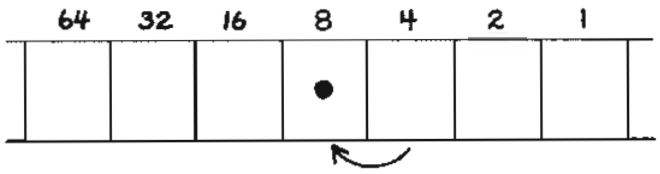
For each abacus, there is a rule governing the valid trading of checkers. If b is the base number, the rule of the Base b abacus is:

> b checkers on any board of the abacus represent the same number as one checker on the next board to the left.

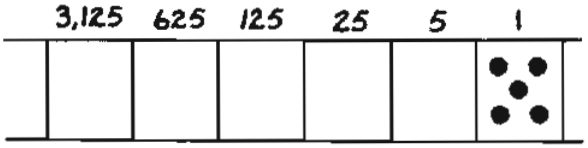
For example, on the Binary abacus two checkers on a board . . .



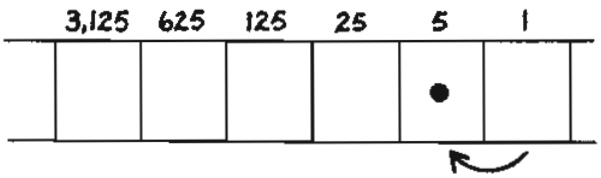
. . . can be traded for one checker on the next board to the left and vice versa.



On the Base Five abacus five checkers on a board . . .

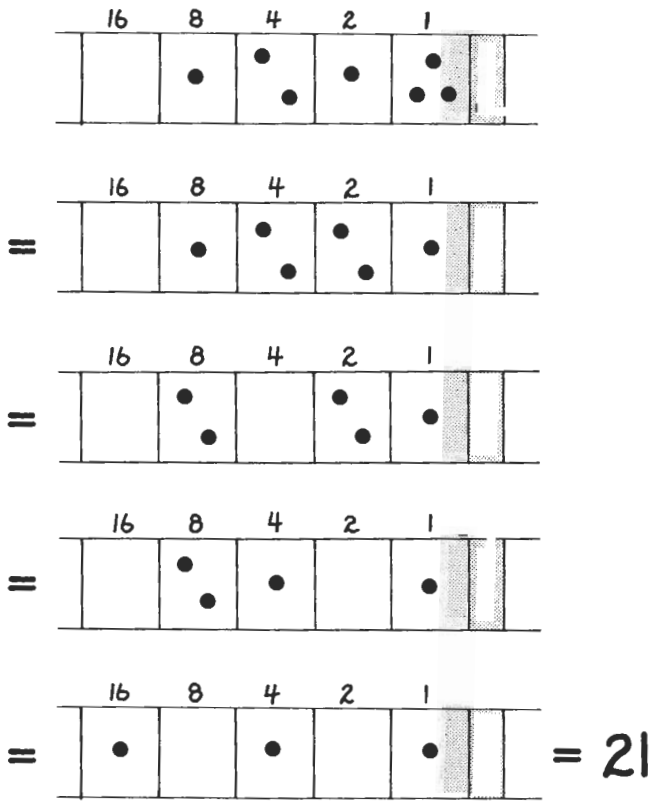


. . . can be traded for one checker on the next board to the left and vice versa.



The standard or usual configuration for a number is the configuration that uses the fewest number of checkers to represent it. By making trades, we can always start with a configuration for a number and arrive at its standard configuration.

For example, the following sequence demonstrates a series of trades on a Binary abacus for simplifying a non-standard configuration for 21 to the standard configuration.



It is clear we can put any number n on a base abacus, for we can simply put n checkers on the ones' board. Furthermore, if we make all of the possible trades, we will arrive at one and only one configuration for n, namely its standard configuration on the given abacus.

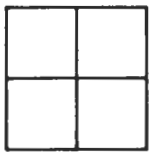
The following three situations are representative of the CSMP philosophy and approach to mathematics, in particular to combinatorics.

RANDOM ART

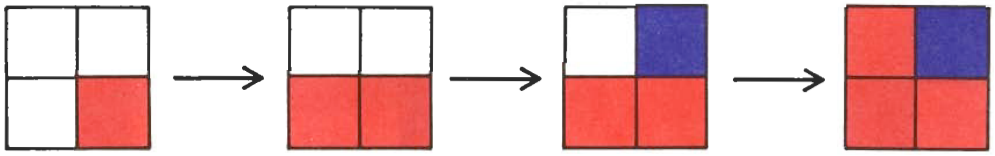
One of Nabu's † interests is painting. He does not paint portraits or landscapes; he paints pictures with red and blue squares, randomly selecting the color for

† Nabu is a fictional character appearing in several CSMP lessons.

each square. To decide the color of each square, Nabu first outlines the picture:

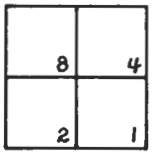


Then for each small square he takes a red marble and a blue marble in his hands and shakes them. He puts them behind his back and brings one marble forward. The color of the marble determines the color of the square. He continues in this way until all four squares are painted.



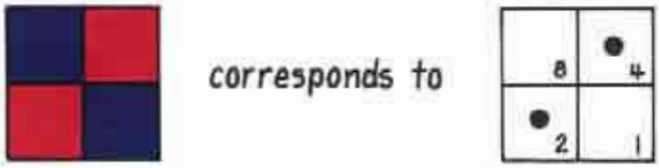
How many different pictures with four squares could Nabu paint?

Students might suggest drawing all of the pictures, but they would need a systematic way of accounting for all possibilities and of finding duplicates. One method that will do both involves imposing a Binary abacus on the picture. We use a Binary abacus rather than an abacus for a different base because there are only two possibilities for each square — either Nabu colors it red or he colors it blue. Since there are four squares in the picture, we need only consider the first four boards of the abacus.



We can set up a correspondence between the paintings and the configurations on these four boards of the Binary abacus in this manner:

- Having a square colored red is equivalent to having a checker on the corresponding board of the Binary abacus.
- Having a square colored blue is equivalent to not having any checkers on the corresponding board of the Binary abacus.

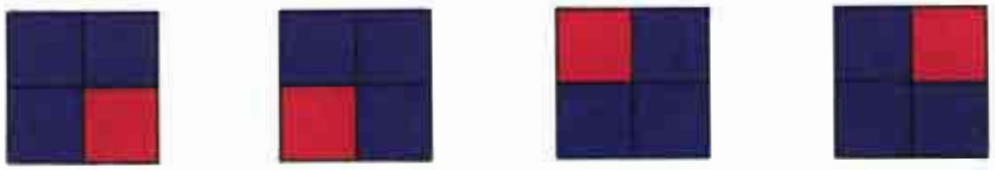


The abacus provides a way of assigning a number to each painting. The code number for a painting is the decimal number represented by the corresponding configuration on the Binary abacus. For example,

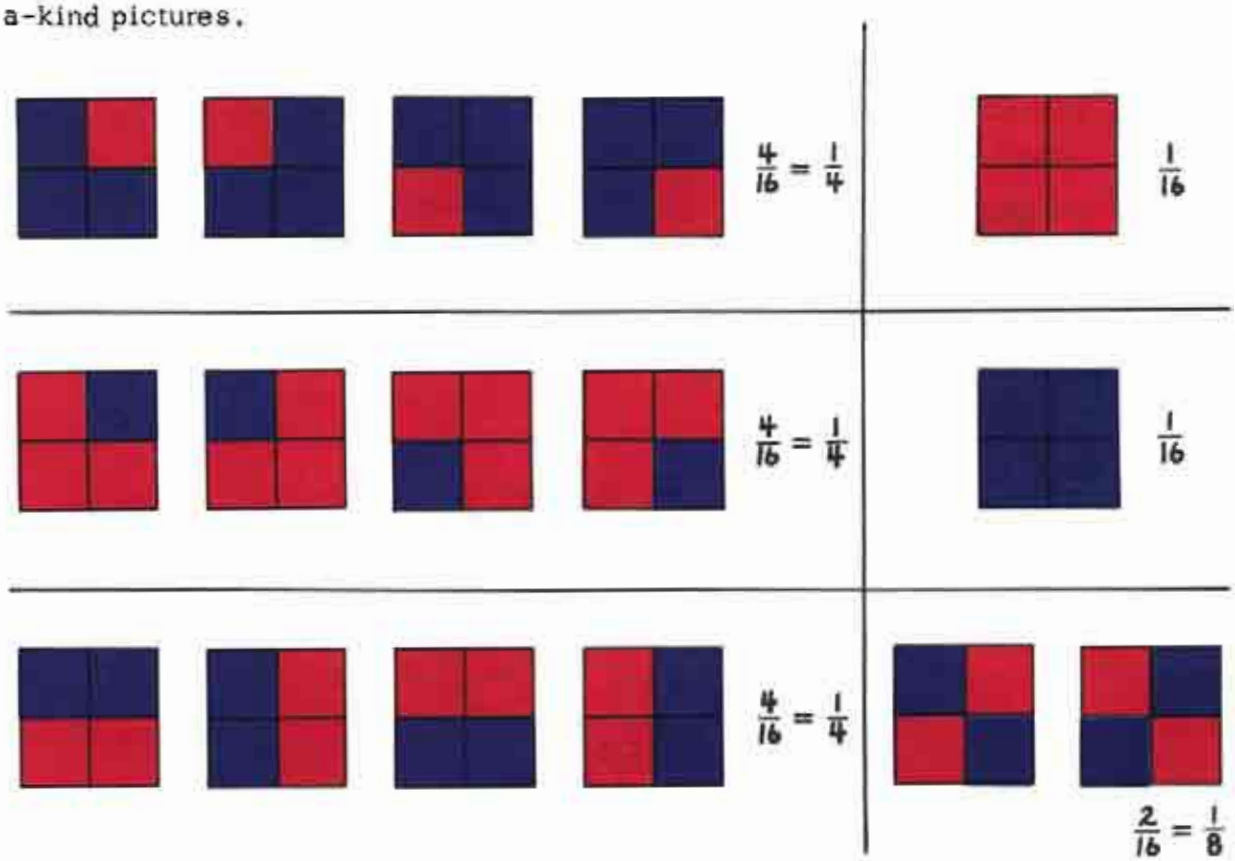
For each painting there is a unique number, and for certain numbers there is a painting. Which numbers are they? The smallest is 0, assigned to the picture with four blue squares; the largest is 15, assigned to the picture with four red squares.

So there are at most sixteen (0 through 15) possible paintings. To be convinced that all sixteen are possible, we could actually do the coloring for each of the numbers 0 through 15; since there are few in number, this is a realistic task. But in fact students are convinced already, being familiar with abaci from previous activities.

There are sixteen possible paintings, but some of them are essentially the same. For example,

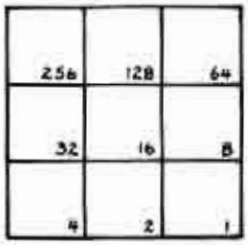


. . . by rotating any one of these four paintings, we can get the other three. Using rotations to partition the set, we get three subsets of four-of-a-kind pictures, two subsets of one-of-a-kind pictures, and one subset of two-of-a-kind pictures.



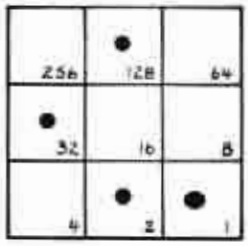
Each of the 16 pictures is equally likely to be painted by Nabu because of the "one red-one blue marble" method used to select the colors. So to find the probability that Nabu will paint a picture from any one of these subsets, we can take the ratio of the number of elements in a subset to the number of elements in the set. The probabilities provide a means for measuring the rarity of Nabu's various pictures.

Similar methods could be used for painting with nine squares. Again Nabu paints each square red or blue. There are many more paintings that are possible, as expected. To count them we can use the Binary abacus similarly to the way we used it for the four-square picture. This time we use nine boards of the abacus.



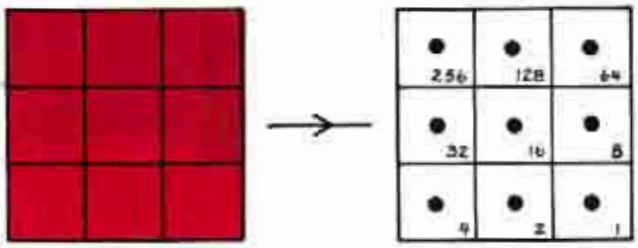
The correspondence between colorings and configurations of these nine boards of the abacus is set up as before; the code numbers are assigned in the same way. For example,

corresponds to



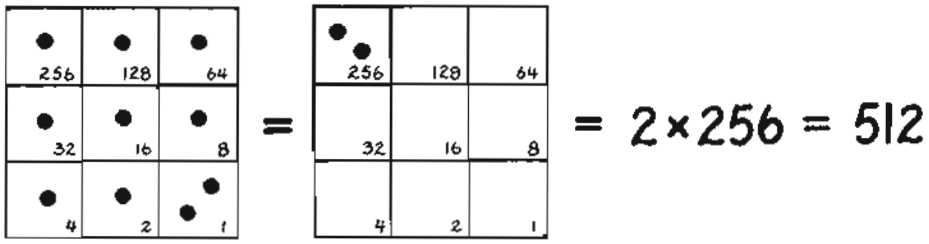
= 163

The smallest code number is 0, corresponding to the picture with nine blue squares. The largest code number corresponds to the picture with all red squares.



To find out which number is on the abacus, we could add the values of the nine checkers. But there is a more clever way!

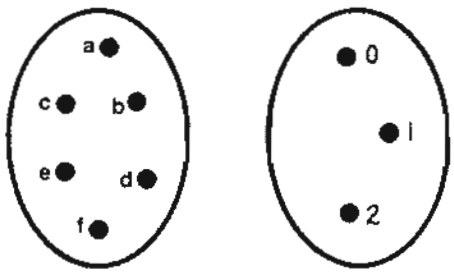
Place an extra checker on the ones' board. The extra checker on the ones' board sets off a chain of trades (1 + 1 = 2; 2 + 2 = 4; 4 + 4 = 8; . . .) by the "two for one" rule of the Binary abacus. The final result is two checkers on the 256-square.



Since an extra checker was added, the largest code number is 511 (512 - 1). Each code number from 0 to 511 represents a different painting. Therefore Nabu can draw 512 different pictures. The one-to-one correspondence between the numbers 0 to 511 and Nabu's paintings need not be proven in any formal sense; previous activities with base abaci build credibility for this correspondence.

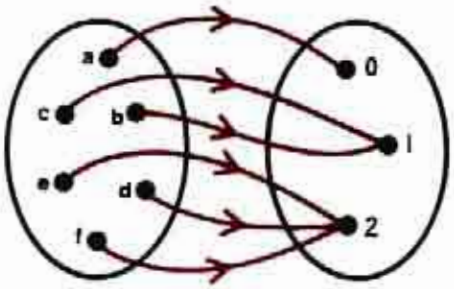
SPIES AND BRIDGES

This is the story of a spy named "Boris." Boris has six helpers whose code names are "a", "b", "c", "d", "e", and "f". Each day Boris's job is to assign each helper to observe one of three bridges. We call the bridges "0", "1", and "2".

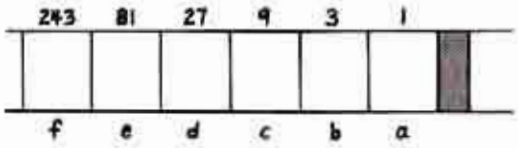


Boris assigns each spy to observe exactly one bridge. He might make the assignment so that all bridges are covered, or just one or two. He uses an arrow picture to record how the helpers are assigned to the bridges.

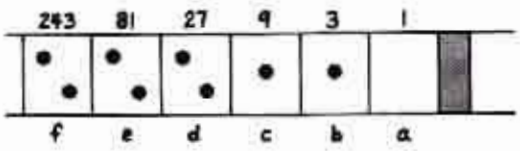
One day Boris makes this assignment:



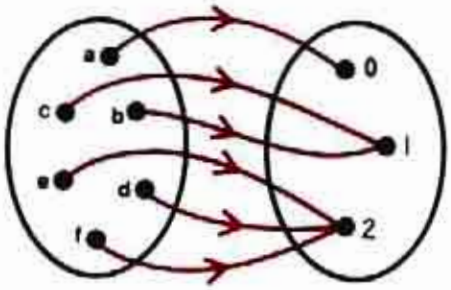
Each day Boris transmits the assignment to headquarters. He must send a secret message, but the arrow picture is certainly not very secret. Since there are three bridges, Boris decides that he could use the Base Three abacus to produce a secret code number for each assignment and lets each of six boards of the abacus be for one of the spies.



The number of checkers (zero, one, or two) on a spy's board indicates to which bridge a spy is assigned. For example,



Corresponds to

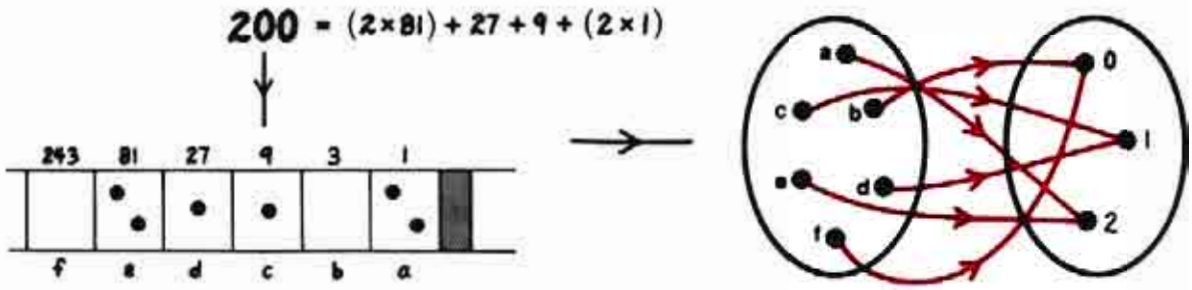


The code number for the assignment is the decimal number represented by the corresponding configuration of checkers on the Base Three abacus. The code number for the preceding arrow picture is 714:

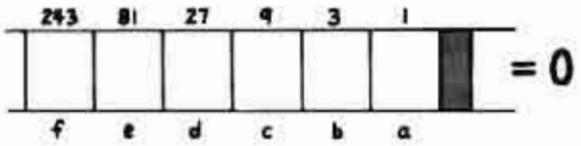
$$(2 \times 243) + (2 \times 81) + (2 \times 27) + 9 + 3 = 714$$

Therefore, instead of sending a picture of the assignment to headquarters, Boris, in this case, sends the message "714 code 3." When received, headquarters knows to put 714 on the Base Three abacus to determine Boris assignment of spies to bridges.

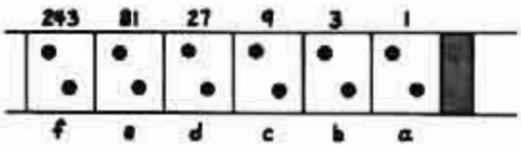
The following diagram indicates how headquarters would decode the message "200 code 3."



For each assignment there is a unique number since the assignment indicates the number of checkers to place on each board of the abacus. For certain numbers there is an assignment. Which numbers could they be? The smallest is clearly 0, corresponding to all six spies watching Bridge 0.

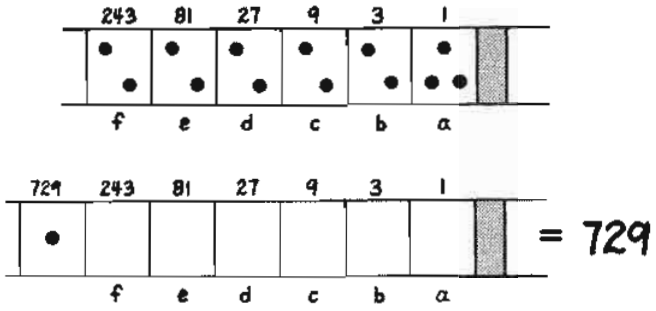


The largest number corresponds to each spy watching Bridge 2.



To find out which number this is, one could add the values of the twelve checkers. But is there a more clever way? Repeat the trick used to determine the largest number on the three-by-three Binary abacus.

Place an extra checker on the ones' board. The extra checker on the ones' board sets off a chain of trades by the "three for one" rule of the Base Three abacus. The final result is a single checker on the next board to the left of the original six.



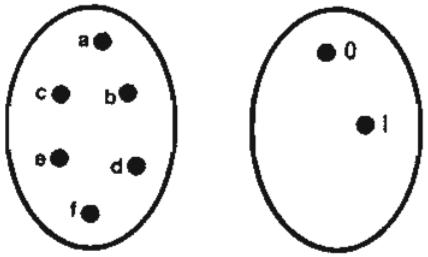
Since an extra checker was added, the largest possible code number is 728 (729 - 1). We conclude that there are at most 729 possible assignments (remember that 0 is a possible code word). In fact, each whole number between 0 and 728 represents a unique assignment, so there are exactly 729 possible assignments. This one-to-one correspondence between the numbers 0 through 728 and the assignments that Boris can make does not need to be shown formally; prior activities with abaci build credibility for the correspondence.

Suppose that one day the enemy plans to blow up Bridge 2 and that Boris, not knowing this, assigns the spies randomly to the bridges. What is the probability that Bridge 2 will be covered by at least one spy?

