P4 SAME OR DIFFERENT #1

Capsule Lesson Summary

Using a set of marbles of two colors, select two marbles at random and note whether the two marbles are of the same color or different colors. Use cord pictures to determine the probabilities and expected frequencies of same and different. Compare expected frequencies to students' experimental results.

Materials

Teacher	• Colored chalk • Marbles	• Paper cup • Paper • Worksheets P4* and **
Student	• Marbles (two blue, one red)	

Description of Lesson

Exercise 1

Initiate a class discussion centered on the following questions.

T: ***What games do you play? Are all your games fair? What do you mean by a "fair" game?***

Which games that you play are fair? Which are unfair? How are they unfair? How can you tell if a game is fair or unfair?

We will examine some marble games to see if they are fair.

Show the class that you have one red and two blue marbles. Tell the class that you will demonstrate how to play a game.

1. Put the marbles in a cup and shake them.
2. Invite a student to take two marbles from the cup without looking.
3. Note whether the two marbles are of the same color (both blue) or of different colors (one red and one blue).

Draw a table on the board and make a tally to indicate the result.

Same	Different
I	III

Repeat the game two or three times so that the students understand the procedure.

T: ***Do you think it's more likely that the two marbles will be of the same color or that they will be of different colors?***

Let students express their opinions. Put the question in terms of a fair game: if one player gets a point for same and another gets a point for different, would it be a fair game? If not, who is favored?

Divide the class into pairs. Give each pair one red and two blue marbles in a cup. You may need to let one student be your partner or form one group of three. One student in each pair should be scorekeeper.

T: ***Each pair will play the game ten times. How many games will we play altogether?***

Adapt the following discussion according to the number of students in your class.

S: ***We have 26 students, or 13 pairs. 13* x *10 = 130, so we will play 130 games.***

T: ***About how many times in 130 games do you think we will get two marbles of the same color? Of different colors?***

Record students' predictions on the board. Encourage students to give explanations, but do not insist that they do so.

Let students play the game. When two students finish their ten games, record the results in a table on the board. Total the number of same and different for all the games.

	Same	Different
	4	6
	2	8
	3	7
	4	6
	1	9
	5	5
	2	8
	6	4
	3	7
	3	7
	5	5
	4	6
	4	6
Totals	46	84

Check that the total for same plus the total for different equals the total number of games played; in this sample data, 46 + 84 = 130.

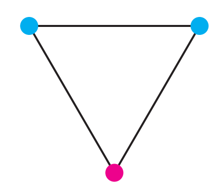
T: ***Which is favored, same or different?***

S: ***Different occurred more often than same.***

Compare the totals, 46 and 84 in the sample, to students' predictions.

T: ***Your predictions varied a lot. Here's a way to make better predictions.***

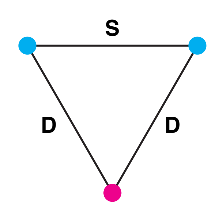
Represent the marbles by drawing one red and two blue dots on the board. Connect the corresponding colored dots with a cord as you say,



T: ***We could pick the two blue marbles.***
… or we could pick this blue marble and the red marble.
… or we could pick the other blue marble and the red marble.

So there are three ways to select two marbles. Of these three ways, how many result in two marbles of the same color?

S: ***One way, only when you select both blue marbles.***



T: ***Of these three ways, how many result in two marbles of different colors?***

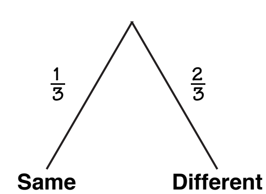
S: ***Two ways.***

T: ***Which is more likely, same or different?***

S: ***Different, because there are two ways to get marbles of different colors and only one way to get marbles of the same color.***

T: ***Here's another way to record what we found out about this game.***

Draw this tree diagram on the board, explaining that $\frac{1}{3}$ means there is one chance out of three of getting same and that $\frac{2}{3}$ means there are two chances out of three of getting different.



T: ***The probability of getting two marbles of the same color is $^1/_3$. What is the probability of getting two marbles of different colors?*** ($^2/_3$)

This picture also tells us that in three games we might expect one same and two different.

Record three games in a table on the board.

Same	Different
I	II
I	II
I	II
I	II
I	II

T: ***In three more games, we might again expect one same and two different.***

Continue adding three games at a time to the table until you have about 15 games recorded.

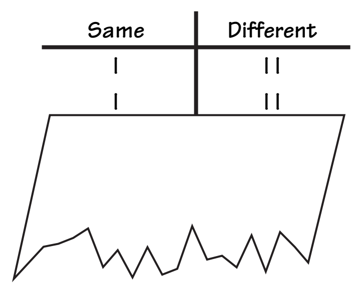
With a piece of paper, cover all but the first line of tallies in the table.

T: ***In three games, we expect same to occur once and different to occur twice.***

Record this in another table on the board.

Number of Games	Same	Different
3	1	2

T: ***In six games, we expect same to occur twice and different to occur four times.***



Record this information. Invite students to continue the table to make predictions for nine, 12, and 15 games.

Number of Games	Same	Different
3	1	2
6	2	4
9	3	6
12	4	8
15	5	10

T: ***What patterns do you notice?***

S: ***For every three games, same increases by 1 and different increases by 2.***

S: ***Different is always double same.***

S: ***Same is always one-third of the total number of games.***

Invite students to use these patterns to make predictions for 30, 60, 90, and 120 games.

Number of Games	Same	Different
30	10	20
60	20	40
90	30	60
120	40	80

P4

Adapt the following discussion to the total number of games played in your class.

T: ***We played 130 games in our class. Using this method, what would have been good predictions for same and for different?***

S: ***43 for same, because $\frac{1}{3} \times 130 = 43\frac{1}{3}$.***

S: ***43 for same, because we except 40 same in 120 games. We expect 3 same in the next 10 games since $\frac{1}{3} \times 10 = 3\frac{1}{3}$. $40 + 3 = 43$.***

S: ***87 for different, because $130 - 43 = 87$.***

S: ***87 for different, because $2 \times 43\frac{1}{3} = 86\frac{2}{3}$, almost 87.***

Compare this theoretical prediction to the actual result in your classroom. Discuss what it means for something to be the best prediction in a situation—that even though the predicted results for a particular set of games does not correspond exactly to the actual results, it is likely that they will come close, especially if many games are played.

Note: If the game is played 130 times, there is a 70% chance that same will occur between 38 and 48 times.

Exercise 2

T: ***When Anita and Bruce play this game, Anita gets one point each time*** **same** ***occurs and Bruce gets one point each time different occurs. Is this a fair game?***

S: ***No, Bruce will usually win.***

T: ***Anita and Bruce would like to make the game fair. Anita suggests changing the number of red marbles and blue marbles. How many red marbles and blue marbles do you think they should use to make this a fair game?***

Let students suggest and discuss various distributions (for example, two blue, two red; three blue, two red). It is likely that many students will suggest an equal distribution of red and blue marbles; that is, one and one, two and two, three and three, and so on.

Draw two red dots and two blue dots on the board.

T: ***Do you think playing with two red marbles and two blue marbles is a fair game?***

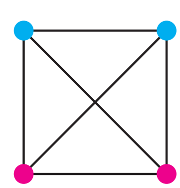
S: ***Yes, there are the same number of marbles of each color.***

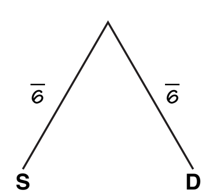
S: ***No, I think that it is more likely to select two marbles of different colors.***

T: ***Let's check it. How many ways can we select two marbles?***

S: ***Six ways.***

Invite a student to draw the six cords. Draw a tree diagram near the cord picture.



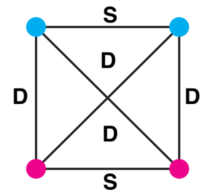


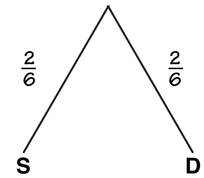
T: ***Of the six ways to select two marbles, how many ways are there to select marbles of the same color?***

S: ***Two ways.***

T: ***Of different colors?***

S: ***Four ways.***





Record the information.

T: ***Is this a fair game?***

S: ***No, there are four chances out of six of getting different and only two chances out of six of getting same.***

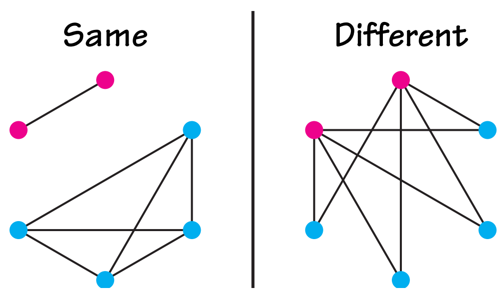
S: ***No, the probability of getting different is greater than the probability of getting same. $^4/_6$ is more than $^2/_6$.***

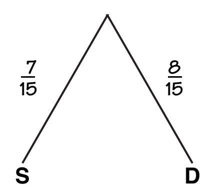
T: ***We thought that using two red marbles and two blue marbles might make this a fair game, but it does not.***

Let's continue looking for a fair same or different game with red marbles and blue marbles. What combination of marbles should we try? (Limit the possibilities to seven or fewer marbles; otherwise, pictures get very complicated.)

Note: The only combinations of seven or fewer marbles that result in a fair game are one red and three blue marbles or one blue and three red marbles. Analyze one of these combinations only if several students strongly insist. Preferably, select a different combination to analyze in this lesson and thereby allow all students to search again for a fair game in the next probability lesson, P5.

Select a combination of marbles favored by several students and analyze the game collectively in a manner similar to the preceding one. If you analyze a game with five, six, or seven marbles, suggest using two identical pictures of marbles, one for same and one for different. For example, suppose you are analyzing a game with two red marbles and four blue marbles.





Comment that the use of two pictures is optional, but that it makes it easier to draw and count all of the cords. Refer to the end of Lesson P5 for a list of solutions for all distributions of between four and seven marbles of two colors.

Optional: Complete a table of best predictions as was done in Exercise 1.

T: ***In 15 games, the best prediction of the results is seven same and eight different. In 30 games, what is the best prediction of the outcome?***

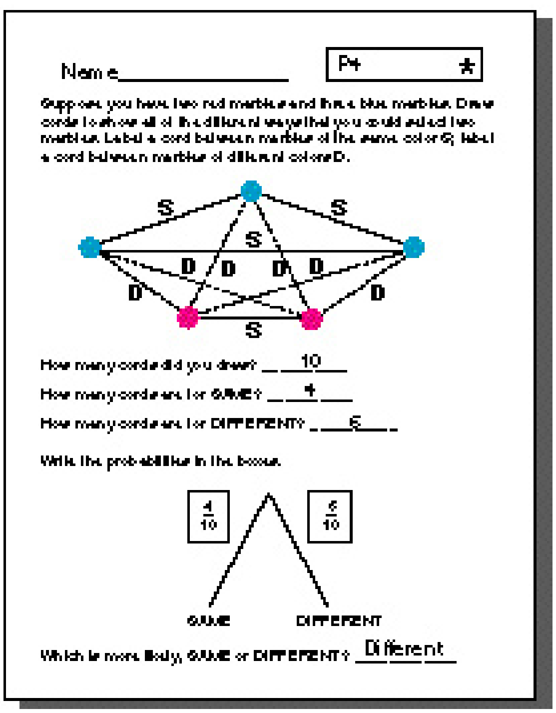
Number of Games	Same	Different
15	7	8
30	14	16
45	21	24
60	28	32
120	56	64

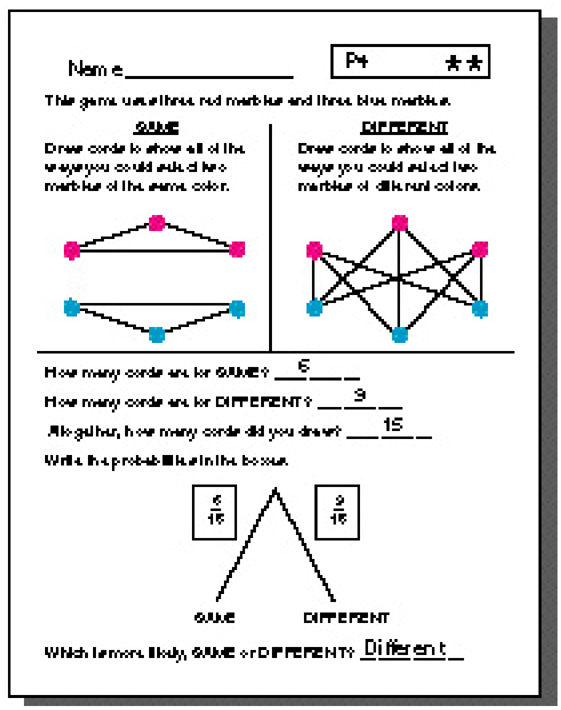
S: ***14 same and 16 different. In another set of 15 games, the most likely result is again seven same and eight different. 7 + 7 = 14 and 8 + 8 = 16.***

Record those results and complete the table for 45, 60, and 120 games.

Worksheets P4* and ** are available for individual work. If your class has already analyzed one of the combinations of marbles in a problem on the worksheet, you may want to direct students to change the marbles in that problem.

You may wish to challenge students who finish P4** to find and analyze combinations of marbles that favor Anita, that is, favor different.





Capsule Lesson Summary

Discover the relationship between a number of dots and the number of cords that can be drawn between pairs of those dots. Review the *Same or Different?* game. Analyze several distributions of marbles of two colors in an attempt to find a fair game.

Materials

Teacher
- Colored chalk
- Marbles

Student
- Colored pencils, pens, or crayons
- Paper
- Worksheet P5

Description of Lesson

Exercise 1

Draw four dots on the board.

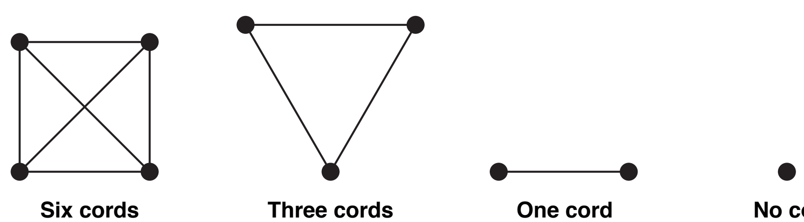
T: ***If we have four marbles, how many different ways can we select two marbles at random?***

S: ***Six ways.***

T: ***How can we check?***

S: ***Draw a cord between each pair of dots and count the total number of cords.***

Invite a student to draw the cords. Similarly, determine the number of cords for three dots, two dots, and one dot.



Record the results in a table. Draw the table in a place where you can save it for use later in the lesson.

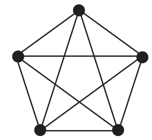
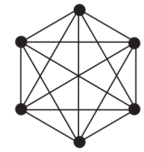
T: ***Can you predict the number of cords that can be drawn given five dots? … given six dots?***

Ask students to explain how they arrive at their predictions.

Instruct students to draw first five dots and all the cords between pairs of dots, then six dots and all the cords between pairs of dots.

Number of Dots	Number of Cords
1	0
2	1
3	3
4	6
5	
6	

T: *How many cords are there for five dots?* (10)
…six dots? (15)

Record the results in the table.

T: ***We won't draw the cord picture for seven dots, but can you predict the number of cords for seven dots?***

S: ***21.***

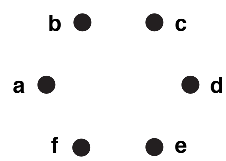
T: ***That's correct. Why did you predict 21?***

S: ***I noticed that the number of cords first increased by 1 (0 + 1 = 1), then 2 (1 + 2 = 3), then by 3 (3 + 3 = 6), and so on. 15 + 6 = 21.***

Number of Dots	Number of Cords	
1	0	
2	1	+1
3	3	+2
4	6	+3
5	10	+4
6	15	+5
7	21	+6
8	$\boxed{28}$	+$\boxed{7}$
9	$\boxed{36}$	+$\boxed{8}$
10	$\boxed{45}$	+$\boxed{9}$

Draw the following arrows to highlight this pattern. Extend the table to include eight, nine, and ten dots. Ask the students for the numbers that are in the boxes before recording them. Save the table for use later in the lesson.

Note: Your students may pose other (perhaps more sophisticated) ways to count the number of cords. For example, suppose there are six dots.



Method 1: Dot **a** connects to dots **b**, **c**, **d**, **e**, and **f** (five cords); dot **b** connects to dots **c**, **d**, **e**, and **f** (four new cords—we already counted the cord between **a** and **b**); dot **c** connects to dots **d**, **e**, and **f** (three new cords); and so on. 5 + 4 + 3 + 2 + 1 = 15, so the total number of cords is 15.

Method 2: Each of the six dots connects to five other dots by cords; 5 x 6 = 30. But every cord is counted twice this way, once at each end. 30 ÷ 2 = 15, so there are 15 cords.

You may wish to explain one of these methods to your class.

Exercise 2

T: ***Who remembers the game we played recently with red and blue marbles?***

Let students describe the game. They should mention that in the game you select two marbles and note whether they are of the same color or of different colors. Also, some students may remember that you can analyze the game by drawing cord pictures.

On the board, draw dots for two red and two blue marbles.

T: ***Many of you thought that the probability of same and the probability of different would be equal if we used the same number of marbles of each color. Was that true when we used two red and two blue marbles?***

S: ***No, different was more likely.***

T: ***Let's check that again. With four marbles, how many ways can we select two marbles at random?***

S: ***Six ways. I used the table on the board.***

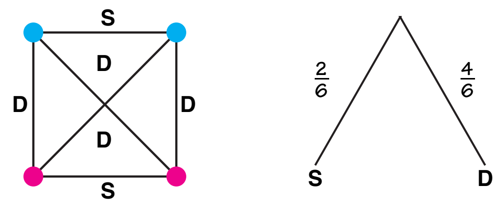
S: ***Six ways. I imagined drawing a cord between each pair of dots.***

Invite students to draw the six cords, to label each of them **S** or **D**, and to draw and label a tree diagram with the probabilities for same and different.

T: ***Which is more likely, same or different?***

S: ***Different; there are four ways to get different and only two ways to get same.***

S: ***The probability for different is greater than the probability for same. ⁴⁄₆ is more than ²⁄₆.***



You may need to adjust the following dialogue slightly if your class or some individuals in the class found a fair game in Lesson P4.

T: ***Last time we checked a few other combinations of red and blue marbles, but we did not find a fair game. We'll try again to find a fair game, but first I have a couple of questions.***

What marbles could we use if we want same to occur always?

S: ***Use two or more marbles, all of the same color. For example, use five red marbles.***

T: ***What marbles could we use if we want different to occur always?***

S: ***One red marble and one blue marble.***

S: ***If you have several colors of marbles, use exactly one marble of each color.***

T: ***Let's try to find a fair game. What combination of red marbles and blue marbles should we try?***

Select one of the combinations suggested by a student and analyze the game in a manner similar to that used in the beginning of this exercise and in Lesson P4. Solutions for all distributions using between four and seven marbles are provided at the end of this lesson.

After analyzing one combination collectively, challenge students to try to find a fair game if they have not already done so. You may like to organize students in cooperative groups to do this. Also, you may want to suggest using at most seven marbles, but allow adventurous students to use more marbles if they wish. Encourage students to use the table from Exercise 1 to check that the total number of cords for each situation is correct.

After about ten minutes, let students or groups present their findings to the class.

T: ***Has anyone found a game that is fair or almost fair?***

It is likely that some students or groups will find that using one blue and three red marbles (or one red and three blue) produces a fair game. If no one has found this solution, give the hint that there is a fair game with only four marbles.

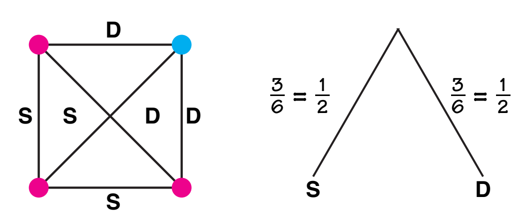


table on the board in which to record expected results of many games.

In 30 games using one blue and three red marbles , how many times should we expect same to occur? How many times should we expect different to occur?

Number of Games	Same	Different
30	15	15
50	25	25
100	50	50
150	75	75
500	250	250

Fifteen each, because it's a fair game.