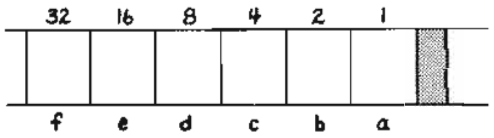
It is because the spies are randomly assigned to bridges that we have a set of equally likely outcomes. There are many ways for Boris to make random assignments; for example, he could use a spinner circle divided into three congruent parts, a set of random six-digit numbers from the set {0, 1, 2}, or three identical marbles labeled "0", "1", and "2".

To answer the question posed, we need to compare the number of possible assignments with the number of possible assignments in which Bridge 2 is covered. We have already found the former to be 729. Let us now proceed to find the latter indirectly by calculating the number of assignments in which Bridge 2 is <u>not</u> covered.

Consider any assignment of spies to bridges in which Bridge 2 is not covered. Then each of the six spies would be assigned either to Bridge 0 or to Bridge 1.



Since for each spy there are only two possibilities to consider, whether a spy is assigned to watch Bridge 0 or Bridge 1, a Binary abacus rather than a Base Three abacus can be used for the counting.



We can argue similarly to the way we did in the case of three bridges. If a spy is assigned to Bridge 0, the corresponding board on the abacus is empty; if a spy is assigned to Bridge 1, one checker is placed on the corresponding board of the abacus. The smallest code number is 0 (all six spies are assigned to Bridge 0). The largest code number is $2^6 - 1$, or 63 (all six spies are assigned to Bridge 1). There is a one-to-one correspondence between the numbers 0 through 63 and the possible assignments to Bridges 0 and 1. We conclude that there are sixty-four possible assignments of six spies to two bridges. The probability that none of the spies will be assigned to Bridge 2 is about 0.09:

$$\frac{2^6}{3^6} = \frac{64}{729} \approx 0.09$$

Therefore the probability that Bridge 2 <u>is</u> being watched is about 0.91:

$$\frac{729 - 64}{729} = \frac{665}{729} \approx 0.91$$

Going one step further, we could ask: What is the probability that all three bridges will be covered if Boris randomly assigns the six spies to bridges?

To find this probability, we need to compare the number of possible assignments in which all three bridges are covered to the total number of possible assignments. We have already found the latter to be 729. To find the former, we can count the possible assignments in which at least one of the bridges is not covered and subtract this number from the number of all possible assignments, 729.

But we have already done most of the work! We have found that the number of possible assignments in which Bridge 2 is not covered is 64. By repeating the argument, there are 64 possible assignments in which Bridge 0 is not covered, and there are 64 possible assignments in which Bridge 1 is not covered.

At first glance it might appear that there are 3×64 possible assignments in which at least one bridge is not covered, but we must not overlook that in counting both the possible assignments in which Bridge 2 is not covered and the possible assignments in which Bridge 0 is not covered, we have twice counted the single assignment of all six spies to Bridge 1. Likewise, we have counted the assignment of all six spies to Bridge 0 twice and the assignment of all six spies to Bridge 2 twice. Therefore, the number of assignments in which at least one bridge is <u>not</u> covered is 3 less than 3×64.

$$(3 \times 64) - 3 = 189$$

The number of assignments in which all three bridges are covered is 540:

$$729 - 189 = 540$$

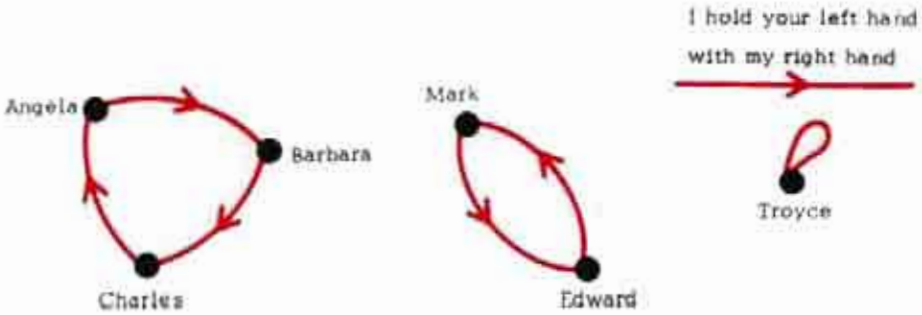
If Boris randomly assigns the bridges to six spies, the probability that all three bridges will be covered is about 0.74:

$$\frac{540}{729} \approx 0.74$$

Note that learning Boris' code for sending messages and counting the number of assignments are worthwhile and interesting combinatorics problems in themselves In the CSMP curriculum, only the possible assignments are counted. Finding the probability that Bridge 2 is not covered and finding the probability that all three bridges are covered are natural extensions of the material and would be appropriate for students in grades 7-9 and possibly as early as grade 6.

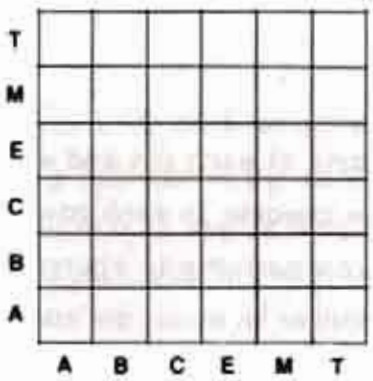
HOW MANY PERMUTATIONS?

Angela, Barbara, Charles, Edward, Mark, and Troyce each randomly select a piece of paper with one of their six names written on it. Unfolding the paper and reading it, each person with his/her right hand takes the left hand of the person named. We use an arrow picture to record the situation that results.

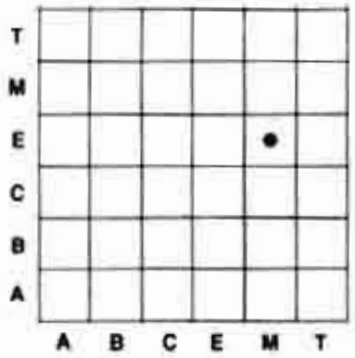


Notice that in such a picture, exactly one arrow starts at each dot and exactly one arrow ends at each dot.

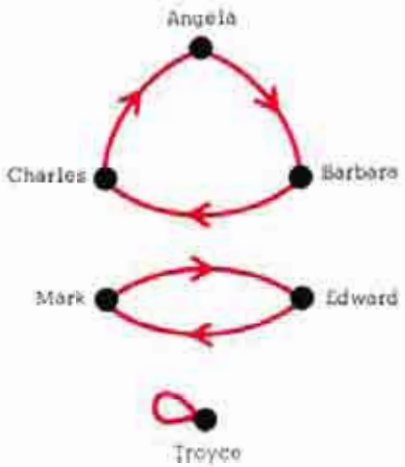
How many different situations could result from six people doing this activity? To answer this question, we will use a grid to count the corresponding arrow pictures. For simplicity, we'll refer to the six persons by their first initials: "A", "B", "C", "E", "M", and "T".

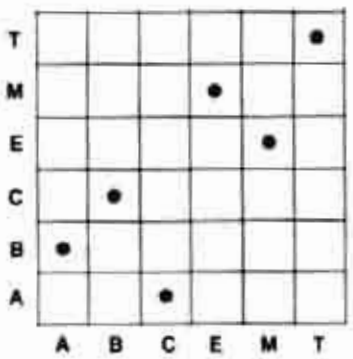


We'll represent arrows on this grid by placing checkers appropriately. For example, there is an arrow from Mark to Edward. We put a checker in the square where the column for M meets the row for E.



Representing arrows in this way, the original arrow picture corresponds to the configuration of checkers to the right of it.





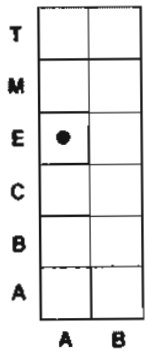
Because exactly one arrow starts at each dot and exactly one arrow ends at each dot, there is exactly one checker in each row and in each column. Counting the number of different arrow pictures is equivalent to counting the number of ways to put exactly one checker in each row and in each column of the grid.

How many choices are there for putting a checker in column A?

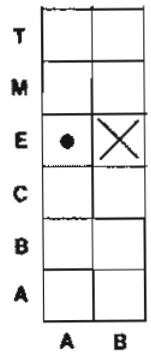
Six.

Suppose we put the checker in row E.

How many choices are there for putting a checker in column B?

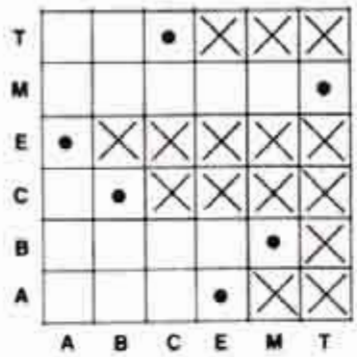


Only five because the checker in column B cannot be put in row E since there already is a checker in that row; in other words, two children cannot get Edward's name.



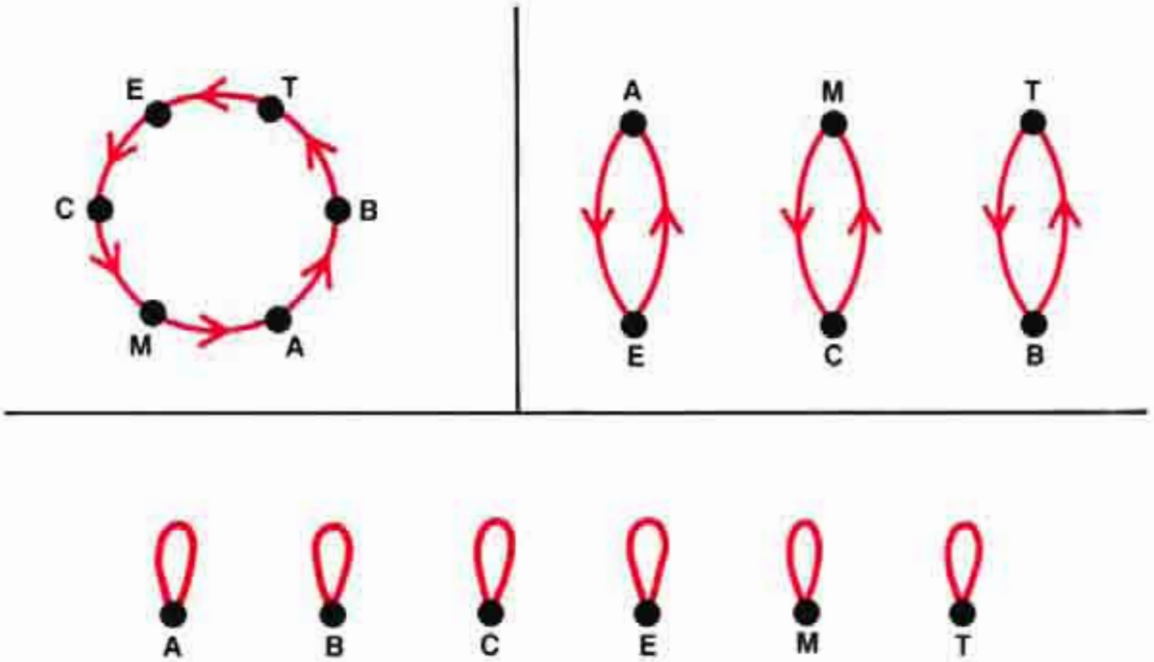
We make use of the product rule for combinations here: namely, if there are six possibilities for putting a checker in the first column and there are five possibilities for putting a checker in the second column, there are 6 x 5 (30) possibilities for assigning checkers to the first two columns.

We continue in this manner until all six columns have been considered. Each time, the number of options for a checker is reduced by one; that is, there are four choices for column C, three choices for column E, and so on.



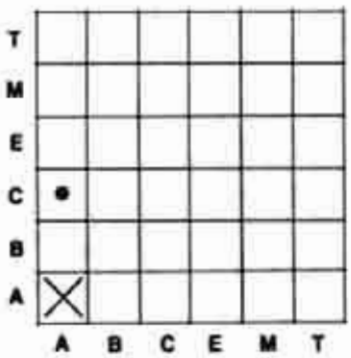
Using the product rule, we find that there are 720 (6 x 5 x 4 x 3 x 2 x 1) different ways to place six checkers on a grid with exactly one checker in each row and in each column. Therefore, there are 720 different arrow pictures and, returning to the original problem, 720 different ways of assigning the six children to hold hands.

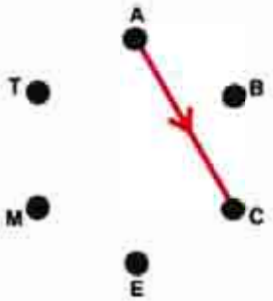
The 720 pictures fall into natural categories involving the number of cycles. Such an arrow picture can have from one to six cycles.



The consideration of cycles leads to a probability question: if the pieces of paper with children's names are given out randomly, what is the probability of getting one cycle — a connected arrow picture? To find out how many of the 720 arrow pictures are connected, we'll determine how to locate six checkers on the grid in such a way that the corresponding arrow picture is connected. Each time, before we place a checker, we'll count the number of choices for that checker.

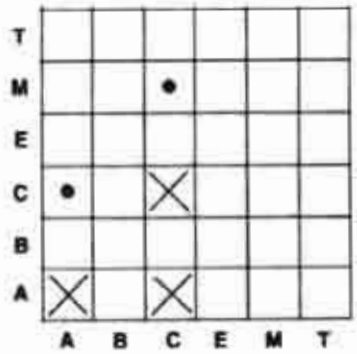
Let's start with A. Where could we place a checker in column A? Anywhere except the first row. We cannot place the checker in row A because we would have a loop at A and eventually more than one piece. We have five squares to choose from. Suppose we choose row C and draw an arrow from A to C, that is, Angela gets Charles' name.

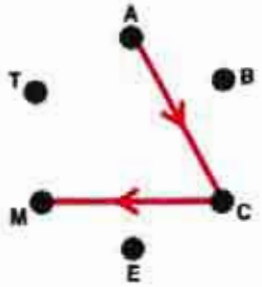




It would seem natural to consider column B next, but that choice leads to later complications in the argument. Since the first arrow ends at C, we consider next the arrow starting at C. Where could we place a checker in column C? We cannot place it in row C because there would be a loop at C. Also we cannot place it in row A because there would be a two-cycle between A and C. We have

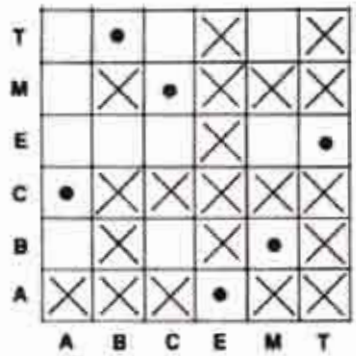
four squares to choose from. Suppose we choose row M and draw an arrow from Charles to Mark.

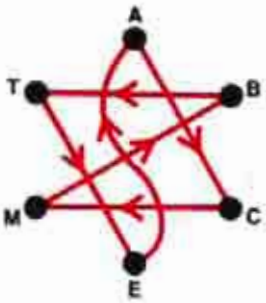




There are five choices for column A and four choices for column C. Using the product rule again, there are 20 (5 x 4) choices for the two columns A and C. Having just drawn an arrow from C to M, we would next consider column M.

We continue in this manner for the remaining four columns. Each time the number of options for drawing an arrow is reduced by one.



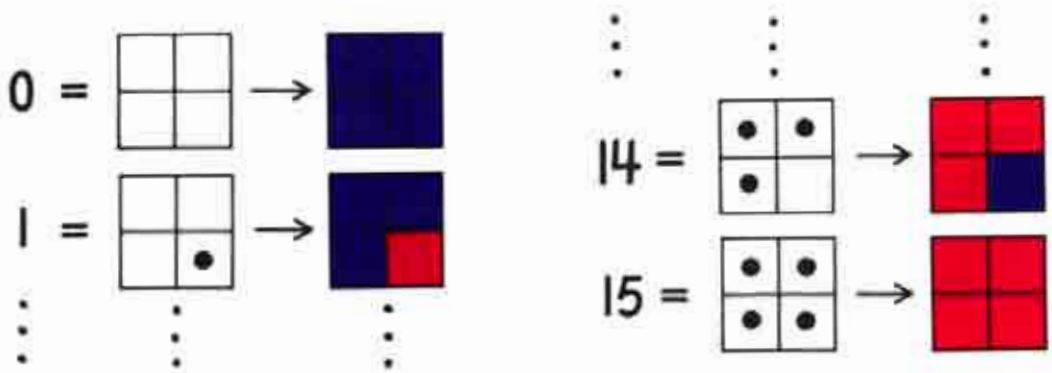


Having counted the choices each time, we use the product rule to conclude that there are 120 (5 x 4 x 3 x 2 x 1 x 1) different ways to have a connected arrow picture. So the probability of getting a connected (one-piece) arrow picture is $\frac{120}{720}$ or $\frac{1}{6}$.

CONCLUDING REMARKS

In this paper you viewed three problem-solving situations from CSMP's Mathematics for the Intermediate Grades and the methods used to solve the problems. These activities illustrate the pedagogical role of both stories and models in the learning of mathematics. The stories add interest to the combinatorial problems and foster the students' understanding of the situation. The various codes demonstrate the power of models to simplify and clarify the solutions to mathematical problems. The models provide a critical link between the problem and its solution. We, the CSMP staff, found these methods to be particularly successful with students in the intermediate grades and also to coordinate well with several themes developed in the CSMP curriculum.

Often a particular method of solving a problem has a side benefit — a bonus of some kind. In "Random Art" and in "Spies and Bridges," the method of using abaci to count the possible outcomes not only accomplishes the enumeration but actually provides a device for generating a complete list of possible outcomes, should ever such a list be desired. (In many combinatoric situations, constructive existence proofs are preferred.) Consider Nabu's artwork. We count Nabu's possible works of art by setting up a one-to-one correspondence between the numbers 0 to 15 and the possible paintings, a correspondence that we set up through the use of the Binary abacus. How many numbers there are from 0 to 15 is evident; the enumeration of the possible paintings is accomplished. But should we wish to see a display of the sixteen possible works of art, we need only to find the corresponding painting for each of the numbers 0 to 15.



In "How Many Permutations?" we use the product rule as the counting device. But recognition of the situation as one in which it is natural to employ the product rule is aided by setting up a correspondence between certain configurations of checkers on a grid and the permutations that we are trying to count.

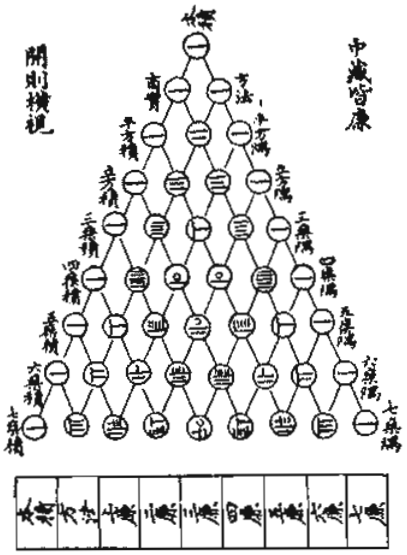
The CSMP curriculum presents many techniques for solving problems. We want to encourage students to meet new situations with a curiosity and an openness toward new solution techniques. Such an attitude does not come readily! We can aid its formation by presenting mathematically rich situations that interest the students and by carefully choosing the techniques to solve the problems that arise. For if we can get to the heart of a situation by building on the students' mathematical experiences and by using tools natural to the situation, the students cannot help but be impressed by the mathematics involved and remember its value.

* * *

In the CSMP curriculum, the activities in this paper appear in the fifth- and sixth-grade lessons from the World of Numbers strand.

Whose Triangle Is It?

Richard D. Armstrong



The arithmetic triangle as depicted by the Chinese mathematician Chu Shih-Chieh in 1303

1 1 1 1 1 1 1 1 1 · · ·
1 2 3 4 5 6 7 8 · · ·
1 3 6 10 15 21 28 · · ·
1 4 10 20 35 56 · · ·
1 5 15 35 70 · · ·
1 6 21 56 · · ·
1 7 28 · · ·
1 8 · · ·
1 · · ·
⋮

The arithmetic triangle as constructed by Blaise Pascal in Treatise on the Arithmetic Triangle, published posthumously in 1665

The arithmetic triangle, commonly known as Pascal's Triangle, has fascinated mathematicians for centuries. In about 1100, Chinese writers and the great Arab poet and scientist Omar Khayyám referred to algebraic patterns that suggest their knowledge of the arithmetic triangle. In 1303, Chu Shih-Chieh depicted part of the triangle in a book on algebra and even then described it as an old method for expanding eighth and lower powers of binomials, for example, $(a + b)^7$. Much later, in the 1550's, the two Italian mathematicians Niccolo Tartaglia and Girolamo Cardano both investigated properties of the number

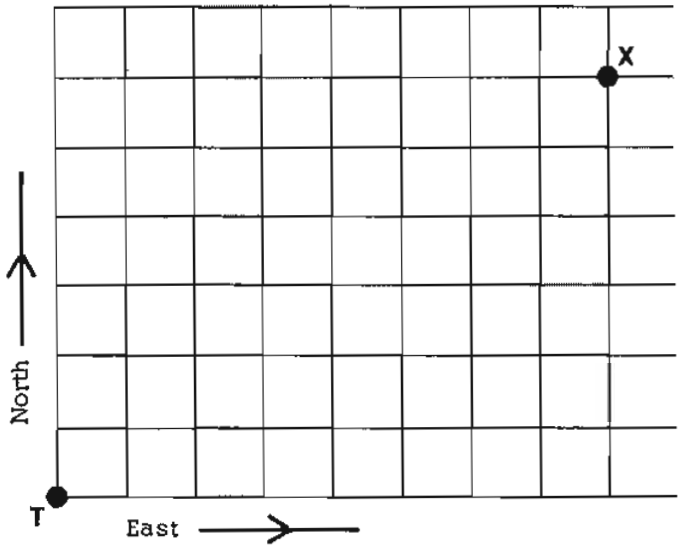
patterns in the arithmetic triangle and appear to have applied it to problems in both algebra and combinatorics.† A century later, Blaise Pascal (1623-1662) wrote Treatise on the Arithmetic Triangle in which he identified and proved interrelationships among numbers in the triangular table. Furthermore, he developed techniques for applying the arithmetic triangle to combinatorial solutions of probability problems.

Whose triangle is it? Chu's? Tartaglia's? Cardano's? Pascal's? Both European and Oriental origins of the arithmetic triangle are obscure. Some historians question the originality and, therefore, the significance of Pascal's contributions. Still, due to his treatise, the title "Pascal's Triangle" seems appropriate.

With its elegance and basic simplicity, the arithmetic triangle can and should also belong to elementary school students. This article presents a detective story from lessons in the fifth-grade CSMP curriculum, prompting students to construct Pascal's Triangle as they solve a problem about locating stolen diamonds. The latter part of the story introduces a code that provides a link between Pascal's Triangle and its application to combinatorial problems. The article concludes with a set of probability problems that demonstrates the use of Pascal's Triangle and the code to determine probabilities.

† Most applications of the arithmetic triangle stem from either binomial expansions or combinations. The appendix to this article provides examples of these two applications.

THE STOLEN DIAMONDS†



After taping a poster to the board, the teacher presents a problem by telling this story. "Here is a street map of part of a city. T is the house of the famous detective, Trek. Some diamonds were stolen from Trek's house and he suspects that they are hidden at X. In order to find clues about who the thieves might be, Trek decides to explore all the routes from his house, T, to the diamonds at X. He must be careful because in this part of town all of the streets are one-way, either north or east. Trek is driving, so he must stay on the streets. About how many different routes from T to X do you think there are for Trek to investigate?"

After tracing several routes from T to X along the one-way streets, students discover that each such route is fourteen blocks long since X is eight blocks east and six blocks north of T. The students' estimates of the number of different routes from T to X typically vary from about 20 to 80. Now the challenge is to count the number of routes.

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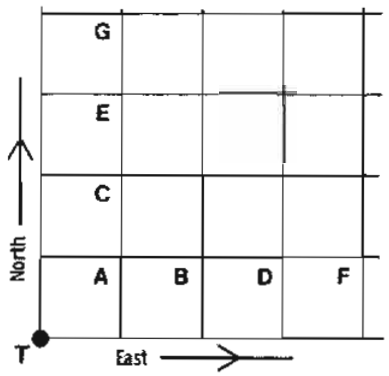
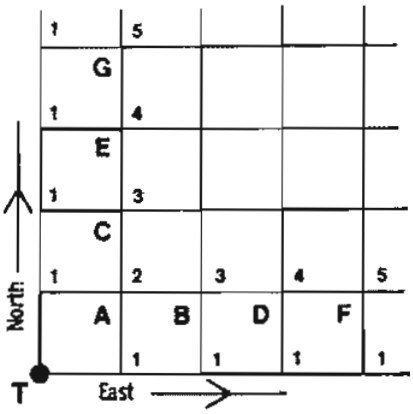
† The two activities, "The Stolen Diamonds" and "The Burglar Suspects," are based on the Storybook "THE HIDDEN TREASURE" by Frédérique Papy. The collection, Stories by Frédérique, is available from CEMREL, Inc., St. Louis, Missouri.

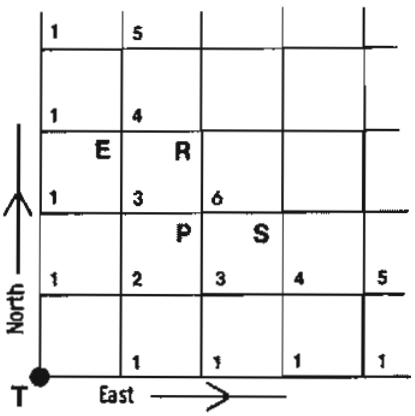
Students first focus on the area near Trek's house and attempt to count the number of routes to each labeled intersection.



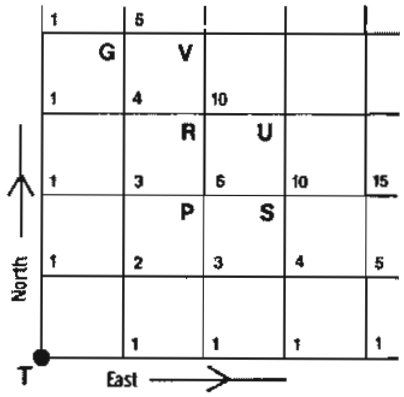
By tracing, they readily find the answers and are able to explain the "2, 3, 4, 5" pattern. They also note that there is only one route to each intersection directly east or directly north of T.



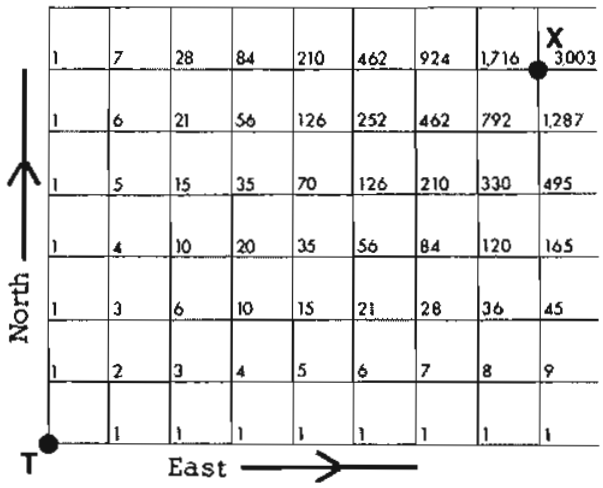
After several attempts, students accurately trace the six routes from T to P. Systematically counting all of the routes from T to R appears formidable. During their experimentation, a few students notice that every route from T to R must pass through E or P, but not both. Since there are four routes from T to E and six routes from T to P, there are ten (4 + 6) routes from T to R. Symmetrically, there are also ten routes from T to S.



The insight concerning intersection R readily generalizes. All routes to U pass through R or S, so there are twenty (10 + 10) routes from T to U; all routes to V pass through G or R, so there are fifteen (5 + 10) routes to V; and so on.



Students use this addition pattern and symmetry to quickly complete the grid. There are 3,003 different routes from T to X! A truly unbelievable result. Yet the simplicity of the pattern quells the doubters in the class.



In solving this problem, students have constructed a part of Pascal's Triangle; though the shape of the array of numbers differs from the more common triangular arrangements. Mathematically the choice of arrangement is unimportant, and so we will continue to call it "Pascal's Triangle." Pedagogically the rectangular array is natural for both the story about Trek and the applications discussed later.

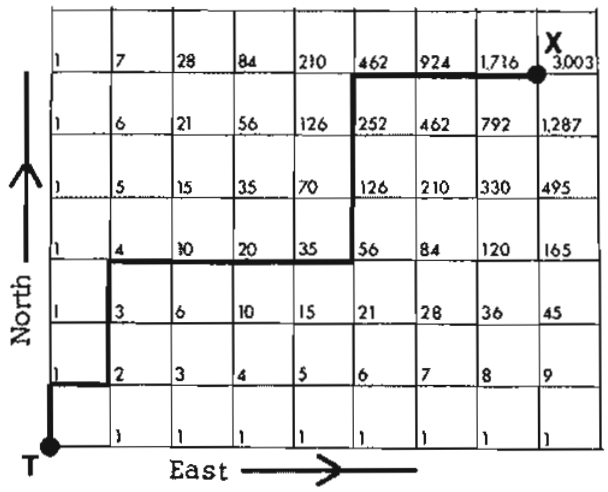
Looking ahead to applications of Pascal's Triangle, it is important to realize that the students have not only determined the number of distinct paths from T to X, but also the number of distinct paths from T to any intermediate intersection. For example, the students' construction indicates that there are 210 distinct routes from T to the intersection six blocks east and four blocks north of T. The students could continue the additive pattern to find any element of Pascal's Triangle.

The above story about Trek following routes does not only prompt the construction of Pascal's Triangle, but also provides a model for many applications of the triangle. In fact, many combinatorial problems can be directly interpreted as problems about counting the number of distinct routes from T to the appropriate intersections on the grid. The continuation of the story about Trek introduces a binary code which provides a key link between Pascal's Triangle and its combinatorial applications.

"For each route Trek travels from T to X, he uses a secret code to record it in his notebook. One day Trek writes **10110000111000** in his notebook. Can anyone guess Trek's secret rule for writing codewords?"

Several students agree on the route for the given codeword:

10110000111000

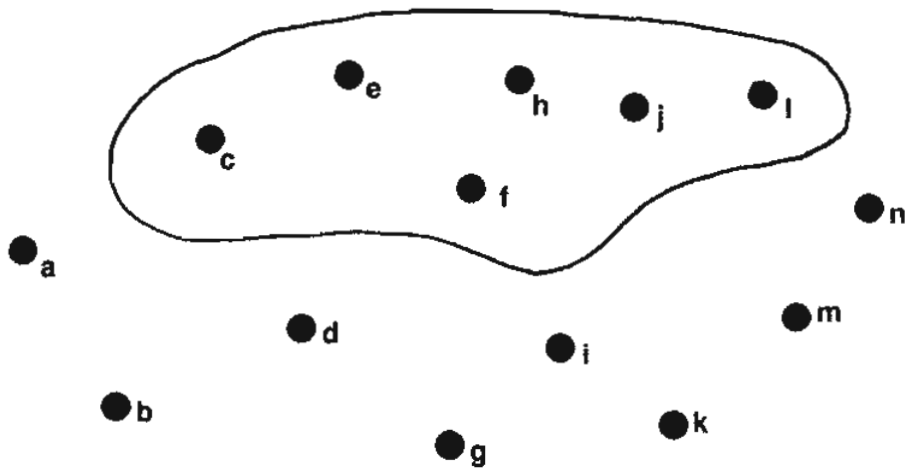


By finding the correct routes for several codewords and by writing their own codewords, most students prove that they have discovered the rule: **0** means to go east one block; **1** means to go north one block. They also notice that each codeword for a path from T to X has exactly fourteen digits: eight **0**'s and six **1**'s. This occurs since X is fourteen blocks from T, eight blocks east and six blocks north.

How many 14-digit codewords are there with exactly eight **0**'s and six **1**'s? 3,003, of course; due to the one-to-one correspondence between these codewords and the routes from T to X. Trek's code itself intrigues students and, most importantly, it prepares students to apply Pascal's Triangle to combinatorial problems. The following episode from Trek's adventures illustrates this role of the **0 - 1** binary code.

THE BURGLAR SUSPECTS

Trek's story proceeds: "During his investigation, Trek learns that a gang of six thieves have stolen the diamonds. Trek has fourteen suspects and is sure that all six thieves are among his suspects. He feels that they would confess if he could interview all six thieves together. So he decides to interview the fourteen suspects in groups of six. Trek draws fourteen dots, labels them "a" through "n", and encircles six dots to represent the first group of six suspects he will interview."



"For each group of suspects he interrogates, Trek decides to write a codeword in his notebook. He writes **00101101010100** for the group of six suspects indicated in the above picture."

Students break the code by aligning the digits of the codeword with the letters of the alphabet.

a	b	c	d	e	f	g	h	i	j	k	l	m	n
0	0	1	0	1	1	0	1	0	1	0	1	0	0

Trek's rule is: write a **1** for each suspect in the group to be interviewed and write a **0** for each of the other suspects.

Students then confront the inevitable combinatorial question, "How many different groups of six suspects could Trek interview?" Many students groan dramatically, but a few spontaneously respond "3,003." They notice that each codeword for a group of six suspects must have fourteen digits: six **1**'s and eight **0**'s. Since they have just determined that there are 3,003 such codewords, they realize that there must also be 3,003 distinct groups of six suspects. The one-to-one correspondence is readily accepted.

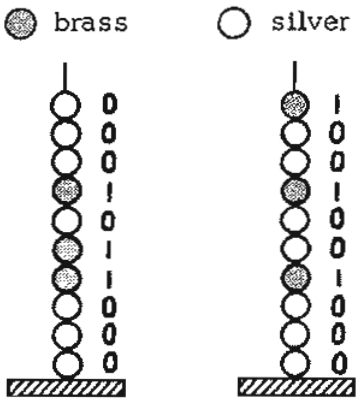
The above solution demonstrates several advantages of imposing a binary code on an appropriate combinatorial problem.

- Pedagogically, the code suggests to students that the current problem might be related to earlier problems involving Pascal's Triangle.
- Mathematically, the binary code defines the one-to-one correspondence between the elements of the problem and the appropriate paths on the grid.
- The codeword identifies the precise entry of Pascal's Triangle that is required for the problem at hand.

The following activity illustrates these features through a further application of Pascal's Triangle to a combinatorial problem. Within a story about a custom in a foreign country, students encounter the following problem:

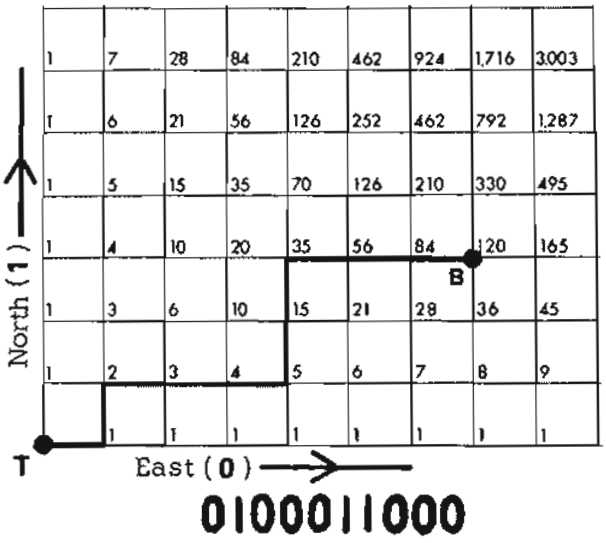
> In how many different ways can three brass rings and seven silver rings be arranged on a pole?

The two types of rings suggest a **0** - **1** binary code. Each codeword will have ten digits: seven **0**'s and three **1**'s.



There is a one-to-one correspondence between the arrangements of the seven silver rings and three brass rings on the pole and the codewords with seven **0**'s and three **1**'s. Therefore the original problem is equivalent to the question: "How many different codewords are there with seven **0**'s and three **1**'s?" These codewords can be applied to Pascal's Triangle.

Each codeword with seven **0**'s and three **1**'s represents a route from T that proceeds, in some order, a total of seven blocks east and three blocks north. All such routes end at B. The route for **0100011000** is shown.



The "120" at intersection B indicates that there are 120 distinct routes from T to B. Hence there are 120 codewords with seven **0**'s and three **1**'s. And therefore, in solution to the combinatorial problem, there are 120 distinct ways to arrange seven silver rings and three brass rings on a pole.

In this problem, the introduction of the binary code recalled earlier applications of Pascal's Triangle, established the required one-to-one correspondences, and indicated which element of Pascal's Triangle was appropriate for the problem. The next section illustrates the application of similar techniques in the solution of probability problems.

FAMILIES

We all know of at least one large family with a preponderance of boys or a preponderance of girls: the Pontello's with six sons and two daughters or the Williams' with seven daughters and no sons. Probability questions naturally arise from such situations. For example, in a family with eight children, what

is the probability that there are exactly six sons? that all are daughters? that at least six are daughters? Through application of Pascal's Triangle, combinatorics provides a means to calculate such probabilities.

A key to a combinatorial approach to these problems is to classify families according to the sex and the order of birth of the children. For example, any family with exactly five children, two young boys and three older girls, is classified **BBGGG**. Two families are considered distinct if their number or order of children differ. For example, **BBGB** and **BGBB** are distinct, as are **GGG** and **GGGG**. This classification assists a combinatorial approach to these problems because any two distinct families with the same number of children are equally probable.[†] For example, **BBBGG**, **BGBGB**, and **GGGGG** are all equally likely families. The following problems employ this classification to provide combinatorial applications of Pascal's Triangle to probability problems.

Problem 1 — Calculate the probability that a family with eight children has exactly six sons.

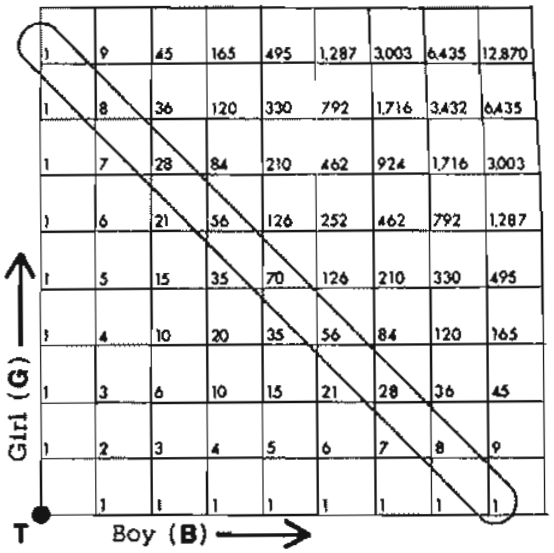
Solution: Determining this probability requires the calculation of:

a. The number of distinct eight-children families in terms of the sequences of boys and girls; and
b. The number of those families with exactly six boys.

The number of distinct families with eight children equals the number of eight-letter codewords using **B** and **G**. By simply changing the coordinate labels, Pascal's Triangle can be used.

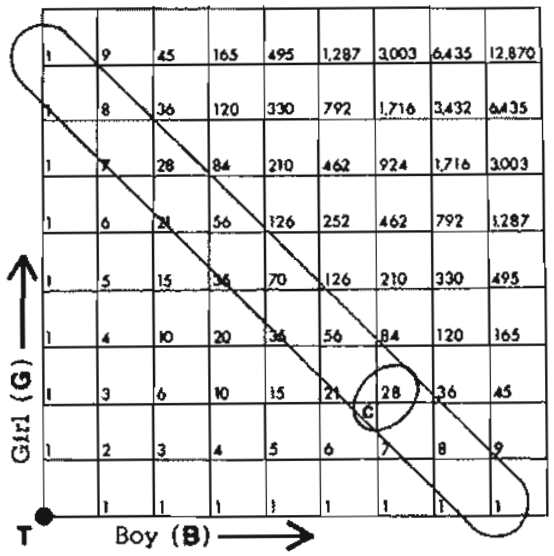
[†] It is assumed that the probability of a child being a girl is 50%.

Each eight-letter codeword refers to a path of length eight, starting at T. It is easily determined that all such paths end on the encircled diagonal.



Therefore, there are 256 (1 + 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1 = 256) paths of length eight. So there are 256 eight-letter codewords with **B**'s and **G**'s and also 256 distinct families with eight children. Of these 256 families, the number of distinct families with six boys equals the number of eight-letter codewords with six **B**'s and two **G**'s.

All such codewords represent paths from T which end at intersection C. There are 28 distinct paths from T to C and 28 codewords with six **B**'s and two **G**'s.

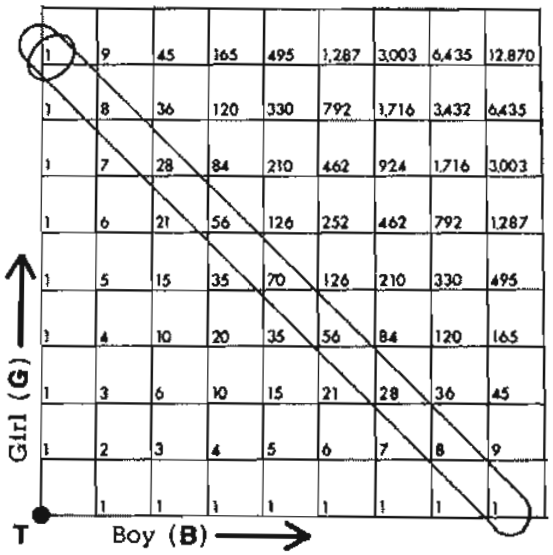


Therefore, there are 28 distinct families with six boys and two girls. So the probability that a family with eight children has exactly six sons is $\frac{28}{256}$ $(= \frac{7}{64})$, or approximately 0.11.

In a similar manner, each of the following problems about children in a family can be interpreted as a problem about codewords consisting of **B**'s and **G**'s and subsequently as a problem about counting paths on the Pascal Triangle grid. For each problem, the appropriate entries on the grid are encircled.

Problem 2 Calculate the probability that a family with eight children has all girls.

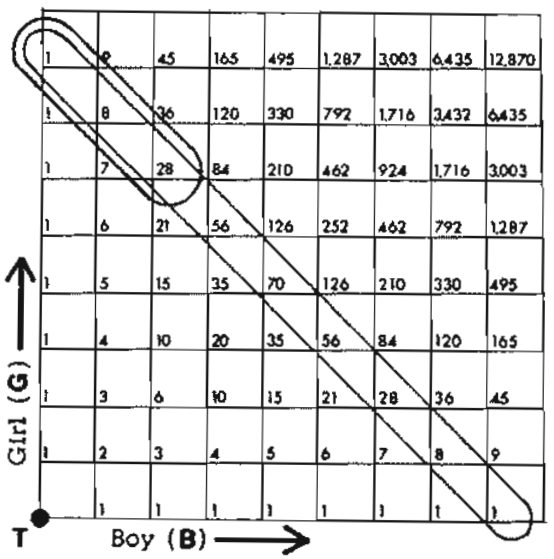
Solution:



$$P = \frac{1}{256} \approx 0.004$$

Problem 3 Find the probability that a family with eight children six or more girls.