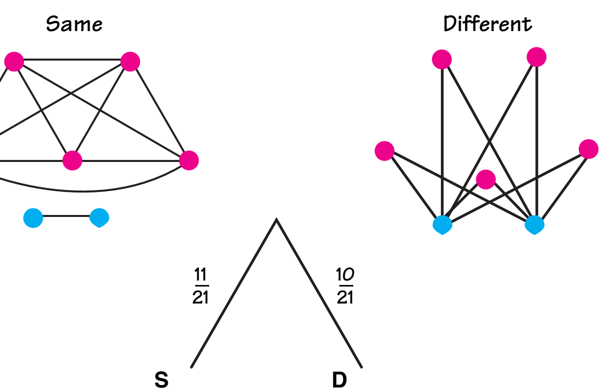
Who found a game that is almost fair?

re several games that are almost fair, for example:

- four red, two blue (7/15 same and 8/15 different)
- five red, two blue (11/21 same and 10/21 different)
- six red, two blue (8/14 same and 6/14 different)

Doubling the number of both colors of marbles in a fair game does not produce another fair Another fair game results if you use six red and three blue (or six blue and three red).

udent or group present a game they believe is nearly fair. For that game, make a table to expected results of many games. For example:



Number of Games	Same	Different
21	11	10
42	22	20
63	33	30
210	110	100
2100	1100	1000

You may observe that students need to play a lot of times before they recognize mentally) that this game is not fair.

se 3

ute copies of Worksheet P5.

Anita and Bruce decide to play the* Same or Different? *game with three colors of marbles*: *red, blue, and white. Find the probabilities for same and different in each case on the worksheet.

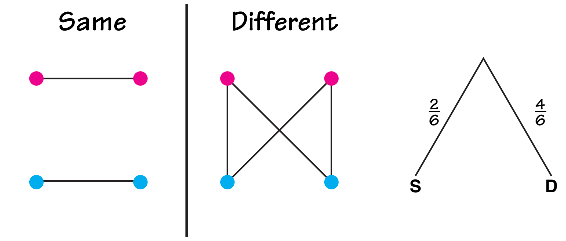
nts finish quickly, suggest they try to find a fair game using marbles of three colors.

There is no fair game using ten or fewer marbles of three colors. Several distributions e games that are nearly fair, for example:

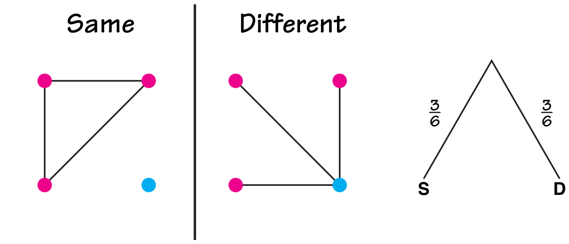
- one red, one blue, five white (10/21 same and 11/21 different)
- one red, one blue, six white (15/28 same and 13/28 different)
- one red, two blue, seven white (22/45 same and 23/45 different)

Solutions for marbles of two colors with 4, 5, 6, or 7 marbles

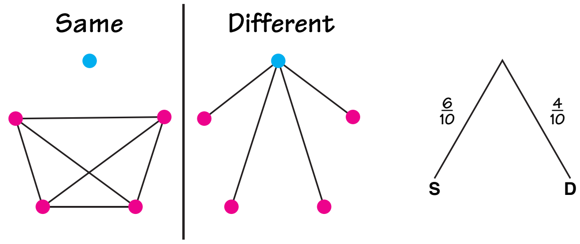
2 red, 2 blue



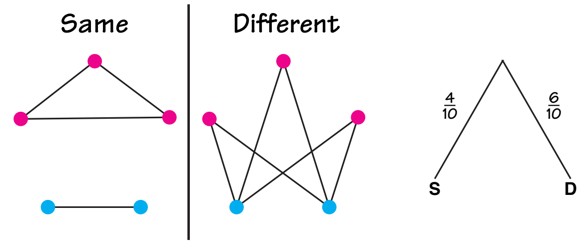
3 red, 1 blue
(equivalent to 1 red, 3 blue)



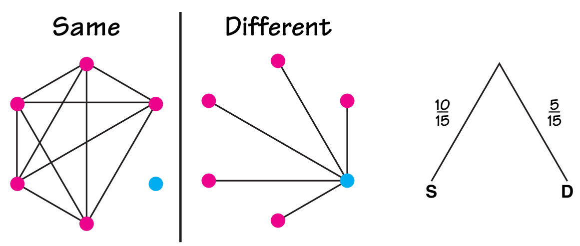
4 red, 1 blue
(equivalent to 1 red, 4 blue)



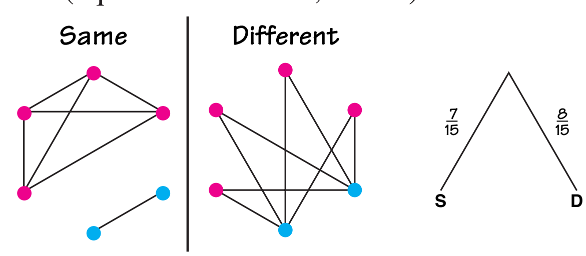
3 red, 2 blue
(equivalent to 2 red, 3 blue)



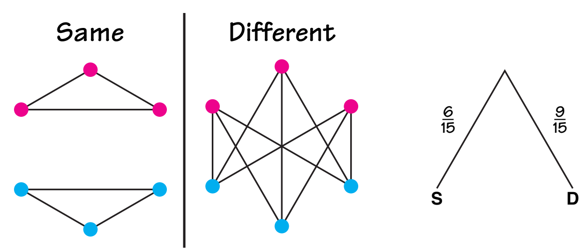
5 red, 1 blue
(equivalent to 1 red, 5 blue)



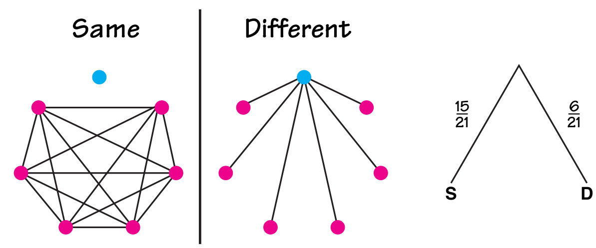
4 red, 2 blue
(equivalent to 2 red, 4 blue)



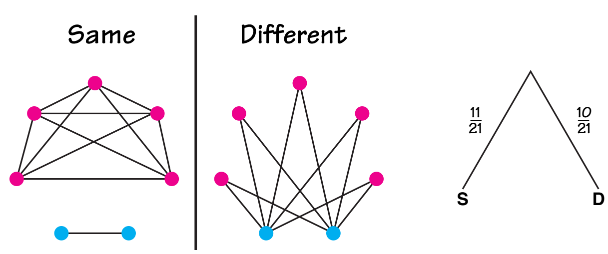
3 red, 3 blue



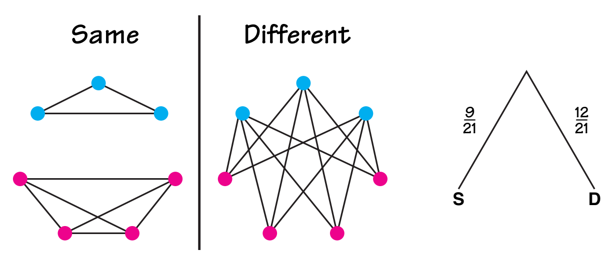
6 red, 1 blue
(equivalent to 1 red, 6 blue)

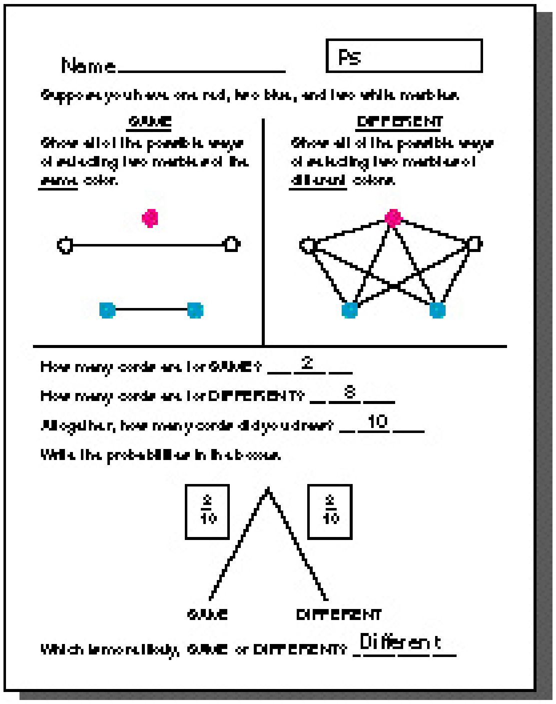


5 red, 2 blue
(equivalent to 2 red, 5 blue)

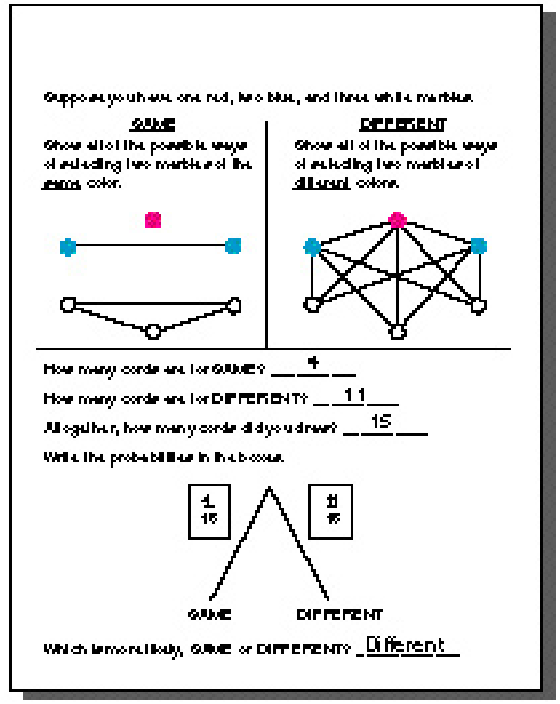


4 red, 3 blue
(equivalent to 3 red, 4 blue)





Name____________
P5
Suppose you have one red, two blue, and two white marbles.
SAME
Show all of the possible ways of selecting two marbles of the same color.
DIFFERENT
Show all of the possible ways of selecting two marbles of different colors.
How many cords are for SAME? ___2___
How many cords are for DIFFERENT? ___8___
Altogether, how many cords did you draw? ___10___
Write the probabilities in the boxes.
2/10
2/10
SAME
DIFFERENT
Which is more likely, SAME or DIFFERENT? Different



Suppose you have one red, two blue, and three white marbles.
SAME
Show all of the possible ways of selecting two marbles of the same color.
DIFFERENT
Show all of the possible ways of selecting two marbles of different colors.
How many cords are for SAME? ___4___
How many cords are for DIFFERENT? ___11___
Altogether, how many cords did you draw? ___15___
Write the probabilities in the boxes.
4/15
11/15
SAME
DIFFERENT
Which is more likely, SAME or DIFFERENT? Different

Capsule Lesson Summary

Discover the relationship between a number of dots and the number of arrows that can be drawn between pairs of those dots. Play the *Same or Different?* game but change one rule: select a marble at random, replace it, and select another marble at random. Investigate what effect the change of rule has on the probability for same and for different with various combinations of marbles.

Materials

Teacher	• Colored chalk • Marbles	**Student**	• Paper • Worksheets P6* and **

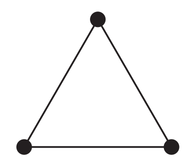
Description of Lesson

Exercise 1

Draw one group of three dots and one group of four dots on the board.

T: ***These dots represent marbles. In an earlier lesson, we counted the number of different ways two marbles could be selected from a collection of marbles. If there are three marbles, how many different ways can we select two marbles?***

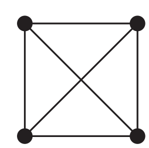
S: ***Three ways.***



Invite students to draw cords to represent selecting two marbles at a time.

T: ***With four marbles, how many ways can we select two marbles?***

S: ***Six ways.***



T: ***Today we are going to change the method of selecting the two marbles. We will select one marble, put it back, and then select another marble.***

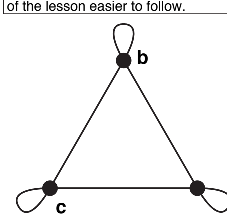
Use three marbles to demonstrate this replacement method. Emphasize the difference between the original way of selecting two marbles (two at a time) and the new way (select one marble, replace it, and then select another marble).

Refer to the cord picture with three dots.

Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: ***How can we change this picture to indicate the new way of selecting two marbles?***

S: ***Add a loop at each dot. You could get the same marble twice.***

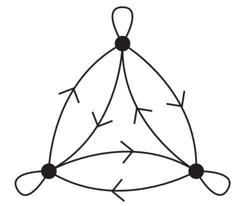


Students might not recognize the need for the following change. Use two different colors of marbles to demonstrate the difference in order of selecting two marbles using this replacement method.

T: ***We must make one more change in this picture.*** (Point to dots **b** and **c**.) ***What are two ways we could select these marbles using the new (replacement) method?***

S: ***We might select this marble (b) first and then the other marble (c), or vice-versa.***

T: ***Yes, the order of selecting the two marbles is now important. To show the two ways of selecting each pair of marbles, we'll replace each cord with two arrows.***

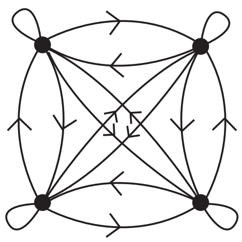


With three marbles, how many ways can we select two marbles using the new method?

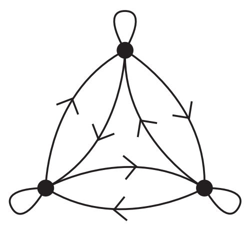
S: ***Nine ways. There are six arrows and three loops.***

Instruct students to draw four dots to represent four marbles on their papers. Then they should draw arrows and loops to show all the ways of selecting two marbles using the new (replacement) method. The picture on the board can then be altered accordingly.

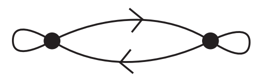
In a similar manner, determine the number of ways to select two marbles with replacement given two marbles, and given one marble.



16 ways



9 ways



4 ways

1 way

Enter the results in a table.

Number of Dots	Number of Arrows and Loops
1	1
2	4
3	9
4	16
5	25
6	

T: ***Can you predict the number of ways to select two marbles from a collection of five marbles?***

S: ***25 ways.***

Suggest students draw five dots and then the arrows and loops to confirm the prediction. Enter the results and extend the table.

T: ***What is your prediction for six dots?***

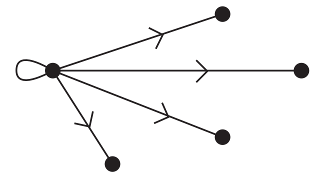
S: ***36 ways.***

T: ***Why?***

S: ***There's a pattern. The numbers in the second column first increase by 3 (1 + 3 = 4), then by 5 (4 + 5 = 9), then by 7 (9 + 7 = 16), and so on. 25 + 11 = 36.***

S: ***There's another pattern. If you multiply the number of dots by itself, you get the number of arrows and loops. 6 x 6 = 36***

Note: You may wish to use a picture on the board to explain why the number of ways (arrows and loops) is the square of the number of marbles (dots). For example, given five dots, consider one of the dots. Exactly one arrow is drawn from that dot to each of the five dots, including itself. This is done at each dot. 5 x 5 = 25, so there are 25 arrows and loops.



Illustrate students' patterns in the table and invite students to extend the table as shown here.

Save the table for use later in the lesson.

Number of Dots	Number of Arrows and Loops	
1	1	
2	4	+3
3	9	+5
4	16	+7
5	25	+9
6	36	+11
7	$\boxed{49}$	+$\boxed{13}$
8	$\boxed{64}$	+$\boxed{15}$
9	$\boxed{81}$	+$\boxed{17}$
10	$\boxed{100}$	+$\boxed{19}$

Exercise 2

With the class, review the *Same or Different?* game from Lessons P4 and P5.

T: ***Let's play the* Same or Different?** ***game again. What do you remember about the game?***

S: ***You select two marbles from a collection of marbles without looking and note whether they are the same color or different colors.***

S: ***We used cord pictures to analyze a game to determine whether same or different was more likely. We tried to find fair games.***

Draw a red dot and a blue dot on the board.

T: ***One day Anita and Bruce want to play, but they only have one red marble and one blue marble. Would it be a fair game if they selected two marbles at a time?***

S: ***No, they will always get different, one red and one blue marble.***

T: ***They decide to try the replacement method of selecting marbles. They select one marble at random, put it back, and then select another marble at random. With this method is it possible to get two marbles of the same color?***

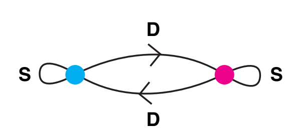
S: ***Yes, you can choose the red (blue) marble twice.***

Refer to the table from Exercise 1 as you ask,

T: ***How many different ways are there to select two marbles using the replacement method?***

S: ***Four ways.***

Invite a student to draw the arrows and loops. Label them **S** and **D** as students observe which represent same and which represents different.



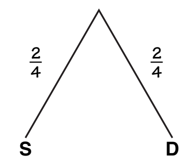
T: ***There are four ways to select the two marbles. How many ways are there to get same?***

S: ***Two ways.***

T: ***Different?***

S: ***Two ways.***

Complete a tree diagram as you announce,



T: ***There are two chances out of four for same and two chances out of four for different. So it is a fair game.***

The probability for same and probability for different are both $\frac{2}{4}$ or $\frac{1}{2}$.

Under the old rules, using one blue and three red marbles produced a fair game. Do you think it will be a fair game with the new rules?

Let your students express their opinions. Refer to the table from Exercise 1 as you ask,

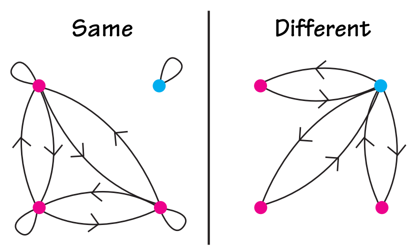
T: ***With four marbles, how many ways can we select two marbles with replacement?***

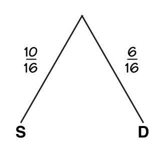
S: ***16 ways.***

Draw two groups of one blue and three red dots on the board.

T: ***Let's use two pictures, one for same and one for different.***

Invite students to draw the loops and arrows for same and different, and to complete a tree diagram.





S: ***This is not a fair game. The probability for same is $\frac{10}{16}$ and the probability for different is only $\frac{6}{16}$. Same is more likely to occur.***

S: ***It's not a fair game. There are ten chances for same and only six chances for different.***

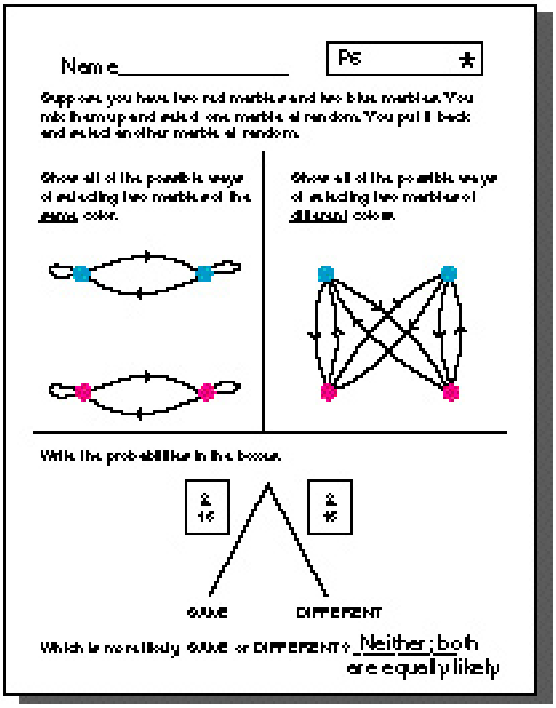
You may like to make a table to record expected results of many games for this situation. Allow time for students to do the worksheets.