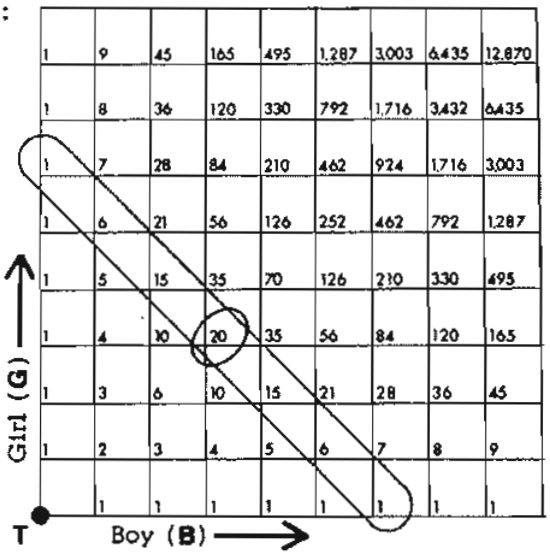
Solution:



$$P = \frac{1+8+28}{256} = \frac{37}{256} \approx 0.14$$

Problem 4 Find the probability that a family with six children has exactly three boys.

Solution:



$$P = \frac{20}{1+6+15+20+15+6+1}$$
$$= \frac{20}{64} = \frac{5}{16} \approx 0.31$$

SUMMARY

Pascal's Triangle is a powerful device for investigating many probability problems through a combinatorial approach. Motivated by a detective story, students can discover the basic additive pattern of the array of numbers. Not only does the story lead to the construction of Pascal's Triangle, but the story also develops a binary code that proves very useful in applying the triangle to solve combinatorial and probability problems. This set of activities demonstrates how a rich problem-solving situation can motivate a mathematical concept, namely Pascal's Triangle, and also lead to understanding and applications of that concept. With this approach, problem solving becomes both a means and an end in mathematics education.

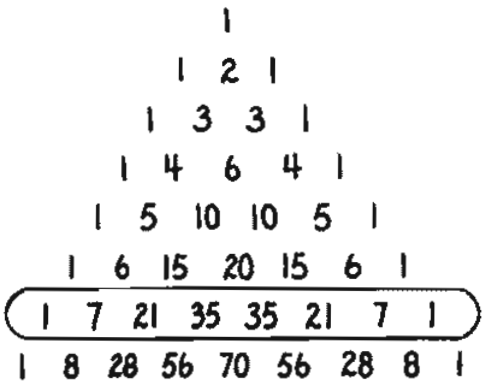
Whose triangle is it? Chu's, Tartaglia's, Cardano's, Pascal's, and any student's who learns its power and recognizes its elegance.

* * *

In the CSMP curriculum, the activities described in this paper appear in the fifth-grade lessons from the Workbook strand, the sixth-grade lessons from the Language of Strings and Arrows strand, and the sixth-grade lessons from the Probability and Statistics strand.

APPENDIX

The earliest and most fundamental applications of Pascal's Triangle involve either binomial expansions or combinations. In algebra, the elements of Pascal's Triangle indicate the coefficients of the expansions of expressions such as $(a + b)^7$.



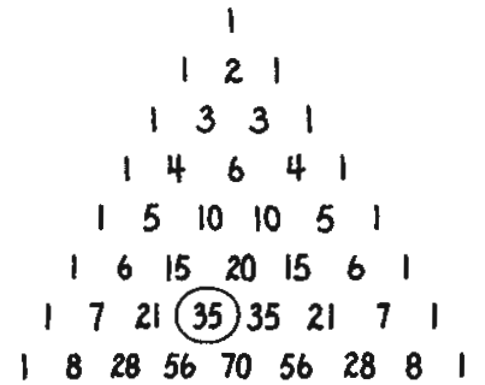
The eighth row of Pascal's Triangle provides the coefficients for $(a + b)^7$.

$(a + b)^7 =$

$1a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^6 + 7ab^6 + 1b^7$

In general, the $(n + 1)^{th}$ row of Pascal's Triangle provides the coefficients for $(a + b)^n$.

A fundamental question in combinatorics is to determine the number of distinct subsets of a specific size of a given set. For example, "How many different three-person subcommittees can be formed from a committee of seven members?"



The answer, 35, is the fourth element of the eighth row of Pascal's Triangle.

In general, the number of subsets with r elements from a set with n elements is the $(r + 1)^{th}$ entry in the $(n + 1)^{th}$ row of Pascal's Triangle. Further combinatorial applications of Pascal's Triangle appear in this paper.

An Area Model for Solving Probability Problems [†]

Richard Armstrong

Most American children have an intuitive concept of randomness, partially due to games involving dice, spinners, and cards. Since probability provides a rich source of problem-solving experiences, we decided to extend our students' informal experiences and include probability as an integral part of our elementary mathematics curriculum. The article, "Probability and Statistics in Grades 1 to 3," in this book describes stories and games for second and third grades which introduce concepts such as expected frequency, equally likely events, and prediction. The students' reactions to these activities indicated to us their capability of progressing to the analysis of one-stage probability experiments through combinatorial methods. In one third-grade lesson, the students considered the thirty-six equally likely outcomes when two dice are thrown and determined that there are six ways for a sum of seven to occur. Thereby they calculated that the probability of rolling a sum of seven is $\frac{6}{36}$, or $\frac{1}{6}$.

Their success and enjoyment in analyzing several one-stage probability situations demonstrated that these students were capable of considering more complex multistage experiments in the intermediate grades. However, traditional arithmetic solution techniques of such problems tend to require either unwieldy combinatorial analysis or a well-developed understanding of the addition and multiplication of fractions. Of course, the consideration of these problems could be postponed to later grades, but even for more mature students the computational aspects of arithmetic solutions often tend to obscure rather than illuminate the underlying probabilistic concepts.

The need for an alternative model for solving probability problems became apparent. To be appropriate for intermediate grade students, we thought the model

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[†] This paper appears in NCTM's 1981 yearbook, Teaching Statistics and Probability.

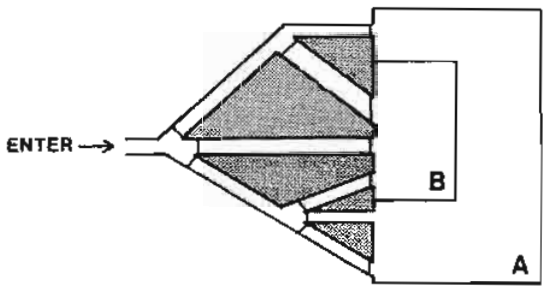
should —

- be sufficiently powerful to handle fairly sophisticated probability problems;
- *rely primarily on mathematical skills that the students have already* acquired;
- be consistent with the students' current understanding of probabilistic *concepts*;
- support the eventual development of more advanced solution techniques.

Considering that most probability situations intrinsically involve fractions and that a common model for fractions involves the partitioning of circular or rectangular regions ("pies" or "cakes"), perhaps it is natural that we developed a geometric model to satisfy the above criteria. In this model, a unit square is divided into regions so that the areas of the regions are proportional to the probabilities involved in the situation. The following three activities indicate the use and development of this model and moreover illustrate its pedagogical and mathematical characteristics.

MARRIAGE BY CHANCE†

Mr. Simons, a fifth-grade teacher, tapes a poster on the board and with appropriate embellishment tells the following story, occasionally allowing students to react and comment.



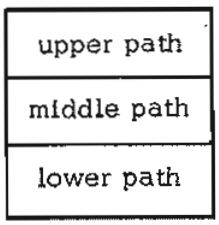
† This story is inspired by a popular short story, "The Lady or the Tiger?" by Frank Stockton which appears in A Storyteller's Pack: A Frank R. Stockton Reader, Scribner, 1968.

"The king and queen of a medieval kingdom arranged a marriage for their daughter and Prince Cuthbert from a neighboring kingdom. The princess accepted this plan without enthusiasm. A short time before the proposed wedding day, she met Reynaldo — handsome, clever, romantic, but only a peasant. Their love developed quickly and secretly, but inevitably the king learned of their relationship. Irate, he ordered that Reynaldo be thrown into a room full of tigers. But in response to his daughter's pleas, he offered a compromise: Reynaldo would walk through a maze, each path leading to one of two rooms. While the hungry tigers wait in one room, the hopeful princess waits in the other room. If Reynaldo enters the latter room, he and the princess could marry."

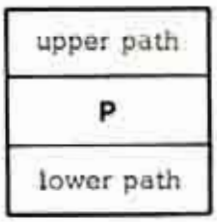
Pointing to the poster, Mr. Simons continues, "The king showed the princess a map like this one of the maze and let her decide in which room to wait. Remember that Reynaldo does not have a copy of the map and can only guess which paths to follow. Which room is he more likely to enter, A or B?"

Some students suggest that Reynaldo's probability for entering each room is $\frac{3}{6}$, or $\frac{1}{2}$, because there are three doors into each room. However, other students realize that the answer is not so obvious, since Reynaldo is more likely to arrive at the third door from the top than at other doors because there is a path which leads directly from the entrance to the third door. After more discussion, the majority of the class votes that the princess should wait in Room B.

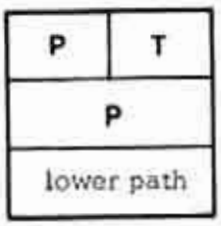
Mr. Simons draws a large square on the board and suggests, "Let's use this square to determine the probability that Reynaldo will enter Room B. When he enters the maze, what is the first choice Reynaldo must make?" When a student responds that Reynaldo must choose to take the upper path, the middle path, or the lower path, Mr. Simons adds some information to the square.



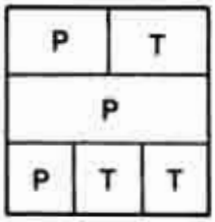
"I've divided the square into three equal parts, since each of the three paths is equally likely to be chosen by Reynaldo," explains Mr. Simons. "What happens if Reynaldo chooses the middle path?" A student observes, "He's lucky and walks straight to the room where the princess is waiting." Mr. Simons shows this by marking "P" in the center section of the square.



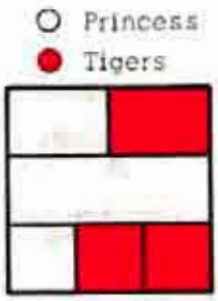
He continues, "What happens if Reynaldo chooses the upper path?" Observing that the upper path splits into two paths, the students state that Reynaldo's chances of reaching each room would then be the same. They agree to indicate this by dividing in half the part of the square labeled "upper path."



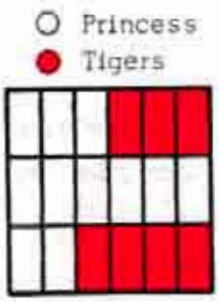
Then the class correctly divides and labels the region for the lower path.



For contrast, Mr. Simons colors gray the regions marked "P" and red the regions marked "T".



Looking at the square convinces the class that they have placed the princess in the correct room, since more than half the square is colored gray. Mr. Simons agrees and inquires how they could calculate exactly Reynaldo's probability of finding the princess. With hints and encouragement, the class decides to divide the square into small pieces all the same size and to count the number of gray pieces and red pieces.



Out of eighteen pieces of the same size, eleven are gray and seven are red. Therefore, Reynaldo has eleven out of eighteen chances to find the princess. His probability of success is $\frac{11}{18}$, or almost $\frac{2}{3}$. His success is not guaranteed, of course, but the class did place the princess in the better room.

Some students at first were intent on finding clever ways for Reynaldo to detect and avoid the tigers. Rather than being out of place, this humorous diversion emphasized the need to accept certain restrictions when a situation is being modeled. As in real-life applications, the situation had to be idealized. An advantage of embedding a problem in a story instead of using a real example is that the necessary restrictions can be minimized and well controlled.

Solving several more probability problems presented in story contexts prepares the students to consider a famous problem from the early history of probability theory — a problem which requires more sophisticated mathematical insights.

A PROBLEM OF POINTS

In the history of mathematics, the first probability questions arose from games of chance. One particularly intriguing problem, now called the "problem of points," appeared as early as the fourteenth century. The following is an example of the problem. Two gamblers play a game for a stake which goes to the first player to gain ten points. If the game is stopped when the score is 9 to 8, how should the stake be divided between the two players? It is assumed that the players have equal chances of winning each point.

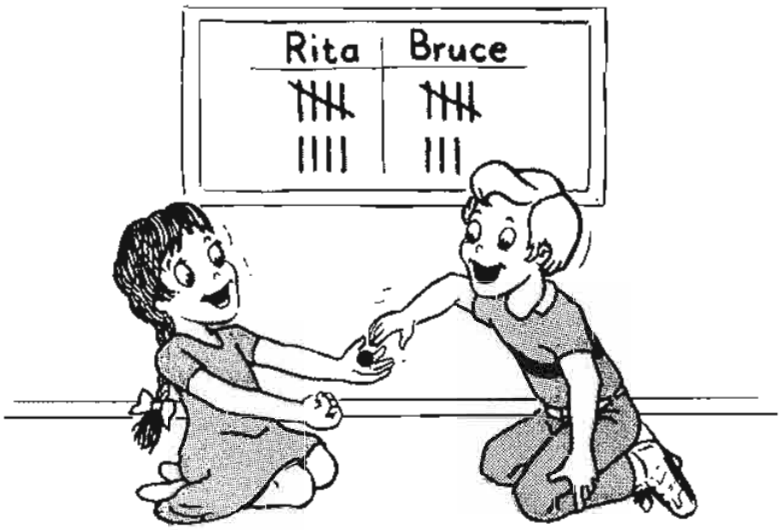
This problem was popular and controversial in Europe in the sixteenth and early seventeenth centuries. In 1556, Tartaglia claimed to have the solution but simultaneously declared that any solution is "judicial rather than mathematical," that is, it must be agreed upon by the two players (an astute commentary on applied mathematics!).

In 1654 Antoine Gombard, the chevalier de Méré and a member of King Louis XIV's court in France, encountered the problem through his interest in mathematics and gambling. Being an unsolved problem, he proposed it to a young mathema-

tician, Blaise Pascal. The ensuing correspondence between Pascal and an older friend, Pierre Fermat, reveals that they developed three distinct techniques for correctly solving this problem. The application of these techniques to other probabilistic questions provided an impetus to mathematicians and eventually led to the development of modern probability theory.

Embedding the problem of points into a children's game and using the area technique allows intermediate grade students to solve this historically significant problem.†

Let's listen to Ms. Kell as she describes a game to her class. "Rita and Bruce play a game. Rita has one red marble and one blue marble. With her hands behind her back, she mixes them and then puts one marble in each hand. Bruce chooses a hand. If he selects the hand with the blue marble, he scores one point. Otherwise Rita scores one point. The procedure is repeated, and the winner is the player to first score ten points."

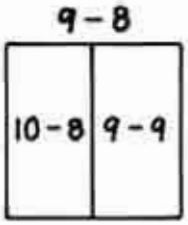


After playing the game a few times in class, Ms. Kell suggests the following situation. "One day, Rita and Bruce must stop a game when the score is Rita 9 and Bruce 8. If they continue the game the next day, what is the probability that Rita will win?"

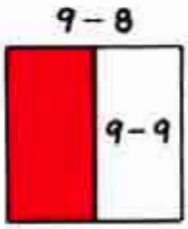
† The Belgian math educators, Frédérique and Georges Papy discovered this solution technique for the "problem of points." Their solution revealed to our staff the potential of the method in many other situations.

After discussing the game and making some estimates, the students use a square to analyze the situation:

If the score is 9-8, the next score will be 10-8 or 9-9 with equal likelihood. Divide the square into halves.



Rita wins if the score is 10-8. Color the appropriate region red for Rita.

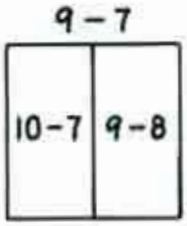


If the score reaches 9-9, the game is fair. Color half the appropriate region red for Rita and half gray for Bruce.

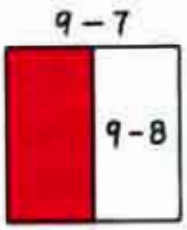
Three-fourths of the square is colored red and one-fourth gray. Therefore when Rita is leading 9 to 8, the probability of her winning is $\frac{3}{4}$ and the probability of Bruce winning is $\frac{1}{4}$. Because of the symmetry induced by using one red marble and one blue marble, we can immediately conclude that if Bruce were leading 9 to 8, his probability of winning would be $\frac{3}{4}$ and Rita's probability of winning would be $\frac{1}{4}$.

The solution for the problem when Rita is leading 9 to 7 is similar and reveals a useful shortcut.

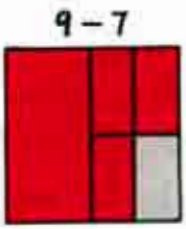
If the score is 9-7, the next score will be either 10-7 or 9-8, with equal likelihood.



Rita wins if the score is 10-7.



For the intermediate score 9-8, we could consider the scores 10-8 and 9-9. But the previous argument shows that if Rita leads 9-8, her probability of winning is $\frac{3}{4}$. Therefore the region for "9-8" can immediately be colored $\frac{3}{4}$ red and $\frac{1}{4}$ gray.



Once the square is divided into regions of the same size, there are seven red pieces and one gray piece. Therefore, if Rita is leading 9-7, her probability of winning the game is $\frac{7}{8}$.

By applying the area technique, students can now determine Rita's and Bruce's winning probabilities for any intermediate score. Such a task appears uninteresting and tedious. However, the use of the area technique has suggested a very natural application of the concept of recursion — for example, using the computed result for the "9-8" problem to shorten the solution of the "9-7" problem. In fact, by detecting patterns, using recursion, and occasionally employing the area technique to check hypotheses, a class is able to determine

quickly Rita's and Bruce's probabilities of winning at any intermediate score.

The following chart indicates the odds (Bruce : Rita) for winning a game to ten points when each player has at least five points. Readers are invited to check the results, detect and confirm patterns, and thereby extend the chart to include lower intermediate scores.

ODDS OF WINNING (BRUCE : RITA)

Bruce's Intermediate Score	Rita's Intermediate Score 5	6	7	8	9
9	31:1	15:1	7:1	3:1	1:1
8	57:7	26:6	11:5	4:4	1:3
7	99:29	42:22	16:16	5:11	1:7
6	163:93	64:64	22:42	6:26	1:15
5	256:256	93:163	29:99	7:57	1:31

This solution of the problem of points by the area method is similar to one of the solutions of Pascal and Fermat in that each depends on a technique of partitioning. However, instead of partitioning a region, Pascal considered the partitioning of a stake of 64 pistoles (units of money). Also, each solution uses a different justification for its partitioning. Of Pascal and Fermat's other two solutions, one relied on combinatorics and the other on the addition of independent probabilities. Secondary school mathematics students could gain some valuable insights into probability by solving the problem of points on their own and then comparing their solution to Pascal and Fermat's three solutions.

An interesting extension to the problem of points occurs if Rita and Bruce use two red marbles and one blue marble in their game. The altered patterns and recursions reflect the influence of the asymmetry induced by the new marble mixture.

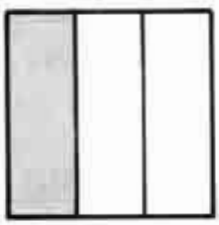
AN ARCHERY GAME

Modeling the analysis on a square provides several pedagogical advantages for solving probability problems. Pictorial representation of the analysis provides visual insights into probability concepts. Reliance on geometric skills allows the development of concepts, which a lack of arithmetic skills would normally impede. Division of a region in proportion to the appropriate probabilities appeals to the students' intuitive understanding of probability. But this solution technique also provides a mathematical advantage by producing a less complex solution for certain types of sophisticated probability problems. For example, the probability problem presented in the following story involves an infinite Markov chain.

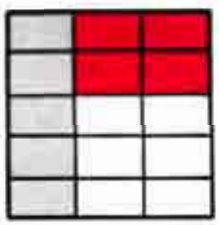
As archers, Rita hits the target $\frac{2}{5}$ of the time and Bruce hits the target $\frac{1}{3}$ of the time. They decide to have a contest. Letting Bruce shoot first since he is the poorer archer, they alternate shots until one wins by hitting the target. Who is favored? What is each contestant's probability of winning?

Use a square to calculate the probabilities.

Bruce shoots first and has a probability of $\frac{1}{3}$ of hitting the target and winning immediately. Color $\frac{1}{3}$ of the square gray.

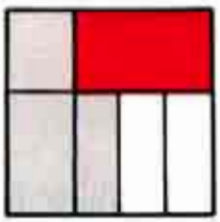


If Bruce misses the target, Rita shoots and wins by hitting the target with a probability of $\frac{2}{5}$. Of the colored region, color $\frac{2}{5}$ of it red.

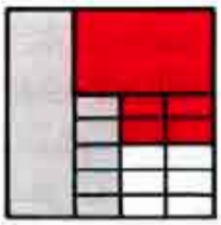


Notice that the ratio of the area of the gray regions to the area of the red regions is 5: 4.

If both shots have missed, Bruce shoots again, and his probability of hitting the target is $\frac{1}{3}$.



If no one has hit the target, Rita shoots again, and her probability of hitting the target is $\frac{2}{5}$.



Notice that for the newly colored regions, the ratio of the area of the gray regions to the red regions is again 5: 4. Therefore, for all the colored regions, the 5: 4 ratio is maintained.

Continuing the process, the uncolored region gradually vanishes and the ratio of the area of the gray regions to the area of the red regions is always 5:4.

Therefore, it is plausible (and correct) to conclude that for this archery contest the ratio of Bruce's chances of winning to Rita's chances of winning is 5: 4. Thus, Bruce's probability of winning is $\frac{5}{5 + 4}$ or $\frac{5}{9}$. Shooting first provides a sufficient advantage to overcome his lesser skill.

The probability problem involved in this archery contest is an example of a Markov chain. The area technique provides a solution that does not require advanced algebraic processes such as matrix multiplication, summation of

infinite series, or the formation and solution of linear equations. Of course, the intuitively appealing conclusion that a particular ratio is maintained throughout an infinite process is assumed but not proven at this level.

PERSPECTIVE

A desire to allow intermediate grade students to progress in their understanding of probability concepts without relying on a comprehension of the multiplication and addition of fractions motivated the development of this area technique. Observing the students' ability to apply this model to solve fairly sophisticated probability problems and listening to their responses convince us that this goal was achieved. Besides its pedagogical advantages, this area technique provides simpler solutions to certain advanced problems such as some Markov chain problems.

However, we do not suggest that this area method should supplant other approaches to probability. Other representations, for example using probability trees, provide further insights into probability topics. As problems become more complex and lead to general theories, the use of variables and algebraic techniques become a necessity. Therefore we suggest that a strong background for probability be built in the intermediate and middle grades by parallel development of these topics: numerical skills with rational numbers; analyses of probability problems by combinatorial methods, by this area technique, and by using trees; statistical experiences which include the concept of expected frequency; and an introduction to variables. Each of these topics by itself is appropriate in the intermediate grades and taken together would provide students the ability to model and solve realistic, fascinating probability and statistical problems in later grades.

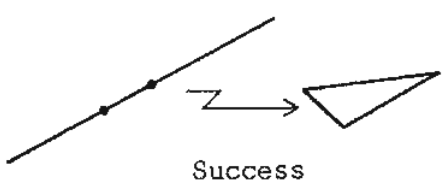
* * *

In the CSMP curriculum, the activities described in this paper appear in the fifth- and sixth-grade lessons from the Probability and Statistics strand.

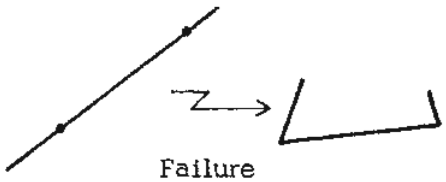
Breaking a Stick: Probability Without Counting

Joel Schneider

Probability in school mathematics curricula commonly occurs in finite situations, for example: What are the chances of picking a white ball from a collection of white balls and black balls? By contrast, consider this problem: If a stick is broken at two points chosen at random, what is the probability that one may construct a triangle with the three pieces? Some breaks yield a triangle; for example, all three pieces might be the same length and give an equilateral triangle. Some breaks do not yield a triangle; for example, two of the pieces might be very short.



Success



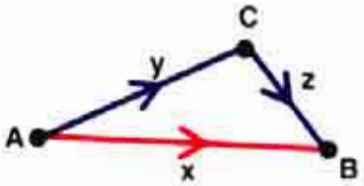
Failure

Of course, there are an infinity of choices for the breaking points and so no simple counting of successes is possible. Our approach of using a geometrical device to represent the problem is based on an idea of Castelnuevo (Proceedings of First International Congress on Mathematical Education, Dordrecht (Holland), D. Reidel, 1969) and modified by G. Papy (1977 seminar at CEMREL Inc., St. Louis). Using a geometrical approach to a probability problem is a particularly attractive example of cross-fertilization among areas of mathematics.

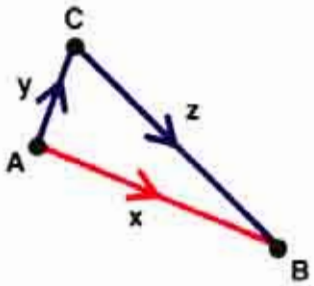
The students' experiences of geometry throughout CSMP is informal and largely based on the use of several tools: the compass, the straightedge, and a device for constructing parallel line segments. One application of the last tool is in effecting parallel projections, providing one of the basic experimental constructions with which to study the problem. Developing the geometrical prerequisites and solving the probability problem occupies weekly lessons for most of a semester for a class of sixth-grade students. This article describes the content of the lessons.

THE TRIANGLE INEQUALITY

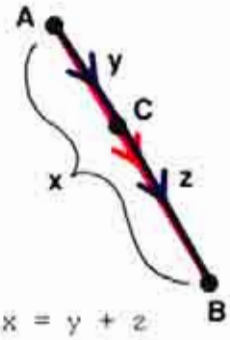
When can three line segments be used to build a triangle? The Triangle Inequality provides a ready test to answer the question in terms of the relative lengths of the line segments. An informal statement of the Triangle Inequality is that the distance (x) from one point (A) directly to another (B) is at most the distance (y + z) taken via a third point (C).



$x < y + z$

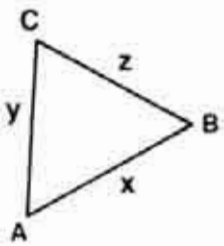


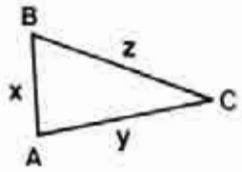
$x < y + z$

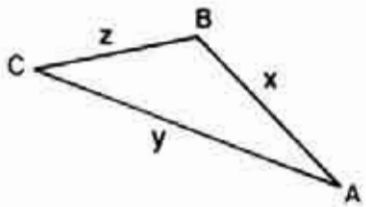


$x = y + z$

In general, for any points A, B, and C, $x \leq y + z$. From this we deduce the Triangle Inequality: Any two sides of a triangle together are longer than the third side.





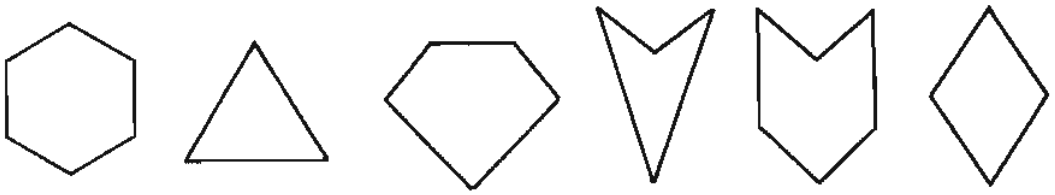


In all cases,

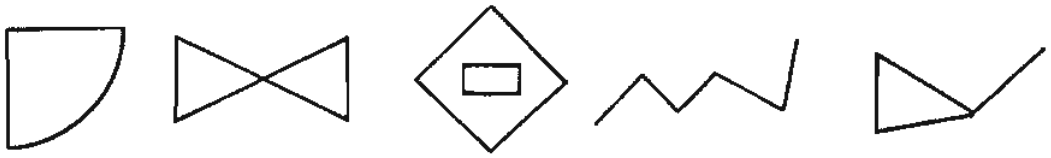
$x + y > z$, $x + z > y$, and $y + z > x$.

The statement of the Triangle Inequality is direct and simple, almost obvious. But appreciation of its significance usually requires experience. In the CSMP curriculum, the students' discovery of the Triangle Inequality follows a sequence of activities in constructing polygons under a variety of constraints on their sides. We begin the sequence with informal sketches of shapes to clarify the idea of polygon. The object of the discussion is to narrow the concept without

resorting to formalism. Examples . . .



and counter examples . . .

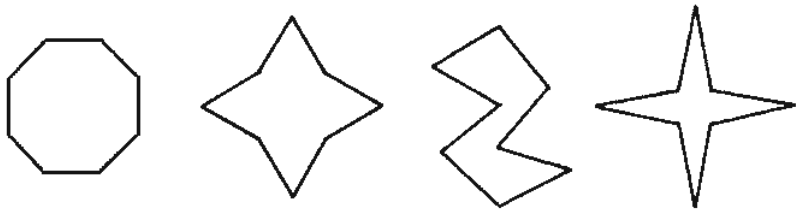


lead to recognition of polygons (more formally, "simple closed polygons"). With the idea of polygon secure, we attempt to construct polygons under constraints on the lengths of the sides.

Duplication of line segments is basic to these constructions. The available tools are a compass and a straightedge. The following sequence of constructions, posed as problems, enables students to develop their facility with the tools as they respond to increasingly restrictive constraints on the number and length of sides of polygons.

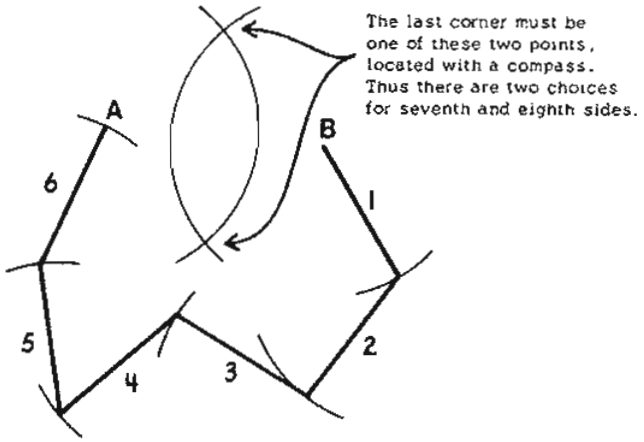
Problem 1 : Construct a polygon with eight sides, all having the same length.

Many solutions are possible; students each construct several.



The key to the construction lies in drawing the sixth side so as to bring the ends of the chain of sides close together. Then the seventh and eighth sides close the shape. There are usually two choices for the location of the last corner and these are located by finding the intersections of the arcs centered on the free

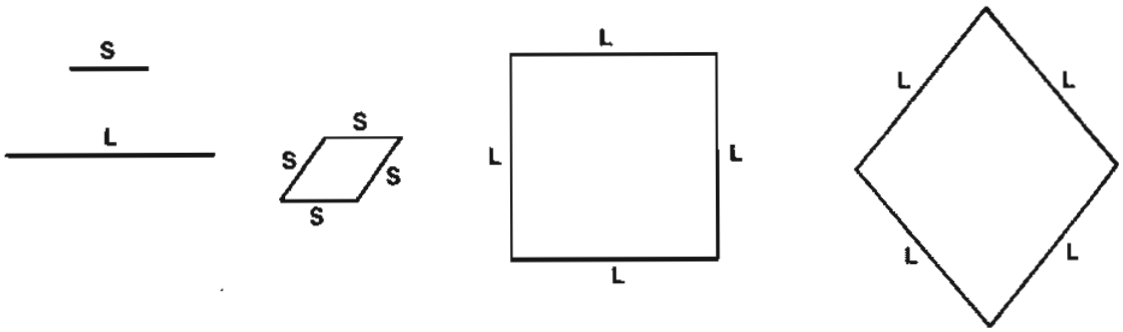
endpoints (A and B) as shown here.



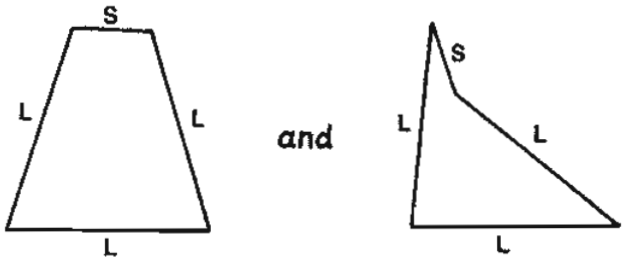
Several experiences with this problem, with varying numbers of sides, provide students an opportunity to develop a good sense of the use of the compass in constructing polygons under a simple constraint. The constraint is so simple as to allow the students to concentrate on developing their techniques. Through studying this problem with several numbers of sides, we discover the fact that while there are many solutions with 8 sides, with 6 sides, and with 4 sides, there is only one solution with 3 sides, namely the equilateral triangle.

Problem 2: Given two line segments, draw a quadrilateral so that each side is the same length as one of the two segments.

Once again there are many solutions. By comparing their solutions, the class discovers that they fall naturally into families. There are five combinations of sides: all short, all long, one short and three long, three short and one long, and two short and two long. The first two cases appeared along with Problem 1; there are many solutions, but each is a rhombus.

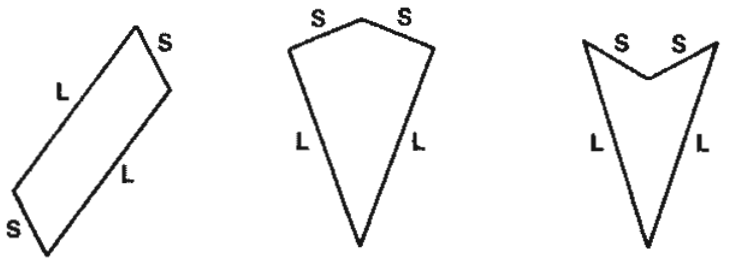


In the case of one short and three long, there is again only one family of solutions. If we classify with respect to the arrangement of sides, there is only short-long-long-long, even though the shape may vary. For example,



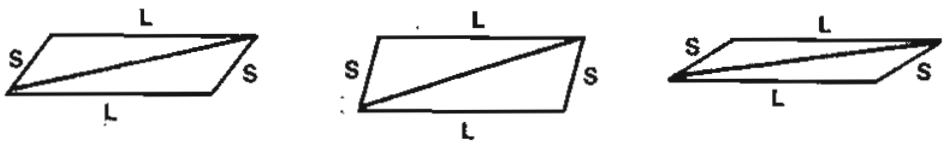
The case of three short and one long is similar; there is only one family of solutions, short-short-short-long. Again the shape may vary; for example,

The case of two short and two long is more interesting since there are two families of solutions, depending on the order of the sides in rotation: short-long-short-long or short-short-long-long. The first sequence results in a parallelogram and the second in a kite or a wedge.



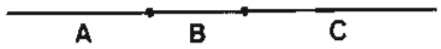
parallelogram kite wedge

Drawing the longer diagonal in red in several examples of the parallelogram suggests a comparison of the diagonal with the two unequal sides of the parallelogram.



BROKEN STICKS

Break a stick into three pieces. Label the pieces "A", "B", and "C".



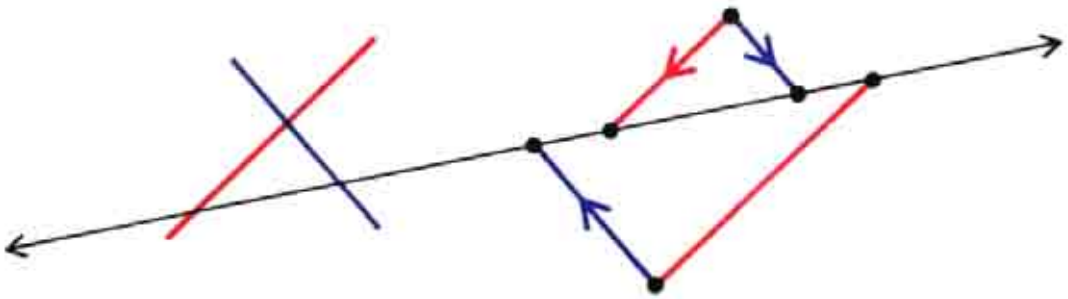
According to the Triangle Inequality, to make a triangle any two sides must be longer than the third side. In particular,

A and B together must exceed C,
A and C together must exceed B, and
B and C together must exceed A.

By examining many broken sticks and comparing the lengths of their pieces, we notice a pattern. Look at the largest piece, say it is C. How long can it be if we are to construct a triangle? If the stick measures 100 cm, whatever the length of C, A and B make up the remainder. C cannot be too long — if C is 80 cm long, then A and B together are 20 cm long, but in order to form a triangle they must exceed C. Hence C cannot be 80 cm long. More generally, if C is more than half the stick, then A and B together are less than half the stick and no triangle can be constructed. But if C is the longest piece and C is less than half the stick, then A and B together are longer than half the stick and the three pieces yield a triangle. This conclusion suggests a modified version of the Triangle Inequality: If a stick is broken at two points chosen at random, one can construct a triangle with the pieces if and only if each piece is less than half of the stick.

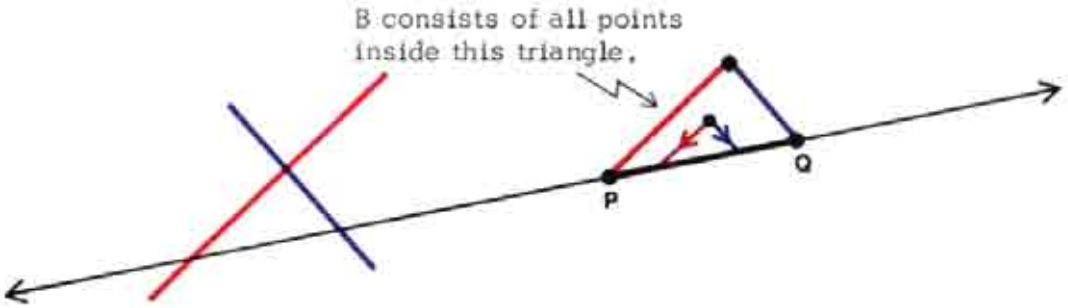
We represent the stick as a line segment. The first task is to choose two breaking points at random. Random choice is familiar from other situations in the probability strand, but the simultaneous random choice of two points is a new problem. For this we recall some ideas from earlier work in geometry. Coordinates on a grid provide a link between a pair of points and a single point. That is, two points (one on each axis) identify a single point on the plane and vice versa. Regardless of the orientation of the axes, the linking mechanism in the constructions is parallel projection along the axes.

Randomly selecting two points on a stick requires a slight modification: we make both projections onto the same line. Thus given a line in the plane and two directions for projection, shown by the red and the blue lines in the following illustration, each point in the plane produces two points on the line through parallel projections.



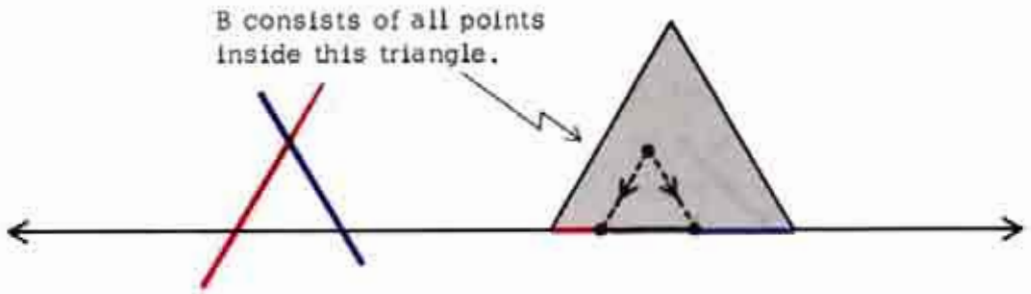
If we consider only one of the half planes, then choosing one point in the half plane at random is equivalent, through the dual projections, to choosing two points on the line at random. This technique provides the random device for breaking our stick at two points.

We mark a segment, PQ, of the line to represent the stick. Points in the plane produce two, one, or zero points on the stick through the dual projections. We experiment to find the set B (shaded in the next illustration) of points in the half plane that corresponds to pairs of points on the stick.



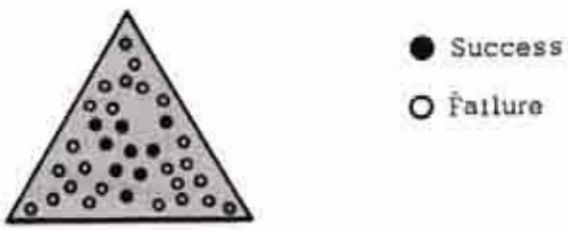
Having developed a means to effect the simultaneous random choice of two points of the stick, the probability question is: Which points in B correspond to points at which to break the stick into three pieces that can form a triangle? The Triangle Inequality provides a useful criterion to apply to the pieces. We need to find a criterion for deciding whether a point in B will yield a triangle.

After sufficient experience with dual projections, we normalize the representation by choosing projections that result in an equilateral triangular region for B. Coloring the segments of the stick red, black, and blue provides a convenient notation for discussing the problem.

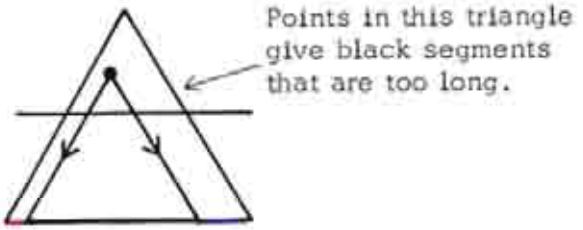


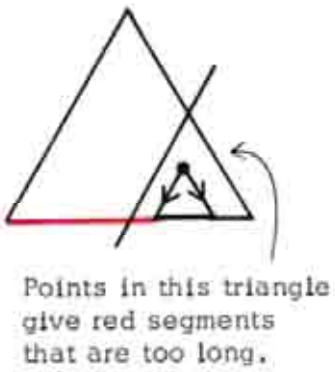
Now we can experiment by choosing a point in B, performing the dual projections and attempting to construct a triangle.