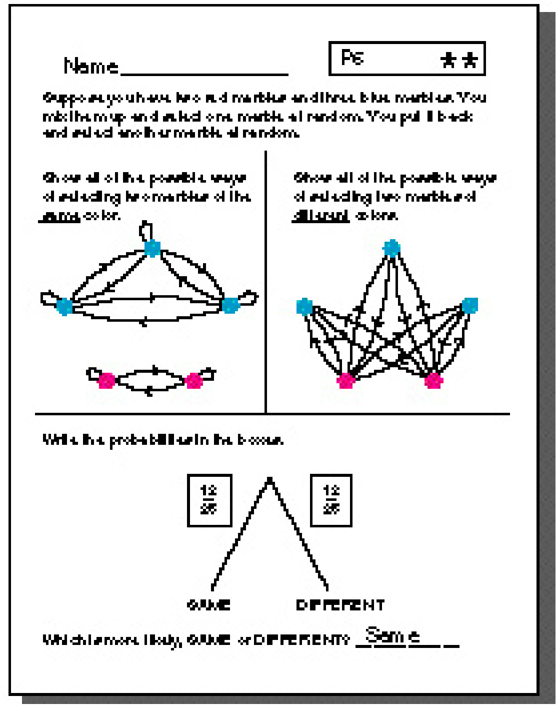
Number of Games	Same	Different
16	10	6
32	20	12
64	40	24
80	50	30
160	100	60

Worksheet P6* and ** are available for independent or group work. Ask students who finish quickly to find other collections of marbles that are fair for the *Same or* marbles with replacement.

Extension Activity

Suggest that students consider what would happen in a *Same or Different?* game with three colors of marbles when selecting two marbles with replacement.

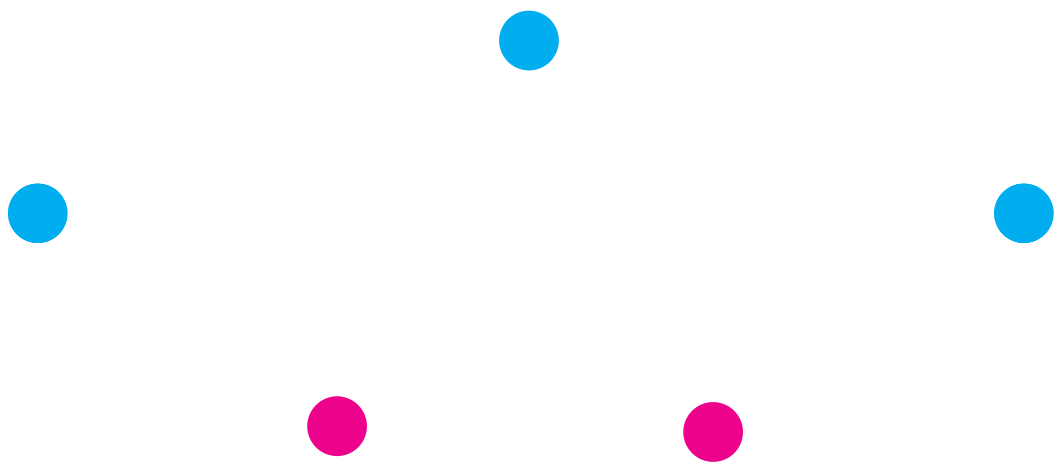

Name
P6 ★
SAME
DIFFERENT
Neither; both
are equally likely


Name
P6 ★★
SAME
DIFFERENT
Same

Name ____________________

P4 *

Suppose you have two red marbles and three blue marbles. Draw cords to show all of the different ways that you could select two marbles. Label a cord between marbles of the same color **S**; label a cord between marbles of different colors **D**.

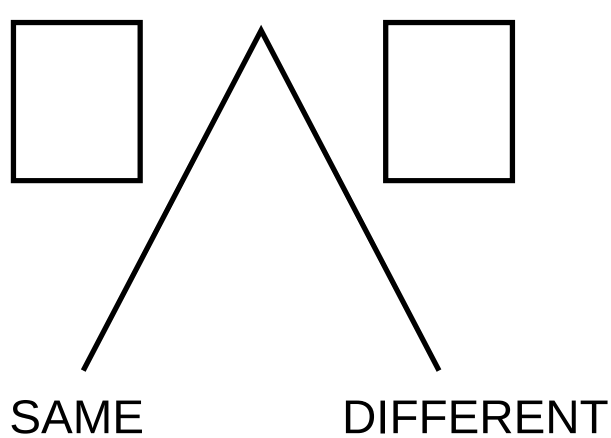


How many cords did you draw? __________

How many cords are for SAME? __________

How many cords are for DIFFERENT? __________

Write the probabilities in the boxes.



Which is more likely, SAME or DIFFERENT? ____________

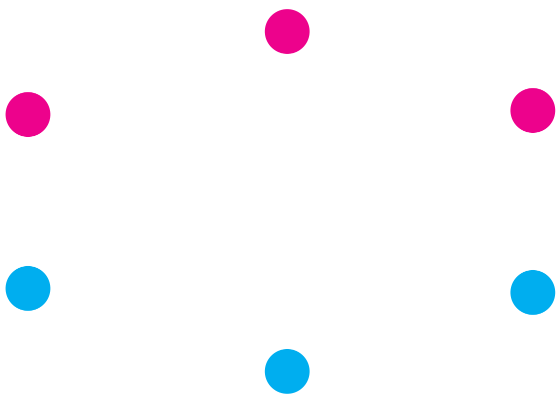
Name ____________________

P4 **

This game uses three red marbles and three blue marbles.

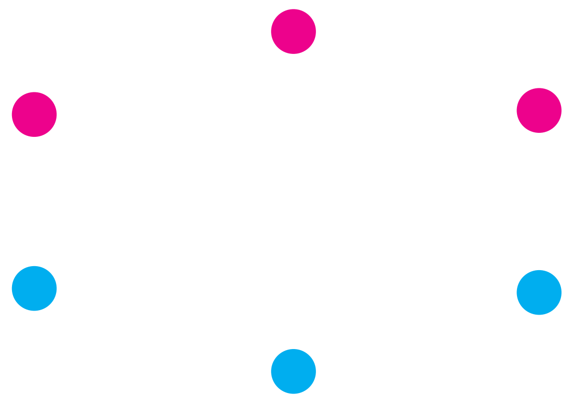
SAME

Draw cords to show all of the ways you could select two marbles of the same color.



DIFFERENT

Draw cords to show all of the ways you could select two marbles of different colors.

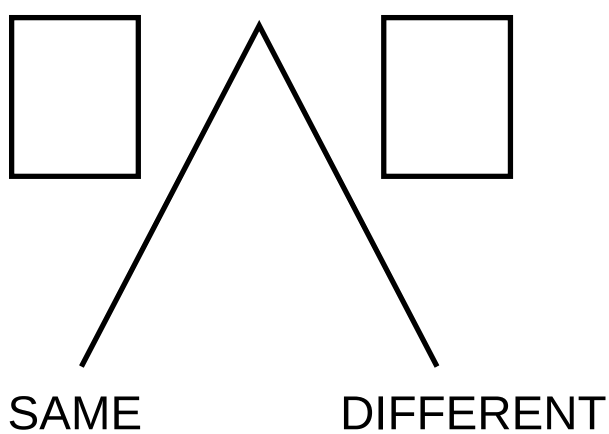


How many cords are for SAME? __________

How many cords are for DIFFERENT? __________

Altogether, how many cords did you draw? __________

Write the probabilities in the boxes.



Which is more likely, SAME or DIFFERENT? ____________

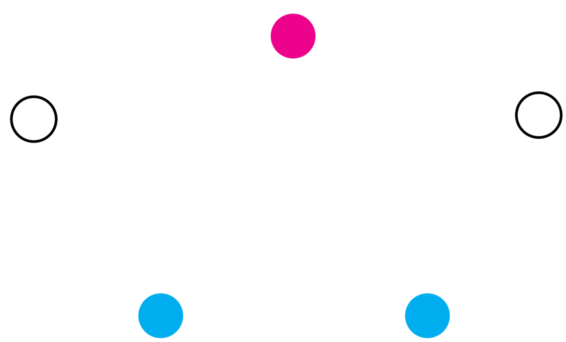
Name ____________________

P5

Suppose you have one red, two blue, and two white marbles.

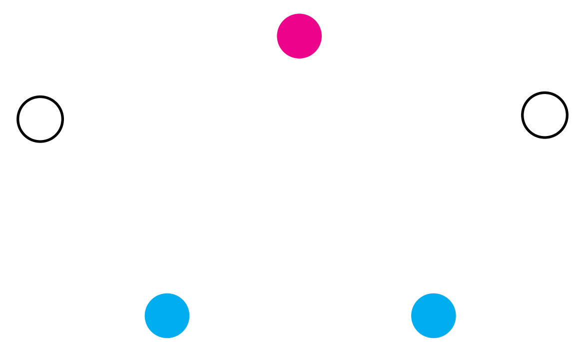
SAME

Show all of the possible ways of selecting two marbles of the same color.



DIFFERENT

Show all of the possible ways of selecting two marbles of different colors.

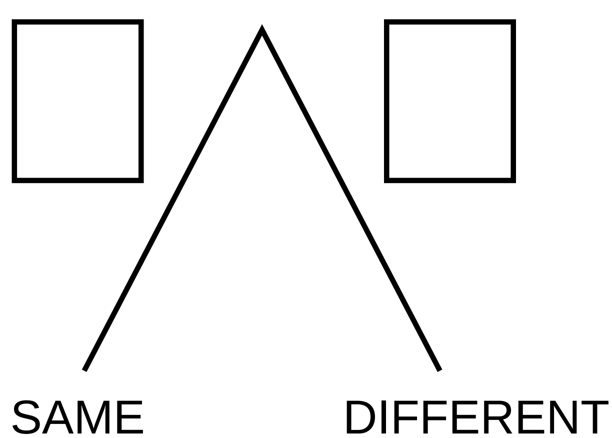


How many cords are for SAME? __________

How many cords are for DIFFERENT? __________

Altogether, how many cords did you draw? __________

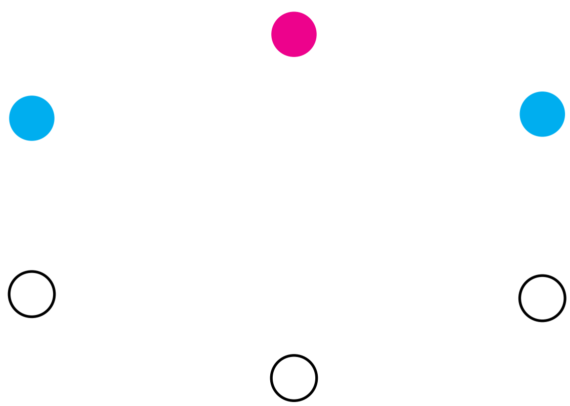
Write the probabilities in the boxes.

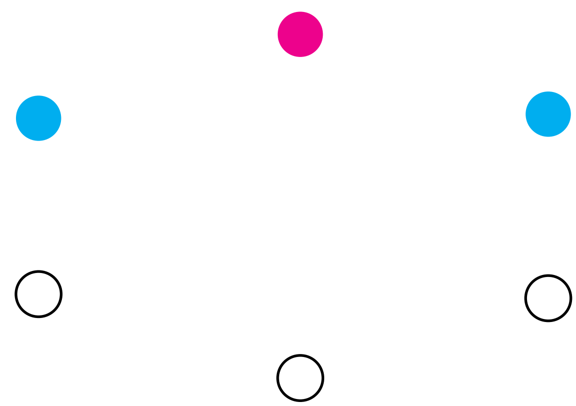


Which is more likely, SAME or DIFFERENT? ____________

Suppose you have one red, two blue, and three white marbles.

SAME	DIFFERENT
Show all of the possible ways of selecting two marbles of the same color.	Show all of the possible ways of selecting two marbles of different colors.



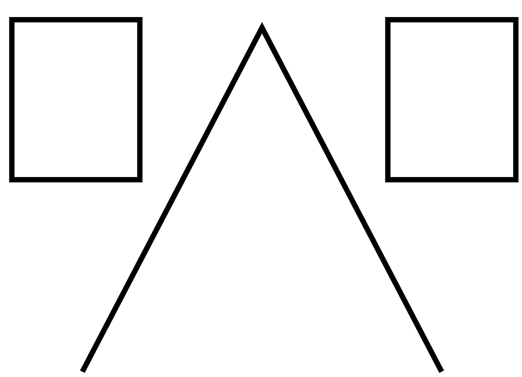


How many cords are for SAME? ___________

How many cords are for DIFFERENT? ___________

Altogether, how many cords did you draw? ___________

Write the probabilities in the boxes.



SAME DIFFERENT

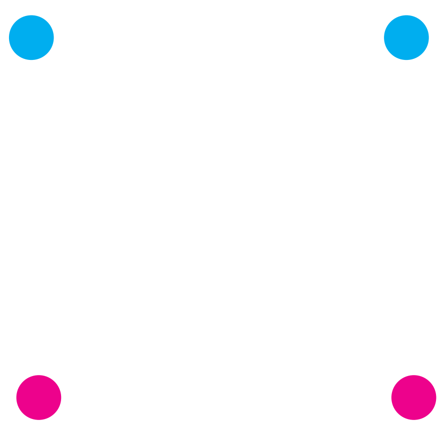
Which is more likely, SAME or DIFFERENT? _____________

Name ____________________

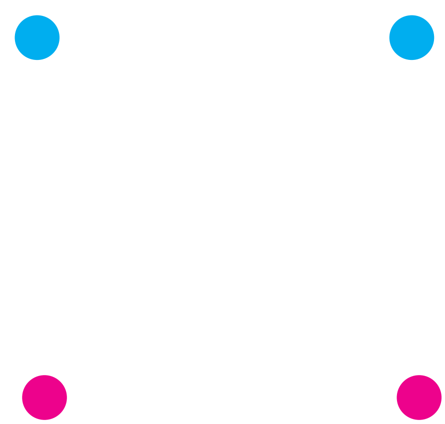
P6 *

Suppose you have two red marbles and two blue marbles. You mix them up and select one marble at random. You put it back and select another marble at random.

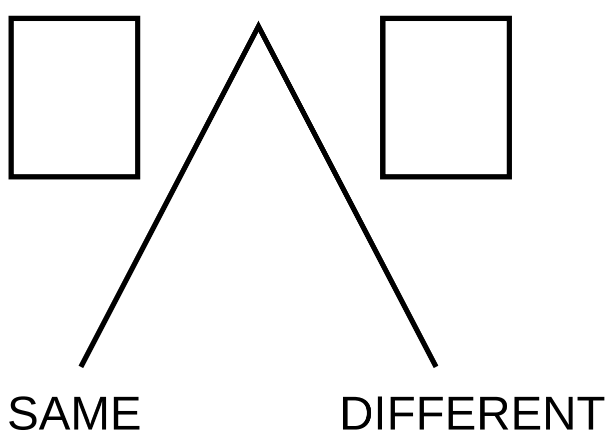
Show all of the possible ways of selecting two marbles of the <u>same</u> color.



Show all of the possible ways of selecting two marbles of <u>different</u> colors.



Write the probabilities in the boxes.



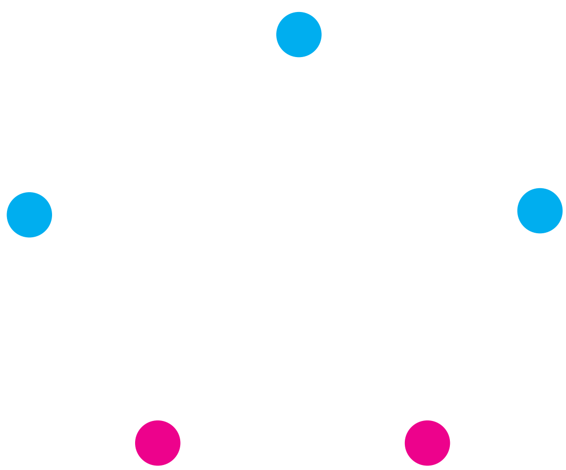
Which is more likely, SAME or DIFFERENT? ____________

Name ____________________

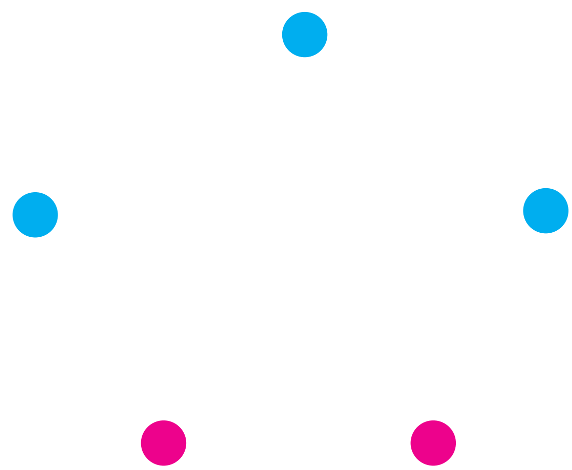
P6 **

Suppose you have two red marbles and three blue marbles. You mix them up and select one marble at random. You put it back and select another marble at random.

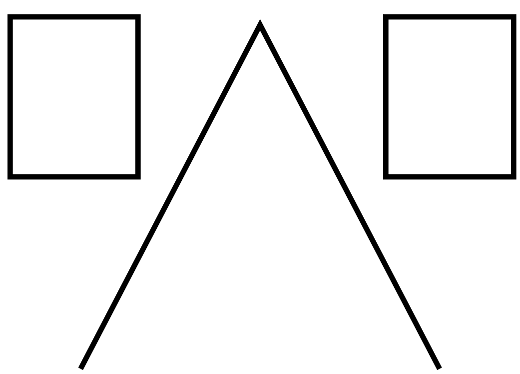
Show all of the possible ways of selecting two marbles of the <u>same</u> color.



Show all of the possible ways of selecting two marbles of <u>different</u> colors.



Write the probabilities in the boxes.



SAME DIFFERENT

Which is more likely, SAME or DIFFERENT? ____________

Probability & Statistics

PROBABILITY AND STATISTICS TABLE OF CONTENTS

PROBABILITY AND STATISTICS INTRODUCTION

In today's world, probabilistic and statistical methods have become a part of everyday life. They are an important part of industry, business, and both the physical and social sciences. Just being an intelligent consumer requires an awareness of basic techniques of descriptive statistics, for example, the use of averages and graphs. Probability and statistics are key to the present day modeling of our world in mathematical terms. The lessons in this strand utilize both fictional stories and real data as vehicles for presenting problems and applications. The problems and questions that arise focus attention on key concepts of probability and statistics such as randomness, equally likely events, and prediction.

Probability stories fascinate most students and encourage their personal involvement in the situations. They often relate the probability activities to games they have encountered outside the classroom. This personal involvement builds students' confidence and encourages them to rely on their intuition and logical thinking to analyze the situations. In *IG-III*, students use marbles or dice to simulate a situation or to play a game. These activities help students understand the story and also form a basis for predicting the likelihood of particular outcomes. Yet simulations produce only estimates of the probabilities, leaving open the question of a true probability. Pictorial techniques make the analysis of theoretical probabilities accessible. This combination of simulation and analysis of situations demonstrates the strong interdependence between probability and statistics.

Some applications of mathematics are more easily investigated than others. The probability and statistics applications in this strand lean towards the easy end of the spectrum, both because of the small amount of theoretical knowledge required, and because the link between the situation and the mathematical model is easily perceived. In particular, the development of pictorial models for analyzing the problems facilitates the ease of solutions.

Content Overview

Probability

There are many methods available for determining probabilities. The simplest techniques, though usually tedious, require listing all possible outcomes. Most powerful techniques rely on formulas involving the multiplication of probabilities. The lessons in this strand review and introduce several efficient pictorial techniques that elementary students can readily apply.

Several lessons in this strand return to Bruce, the boy who invents games of chance to play with his friends. At first glance the games appear to be fair, but students soon begin to suspect the games favor Bruce since he wins more often that either of his two friends. In Bruce's first game, a paradox arises from the fact that when two dice are rolled, the sums of 6, 7, and 8 are each more likely than any other sum. Students determine this by using a grid to list the 36 equally likely outcomes when two dice are rolled. Besides being more compact than simply listing all possible outcomes, the grid strongly suggests that there are exactly 36 outcomes.

The analyses of Bruce's other games rely on an innovative geometrical method of solving probability problems. The method makes use of a graphical representation in which a square is divided into regions according to the probabilities present in the problem. This technique allows the solution of problems dealing with multi-stage random experiments in an elegant and concrete way that avoids multiplication and addition of fractions.

PROBABILITY AND STATISTICS INTRODUCTION

The two “random art” lessons in this strand feature Nabu as a modern artist who flips a coin to determine which colors to use in his paintings. Nabu’s painting technique leads to combinatorial problems that are solved with a clever choice of an appropriate abacus. This abacus, similar to the Minicomputer which students know from lessons in the World of Numbers strand, allows students to solve quite complex counting problems.

While the major goal in these activities is the development of efficient and accessible pictorial techniques for determining probabilities, the lessons also reflect the continual development of other probabilistic themes; randomness, equally likely events, simulation, fair games, and predictions.

Lessons: P1, 2, 3, 6, 7, and 8

Statistics

Using standard relative frequencies to decipher a message presented as an unknown permutation of letters is a usual decoding technique and one that underlies even more sophisticated methods. In one *IG-III* lesson the students investigate the problem of breaking the code for a secret message using relative frequency data, and in doing so they experience its value and limitations.

Several lessons in this strand include descriptive statistics—the use of numerical and graphical techniques to summarize and compare sets of data. These activities continue to develop students’ abilities to use averages and to read, draw, and interpret bar graphs. The goal is to increase the students’ familiarity with these topics through rich experiences rather than simply to drill the techniques of computing an average or drawing a graph.

Lessons: P1, 4, and 5

Capsule Lesson Summary

Roll two dice and represent all possible outcomes in a grid. Make frequency graphs for sums and for differences. Determine the probabilities of certain sums and differences appearing. Use the information to analyze the fairness of two dice games.

Materials

Teacher		Student	
	• Two dice of different colors • Colored chalk		• Worksheets P1(a) and (b); P1* and ** • Colored pencils, pens, or crayons

Note: If you have access to an overhead projector, you may wish to prepare transparencies of the graphs and grids in this lesson instead of drawing them on the board.

Description of Lesson

Exercise 1

Show the class two different colored dice. Explain that *die* is singular for dice.

T: ***Each side of a die is called a face. How many faces does a die have?*** (Six) ***What is on the faces of a die?***

S: ***One dot, two dots, three dots, four dots, five dots, or six dots.***

T: ***When we roll a die, what is the probability of getting 4? … of getting 6?***

S: ***$\frac{1}{6}$; there is one chance out of six for any number 1 to 6.***

Based on their experiences in games, some people believe it is less likely to roll a 6 than other numbers. As appropriate, lead a discussion on the equal likelihood for each of the six faces.

T: ***What is the probability of rolling a number less than 3?***

S: ***$\frac{2}{6}$; there are two chances out of six of rolling a number less than 3.***

T: ***What is the probability of rolling an odd number?***

S: ***$\frac{3}{6}$ or $\frac{1}{2}$.***

T: ***Bruce, Helen, and Victor are friends. Bruce often invents games for them to play. Usually Bruce invents fair games, but sometimes he likes to make a game that favors him. Helen and Victor must always be alert for Bruce's tricks.***

In one game that Bruce invents, two dice are rolled and the sum of the numbers on the two dice determines the winner. What sums are possible?

S: ***2 through 12.***

Write this information on the board as you describe Bruce's game.

Helen	Bruce	Victor
2 3 4 5	6 7 8	9 10 11 12

P1

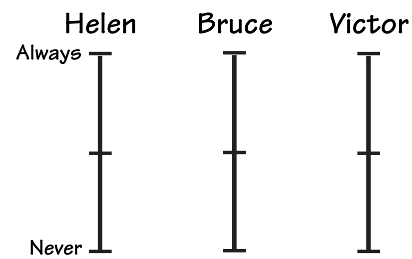
T: ***After rolling two dice, Helen wins if the sum is 2, 3, 4, or 5; Bruce wins if the sum is 6, 7, or 8; and Victor wins if the sum is 9, 10, 11, or 12. Do you think this is a fair game?***

S: ***No, I think Helen and Victor are favored. They each have four winning numbers, while Bruce has only three.***

S: ***No, I think Bruce is favored. It's easier to get the sums in the middle than to get Helen's or Victor's sums.***

S: ***It could be fair.***

On the board, draw three line segments of equal length, and label them for the three players.

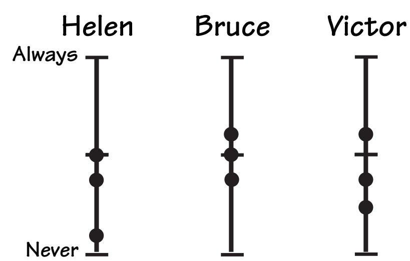


T: ***Do you remember how we use segments like these to record probabilities?***

S: ***Yes, a dot at the top of the line segment is for an event that is certain to happen. A dot at the bottom is for an event that can never happen. The probability of an event that might happen is represented by a dot somewhere along the line segment.***

T: ***Where should we draw dots for each player's probability of winning?***

Suggest that students do this on their papers, and invite several students to put their guesses on the segments on the board. For example:



Make observations appropriate to students' estimates such as the following.

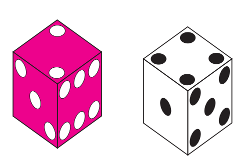
T: ***Twyla thinks that Bruce is favored. Derrick thinks that Helen and Victor have equal chances. Julie thinks that Bruce's probability of winning is exactly ½.***

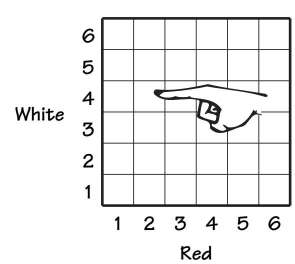
Distribute copies of Worksheet P1(a), and sketch a similar six-by-six grid on the board or use an overhead transparency.

Explain that each square in the grid is for a way the dice can land when tossing two dice; it indicates what is on the red die and what is on the white die. **Note:** Use the colors of your dice.

Toss the two dice and ask,

T: ***Which square is for this way the dice land?***





Let a student locate the square while you point to 2 at the bottom edge of the grid and 4 along the side edge.

T: ***What is the outcome (sum)?***

S: ***6, because 2 + 4 = 6.***

Direct students to write 6 in the square. Call on students to suggest other possible ways for the dice to land, or allow them to experiment by tossing the dice; then ask them to locate the appropriate squares. Write outcomes (sums) in the squares. You may also point to a square in the grid and ask how the dice land. After three or four squares have been filled in, ask for a way for the dice to land, given a specific sum.

T: ***What could be on the red die and on the white die if the sum is 8.***

S: ***4 on both dice.***

S: ***3 on the red die and 5 on the white die.***

S: ***5 on the red die and 3 on the white die.***

As the ways are mentioned, locate and fill in the appropriate squares.

White						
6		8			11	
5			8			
4		6	7	8		
3					8	
2						8
1			4			
	1	2	3	4	5	6
	Red					

T: ***How many different ways can we get a sum of 8?***

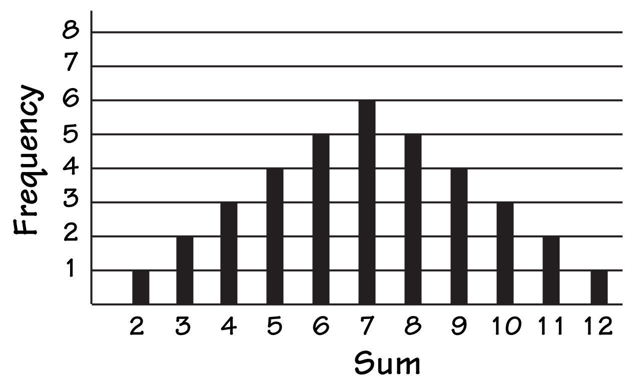
S: ***Five; there are five 8s in the table.***

T: ***The frequency of a sum is the number of different ways to roll that sum. For example, we found that there are five ways to roll a sum of 8. A bar in this graph*** (indicate the bottom half of the worksheet) ***shows the frequency of 8 is 5.***

Direct students to fill in all 36 squares in the grid and then to complete the frequency bar graph. Encourage students to observe and use patterns.

Sum of Two Dice

White						
6	7	8	9	10	11	12
5	6	7	8	9	10	11
4	5	6	7	8	9	10
3	4	5	6	7	8	9
2	3	4	5	6	7	8
1	2	3	4	5	6	7
	1	2	3	4	5	6
	Red					



T: ***There are three ways to get a sum of 10. What is the probability of rolling a sum of 10?***

S: ***3⁄36. There are three ways to get a sum of 10. There are 36 ways to roll the two dice since there are 36 (6 x 6 = 36) squares.***

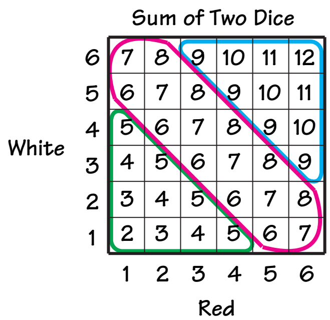
Use the grid and/or frequency graph to discuss probabilities of various outcomes. Your class may observe symmetry; that is, 2 and 12 have the same probabilities, as do 3 and 11, as do 4 and 10, and so on.

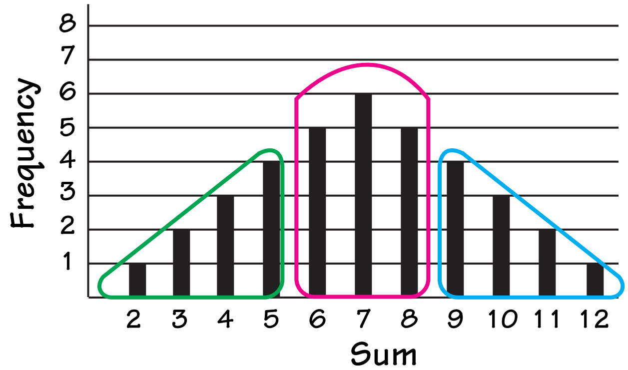
T: ***Let's check whether Bruce's game is fair. When does Helen win? In how many ways can Helen win?***

S: ***If the sum is 5 or less (sums of 2, 3, 4, and 5).***

S: ***Ten ways; ten squares have a sum of 5 or 4 or 3 or 2.***

Suggest students circle the winning sums for each player in their grid or frequency graph, and then determine the number of ways to win for each player.





Helen: 10 **Bruce: 16** **Victor: 10**

T: ***What is each player's probability of winning?***

S: ***Helen's probability of winning is $\frac{10}{36}$, Bruce's is $\frac{16}{36}$, and Victor's is $\frac{10}{36}$.***

T: ***Is the game fair?***

S: ***No, Bruce is favored.***

T: ***Why is Bruce favored even though he wins on only three sums?***

S: ***There are more ways to roll a sum of 6, 7, or 8 than to roll any of the other sums.***

Refer back to the three line segments on which students indicated their estimates of each player's probability of winning.

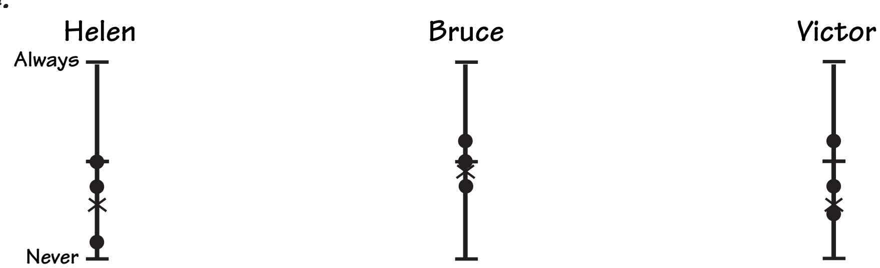
T: ***Let's put an X to show each person's actual probability of winning.***

Let the students discuss where to place Xs You may need to direct the discussion toward the following line of reasoning.

S: ***Draw Bruce's X just below the middle. His probability of winning is $\frac{16}{36}$, which is a little less than $\frac{18}{36}$ or $\frac{1}{2}$.***

S: ***Helen's and Victor's probabilities of winning are equal, so their Xs should be at similar locations. Their Xs should be lower than Bruce's since Bruce is favored.***

S: ***Helen's and Victor's probabilities of winning are both $\frac{10}{36}$. $\frac{9}{36} = \frac{1}{4}$, so draw their Xs just above $\frac{1}{4}$.***



Determine which dots indicated better estimates.

T: ***Victor likes this dice game but would like to make it fair. He thinks they should share the possible sums in a different way. How could Helen, Bruce, and Victor share the sums to make it a fair game?***

Let students explore and make suggestions. You may like to let students work with partners or in small groups to find fair games. If no one finds a fair game, give the following hints.

T: ***How many ways are there to roll two dice?***

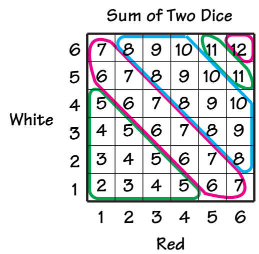
S: ***36 ways, there are 36 squares.***

T: ***To be a fair game, how many ways must each person have of winning?***

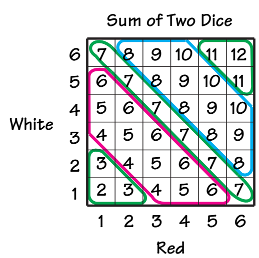
S: ***12 ways; with three players each person should have*** $\frac{1}{3} \times 36 = 12$.

T: ***So we must find three sets of sums with each set giving 12 ways to win. How could we do that?***

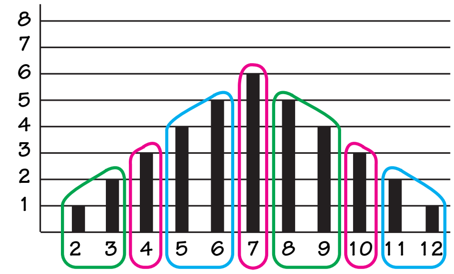
There are many ways to share the sums among three people to produce a fair game, for example:



Helen {2, 3, 4, 5, 11}
Bruce {6, 7, 12}
Victor {8, 9, 10}



Helen {2, 3, 7, 11, 12}
Bruce {4, 5, 6}
Victor {8, 9, 10}



Helen {2, 3, 8, 9}
Bruce {4, 7, 10}
Victor {5, 6, 11, 12}

As a homework assignment, you may wish to challenge students to find other solutions to this problem.

Erase the board before going on to Exercise 2.

Exercise 2

Explain that the difference between two numbers is the result after subtracting the smaller from the larger. Roll the two dice a few times and ask students to announce the difference between the two numbers rolled.

T: ***What is the greatest possible difference when we roll two dice?***

S: ***5; roll a 6 and a 1.***

T: ***What is the least possible difference?***

S: ***0; roll the same number on each die.***

T: ***Are all differences between 0 and 5 possible?*** (Yes)

Write this information on the board.

P1

Helen	Bruce	Victor
0 1	2 3	4 5

T: ***Helen suggests a new dice game. One player rolls the two dice. The difference between the two numbers determines the winner. Do you think Helen's game is fair?***

Let students express their opinions about the fairness of this game.

Direct students to complete Worksheet P1(b) to determine if Helen's game is fair. You may like to let students work with partners or in small groups.

After a while, hold a collective discussion of the findings. Decide that this game is unfair since Helen's probability of winning is $\frac{16}{36}$, Bruce's is $\frac{14}{36}$, and Victor's is $\frac{6}{36}$. Similar to Exercise 1, let the class determine how to make a different game fair for the three players. One possibility is to give each player two differences: {0, 3}, {1, 5}, and {2, 4}.

Worksheet Pl* and ** are available for individual work.

Writing Activity

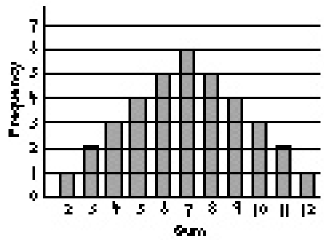
You may like students to take lesson notes on some, most, or even all their math lessons. The "Lesson Notes" section in Notes to the Teacher gives some suggestions and refers to forms in the Blacklines you may provide to students for this purpose. In this lesson, for example, students can note how to use probabilities to determine when a game is fair. They may also review how to calculate probabilities of various sums and differences when tossing two dice.

Name________________ P1(a)

Sum of Two Dice

6	7	8	9	10	11	12
5	6	7	8	9	10	11
4	5	6	7	8	9	10
3	4	5	6	7	8	9
2	3	4	5	6	7	8
1	2	3	4	5	6	7
	1	2	3	4	5	6

Draw a bar graph to show the frequency of each sum. One is done for you.



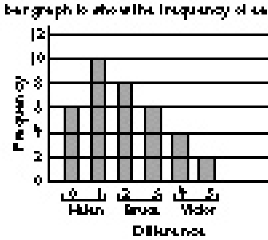
Name________________ P1(b)

Complete. Two squares are filled in for you.

Difference of Two Dice

6	5	4	3	2	1	0
5	4	3	2	1	0	1
4	3	2	1	0	1	2
3	2	1	0	1	2	3
2	1	0	1	2	3	4
1	0	1	2	3	4	5
	1	2	3	4	5	6

Draw a bar graph to show the frequency of each difference.



How many ways does Helen have to win? __16__

How many ways does Bruce have to win? __14__

How many ways does Victor have to win? __6__

Name________________ P1 ★

Use the information on Worksheet P1(a) to answer these questions.

What is the probability that the sum is 6? __ $\frac{5}{36}$ __

What is the probability that the sum is not 6? __ $\frac{31}{36}$ __

What is the probability that the sum is more than 6? $\frac{21}{36}$ or $\frac{7}{12}$

What is the probability that the sum is less than 6? $\frac{10}{36}$ or $\frac{5}{18}$

When Bruce goes home, Helen and Victor decide to continue playing the sum game. They wish to play a fair game. List the sums each person could take to make the game fair.

Helen __2, 4, 6, 8, 10, 12__

Victor __3, 5, 7, 9, 11__

Many solutions are possible.

Explain why your solution produces a fair game.

There are 18 ways to get an odd sum and 18 ways to get an even sum.

Many explanations are possible.

Name________________ P1 ★★

Use the information on Worksheet P1(b) to answer these questions.

What is the probability that the difference is 1? __ $\frac{10}{36}$ or $\frac{5}{18}$ __

What is the probability that the difference is not 1? __ $\frac{26}{36}$ or $\frac{13}{18}$ __

What is the probability that the difference is 0? __ $\frac{6}{36}$ or $\frac{1}{6}$ __

What is the probability that the difference is not 0? __ $\frac{30}{36}$ or $\frac{5}{6}$ __

When Helen goes home, Bruce and Victor decide to continue playing the difference game. They want to play a fair game. List the differences each person could take to make a game fair.

Bruce __0, 1, 5__

Victor __2, 3, 4__

Many solutions are possible.

Explain why your solution produces a fair game.

Bruce has 18 ways to win and Victor has 18 ways to win.

Many explanations are possible.

Capsule Lesson Summary

Tell a story about a princess who can help the one she loves in a situation that could mean his death or their marriage. Use marbles to simulate a walk through a maze. Use an area technique to determine the probabilities involved in the choice that confronts the princess.

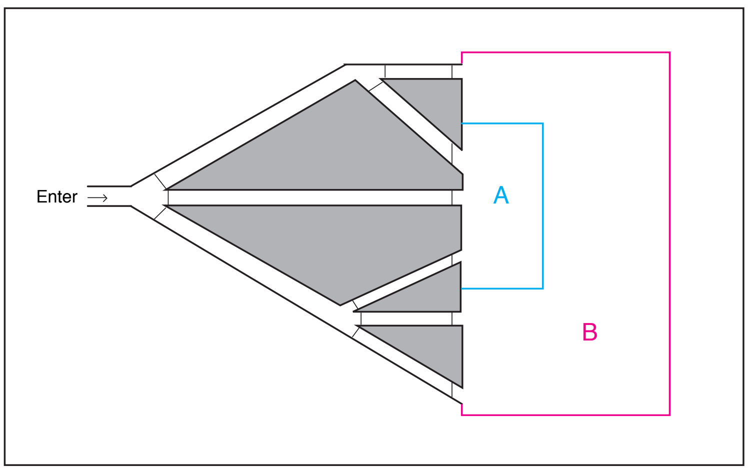
Materials

Teacher
- Meter stick
- Colored chalk
- Three marbles of different colors
- *IG-III Probability Poster* #1
- Crayon or marker

Student
- Colored pencils, pens, or crayons
- Metric ruler
- Worksheets P2* and **

Description of Lesson

Display *IG-III Probability Poster* #1.