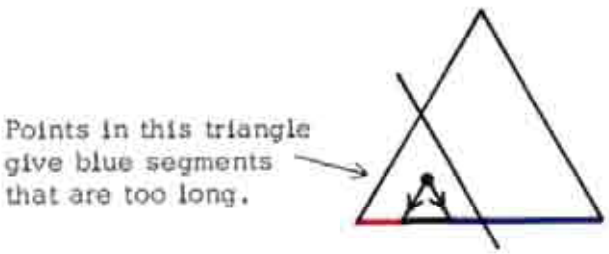
In experimenting with many choices of points in B, students classify points in B as "successes" or "failures" and discover that they fall in clusters.



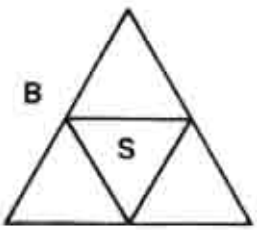
The dots representing failures appear in three clusters; three clusters — three segments — three colors. We explore the result in terms of the red, blue, and black segments. Of course, varying the choice of a point in B results in varying lengths for red, blue, and black segments. When can the three segments be used to form a triangle? The Triangle Inequality, discovered earlier, reveals that each must be less than half of the stick. Through many experiments, choosing a point in B, performing the parallel projections, and testing the segments, we find the points that give too long of a black segment appear to cluster together. And the same is true for points giving too long of a blue segment and points giving too long of a red segment.







We can certify that the three sets of points do cluster and locate the boundaries of the clusters. The points of B that yield red, black, and blue segments, all of proper length, are those in the central triangular region S.



The ratio of the area of S to that of B is one to four. The problem is solved. The probability of breaking a stick into three pieces with which we can form a triangle is $\frac{1}{4}$.

* * *

In the CSMP curriculum, the activities described in this paper appear in the sixth-grade lessons from the Geometry and Measurement strand.

Shunda's Newsstand

Clare Heidema

The business world offers many opportunities for statisticians, especially in advising decision-making processes. Elementary school age youngsters in their everyday lives also are confronted with decision situations, some with a business flavor, where a rudimentary understanding of statistics may prove useful. However, the pedagogical concerns of making the study of these situations accessible at an early age often prohibits consideration until the problems can be discussed on a high mathematical level. Consider, for example, the classic "Newsboy's Problem" concerning a newspaper seller attempting to maximize profit. Children can appreciate such a problem; indeed they may have paper routes or operate newspaper stands themselves. SHUNDA'S NEWSSTAND provides an excellent example of how an operations research problem involving statistics in a decision-making context can be presented at the elementary school level.

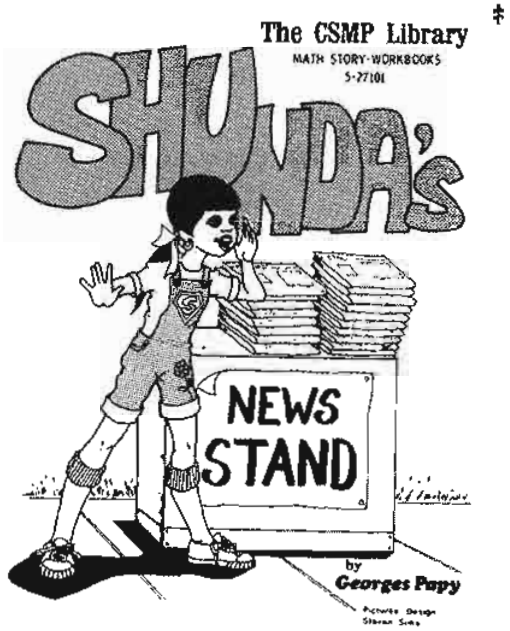
Crucial to the presentation of the problem in terms that fifth-grade students can understand is the use of stimulating pictures to view the sample data in a variety of ways. The pictures and graphs provide an alternative to technical numerical methods; an alternative that is both aesthetically and pedagogically appealing. This approach illustrates two goals of CSMP: (1) to present the best of mathematical content essential for understanding the nature of mathematics and its ever-increasing applications to diverse situations in the real world; and (2) to engage youngsters immediately and naturally with the content and applications of mathematics, making mathematical ideas accessible to young children through the use of non-verbal languages.

SHUNDA'S PROBLEM

A common problem in the business world is to determine the optimal quantity of items to buy or produce in order to make the most profit. Many factors affect such decisions and no one can be guaranteed of always making the best choice. Still, educated decisions can be made by doing a careful study of sample data and by assuming that past behavior of the consuming public is a good predictor of future behavior. There is always a risk; all one can do is decide what is the most reasonable prediction. SHUNDA'S NEWSSTAND is concerned with such a business world problem as it affects Shunda, a young newspaper seller.

We present the situation in a story-workbook†, that is, as a story told in the pages of a "comic book." Along the way the students respond to questions and solve intermediate problems. They become familiar with a variety of pictorial representations of information and supply the necessary results needed to understand the main problem. Prompted by a series of stimulating pictures for recording data, the students use the daily demand to determine the most profitable inventory of newspapers.

Our story is about Shunda, a newspaper seller, who has a newsstand on the corner of Hamilton Street and Euler Avenue. Each day between 4:00 and 6:00 PM, she sells newspapers to people passing by her stand.

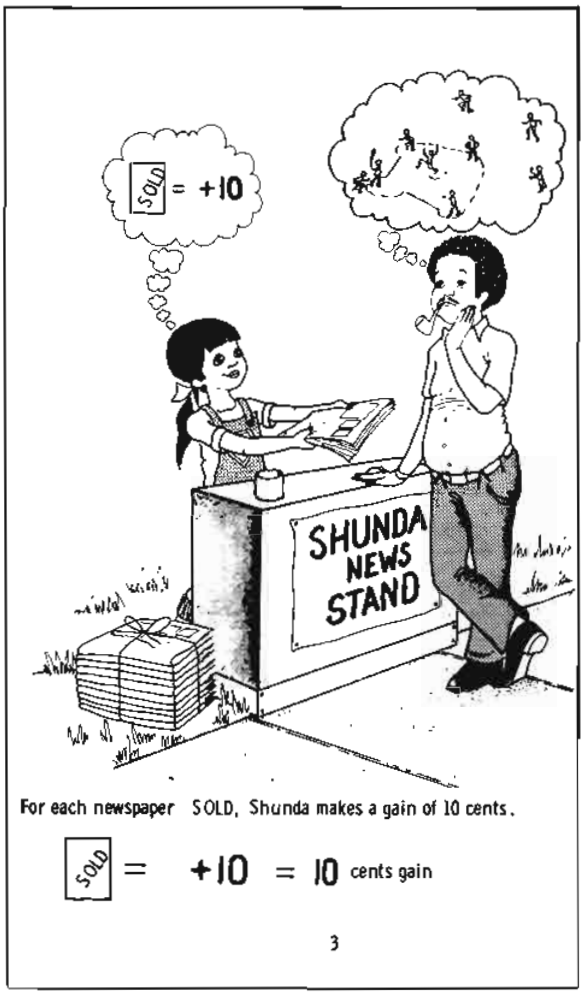


‡

† SHUNDA'S NEWSSTAND is in The CSMP Library, a collection of math story-workbooks providing fanciful excursions into the colorful world of mathematics.

‡ In this paper, we use actual reproductions (in reduced form) from the booklet, omitting color except where it is essential to the presentations.

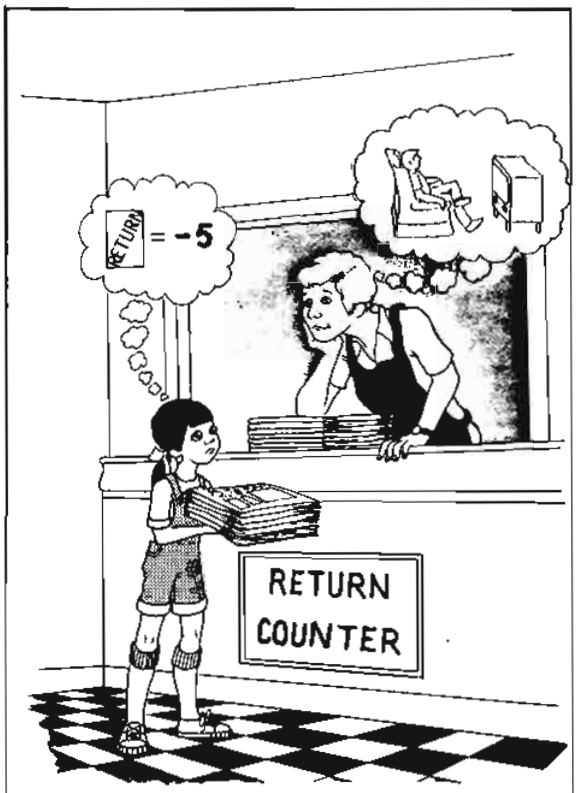
Several pages at the beginning of the booklet present basic information about Shunda's job. Shunda buys newspapers from a dealer for 10 cents each, and she sells them to her customers for 20 cents each. Since the number of Shunda's customers varies from day to day, the dealer agrees to buy back the unsold papers for 5 cents each.



For each newspaper SOLD, Shunda makes a gain of 10 cents.

SOLD = +10 = 10 cents gain

3

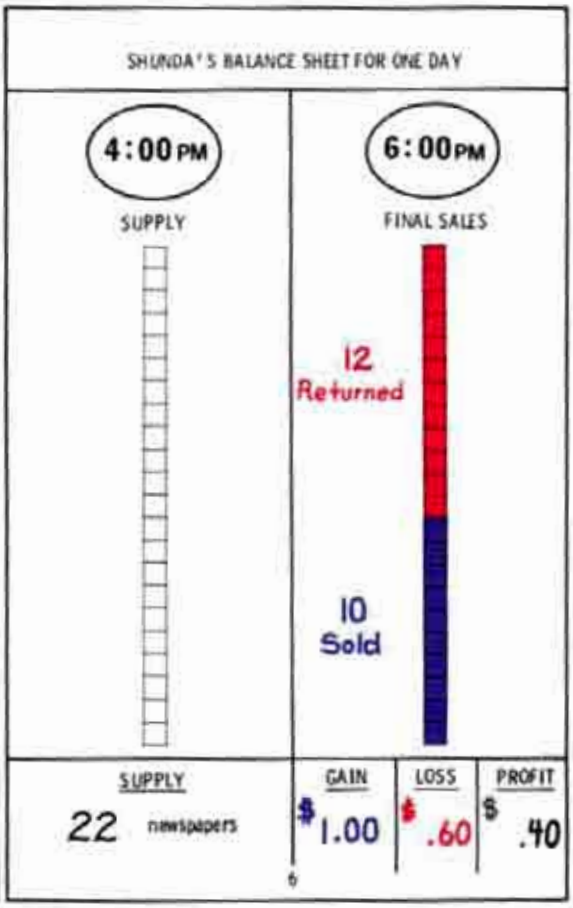


For each newspaper RETURNED, Shunda has a loss of 5 cents.

RETURN = -5 = 5 cents loss

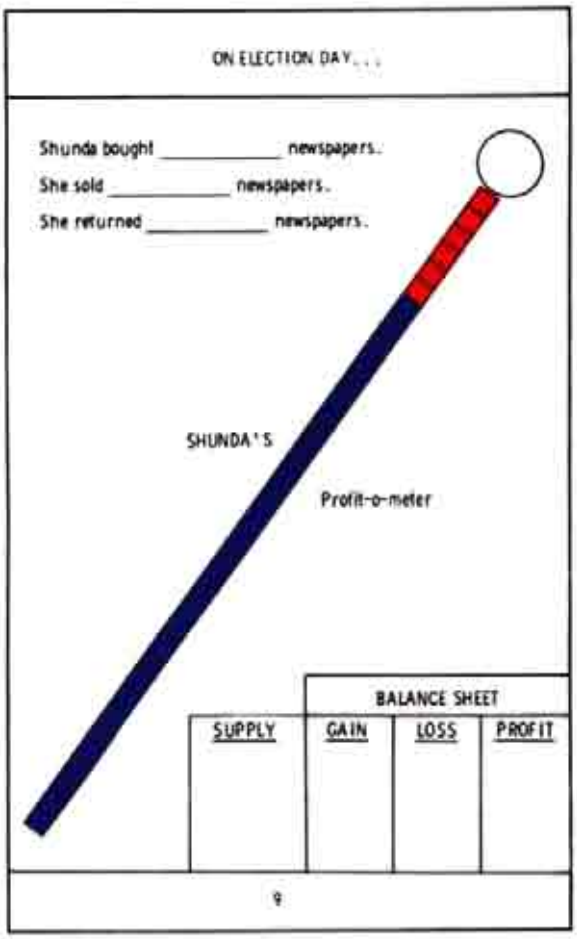
4

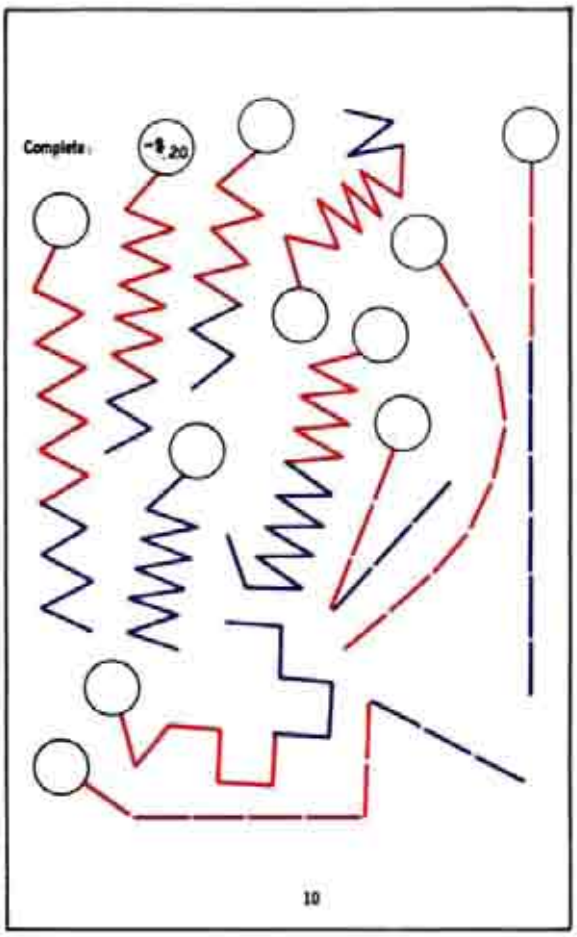
Shunda wants to have a successful business and to make as much money as possible. She devises procedures for keeping records of her business from day to day.



The students now work through several pages of practice using Shunda's record keeping procedures. We will display only selected pages from the booklet to demonstrate the variety of pictures and the nature of the individual work.

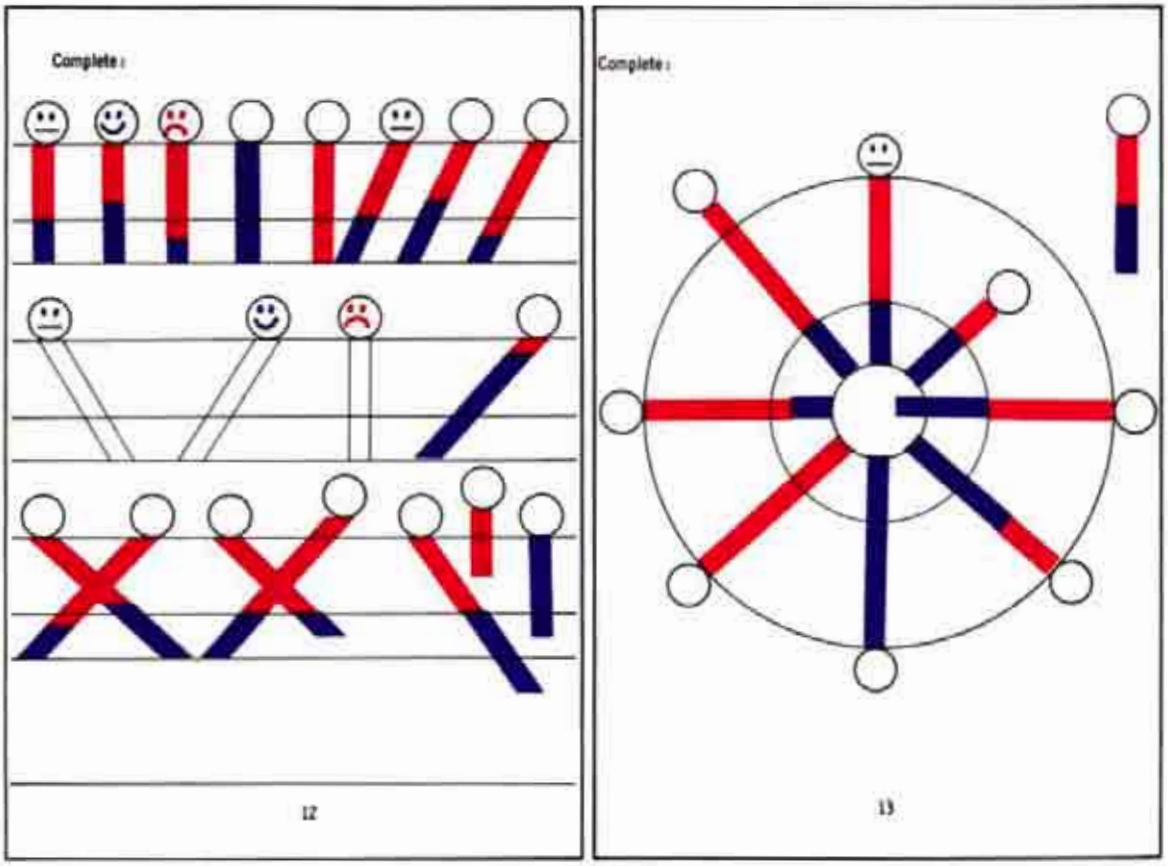
Shunda begins her newspaper business as an apprentice and so is allowed to buy differing numbers of papers from day to day depending on what she expects the demand to be.





These pages ask students to read graphs and to calculate Shunda's daily profit. The stylization on page 10 reflects Shunda's whimsical side. Such artistic freedom requires little or no explanation, and it contributes to the students' experience viewing data in a variety of ways.

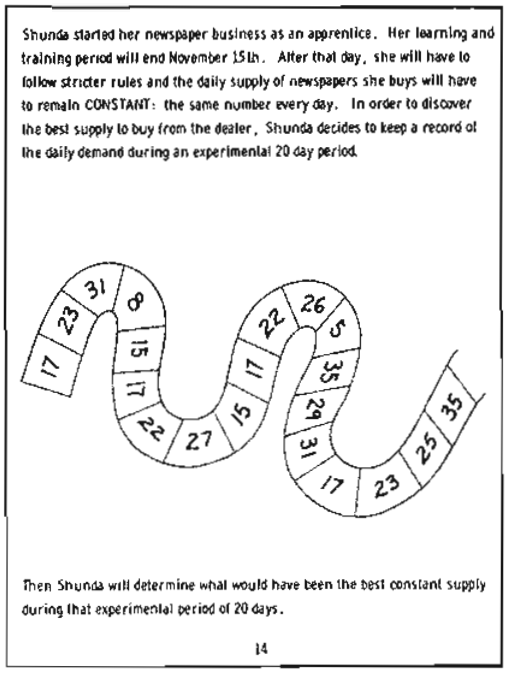
On days when Shunda has a positive profit, she is happy; when she has a negative profit, she is unhappy; and when she breaks even, she shrugs her shoulders and hopes for a better next day.

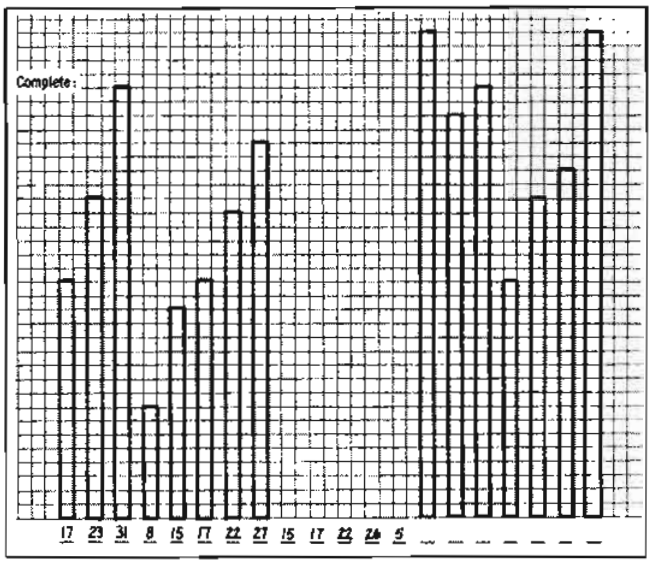


Students must use only the red:blue ratio to determine Shunda's mood or sales on several days. This step suggests the critical role in Shunda's business of a one-third, two-thirds ratio of returns to sales. Later we will see how these pictures contribute to a way of determining a most profitable inventory.

The story continues with new information that presents Shunda's main problem. The training period for Shunda will end shortly. Then she will have to follow stricter rules; the daily supply of newspapers will have to remain constant: the same number every day. Shunda must decide what constant daily supply would be the best to buy from the dealer. How can she make such a decision? In order to determine the best daily supply of newspapers to buy from the dealer, Shunda keeps a record of the daily demand during an experimental twenty-day period.