Tell the following story in a manner appealing to your class. Stop periodically to let students comment or contribute to the story. (This story is inspired by a popular short story, "The Lady or the Tiger?" by Frank R. Stockton in *A Storyteller's Pack: A Frank R. Stockton Reader*.)

T: ***This is a love story about a beautiful princess. Her parents, the king and queen, had arranged for her to marry Prince Cuthbert from a nearby kingdom. The princess did not consider Cuthbert very appealing, but in the royal tradition she had to accept the marriage planned by her parents.***

As the wedding day approached everything was fine until the princess met Reynaldo, a poor farmer from the village. Reynaldo was handsome, clever, and romantic. It was love at first sight. But being a poor farmer, Reynaldo was not even allowed to talk to the princess. Despite the laws, they began to meet secretly. Reynaldo and the princess were careful, but one day the king caught them together. The king was extremely angry that Reynaldo, a commoner, would dare to fall in love with his daughter.

What do you think was the punishment for this?

Let students suggest possible punishments.

T: ***In this country, the punishment was to put the person in a room full of tigers. So the king ordered Reynaldo sent to the tigers. But when the princess heard this, she wept and protested that she now loved Reynaldo and would never marry Cuthbert. The king was very upset and confused. According to tradition, Reynaldo needed to be punished. If the king allowed the princess to marry Reynaldo instead of Cuthbert he would lose face. Still, the king loved his daughter and did not wish to hurt her. Because of his daughter's plea, the king agreed to discuss the problem with the queen. For the night, he sent Reynaldo to the dungeon instead of giving him to the tigers.***

What should the king do?

Encourage the students to comment on the king's dilemma.

T: ***The next morning the king called for Reynaldo and the princess to tell them of his plan. "The queen and I cannot decide whether to let you, Reynaldo, marry our daughter or to send you to the tigers. So we will let the two of you determine your own fate." The king continued, "Tonight Reynaldo will walk through a maze. At the end of the maze he will arrive at a door that leads to one of two rooms. He must open the door and enter the room."***

Tell your class that the poster is a picture of the maze, but that Reynaldo, of course, did not get to see this picture. Ask your class what they think is waiting in each room.

T: ***The king told Reynaldo that the princess would be waiting in one room and hungry tigers would be waiting in the other. He then sent Reynaldo back to the dungeon to wait until it was time for him to walk the maze.***

The king turned to his daughter and said, "If you love Reynaldo and are wise, you can help him. Here is a map of the maze." The king showed the princess a picture like this one (point to the poster)***. He then explained to her, "Reynaldo has not seen this maze. He must enter, choose a door, and walk down one of the three paths. Each door is made so that it springs open even if he opens it only slightly. And each door closes behind him so he cannot turn back. He must continue until he enters one of the rooms and finds either you or the tigers. The rooms are soundproof, so Reynaldo will not be able to hear you or the tigers. If Reynaldo finds you, I will allow him to marry you. Otherwise, he will go to the tigers and you will marry Cuthbert."***

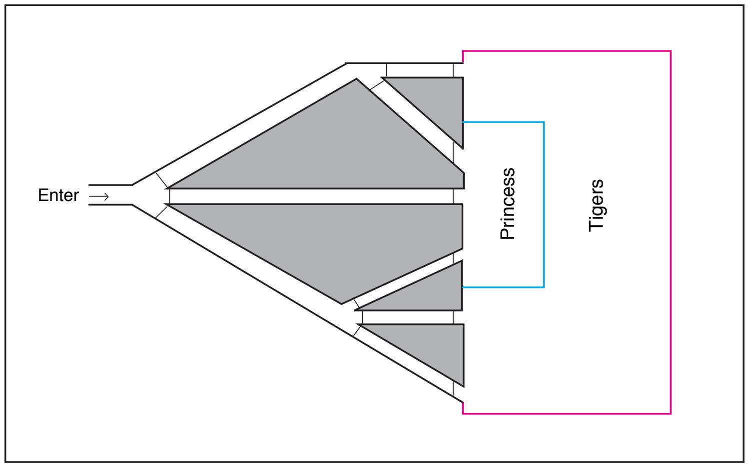
The princess agreed to this plan, understanding that she could choose which room to wait in, **A** ***or*** **B*****, and that the tigers would then be in the other room.***

Discuss the princess's choice with the class. You might ask the following questions.

T: ***Having no map, Reynaldo can only guess which paths to follow. But do you think that he is more likely to enter room*** **A** ***or room*** **B*****? Or are the chances for each the same? If you were the princess, which room would you choose to stay in?***

Encourage discussion. Some students might suggest that Reynaldo's probability for entering each room is $\frac{3}{6}$ or $\frac{1}{2}$ because there are three doors into each room. If no one questions this response, ask whether Reynaldo is equally likely to arrive at each of the six doors. The class should realize that Reynaldo is more likely to arrive at some doors (for example, the third door from the top) than at other doors (for example, the fifth door from the top) due to the layout of the maze. Therefore, we cannot easily tell if Reynaldo's probability of entering each room is $\frac{1}{2}$.

After a class discussion, take a vote on which room the princess should choose. Then mark the rooms as decided by the vote. The dialogue in the remainder of this lesson is based on placing the princess in room **A**. If your class decides to put the princess in room **B**, use the same dialogue and pictures, but reverse all references to the princess and to the tigers.

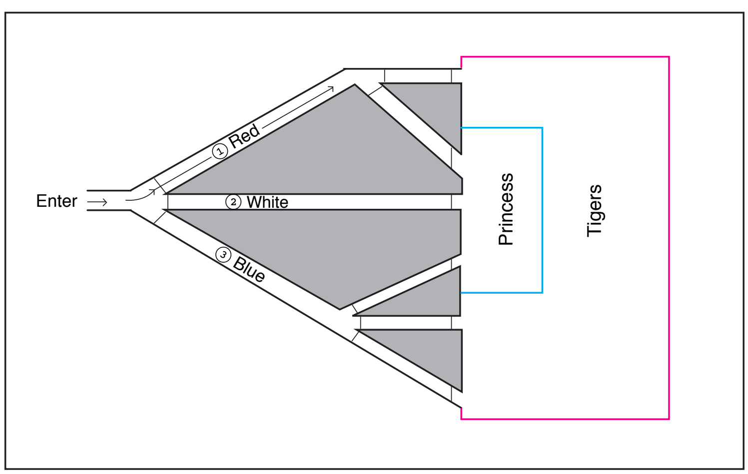


T: ***Let's pretend that we are Reynaldo and must walk through the maze. Reynaldo hates to make life and death decisions at each door; he prefers to randomly decide which door to open. Reynaldo happens to have some marbles in his pocket. How could he use marbles to decide which doors to open?***

After discussing students' suggestions, lead to the following technique.

S: ***Reynaldo could use three marbles of different colors. At the first junction, just beyond the entrance, he could assign one marble color to each of the three doors. Then he could randomly select a marble and open the corresponding door.***

Show the class your three marbles. (This discussion assumes you have one red marble, one white marble, and one blue marble.) On the poster, label each of the three doors near the entrance with a color. Mix the marbles behind your back and, without looking, select one marble. Draw the appropriate path on the poster. Suppose you select the red marble.



T: ***Reynaldo proceeds to the next set of doors. How could he use the marbles to determine which door to open?***

S: ***Use only two of the three marbles since Reynaldo is facing only two doors. Assign one color to each door and randomly select a marble.***

P2

Complete this simulation to determine whether Reynaldo meets the princess or the tigers. Repeat the simulation several times, asking students how Reynaldo should use the marbles at each choice of doors.

After completing several trials, ask if anyone has changed his or her opinion of the room in which the princess should wait. Then adapt the following dialogue to the results in your class.

T: ***In our simulations, Reynaldo found the princess three times and the tigers twice. If we did this five more times, would we get the same results?***

S: ***Maybe, but they could be different.***

T: ***Five simulations are not enough to tell us if the princess made the better choice of rooms. Perhaps Reynaldo just hit a lucky (or unlucky) streak in these five tries. We could do many more simulations. But instead, let's look at a new way to calculate the probability that Reynaldo will find the princess.***

Draw a 60-cm square on the board.

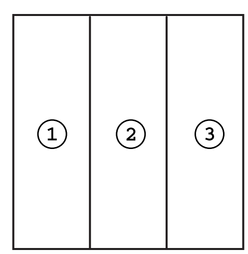
T: ***We will use this square to show what could happen to Reynaldo. When he enters the maze, he must choose one of three paths. On the square, we show this choice by dividing the square into three parts of equal size. How could we do this?***

S: ***Measure the length of one side of the square.***

T: ***It is 60 cm.***

S: ***Draw dividing lines at 20 cm and 40 cm since 60 ÷ 3 = 20.***

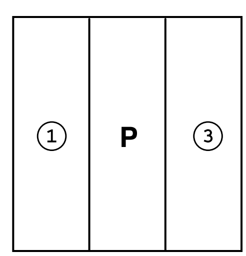
Using the meter stick, carefully divide the square into thirds and label the pieces.



T: ***Reynaldo is equally likely to choose each of the three doors (①, ②, and ③), so we've divided the square into three parts of equal size.***

Suppose Reynaldo chooses path ②. What happens?

S: ***He goes straight to the room with the princess.***



T: ***Let's mark the region for path ② with a* P*.***
Now, suppose he chooses path ①. What happens?

S: ***His chances of going to the princess are the same as his chances of going to the tigers.***

T: ***Why?***

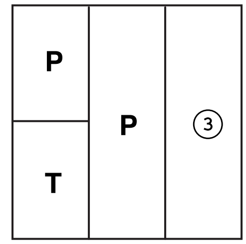
S: ***After following path ①, he comes to a junction and must select one of two paths. One path leads to the princess and one path leads to the tigers.***

T: ***How can we show that on the square?***

S: ***Divide the region for path ① into two parts of the same size. Mark one part* P *for princess and one part* T *for tigers.***

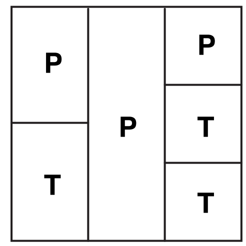
Use a meter stick to divide the region for path ③ in half.

T: ***Let's see what happens if Reynaldo chooses path ③.***



S: ***He comes to a junction with three doors. He has two chances to go to the tigers and one chance to go to the princess. On the square, divide the region for path ③ into three parts of the same size. Mark two parts with a* T *and one part with a* P*.***

If no student gives the above suggestion, do so yourself. Accurately divide the region for path ③ into thirds and mark the parts appropriately.



T: ***Is Reynaldo more likely to find the room with the princess or the room with the tigers?***

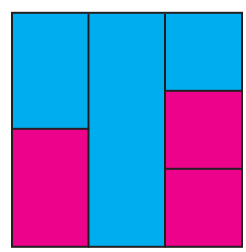
The students are likely to observe that more of the square is marked **P** than is marked **T**.

T: ***Look at the paths in the maze. Can anyone explain why Reynaldo is more likely to find the room with the princess than the room with the tigers?***

S: ***Reynaldo may be very lucky and choose path ② which leads directly to the princess. If he chooses path ① or path ③, he could go to the princess or to the tigers.***

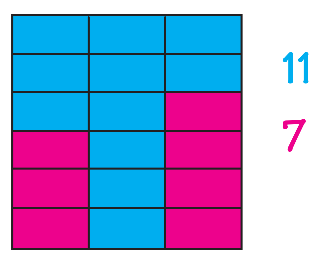
Color the regions of the square marked **P** one color (blue) and those marked **T** another color (red).

T: ***I colored the regions for the princess blue and the regions for the tigers red. Let's calculate exactly Reynaldo's chances of finding the princess and his chances of finding the tigers. Can anyone divide the square into small pieces all the same size so we can count the red pieces and the blue pieces?***



Invite students to the board to explain methods of dividing the square. Several methods are possible. The most natural division is shown here.

T: ***How many blue pieces?*** (11)
How many red pieces? (7)



T: ***Reynaldo has 11 chances to find the princess and 7 chances to find the tigers. What is Reynaldo's probability of finding the princess? The tigers?***

S: ***His probability of finding the princess is $^{11}/_{18}$; he has 11 chances out of 18. His probability of finding the tigers is $^{7}/_{18}$; he has 7 chances out of 18.***

Write the probabilities as fractions near the square.

T: *What would the results have been if you had placed the princess in the other room?*

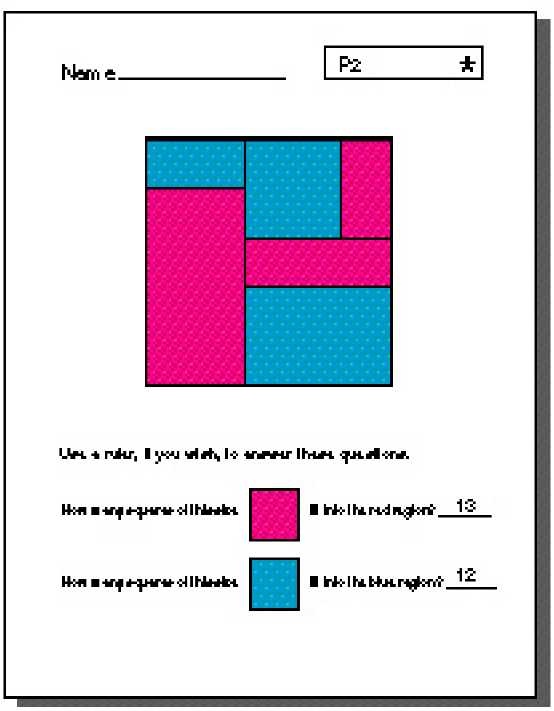
S: *Just the opposite, Reynaldo would have had 11 chances to find the tigers and 7 chances to find the princess.*

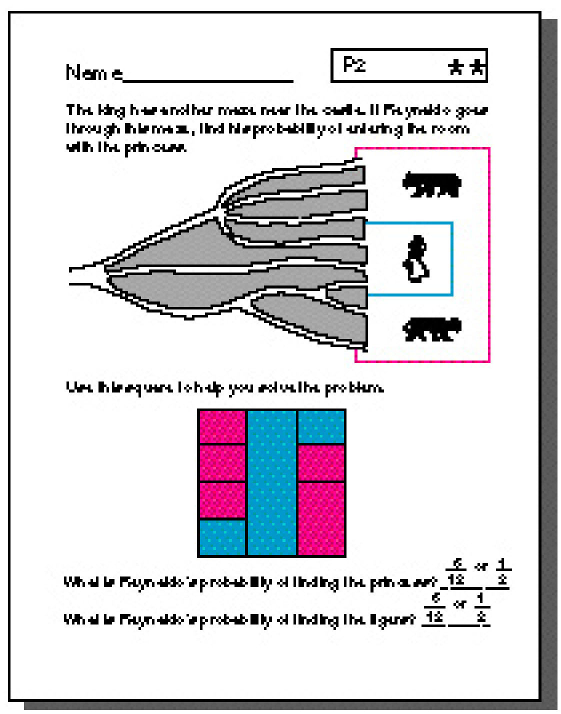
Many students are likely to be curious how the story ended, that is, what really happened to Reynaldo when he walked the maze. We suggest that you either invent an appropriate ending to the story or assign writing an appropriate ending as homework.

Worksheets P2* and ** are available for individual work.

Extension/Writing Activity

Suggest that students create their own mazes for Reynaldo to walk through, and then calculate probabilities of finding the princess (or tigers). You may give them specific probabilities to attain. For example, create a maze where Reynaldo's probability of reaching the princess if $\frac{9}{12}$ or $\frac{3}{4}$.





Capsule Lesson Summary

Use a probability model to generate equivalent fractions. Use a cord diagram and an area method to determine the fairness of several two-stage probability games. Investigate the effect that changing an aspect of the game has on the probabilities of winning for each player.

Materials

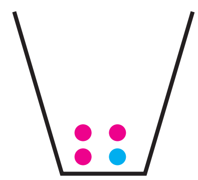
Teacher
- Colored chalk
- Meter stick

Student
- Paper
- Metric ruler
- Worksheet P3

Description of Lesson

Exercise 1

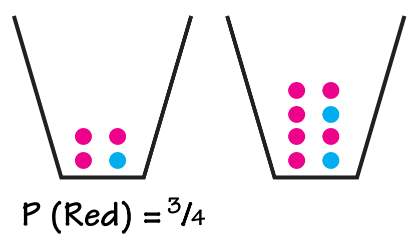
Draw a corresponding picture on the board as you ask,



T: ***If we randomly select one marble from a cup with three red marbles and one blue marble, what is the probability of our selecting a red marble?***

S: ***$\frac{3}{4}$; we'd have three chances out of four of selecting a red marble.***

Record the probability and draw another picture as you ask,



T: ***If we add another three red marbles and one blue marble, what is the probability of selecting a red marble?***

S: ***$\frac{6}{8}$; we'd have six chances out of eight of selecting a red marble.***

Record the probability on the board.

T: ***With which cup are we more likely to randomly select a red marble?***

S: ***Neither; the probability of selecting a red marble is the same for both.***

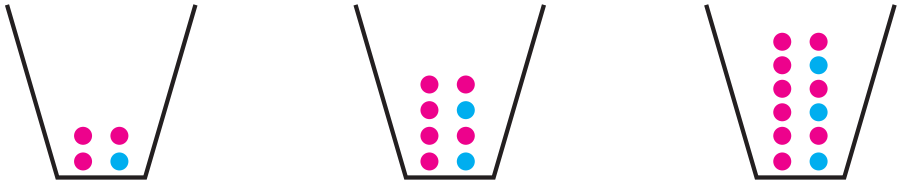
T: ***Why?***

S: ***You just doubled the number of marbles of each color.***

S: ***The probabilities of selecting a red marble, $\frac{3}{4}$ and $\frac{6}{8}$, are equal.***

P3

Draw a third picture on the board as you ask,



T: ***If we add three more red and one more blue marble to the cup, what is the probability of randomly selecting a red marble?***

S: ***9/12 or 3/4; since you just added three red marbles and one blue marble, you did not change the probability of selecting a red marble.***

S: ***There are three red marbles for each blue marble.***

T: ***Can you suggest other probabilities equal to 3/4?***

Let students generate fractions equivalent to 3/4. For example:

$$\frac{3}{4} = \frac{6}{8} = \frac{9}{12} = \frac{12}{16} = \frac{24}{32} = \frac{48}{64} = \frac{96}{128} = \frac{15}{20} = \frac{21}{28}$$

Ask students to describe the patterns they used to generate equivalent fractions. For example:

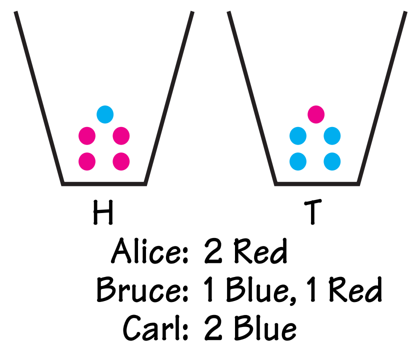
- Add 3 to the numerator and 4 to the denominator:
  $\frac{9}{12} = \frac{9+3}{12+4} = \frac{12}{16}$ and $\frac{12}{16} = \frac{12+3}{16+4} = \frac{15}{20}$.
- Add numerators and add denominators:
  $\frac{9}{12} = \frac{12}{16} = \frac{9+12}{12+16} = \frac{21}{28}$ and $\frac{6}{8} = \frac{24}{32} = \frac{6+24}{8+32} = \frac{30}{40}$.
- Multiply both numerator and denominator by the same number:
  $\frac{3}{4} = \frac{5 \text{ x } 3}{5 \text{ x } 4} = \frac{15}{20}$ and $\frac{3}{4} = \frac{7 \text{ x } 3}{7 \text{ x } 4} = \frac{21}{28}$.

Do not expect students to mention all of the above methods. However, if the whole class tends to use only one method, for example, doubling, suggest problems that force another method by providing either the numerator or denominator of a new fraction.

$$\frac{3}{4} = \frac{\square}{20} \quad \text{or} \quad \frac{3}{4} = \frac{21}{\square}$$

Exercise 2

Draw this picture on the board.



T: ***Do you remember any stories about Bruce?***

S: ***Bruce likes to invent games to play with his friends. The games often look fair but actually favor Bruce.***

T: ***Here is Bruce's new game for three people. He has two cups. An* H *cup has four red marbles and one blue marble. A* T *cup has one red marble and four blue marbles. One player flips a coin to determine which cup to use:* H *for heads,* T *for tails. Then a player selects two marbles from that cup. Alice wins if both marbles are red, Bruce wins if one marble is red and one marble is blue, and Carl wins if both marbles are blue. Do you think this game is fair or that one player is favored?***

Let students express their opinions.

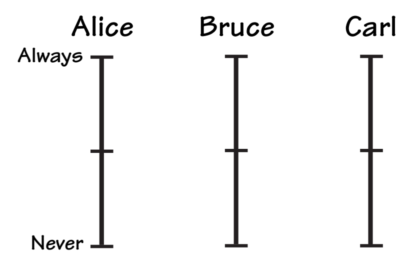
S: ***I think Bruce is favored because he can win with either cup.***

S: ***Carl is favored with tails.***

S: ***Alice is favored with heads.***

S: ***It could be a fair game.***

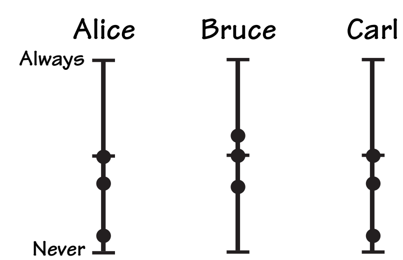
Draw three line segments of the same length on the board, and label them for the three players.



T: ***Do you remember how to use segments like these to record probability estimates?***

S: ***Draw dots for estimates of probabilities on a scale from* Never *to* Always*. A dot at the bottom is for an event that can never happen. A dot at the top is for an event that always occurs. Dots between* Never *and* Always *are for events that might occur.***

Suggest that students copy these segments on their papers and draw dots to indicate their estimates of each player's probability of winning. Invite several students to put their guesses on the board, as illustrated here.



Encourage students to explain their estimates.

Draw a 60-cm square on the board.

T: ***Let's use this square to help calculate each player's probability of winning. Do you remember another time when we used a square to help calculate probabilities?***

S: ***In the story about the princess and the tigers; we calculated Reynaldo's probability of finding the princess when he went through a maze.***

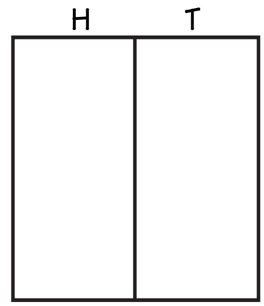
P3

T: ***In this game, a player first flips a coin. How should we divide the square to show the possible outcomes of the coin flip?***

S: ***Divide the square in two equal parts: half for heads and half for tails.***

T: ***The square is 60 cm on each side. How could we divide it in half?***

S: ***$\frac{1}{2} \times 60 = 30$, so draw a line (segment) at 30 cm.***



T (pointing to the **H** region): ***Suppose that heads is flipped. Who could win the game?***

S: ***Only Alice or Bruce.***

T: ***How can we calculate Bruce's probability of winning when heads is flipped? How can we calculate Alice's probability of winning?***

You may need to remind students of the following method used in some *IG-II* lessons.

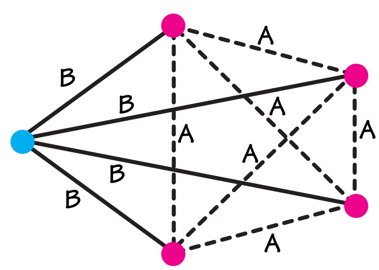
S: ***Draw colored dots for the marbles. Draw cords to show how each person could win.***

Draw one blue dot and four red dots on the board.

T: ***In how many ways can Bruce win?***

S: ***In four ways.***

Invite a student to draw the appropriate cords (solid).



T: ***In how many ways can Alice win?***

S: ***In six ways.***

Invite a student to draw the appropriate cords (dotted).

T: ***Who is favored if heads is flipped?***

S: ***Alice; she has six ways to win and Bruce has only four ways to win.***

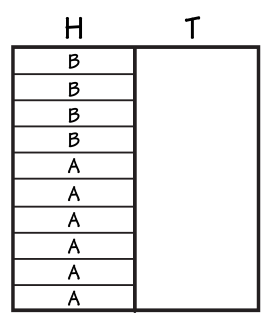
T: ***How can we show this on the square?***

S: ***Divide the H region in ten equal parts since $4 + 6 = 10$; four parts for Bruce and six parts for Alice.***

T: ***The H region is 60 cm by 30 cm. How could we divide it into ten equal parts?***

S: ***Make ten strips each 6 cm wide since $10 \times 6 = 60$.***

Accept any correct method, as there are many.



T (pointing to the **T** region)**:** ***Suppose that tails is flipped. Who could win the game?***

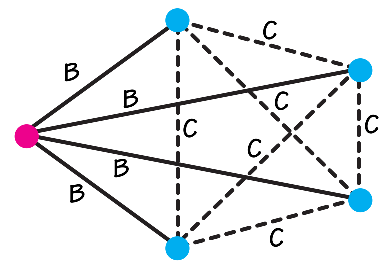
S: ***Bruce or Carl.***

T: ***In how many ways can each person win?***

S: ***Bruce can win in four ways, and Carl can win in six ways.***

T: ***Why?***

S: ***In the picture with colored dots and cords, just switch the colors of the marbles. Then change* A *for Alice to* C *for Carl.***



Make changes to the earlier picture as indicated, or redraw it.

T: ***How can I show these results on the square?***

S: ***Divide the* T *region into ten equal parts; four parts for Bruce and six parts for Carl.***

H	T
B	B
B	B
B	B
B	B
A	C
A	C
A	C
A	C
A	C
A	C

T: ***Who is favored in this game? Why?***

S: ***Bruce. In the square, we see that Bruce has eight chances of winning while Carl and Alice each have only six chances of winning.***

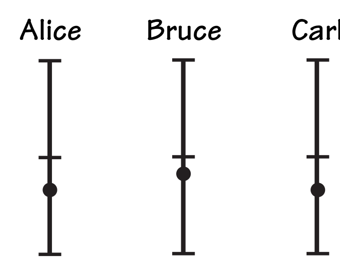
T: ***What is each player's probability of winning?***

S: ***Since there are 20 equal parts in the square, $8/20$ for Bruce and $6/20$ each for Alice and Carl.***

Record these results and check where to locate dots for the actual probabilities on the line segments.

$$P\,(\text{Alice}) = 6/20 \qquad P\,(\text{Bruce}) = 8/20 \qquad P\,(\text{Carl}) = 6/20$$

Encourage students to use good estimation techniques in locating the dots.

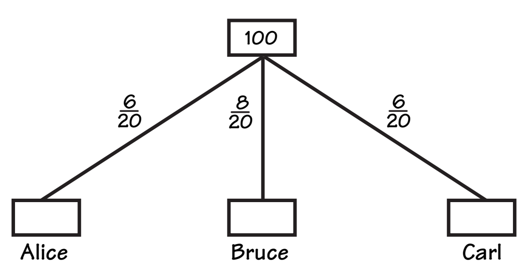


S: ***$8/20$ is just less than $10/20$ or $1/2$, so Bruce's dot is just below the midpoint.***

S: ***Place Alice's dot and Carl's dot lower than Bruce's dot since $6/20$ is less than $8/20$.***

You may want to make the line segments a length divisible by 20 to make more accurate locations. Compare these results to the students' earlier estimates. Highlight the fact that Bruce is favored even though his probability of winning is less than ½.

Draw this probability tree on the board.



T: ***Pretend that Alice, Bruce, and Carl play this game 100 times. Using our results, what is the best prediction for the number of games each player will win?***

S: ***Alice should win about 30 games since $6/20 \times 100 = 30$.***

T: ***How did you calculate $6/20 \times 100$?***

S: ***$100 \div 20 = 5$ (or $1/20 \times 100 = 5$) and $6 \times 5 = 30$.***

S: *$6 \times 100 = 600$ and $600 \div 20 = 30$.*

S: *Carl should also win about 30 games since his probability of winning is the same as Alice's, $6/20$.*

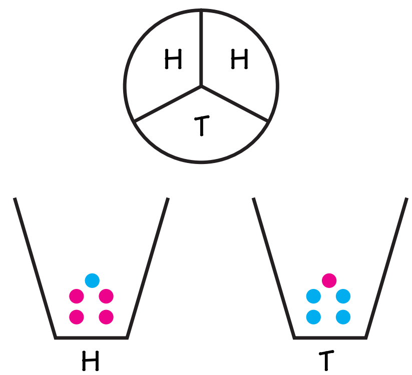
T: *About how many games should Bruce win?*

S: *40, since $100 - 30 - 30 = 40$*

S: *40, since $8/20 \times 100 = 40$.*

Exercise 3

Draw this picture on the board.



T: *Bruce and Carl agree to let Alice change the game since the original game favors Bruce. Alice decides to use the same marble distribution but to use a spinner to decide which cup to select. Alice's spinner is divided into three equal parts: two parts are for cup* H*, one part for cup* T*. How does this spinner change the game? Who does it help? Who does it hurt?*

S: *It helps Alice because the probability that cup* H *will be selected is $2/3$ and Alice is more likely to win when cup* H *is used.*

S: *It hurts Carl because the probability that cup* T *will be selected is less.*

As in the previous activity, ask students to predict each player's probability of winning by drawing dots on segments. Then direct the class to calculate probabilities using the method of dividing up a square. You may like to suggest that students first try to do this calculation themselves on papers, or at least draw a squares on their papers and follow along.

T: *How should we first divide the square now that we are using this spinner?*

S: *Divide the square into three equal-sized parts; two parts for* H *and one part for* T*.*

Let students measure and divide the square into three parts all the same size.

T: *Now what should we do?*

S: *Divide each region into ten parts just like last time since the distribution of marbles has not changed.*

H	H	T
A	A	C
A	A	C
A	A	C
A	A	C
A	A	C
A	A	C
B	B	B
B	B	B
B	B	B
B	B	B

S: *In each* H *region, six parts are for Alice and four parts are for Bruce.*

S: *In the* T *region, six parts are for Carl and four parts are for Bruce.*

T: *What is each player's probability of winning?*

S: *$12/30$ for both Alice and Bruce and $6/30$ for Carl.*

Record the probabilities on the board.

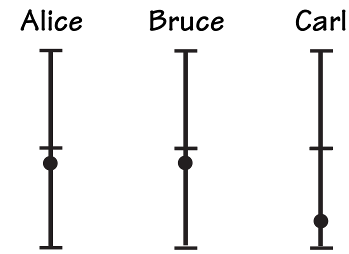
T: *What has changed from last time?*

S: ***Alice and Bruce have equal probabilities of winning. Both are favored over Carl.***

S: ***Carl's chances have decreased from 6 out of 20 to 6 out of 30.***

S: ***Bruce's probability of winning has stayed the same since he still has four out of ten chances to win in each third of the square.***

Locate the winning probabilities for the three players on their respective segments, and let students check their predictions.



Exercise 4

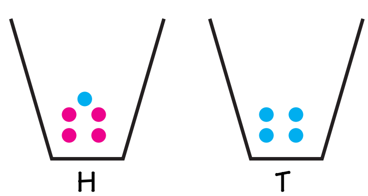
T: ***Bruce and Alice now let Carl change the game. How could Carl change the game so that it is more likely that he will win?***

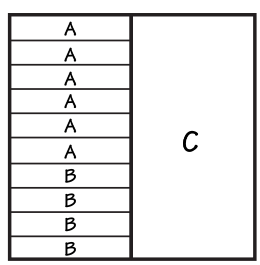
Encourage students to suggest several changes that would favor Carl. For example:

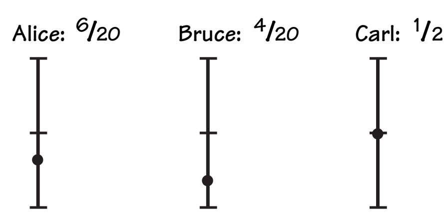
- Switch the **H** and **T** regions on the spinner, or switch the cups.
- Use a die to determine from which cup to select. Select from cup **H** if 6 is rolled, otherwise select from cup **T**.
- Put four blue marbles and one red marble in each cup.

If some of the games suggested by students can be quickly analyzed using a square, you may wish to stop to do the analysis. For example:

S: ***Use the coin again, not the spinner, and remove the red marble from the second cup.***



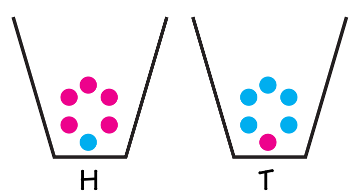




Exercise 5 (optional)

T: ***Instead of trying to favor himself, Carl decides to try to make the game fair. He chooses to use a coin again, not a spinner. And he decides to add a red marble to the first cup and a blue marble to the second cup.***

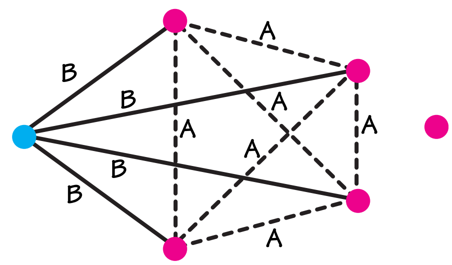
Compared to the first game, who should this change help?



S: ***Both Alice and Carl; each is more likely to win if their cup is selected.***

P3

Redraw the cord picture from Exercise 2 and add another red dot (marble). Divide a square in half as in Exercise 2.



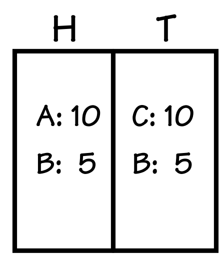
T: *We've solved the problem with four red marbles and one blue marble in the cup. Now I've added one extra red marble. Who will this help more, Alice or Bruce?*

S: *Alice; we will have only one new cord for Bruce (from the new red dot to the blue dot) but we will have four new cords for Alice.*

S: *There will be ten cords (6 + 4 = 10) for Alice and five cords (4 + 1 = 5) for Bruce.*

S: *Divide the* H *region into 15 equal-sized parts: Ten for Alice and five for Bruce.*

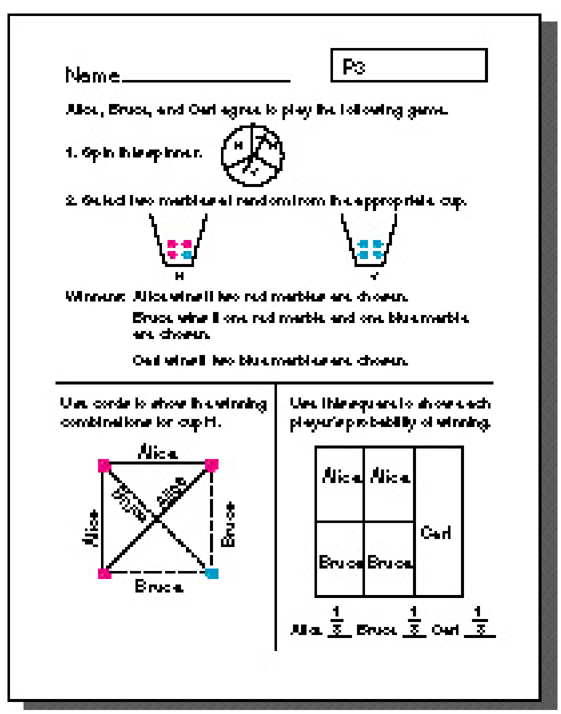
T: *I won't actually divide the region; I will just record the results.*



S: *Similarly, if tails is flipped, Carl has ten ways to win and Bruce has only five ways to win.*

S: *The game is fair! Each player has ten ways to win. Each player's probability of winning is $^{10}/_{30}$ or $^{1}/_{3}$.*

Worksheet P3 is available for individual work.


Name
P3
Alice, Bruce, and Carl agree to play the following game.
1. Spin the spinner.
2. Select two marbles at random from the appropriate cup.
Winners: Alice wins if two red marbles are chosen.
Bruce wins if one red marble and one blue marble are chosen.
Carl wins if two blue marbles are chosen.
Use cords to show the winning combinations for cup H.
Use this square to show each player's probability of winning.
Alice
Alice
Bruce
Bruce
Alice Alice
Bruce Bruce
Carl
Alice $\frac{1}{3}$ Bruce $\frac{1}{3}$ Carl $\frac{1}{3}$

Capsule Lesson Summary

Decode a message written in a known code. Decode another message written in an unknown code by comparing letter frequencies to already determined standards.

Materials

Teacher
- Colored chalk
- Tape
- *IG-III Probability Poster #2*
- Colored markers
- Blacklines P4(a) and (b)
- Colored pencils, pens, or crayons

Student
- Worksheets P4(a) and (b)

Advance Preparation: Use Blacklines P4(a) and (b) to prepare transparencies for display or to make multiple copies to distribute.

Description of Lesson

With the class, recall Boris, the spy who specializes in codes.

T: ***Sometimes Boris receives coded messages, other times he sends coded messages, and sometimes he even intercepts coded messages from the enemy. Here is a message that Boris received.***

Write the message on the board. Leave enough space below each letter to write the decoded message.

GVZFZ UB KP ZPZOJ
OZBBZPWZF QP GVZ
PZIG OUYPUWVG GFKUP.

T: ***Why did headquarters write the message this way?***

S: ***So no one except Boris could read it.***

Display or distribute the code from Blackline P4(a). You may make a transparency or give students copies of the code. To give more visual distinction, you may like to color over the gray letters on the blackline to make them red.

CODE

A	B	C	D	E	F	G	H	I
Q	S	U	W	Z	R	T	V	X
J	K	L	M	N	O	P	Q	R
Y	A	B	C	L	M	N	O	P
S	T	U	V	W	X	Y	Z	
K	J	I	H	G	F	D	E	

T: ***This is Boris's secret code. What does the letter G represent in the secret message?***

S: ***T.***

Ask a student to write **T** in red below each **G** in the message.

Invite students to continue in this way until the message is fully decoded.

G V Z F Z   U B   K P   Z P Z O J
T H E R E   I S   A N   E N E M Y

O Z B B Z P W Z F   Q P   G V Z
M E S S E N G E R   O N   T H E

P Z I G   O U Y P U W V G   G F K U P
N E X T   M I D N I G H T   T R A I N

T: ***Boris meets the midnight train, and lo and behold who is on the train? Medussa, his archenemy. Medussa is the most clever and most important of all of the enemy spies. Boris knows immediately that she is carrying an important message. A great chase begins. First Boris chases Medussa inside the train, then on top of the train, then in and out of the train. Finally he leaps up and grabs her just in time, before she can swallow a secret message. But unfortunately, she has already swallowed the secret code. This is what the message looks like.***

Display *IG-III Probability Poster* #2 and distribute copies of Worksheet P4(a).

W	E		J	E		J	X	U		S	H	U	U	A		D	U	Q	H			
J	X	U		B	Q	I	J		J	H	Q	Y	D		I	J	Q	J	Y	E	D	
M	X	U	H	U		Q		J	H	U	U		X	Q	I		V	Q	B	B	U	D
E	L	U	H		J	X	U		M	Q	J	U	H		Y	D	I	Y	T	U		
J	X	U		J	H	U	U		O	E	K		M	Y	B	B		V	Y	D	T	
Q		V	E	H	C	K	B	Q		V	E	H		Q		I	U	S	H	U	J	
F	E	J	Y	E	D		M	Y	J	X		J	X	Y	I		F	E	J	Y	E	D
M	U		M	Y	B	B		H	K	B	U		J	X	U		M	E	H	B	T	

T: ***What does the message say?***

Let the class comment on how they could decipher this message.