Students draw a graph of Shunda's sales record.





Shunda hopes that if she can determine what would be the best constant supply during that experimental period, then she can expect it also to be the best constant supply in the future.

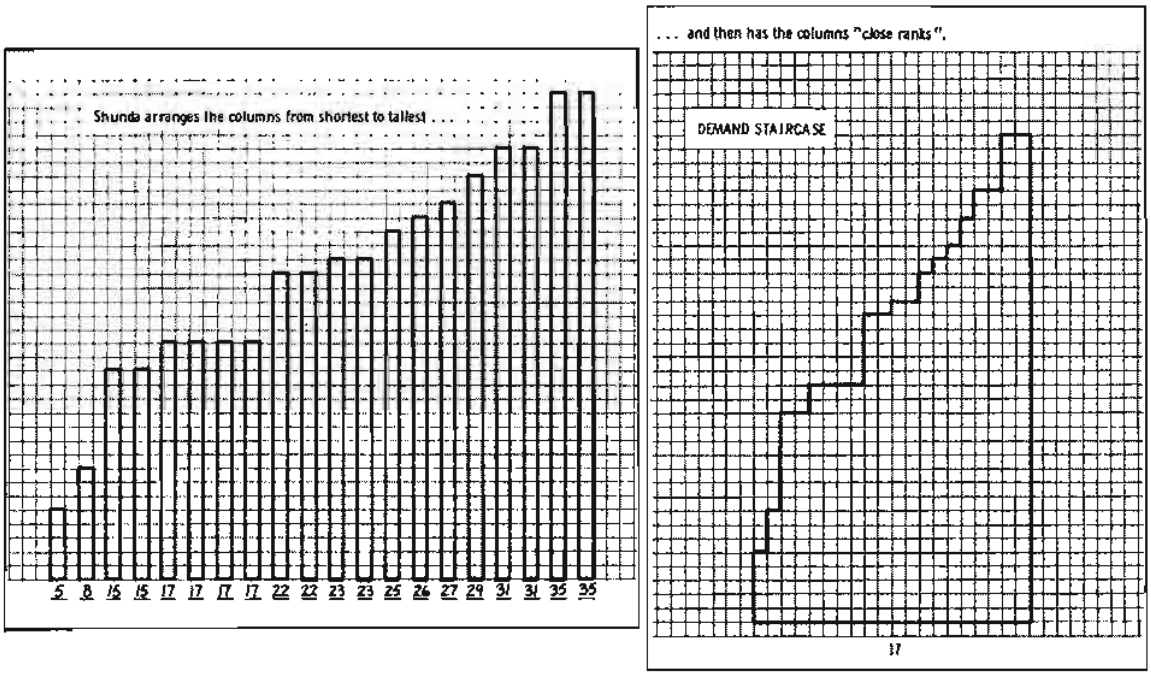
Before proceeding towards a solution, students stop to express their opinions on Shunda's decision. Such a problem interests 10 to 12 year olds; they may be just beginning to earn their own spending money. This discussion can bring out the students' natural curiosity about business matters; it tests their intuition about implications in the statistics for a twenty-day experimental period. Our experience suggests that students will think of a variety of ways of viewing the data; for example,

- Shunda should get only as many papers as she is sure to sell; then she is sure to have a positive profit every day.
- Shunda should get about 17 newspapers every day because she most often sold that number of papers.
- Shunda should get about 22 newspapers every day because half the time she sold more than 22 newspapers and half the time she sold less than 22 newspapers.

- Shunda should get about 20 newspapers every day because that is halfway between 35 (the most she sold) and 5 (the least she sold).
- Shunda should get 22 newspapers every day because that is the average number she sold daily during the experimental period.

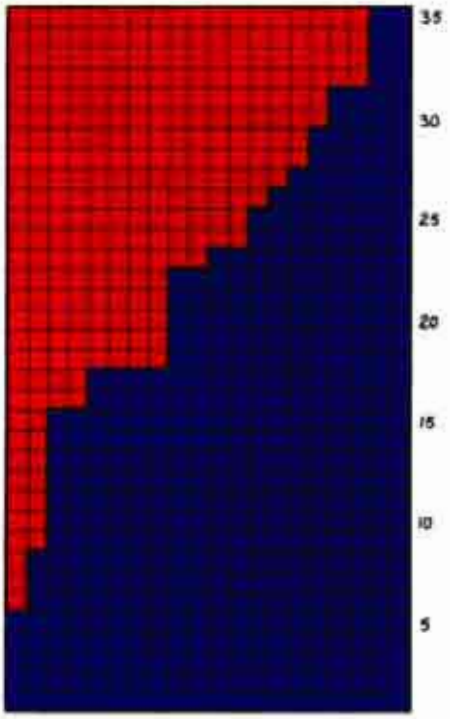
SHUNDA'S SOLUTION

We continue our story with Shunda organizing and studying the demand process as reflected in her records for the experimental twenty-day period.



Students follow along stopping to provide calculation results as needed. This contribution and their previous practice with completing balance sheets or "profit-o-meters" provide opportunities for students to find various calculation methods.

The largest demand during the twenty-day experimental period was 35. Suppose Shunda would decide on a constant daily supply of 35 newspapers. Page 19 of the booklet shows a record for the twenty-day experimental period with the constant supply of 35 newspapers. We can view this picture as putting Shunda's profit-o-meters for the twenty days side by side and as highlighting the demand staircase.



19

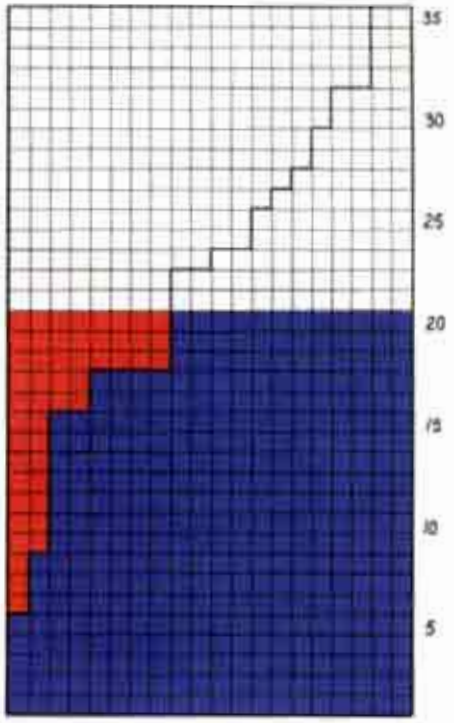
Students complete a balance sheet . . .

BALANCE SHEET for 20 day experimental period			
DAILY SUPPLY	TOTAL GAIN	TOTAL LOSS	TOTAL PROFIT
35	$44.00	$13.00	$31.00

and then question the choice of 35 as a constant daily supply. With such a supply, Shunda would satisfy her customers every day, but is it her most profitable choice? It seems that there is considerable loss (red) in the previous picture.

With the above picture and previous experience of finding daily gain, loss, and profit, students explore "what if" questions in preparation for determining a most profitable constant daily supply.

Suppose Shunda would decide on a constant daily supply of 20 newspapers. Page 21 of the booklet shows a record for the twenty-day experimental period with the constant daily supply of 20 newspapers. Would this choice produce more or less profit than a constant daily supply of 35 newspapers? There is much less loss (red) but also less gain (blue).



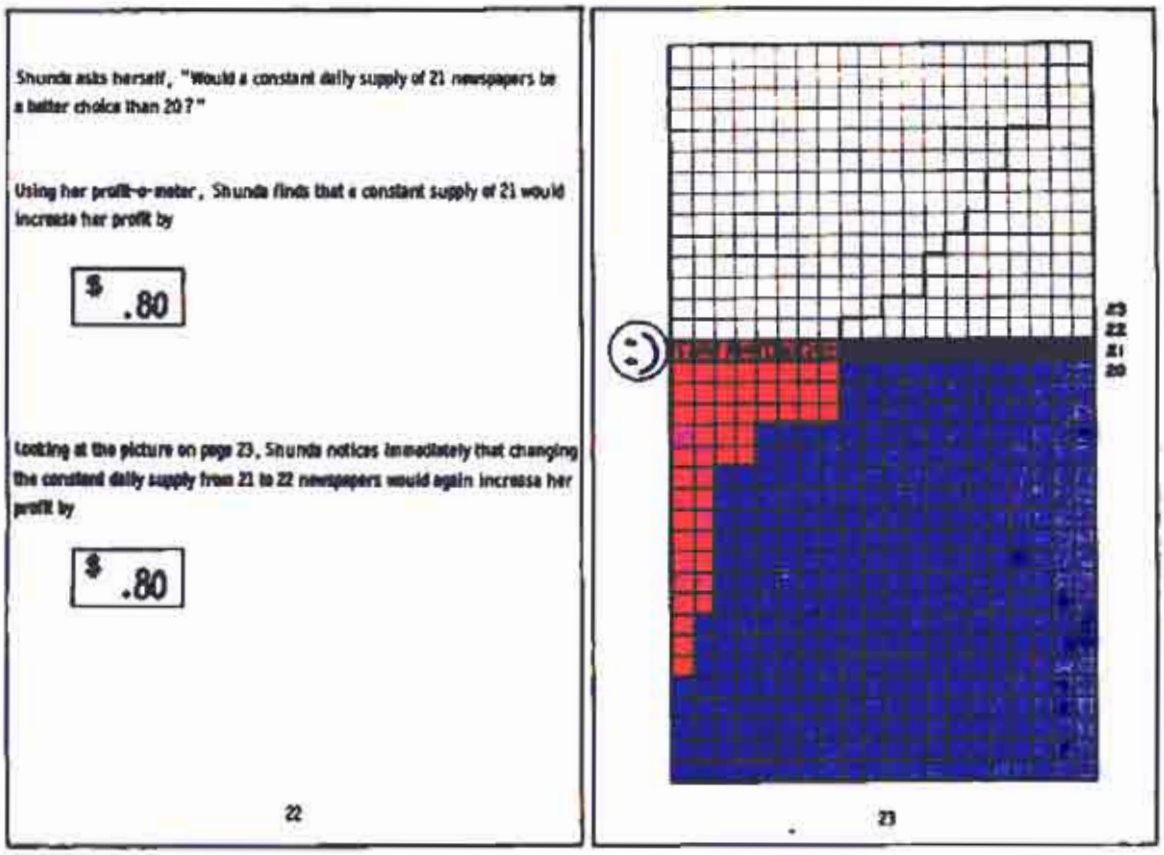
21

Students complete a balance sheet . . .

BALANCE SHEET for 20 day experimental period			
DAILY SUPPLY	TOTAL GAIN	TOTAL LOSS	TOTAL PROFIT
20	$35.10	$2.45	$32.65

and find that 20 is a better choice than 35 for the constant daily supply. But is it the best possible choice?

The next two pages of the booklet suggest a way to view the effect of changing the constant daily supply.



Shunda asks herself, "Would a constant daily supply of 21 newspapers be a better choice than 20?"

Using her profit-o-meter, Shunda finds that a constant supply of 21 would increase her profit by

$ .80

Looking at the picture on page 23, Shunda notices immediately that changing the constant daily supply from 21 to 22 newspapers would again increase her profit by

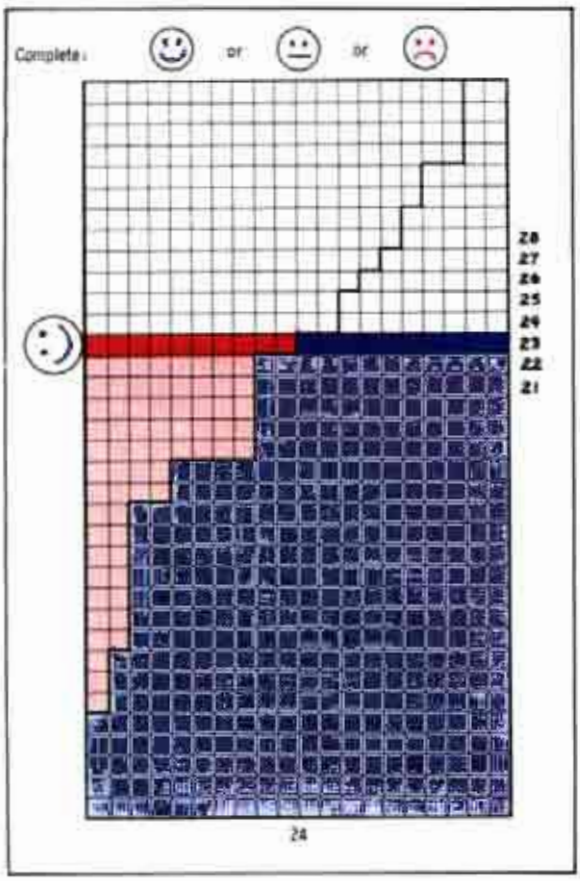
$ .80

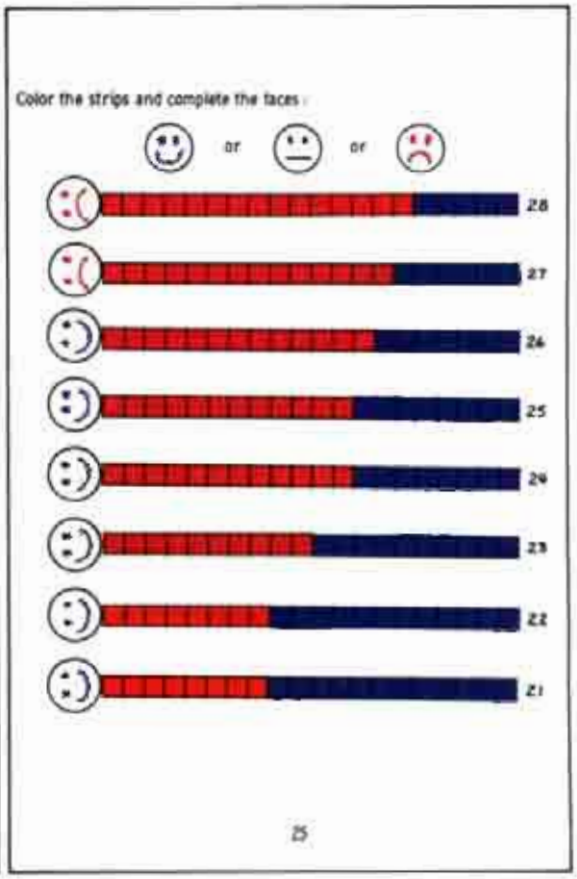
22

23

Therefore, with a constant daily supply of 22, Shunda's total profit would be $34.25 ($32.65 + $.80 + $.80). This procedure demonstrates how the earlier stylizing of Shunda's profit-o-meter contributes appropriate experience for studying the graph that describes the demand process.

Students complete additional profit-o-meters on pages 24 and 25 of the booklet (shown here with solutions) . . .





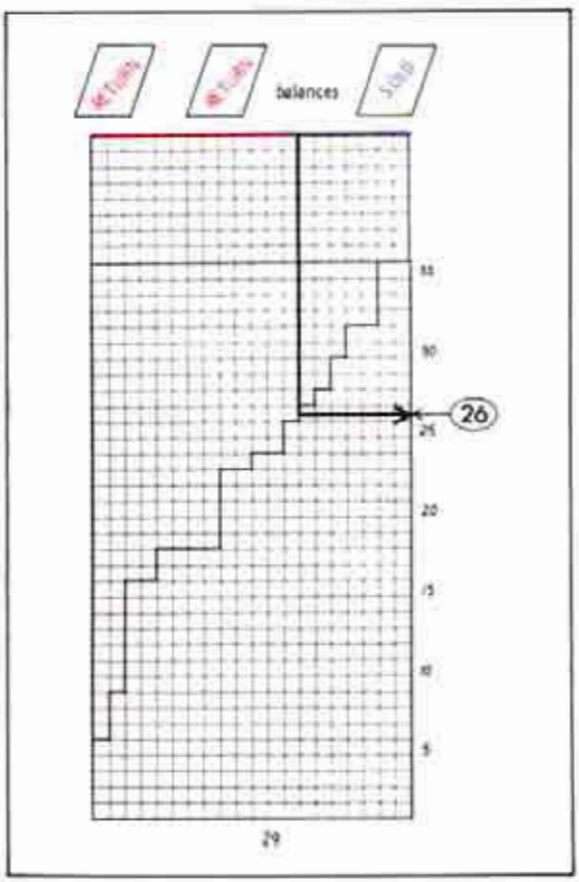
to find that Shunda could continue to increase her profit by increasing the constant daily supply until the supply is 26. Changing the constant daily supply from 26 to 27 would result in less profit as suggested by the frown on the horizontal line for 27.

SHUNDA'S CONCLUSION

During the twenty-day experimental period, the largest profit would have been obtained with a daily supply of 26 newspapers.

BALANCE SHEET for 20 day experimental period			
DAILY SUPPLY	TOTAL GAIN	TOTAL LOSS	TOTAL PROFIT
26	$40.80	$5.60	$35.20

Returning to the early observation that one newspaper sold balances two newspapers returned, Shunda takes another look at the graph describing the demand process for the twenty-day experimental period.



The story-workbook ends with a discussion of Shunda's conclusion. Since 26 papers would have been the best constant supply for the twenty-day experimental period, Shunda decides to adopt it. Of course, she cannot be sure this will remain the most profitable choice. She does not try to forecast future demand; what she does is make the best choice based on the limited evidence of the experimental period. As in most business ventures, she cannot avoid taking some risk.

This example of statistics activities for the elementary school intends to involve students in a real world application of mathematics. There is considerable calculation practice in a meaningful context. There is the opportunity to use bar graphs in a dynamic way to organize data. And, most importantly from a mathematical point of view, there is the experience of using statistics for problem solving.

* * *

In the CSMP curriculum, the activities described in this paper appear in the fifth-grade lessons from the Probability and Statistics strand.

Population Growth

Tom M. Giambrone

> Twenty-five percent of all of the people who ever walked the face of the earth are living now. Ninety percent of the scientists of all time are living now.

Population statistics can be a rich source of surprising information that suggests many implications about the world around us. The above statistics may indicate why population growth is of worldwide concern and may reflect one reason for the continuing technological explosion of this century.

Population statistics appeal to students' natural curiosity about the world around them and the future that lies ahead. The sixth-grade CSMP curriculum includes a series of lessons on population growth. The lessons provide the opportunity to organize, interpolate, and analyze real life statistics. More importantly, students use the data to make inferences about the past, present, and future — a rare activity at the elementary level in the study of statistics.

Can elementary school students handle such a sophisticated topic? Can they form inferences based upon statistical data? Students' experiences in the four lessons outlined below reveal that the answer to each question is a definite "yes."

POPULATION GROWTH RATES

The first lesson introduces students to the concept of population growth rate. Using their intuitive sense of ratio, students develop techniques to compute the growth rates and net population gains of several United States' cities.

The lesson begins with a discussion focused on the question: "What factors affect population growth?" Students conclude that the factors affecting population

fall into four categories: births, deaths, immigration, and emigration.

The discussion then moves to the meaning of the following statistic:

> In 1977, the United States showed a net gain in population. Its rate of growth was an additional 7.5 people for every 1,000 people.

Able to interpret this statistic, students proceed to develop several methods for computing growth rate and net gains. Specifically, they complete this chart of data on five U.S. cities.

City	Population	Annual Population Growth Rate per 1,000 People	Net Population Gain - One Year
Hondo, Texas	5,000	9.8	
Harrisburg, Pennsylvania	79,697	-14.7	
Honolulu, Hawaii	705,381	23.7	
New York, New York	7,895,563	-10.5	
Sunnyside, Oregon	6,208		54
Bogufusa, Louisiana	21,823		-301

Some methods that students might suggest are described below.

- Imagine the City of Hondo divided into five groups of 1,000 people. Each group gains about 9.8 people, so the net gain is about 49 people (5 x 9.8).
- Harrisburg's population is about 80,000, so the net loss is approximately 1,176 people (80 x 14.7).
- Sunnyside's population is about 6.2 thousands. Consider 6.2 x ☐ = 54 and fill in the box by trial and error or by calculating 54 ÷ 6.2. The growth rate is approximately 8.7 people per 1,000.

By not providing specific procedures for computing the net growth or the growth rate, the teacher allows students to create their own techniques. The strategies

they develop increase their understanding of the concept "population growth rate" — the key for later lessons in this sequence.

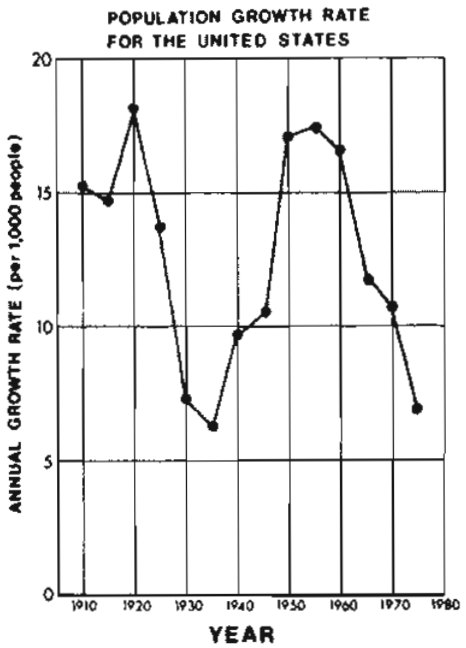
TABLES AND GRAPHS

The activities in a second lesson demonstrate that appropriate organization of data facilitates both the interpretation of the data and inferences based upon the data. Students begin by examining a mock newspaper article:

> The United States population growth rate has been declining lately. It also declined for a while before World War II. We can easily see this from the following data. For each year, the population growth rate per 1,000 people is given: 1950, 17.1; 1935, 6.3; 1965, 11.8; 1920, 18.8; 1955, 17.5; 1910, 15.2; 1975, 7.0; 1930, 7.3; 1940, 9.8; 1915, 14.9; 1960, 16.6; 1925, 13.8; 1970, 10.7; 1945, 10.7.