Display the chart from Blackline P4(b) on letter frequency in English. You may put this on a transparency, or write the list on the board.

T: ***Boris knows that letters normally occur with these frequencies in books written in English. How can Boris use this information to help him break the code?***

S: ***Find out how often letters occur in the message, and list them in order from the most frequent to the least frequent.***

T: ***Good! Let's do that.***

Letter	Frequency in English
E	12.3
T	9.6
A	8.1
O	7.9
N , I	7.2
S	6.6
R	6.0
H	5.1
L	4.0
D	3.7
C, U	3.2
F, P, M	2.3
W, Y	1.9
B, G	1.6
V	0.9
K	0.5
Q, X	0.2
J, Z	0.1
	per 100

You may like to let students work with in pairs or in small groups on this task. Instruct them to find the frequency of letters in the message and to record the results on Worksheet P4(b). When most groups are done, collect the results as a class on the right side of Blackline P4(b). You may want groups to have their own copies of the chart on Blackline P4(b) on which to record the results.

T: ***Which letters appear most often in the message?***

S: ***U appears 22 times.***

S: ***Next, J appears 18 times.***

Continue until all of the letters are recorded. List letters with the same frequency together. In this activity, groups may check each other's counts.

Letter	Frequency in English	Letter	Frequency in Message
E	12.3	U	22
T	9.6	J	18
A	8.1	H	13
O	7.9	E	12
N , I	7.2	Y, Q	11
S	6.6	B	10
R	6.0	X	9
H	5.1	D	8
L	4.0	M	7
D	3.7	I	6
C, U	3.2	V	4
F, P, M	2.3	T, K	3
W, Y	1.9	F, S	2
B, G	1.6	A, C, L, O, W	1
V	0.9	G, N, P, R, Z	0
K	0.5		
Q, X	0.2		
J, Z	0.1		
	per 100		per 146

T: ***How can this information help Boris decode the message?***

S: ***U most likely corresponds to E because it appears the most often.***

S: ***J most likely corresponds to T because it appears the second most often.***

Ask students to write **E** below each **U**, and **T** below each **J** in the message.

At this point your students may have some difficulty determining how to continue decoding the message. Encourage suggestions, and note that a perfect frequency match should not be expected.

S: ***I think X corresponds to H because a common three-letter word that starts with T and ends with E is THE.***

S: ***I think E in the message corresponds to O because TO is a two letter word and E has about the right frequency for O.***

Let other students comment on these suggestions before entering them in decoding the message.

W	E		J	E		J	X	U		S	H	U	U	A		D	U	Q	H			
	O		T	O		T	H	E				E	E				E					
J	X	U		B	Q	I	J		J	H	Q	Y	D		I	J	Q	J	Y	E	D	
T	H	E					T		T							T		T		O		
M	X	U	H	U		Q		J	H	U	U		X	Q	I		V	Q	B	B	U	D
	H	E		E				T		E	E		H								E	
E	L	U	H		J	X	U		M	Q	J	U	H		Y	D	I	Y	T	U		
O		E			T	H	E				T	E								E		
J	X	U		J	H	U	U		O	E	K		M	Y	B	B		V	Y	D	T	
T	H	E		T		E	E			O												
Q		V	E	H	C	K	B	Q		V	E	H		Q		I	U	S	H	U	J	
			O								O						E			E	T	
F	E	J	Y	E	D		M	Y	J	X		J	X	Y	I		F	E	J	Y	E	D
		T		O					T	H		T	H						T		O	
M	U		M	Y	B	B		H	K	B	U		J	X	U		M	E	H	B	T	
	E										E		T	H	E			O				

Note: It is helpful to note in the frequency chart the number of times a letter occurs in the message, and to compare it to the number of times you decode that letter in the message to check that you find them all.

Let students work with in pairs or in small groups to continue decoding the message by trial and error.

W	E		J	E		J	X	U		S	H	U	U	A		D	U	Q	H			
G	O		T	O		T	H	E		C	R	E	E	K		N	E	A	R			
J	X	U		B	Q	I	J		J	H	Q	Y	D		I	J	Q	J	Y	E	D	
T	H	E		L	A	S	T		T	R	A	I	N		S	T	A	T	I	O	N	
M	X	U	H	U		Q		J	H	U	U		X	Q	I		V	Q	B	B	U	D
W	H	E	R	E		A		T	R	E	E		H	A	S		F	A	L	L	E	N
E	L	U	H		J	X	U		M	Q	J	U	H		Y	D	I	Y	T	U		
O	V	E	R		T	H	E		W	A	T	E	R		I	N	S	I	D	E		
J	X	U		J	H	U	U		O	E	K		M	Y	B	B		V	Y	D	T	
T	H	E		T	R	E	E		Y	O	U		W	I	L	L		F	I	N	D	
Q		V	E	H	C	K	B	Q		V	E	H		Q		I	U	S	H	U	J	
A		F	O	R	M	U	L	A		F	O	R		A		S	E	C	R	E	T	
F	E	J	Y	E	D		M	Y	J	X		J	X	Y	I		F	E	J	Y	E	D
P	O	T	I	O	N		W	I	T	H		T	H	I	S		P	O	T	I	O	N
M	U		M	Y	B	B		H	K	B	U		J	X	U		M	E	H	B	T	
W	E		W	I	L	L		R	U	L	E		T	H	E		W	O	R	L	D	

Extension/Writing Activity

Suggest that students write a coded message similar to the ones in this lesson and give it to another student to decode.

Home Activity

Suggest that parents/guardians try to solve a cryptogram puzzle with their child.

Capsule Lesson Summary

Use bowling scores to define the concept of *mean average*. Use averages to determine the winner of a game and to interpret the results of a sampling experiment.

Materials

Teacher
- Colored chalk
- A large bag
- 30 red marbles
- 20 blue marbles

Student
- Paper
- Calculator

Advance Preparation: Before class put 30 red marbles and 20 blue marbles into a large bag. You may substitute beads, bottle caps, dried beans, or any other items that have uniform shape and two distinct colors.

Description of Lesson

Note: Mean, median, and mode are three types of statistical averages. In common usage, the word *average* often refers to the mean. For example, when one gives a bowling average, one is giving the mean of some bowling scores. In this lesson, since only one type of average is referred to, use the words *average* and *mean* interchangeably.

Exercise 1

Students will need calculators and paper.

T: ***How many of you have gone bowling? What is the highest score a bowler can get in one game?*** (300)
Zach is a bowler. He bowled 300 one day. Is that good? (Yes)
Only Zach took two games to do it.

Begin a chart on the board.

T: ***What scores could he have had?***

S: ***150 for both games.***

Encourage students to suggest several possible pairs of scores and record them in the chart. If no one suggests scores of 150 for both games, ask what Zach's scores would have been if he had the same score in both games.

Zach

1st Game	2nd Game	Total
150	150	300
1	299	
200	100	
125	175	
160	140	

S: ***The scores of 150 and 150 are special because Zach would have bowled the same score in each game.***

T: ***150 is the* mean *of Zach's bowling scores; 150 is Zach's bowling* average.**

P5

Write *mean average* on the board.

T: ***A person's bowling average may be calculated by pretending his or her score was the same in each game. Zach has a friend named Zelda. She bowled a total of 444 in three games. What was Zelda's bowling average for those games?***

S: ***148.***

T: ***What calculation did you do?***

S: ***444 ÷ 3.***

Record the information on the board.

T: ***What are some other scores Zelda could have bowled?***

Record several suggestions in a chart. For example:

Zelda

1st Game	2nd Game	3rd Game	Total
148	148	148	444
222	111	111	
0	222	222	
300	100	44	

T: ***If Zelda bowled 150 in the first game and 148 in the second game*** (record 150 and 148 in the chart), ***what was her score in the third game? Why?***

S: ***146, because 150 + 148 = 298, and 444 – 298 = 146.***

S: ***150 is two more than her average, and 148 is the same as her average; so the last game is two less than her average, or 146.***

T: ***If Zelda bowled 140 in the first game, what could her scores have been for the last two games?***

S: ***148 and 156.***

Many answers are possible.

T: ***Zach and Zelda have a friend named Zeno. This is what Zeno bowled in three games.***

Record the information on the board.

Zeno

1st Game	2nd Game	3rd Game
120	86	130

T: ***What was Zeno's bowling average for the three games?***

S: ***112; I added the scores and divided by 3.***

S: ***112; I balanced the scores for three games.***

T: ***Who is the best bowler?***

S: ***Zach, because he has the highest average.***

S: ***Zelda, because she has almost the same average as Zach but for three games, not two.***

T: ***One day Zeno, Zelda, and Zach decide to have a contest to see who is the best bowler. Zeno is late, as usual, and only bowls in the last two games. These are their scores.***

Record this information on the board.

	1st Game	2nd Game	3rd Game	Total
Zach	140	162	130	
Zelda	160	170	60	
Zeno	X	140	170	

T: ***When they finish bowling, they all think they've won. Can you figure out why?***

Allow several minutes for students to consider this problem.

T: ***Why does Zach think he is the winner?***

S: ***Because he has the highest total.***

Invite students to enter each person's total in the chart on the board.

	1st Game	2nd Game	3rd Game	Total
Zach	140	162	130	432
Zelda	160	170	60	390
Zeno	X	140	168	308

T: ***Why do you think Zelda considers herself the winner?***

S: ***Because she won two of the three games, and she had the highest score.***

T: ***What about Zeno?***

S: ***Zeno has the best average.***

T: ***Let's see. What is Zach's average?***

S: ***144; add his scores and divide by 3.***

T: ***What is Zelda's average?***

S: ***130; divide her total 390 by 3.***

T: ***What is Zeno's average?***

S: ***154; divide Zeno's total by 2 because he only bowled two games.***

Exercise 2

Students will need paper and pencil.

T: ***Today we are going to play a tally game. The object of the game is to see which team can make the most tally marks (𝍸) in ten seconds.***

Divide the class into four teams with different numbers of students in each team, for example, with twelve, seven, three, and eight members.

T: ***Raise your hands in the air. When I say, "Go," write as many tally marks as you can. When I say, "Stop," put your hands up again. Go!***

Stop the students after ten seconds and ask two students from each team to count the number of tally marks for their team. Record the results on the board. For example:

	Teams			
	A	B	C	D
Number of Tally Marks	342	184	88	245

T: ***Who won?***

S: ***Team A.***

T: ***Was this a fair game?***

S: ***No! Team A had more people.***

T: ***How could we make the game fair using these scores?***

S: ***Find the (mean) average for each team and let the team with the highest average be the winner.***

T: ***What information will that give us?***

S: ***About how many marks each person on the team made.***

Invite students to use calculators to find the (mean) average for each team. Record each team's average in the chart on the board.

	Team			
	A	B	C	D
Number of Tally Marks	342	184	88	245
Number of People	12	7	3	8
Average	28.5	26.285714	$29.\overset{*}{3}$	30.625

Compare the teams' averages and order them from highest to lowest.

Exercise 3

You will ne~~ed the bag of marb~~les or other objects for this exercise.

T: ***A farmer stocked a pond with equal numbers of catfish and trout. A year later he had the pond tested to see which of the two types of fish was doing better. What information would he know about his pond if the test showed that there were more catfish then trout?***

Let the class discuss the situation briefly.

S: ***The pond is polluted.***

S: ***The temperature of the water is high.***

S: ***There is more food for the catfish.***

S: ***The water is muddy.***

T: ***How might the farmer discover which of the two types of fish is doing better?***

Let students offer suggestions. It is unlikely that they will suggest the method that follows, but it is important to discuss other methods in order to realize the significance of this sampling method.

After students have made several suggestions, hold up the bag with marbles in it.

T: ***The farmer's method was to catch five fish with a net each day for two weeks; record how many of each type of fish were caught; and then return all five of the fish to the pond at once. Let's see how this would work. In this bag, there are red and blue marbles. Red marbles will represent trout, and blue marbles will represent catfish. Who would like to go fishing?***

Let a student reach into the bag with both hands and pick out five marbles without looking. Announce the results of the pick and return the marbles to the bag. Shake the bag and repeat the experiment a total of 14 times to simulate two weeks. You may ask students to record the results of fishing on the board. Perhaps your results will be similar to those listed here.

T: ***How do the numbers of trout and catfish compare?***

S: ***There are probably more trout.***

T: ***What is the total number of trout caught during the two weeks? What is the total number of catfish?***

S: ***40 trout and 30 catfish.***

T: ***What is the (mean) average number of trout caught each day?***

S: ***2.8571428; I divided 40 by 14 using the calculator.***

T: ***Is 2.8571428 closer to 2.8 or to 2.9?***

S: ***2.9.***

T: ***What is the average number of catfish caught each day?***

S: ***2.1428571.***

T: ***Is the (mean) average closer to 2.1 or 2.2?***

S: ***2.1.***

Trout	Catfish
3	2
2	3
2	3
3	2
3	2
3	2
3	2
2	3
1	4
4	1
4	1
3	2
4	1
3	2

Record the size of the sample and the (approximate) average caught per day in a chart.

Number of Fish in the Sample	Number of Trout	Number of Catfish
5	2.9	2.1
10		

T: ***The (mean) average tells us that if there were five fish caught, about 2.9 would be trout and 2.1 would be catfish. What if there were ten fish caught?***

S: ***There would be about 5.8 trout and 4.2 catfish.***

Put these numbers in appropriate columns in the chart on the board. Continue in the same manner for samples of 100, 500, and 50 fish in the pond.

P5

Number of Fish in the Sample	Number of Trout	Number of Catfish
5	2.9	2.1
10	5.8	4.2
100	58	42
500	290	210
50	29	21

T: ***There are 50 marbles in this bag for 50 fish. Let's see how close our prediction of 29 trout (red marbles) and 21 catfish (blue marbles) is to the actual number.***

Give the bag of marbles to some students to count. Ask one of the students to announce the results.

S: ***30 red marbles and 20 blue marbles.***

T: ***In this case our prediction was only off by one marble. Will our results always be this good?***

S: ***No, but most of the time they should be close.***

Note: There is a 15% chance that your class's predictions will differ by five or more than the actual number of marbles.

T: ***What could the farmer do to make his results more accurate?***

S: ***Catch more fish each day.***

S: ***Continue sampling the fish for more than two weeks.***

Capsule Lesson Summary

Determine how Bruce can increase his probability of winning over an opponent in a game he invents for a school fair.

Materials

Teacher
- Six marbles (three red and three blue)
- Two cups
- Meter stick
- Colored chalk

Student
- Colored pencils, pens, or crayons
- Worksheet P6

Description of Lesson

Exercise 1

T: ***Do you remember the stories about Bruce?***

Let students briefly recall that Bruce invents games that appear to be fair but usually favor Bruce.

T: ***Bruce's fifth-grade class is planning a weekend trip to the state capitol. They must earn most of the money to pay for the trip. The class decides to have a school fair and to invite the families of all the students in the school. What are some ways the class could make money from a school fair?***

S: ***Sell food.***

S: ***Raffle off a prize.***

S: ***Have carnival rides and play games.***

T: ***Bruce suggests a game for the school fair. It is played with three red marbles, three blue marbles, and two cups. Bruce chooses how to put the marbles into the two cups. Then, without looking, the player selects a cup and a marble from that cup. The player wins 25¢ if the marble is red and loses 25¢ if the marble is blue.***

Record this information on the board.

- Bruce wins
- Player wins

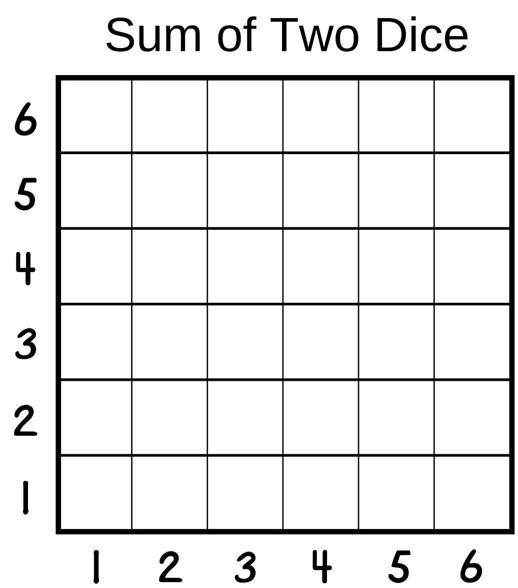
T: ***Let's try Bruce's game.***

Invite one student to be Bruce and another student to play the game. Demonstrate the game by following these steps:

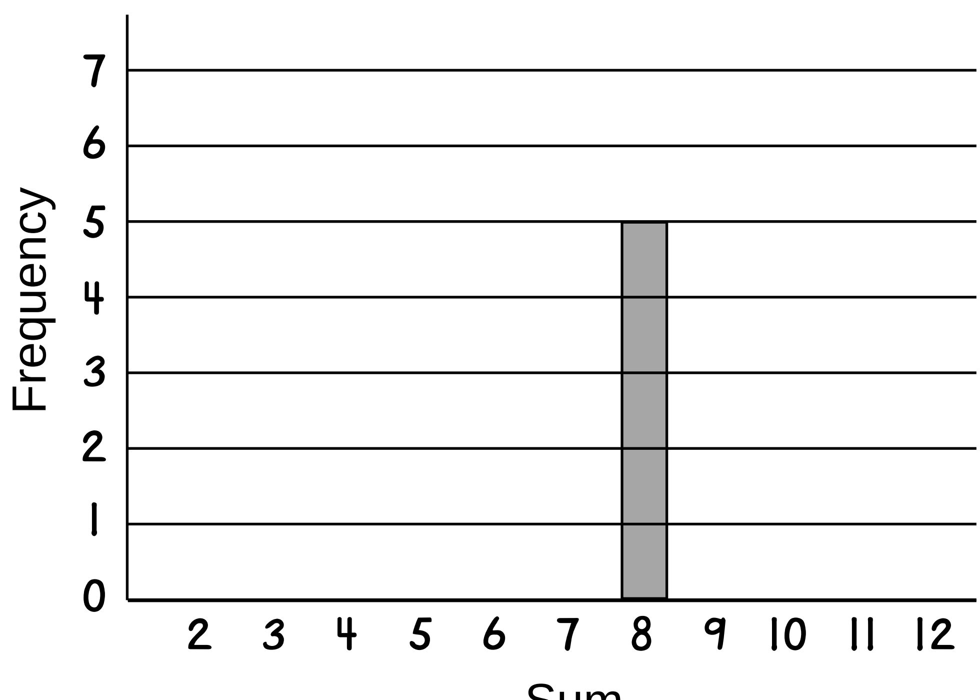
- Bruce puts the six marbles into the two cups in any way he wishes, as long as there is at least one marble in each cup.
- Bruce shuffles the two cups behind his back.
- The player chooses one of the cups (one of Bruce's hands).
- The player selects, without looking, one marble from the cup.
- The player wins 25¢ by choosing a red marble or loses 25¢ by choosing a blue marble.

Name ______________________

P1(a)



Draw a bar graph to show the frequency of each sum. One is done for you.



Name ____________________

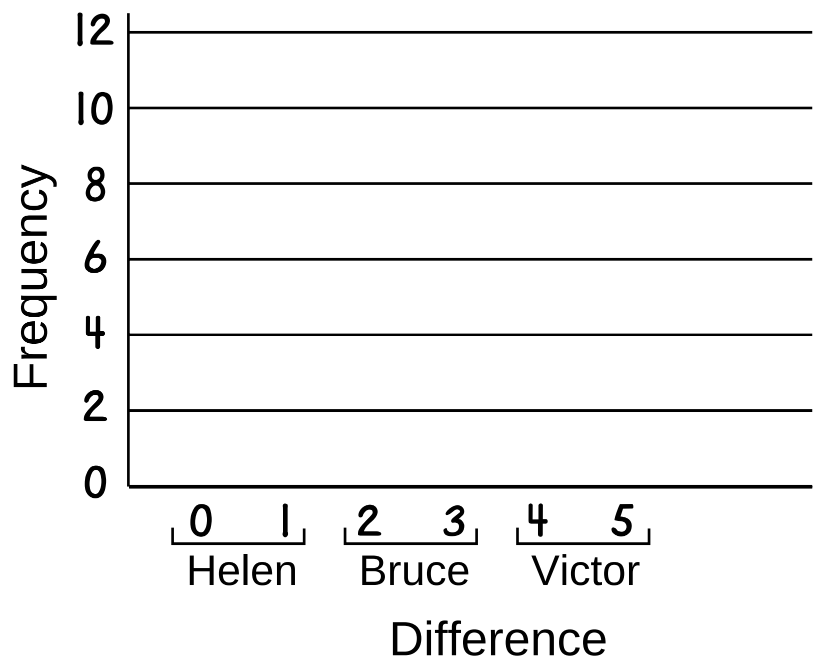
P1(b)

Complete. Two squares are filled in for you.