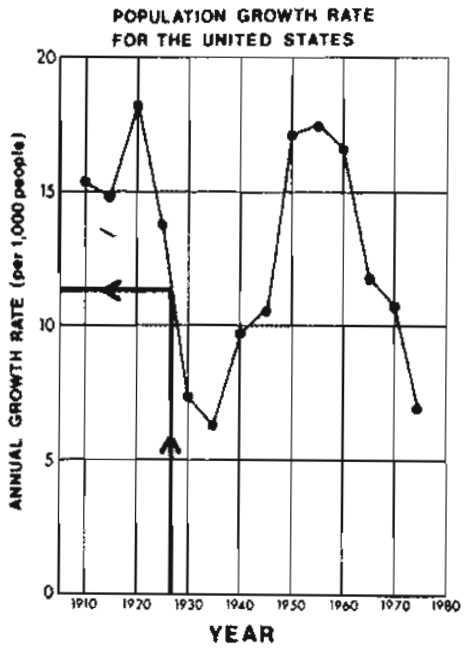
The teacher highlights the effect of the disorganized nature of the data by asking several questions such as: When was the growth rate the highest? lowest? When did it decline? The students' difficulty in answering these questions motivates the central theme of the lesson: Are there better ways in which to present this data? The students suggest a table and a graph.

Year	Annual Growth Rate (per 1,000 people)
1910	15.2
1915	14.9
1920	18.8
1925	13.8
1930	7.3
1935	6.3
1940	9.8
1945	10.7
1950	17.1
1955	17.5
1960	16.6
1965	11.8
1970	10.7
1975	7.0



After constructing the table and the graph, the class reconsiders the questions they previously found so difficult. The ease of answering the same questions accentuates the benefit of organizing data.

Further questions reveal the advantage of the graph over the table. For example, what was the growth rate in the year 1927? Through interpolation, the students can estimate a 1927 growth rate of 12 people per 1,000 (see the arrows below).



The graph also shows the large fluctuations in the U.S. population growth rate. The remainder of the lesson focuses on the historical events that could have caused such fluctuations. Students suggest the Great Depression and World War II as possible causes for the low rate of growth from 1930 to 1945. However, changing sociological views, such as people deciding to marry later or electing to have fewer children, seem a more likely cause of the recent low growth rate from 1970 to 1975.

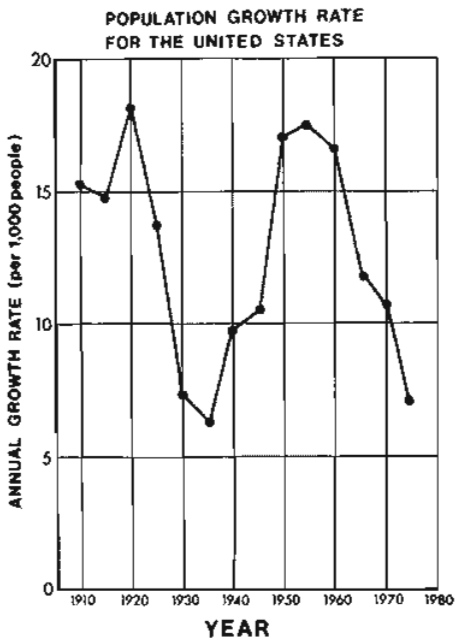
The activity of organizing data, found in the first part of this lesson, is fairly commonplace in the study of statistics. However, the discussion of factors that may have caused the large fluctuations in the population growth rate repre-

sents an important shift in the lesson. The students move from simply reading the data to interpreting and making inferences based upon the data. These attempts to interpret data allow students to appreciate the purpose of organizing data into graphs, rather than rotely practicing techniques of data organization. Correlating data with known historical and sociological factors provides a framework for later using current data to predict future events.

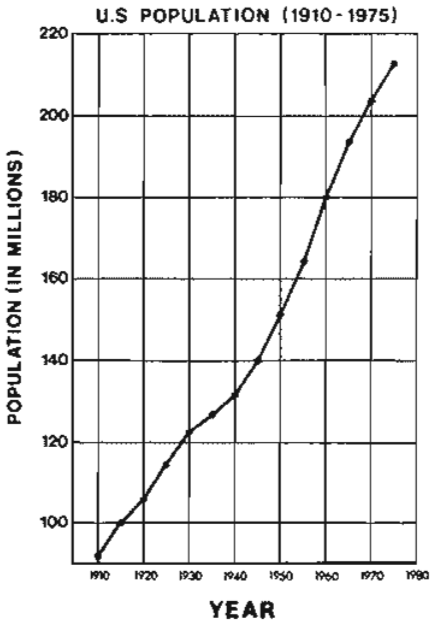
POPULATION PYRAMID

There exist many ways to graph a set of population data. The choice of a particular graph reflects the feature of the data that the statistician wishes to highlight. In a third lesson, students explore several population graphs. First, the comparison of two distinct graphical representations of population growth during the period 1910 to 1975 allows students to observe the different ways that each graph portrays the same information. Then students also analyze another graph, the population pyramid. The lesson begins with a comparison of the following graphs.

A: Population Growth Rate for the U.S. (1910-1975)



B: U.S. Population (1910-1975)



Since these two graphs display the same census data, they reflect the same trends in different ways. For example, students notice the fact that the population is always increasing, shown in graph A by the rate always being positive and in graph B by the total population always rising. Also, the low period in growth from 1930 to 1940 shown in graph A corresponds to a "leveling off" in graph B. Conversely, the sharp rise from 1950 to 1960 in graph B indicates the high growth rate that is recorded in graph A. This activity encourages students to observe different ways that graphs can display rates of growth. In particular, the graphic comparison demonstrates that a declining (but positive) growth rate and a rising total population can occur simultaneously.

Besides studying the overall growth rate, demographers also analyze the United States population according to various factors: age, race, sex, religion, etc. With this in mind, the following statistic motivates the idea of population distributions — the second theme of this lesson.

United States : K-8

1970 : 34,300,000 students
1975 : 32,000,000 students

The decline in total elementary school enrollment surprises most students. Who might be concerned about this decline in school enrollment? Teachers are most certainly concerned. Implications of this single statistic include school closings, teachers being laid off, and many other problems all too familiar to the reader. An open student discussion of this statistic focuses on such implications.

When first confronted with the above statistic, students assume that the decline in school enrollment is due to an overall decline in the population. This conjecture, however, contradicts the previous observation that the United States always had a positive growth rate from 1910 to 1975. To resolve this dilemma, the

teacher presents the idea of population distribution by age group.

U. S. Population by Age Group - 1976

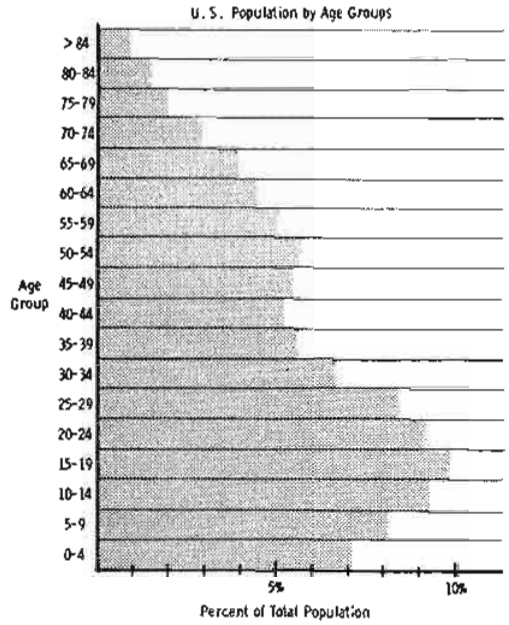
Age	Percent of Total Population
0-4	7.1
5-9	8.1
10-14	9.2
15-19	9.8
20-24	9.1
25-29	8.3
30-34	6.6
35-39	5.5
40-44	5.2
45-49	5.4
50-54	5.6
55-59	5.0
60-64	4.3
65-69	3.9
70-74	2.8
75-79	1.9
80-84	1.3
>84	0.9

Total U. S. Population: 214,649,000

By comparing the percent of population in the 5-to-14 age groups (the approximate K-8 enrollment in 1975) with the 10-to-19 age groups (the K-8 enrollment in 1970), students note that the enrollment loss appears due to a decrease in the number of people in that age group. The table reveals how the population in a particular age group can decline even as the total U.S. population continues to increase.

The introduction of population distribution by age suggests other questions of human interest such as: How many Americans remember World War II? World War I? How many Americans are over the age of 65?

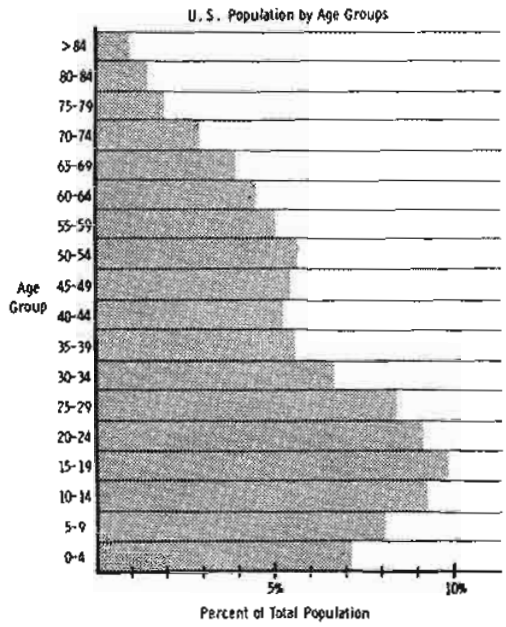
After exploring these questions, students conclude the lesson by constructing a population pyramid of the United States population.



COMPARATIVE POPULATION GRAPHS

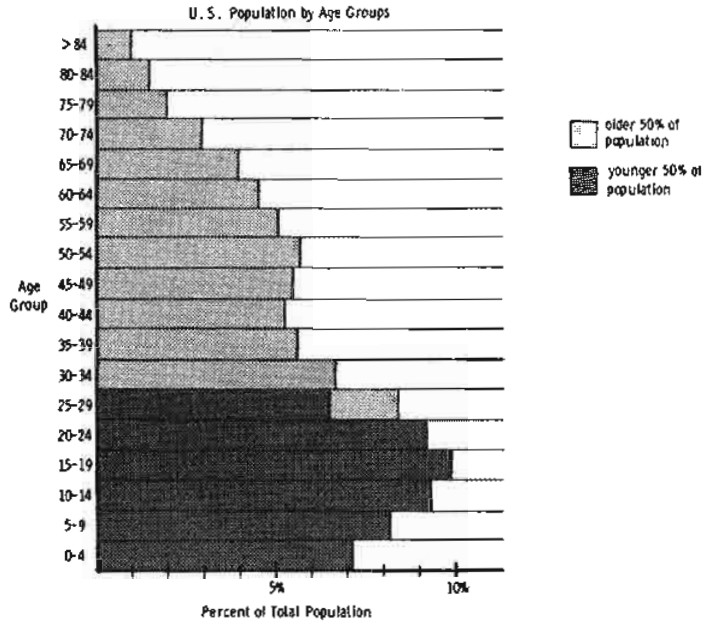
Upon graphing and interpreting a population pyramid for the United States, students naturally assume that pyramid to be "normal" for countries. In this fourth lesson, the very dissimilar shapes of both Sweden's and Mexico's population distribution graphs as compared to the United States' graph, conclusively disproves the students' assumption. A goal of this lesson is to interpret the three countries' graphs as a means for conjecturing the political priorities of the countries and what problems may lie in the future for each nation.

The beginning of the lesson returns to a conjecture made in the first lesson. Between the years 1930 and 1945 the United States experienced a low rate of growth. Two historical occurrences were given as the cause of this: The Great Depression and World War II. The teacher challenges the students to use the population pyramid to determine which event appears to have had greater impact.

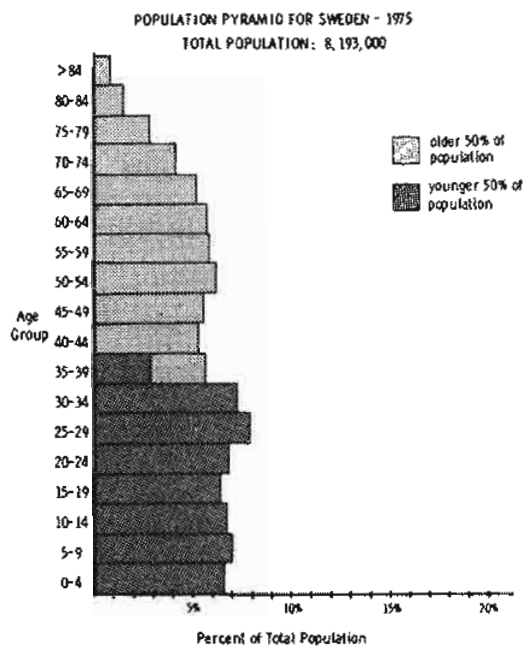


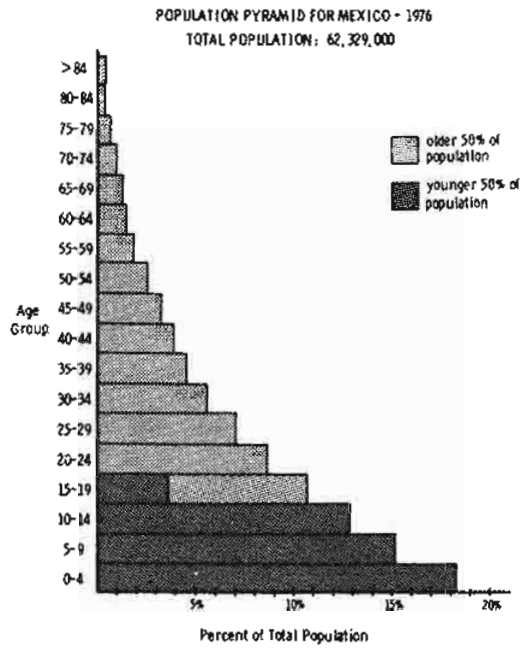
A notable feature of the above graph is the relatively low percent of people in the 40-to-44 age group compared to neighboring age groups. Being of age somewhere between 40 and 44 in 1976 means being born sometime in the years from 1932 to 1936 — the midst of the Depression. Apparently, either people during the Depression chose to have fewer children or their children had a much lower life expectancy. The graph does suggest that the Depression was a greater contributing factor to the lower population growth rate than was World War II.

Besides graphs, another simple statistical tool for comparing national populations is averaging. The population pyramid does not allow easy computation of the mean age of the population; however, the median age is appropriate and is easily computed. (The median age of a population is the age that 50% of the population is younger than and 50% is older than.) The dark gray shading on the following graph indicates the younger 50% of the population.



Most of the 25-to-29 age group is shaded dark gray. Therefore the median age of the U.S. population is about 28. After completing this computation, students discuss whether other countries are likely to have similar median ages. Even at first glance, the population pyramids for Mexico and Sweden (see below) reveal the dramatically different population distributions in different countries.





With Sweden's uniform population distribution, its median age is about 35, higher than that of the United States. In Mexico, a greater portion of the population is in the younger age groups and, therefore, the country has a low median age, about 17.

What insights into areas of concern for these governments do these population graphs and medians suggest? The remainder of the lesson focuses on a comparison of some political and social issues these countries may soon face such as:

- United States and Sweden have a large population in the older age groups who need support.
- United States is closing schools while Mexico needs to build more schools.
- Mexico's rapidly increasing population could cause shortages of food, housing, and health facilities.
- Sweden has the largest percent of older people, while Mexico has the smallest.

As in previous lessons, the lesson has moved to a discussion about the data. The importance is not so much the statistical tools that the students acquire, but the important realization that these tools enhance the discovery and discussion of events in the world around them.

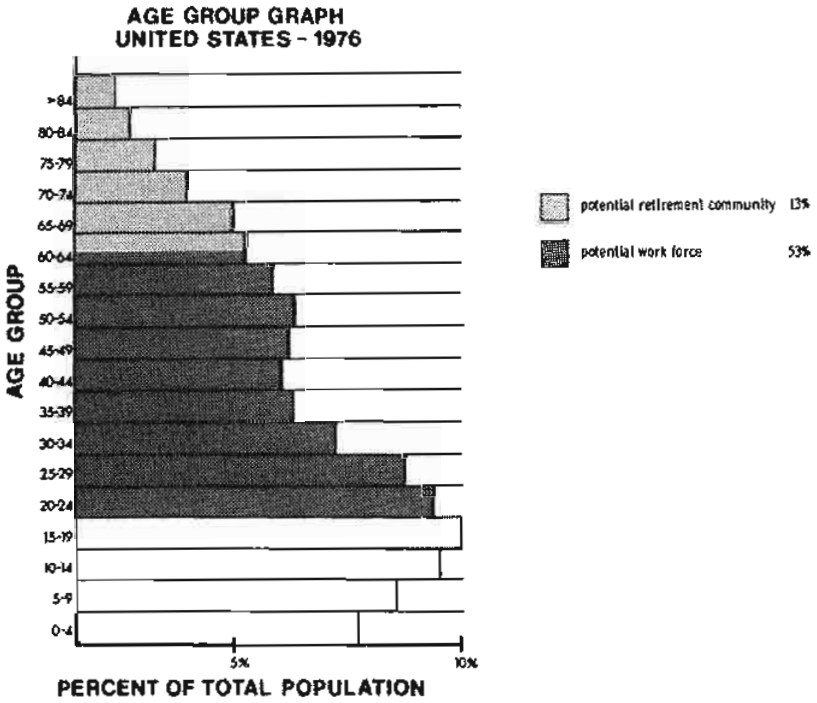
AN EXTENSION

One can extend a lesson either by embellishing one of the topics covered or by applying the same statistical techniques to a different context. There are numerous avenues for extending these lessons. One possible extension is to study the impact of the current population distribution of the United States to a topic of recent controversy: the Social Security System.

There have been several raises in the Social Security tax in recent years, each one promising to set the system on firm ground. As we shall see, the problem

with the system may lie in its basic design, therefore calling for more creative measures to rectify the system.

Some citizens assume that the Social Security System invests the money they collect and later returns the money to the original contributors. In reality, the system is designed so that the current work force generates the revenue for the retirees currently on social security. In order to gain only a crude overview and to simplify the analysis, we will contrast the total potential work force (ages 20-62) to the total potential social security receivers (over 62).



The following questions highlight some difficulties in the Social Security System:

- How many people in the potential work force does it take to maintain one person on social security?
- In 1981, how many people entered the work force for every person who entered the retirement system?
- Will the situation get better or worse? in 1990? (Assume life expectancy of 80.)

The graph provides answers to these questions. In 1976, approximately four people (4 x 13 is about 53) in the potential work force were needed for every one person on social security (a ratio of 4 to 1). In 1981, however, approximately 5% of the population entered the social security system while only 10% of the population entered the work force (a ratio of 2 to 1); thus the potential work force is not growing as fast as the potential retirees. And as we can see from the graph, the problem will continue to worsen in the future since the population entering the work force is decreasing. Questions such as those above could be used as a beginning of another lesson on statistics and economics. The important statistical activity again gives the students an opportunity of not just taking the data at face value but drawing implications from the data in regard to some social issues.†

SUMMARY

The series of lessons on population growth exhibits several valuable features for the teaching of statistics, namely:

- topics that interest students at a particular grade level,
- a problem-solving atmosphere, and
- a unifying factor for a variety of mathematical tools and concepts.

Students' curiosity about the world around them begins to emerge in the intermediate grades. Utilizing this curiosity is valuable in teaching, in particular teaching mathematics, effectively. Students' interest in certain situations can be employed to create problem-solving activities that they are intrinsically motivated to resolve with the use of mathematical tools. The Population Growth lessons are an example of such activities, using interest to explore content.

Building on a foundation of interest, the mathematical content is then introduced as a means to resolve questions and to generate a source of new inquiries, thus

† Even this simple analysis sheds light on problems affecting the Social Security System and can serve to produce many conjectures which can be tested with a more fine grained analysis. Statistics for such a treatment can be found in "Statistical Abstracts of the United States."

creating a true problem-solving atmosphere. Students learn new statistical methods as well as gain an immediate appreciation of their application.

Lastly, an important lesson can be gained for curriculum design from the way that population growth is used as a unifying theme for a variety of mathematical concepts. For example, students encounter: reading graphs, estimating, percents, ratios, as well as many other concepts and procedures that might otherwise be taught separately as disjoint pieces of mathematics. The Population Growth lessons are just one example of CSMP's curriculum design that engages students in a variety of mathematical ideas presented in an interesting and informative context.

* * *

In the CSMP curriculum, the activities described in this paper appear in sixth-grade lessons from the Probability and Statistics strand.

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