Difference of Two Dice

6						
5		3				
4						
3						
2			1			
1						
	1	2	3	4	5	6

Draw a bar graph to show the frequency of each difference.



How many ways does Helen have to win? __________

How many ways does Bruce have to win? __________

How many ways does Victor have to win? __________

Name ___________________

P1 *

Use the information on Worksheet P1(a) to answer these questions.

What is the probability that the sum is 6? ____________

What is the probability that the sum is not 6? ____________

What is the probability that the sum is more than 6? ________

What is the probability that the sum is less than 6? ________

When Bruce goes home, Helen and Victor decide to continue playing the sum game. They wish to play a fair game. List the sums each person could take to make the game fair.

Helen ____________________

Victor ____________________

Explain why your solution produces a fair game.

Name ____________________

P1 **

Use the information on Worksheet P1(b) to answer these questions.

What is the probability that the difference is 1? ____________

What is the probability that the difference is not 1? ____________

What is the probability that the difference is 0? ____________

What is the probability that the difference in not 0? ____________

When Helen goes home, Bruce and Victor decide to continue playing the difference game. They want to play a fair game. List the differences each person could take to make a game fair.

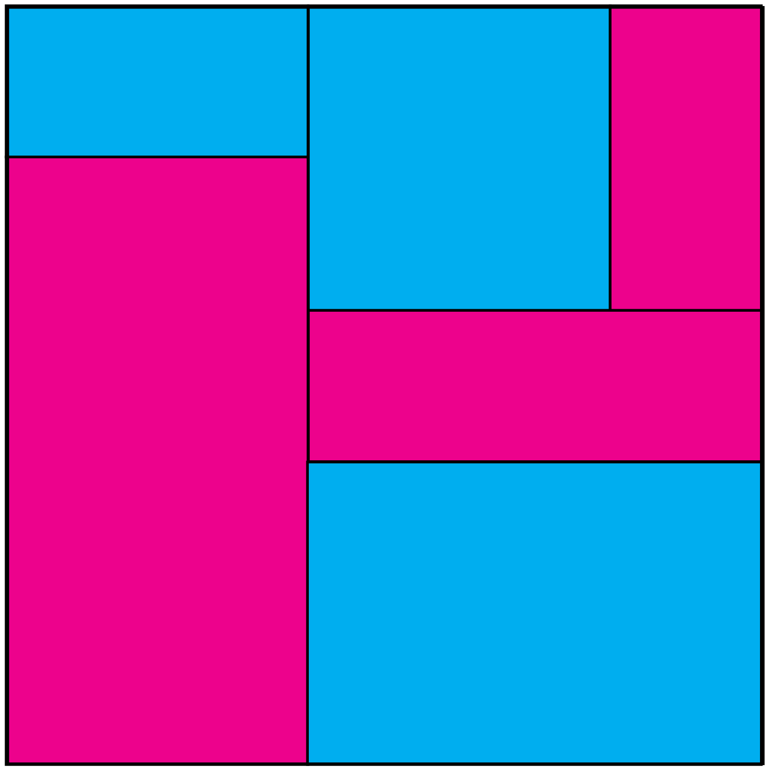
Bruce ________________________

Victor ________________________

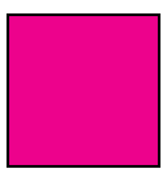
Explain why your solution produces a fair game.

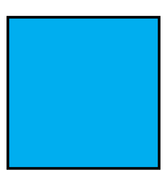
Name ____________________

P2 *



Use a ruler, if you wish, to answer these questions.

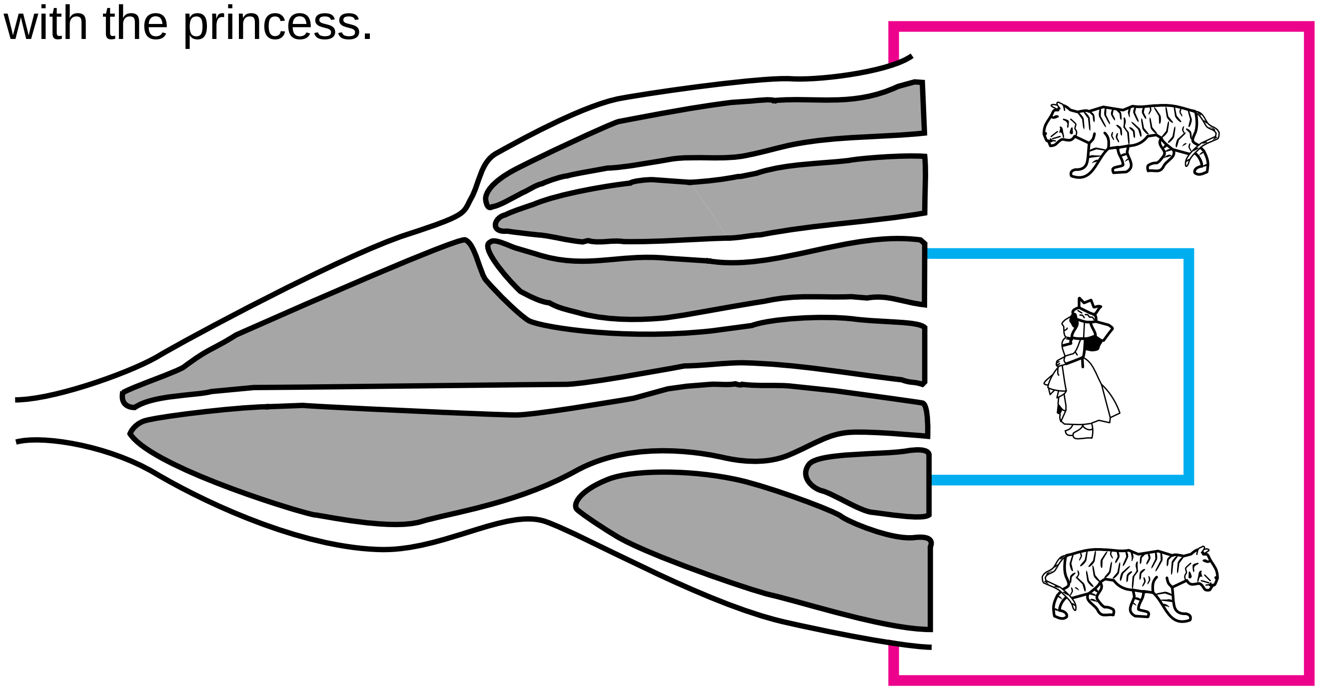
How many squares of this size  fit into the red region? ________

How many squares of this size  fit into the blue region? ________

Name ___________________

P2 **

The king has another maze near the castle. If Reynaldo goes through this maze, find his probability of entering the room with the princess.



Use this square to help you solve the problem.

What is Reynaldo’s probability of finding the princess? ________

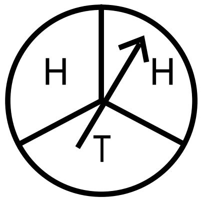
What is Reynaldo’s probability of finding the tigers? ________

Name ____________________

P3

Alice, Bruce, and Carl agree to play the following game.

1. Spin this spinner.



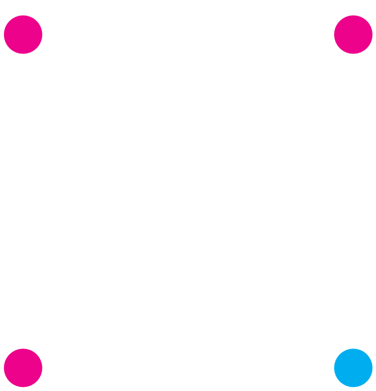
2. Select two marbles at random from the appropriate cup.

Winners: Alice wins if two red marbles are chosen.

Bruce wins if one red marble and one blue marble are chosen.

Carl wins if two blue marbles are chosen.

Use cords to show the winning combinations for cup H.



Use this square to show each player's probability of winning.

Alice ___ Bruce ___ Carl ____

Name ____________________

P4(a)

Decode this message.

W	E		J	E		J	X	U		S	H	U	U	A		D	U	Q	H			
J	X	U		B	Q	I	J		J	H	Q	Y	D		I	J	Q	J	Y	E	D	
M	X	U	H	U		Q		J	H	U	U		X	Q	I		V	Q	B	B	U	D
E	L	U	H		J	X	U		M	Q	J	U	H		Y	D	I	Y	T	U		
J	X	U		J	H	U	U		O	E	K		M	Y	B	B		V	Y	D	T	
Q		V	E	H	C	K	B	Q		V	E	H		Q		I	U	S	H	U	J	
F	E	J	Y	E	D		M	Y	J	X		J	X	Y	I		F	E	J	Y	E	D
M	U		M	Y	B	B		H	K	B	U		J	X	U		M	E	H	B	T	

Name ____________________

P4(b)

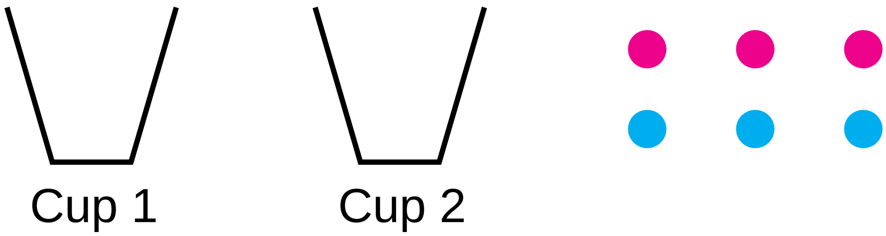
Determine the number of times each letter appears in the message.

Letter	Frequency	Letter	Frequency
A	________	N	________
B	________	O	________
C	________	P	________
D	________	Q	________
E	________	R	________
F	________	S	________
G	________	T	________
H	________	U	________
I	________	V	________
J	________	W	________
K	________	X	________
L	________	Y	________
M	________	Z	________

Name ____________________

P6

Distribute 3 red marbles and 3 blue marbles into the two cups. Use all 6 marbles and put at least one marble in each cup.



Use the square below to calculate the probabilities of winning with this distribution of marbles.

Bruce wins

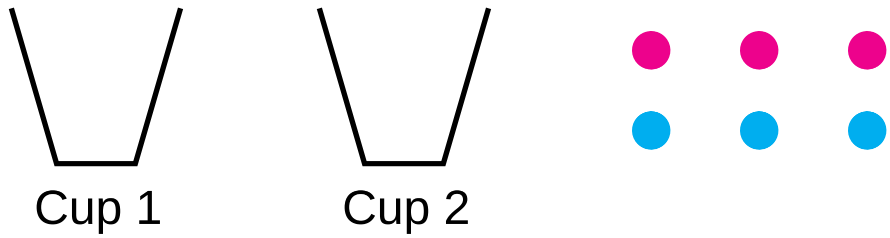
Player wins

What is Bruce’s probability of winning? ________

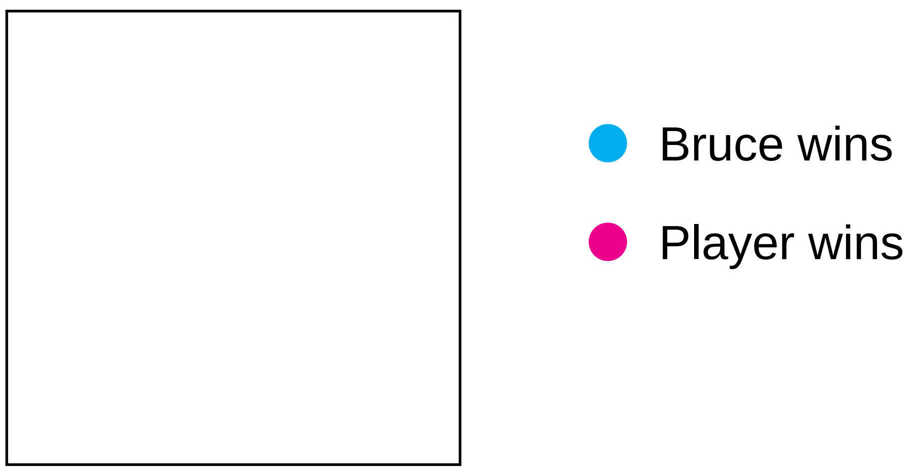
What is the player’s probability of winning? ________

Who is favored, Bruce or the player? ________

Distribute 3 red marbles and 3 blue marbles into the two cups.
Use all 6 marbles and put at least one marble in each cup.



Use the square below to calculate the probabilities of winning with this distribution of marbles.



What is Bruce’s probability of winning? _________

What is the player’s probability of winning? _________

Who is favored, Bruce or the player? _________

P5 AVERAGES

P5

Capsule Lesson Summary

Identify three numbers using their mean as a clue. Draw a cross-section of a lake when only the mean depth is given, and then when the mean, mode, and range of depth are given.

Materials

Teacher

- Grid board
- Colored chalk
- Meter stick

Student

- Worksheets P5(a) and (b)
- Colored pencils, pens, or crayons

Description of Lesson

Exercise 1

Note: Mean, median, and mode are three types of averages. In common usage, the word *average* often refers specifically to the mean.

T: ***I am thinking of three numbers whose mean (average) is 100. What numbers could they be?***

S: ***80, 70, and 150.***

Mean (Average) is 100		
80	70	150

Record a response on the board.

T: ***How could we check to see that the mean (average) of these numbers is 100?***

S: ***Add the three numbers and then divide by 3.***

S: ***Take 50 from 150, and give 20 to 80 and 30 to 70.***

Ask for additional sets of numbers whose mean is 100. List correct responses. Then put 90 in one column.

Mean (Average) is 100		
80	70	150
100	100	100
99	101	100
0	200	100
40	130	130
100	130	100
	90	

T: ***If one of the numbers is 90, what could the other two numbers be?***

S: ***110 and 100.***

S: ***80 and 130.***

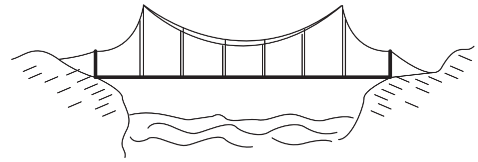
S: ***10 and 200.***

Record these responses in the list, and repeat the activity using 65 as one of the three numbers.

T: ***Here is another clue about my three numbers: One of the numbers is 100 and the other two differ by 60. Can you find my three numbers?***

Allow a few minutes for students to work on the problem. Then ask a student to announce the numbers.

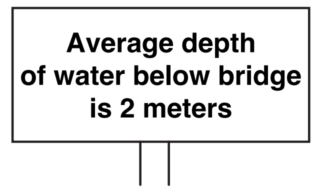
S: *70, 100, and 130.*



Exercise 2

Draw a picture of a bridge on the board.

T: ***Giambrone Park has a lake where people swim. In the lake there is an island connected to the shore by a bridge. This sign is on the bridge.***



Write the sign information on the board.

Solicit comments on the meaning of the sign. Use a meter stick to show how deep 2 meters is. Take a vote to determine how many students would be willing to dive into the water below the bridge.

Lead the discussion to consider how the average depth could have been computed. Students may suggest measuring the wet part of a stick or of a weighted rope that has been lowered to the bottom of the lake.

T: ***If the average depth was found by computing the n these measurements have been?***

Use student responses to record possible measurements on the board. Several possibilities are listed here.

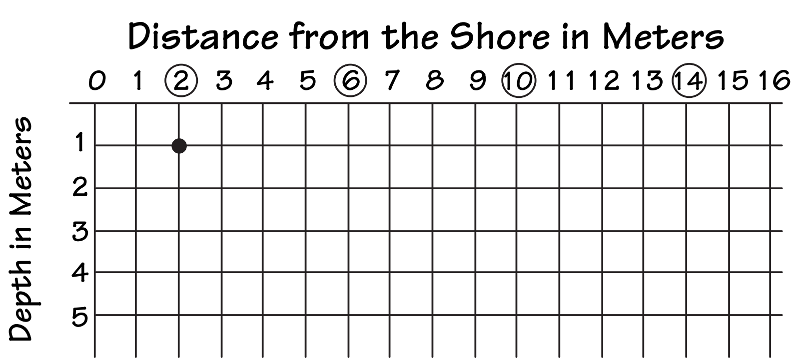
Average is 2 Meters

2 m	2 m	2 m	2 m
1 m	3 m	2 m	2 m
4 m	2 m	1 m	1 m
5 m	1 m	1 m	1 m
6 m	1 m	0.5m	0.5m
2 m	3 m	1 m	2 m

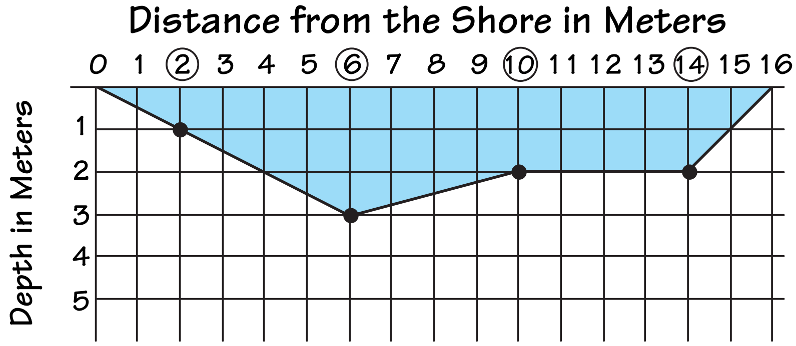
T: ***Let's use one of these sets of measurements to draw a picture of what the profile of the lake under the bridge might look like.***

On a grid, draw the graph in the next illustration as you explain,

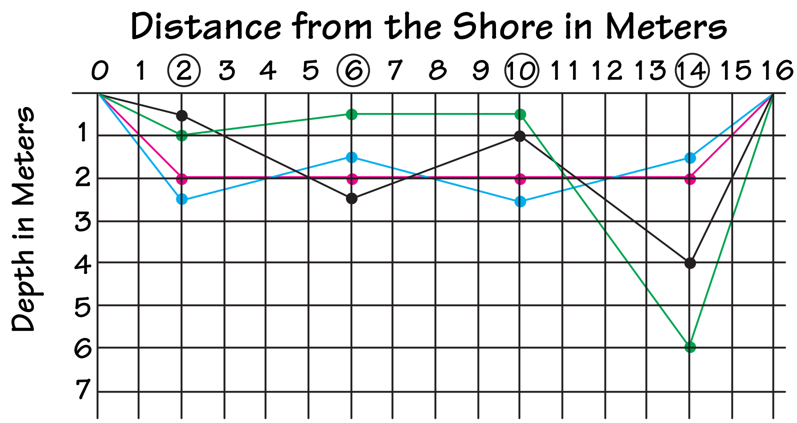
T: ***The bridge is 16 meters long. Let's assume the four measurements were taken at the circled locations. If the measurements taken were 1 m, 3 m, 2 m, and 2 m, who can place a dot to show the depth of the lake 2 meters from shore?***



Continue the activity until all four points have been plotted. Mention to the class that the depth of the lake at 0 meters and 16 meters from the shore is surely 0 meters, and draw a zigzag from (0,0) to (16, 0) that connects the four points. Ask a student to shade where the water is in the picture.



Distribute copies of Worksheet P5(a). Ask students to choose one of the other sets of data and to draw on their worksheets a picture of what the profile of the lake under the bridge might be with that data. Challenge students who finish quickly to find four possible measurements with an average of 2 m that are not listed on the board and to draw a picture of that data. Invite several students to draw their pictures of the profile of the lake under the bridge on the grid board. For example:



T: ***Does the average depth give enough information for us to know the profile of the lake under the bridge?***

S: ***No, there are many possibilities.***

T: ***What other information would be helpful?***

Accept several comments before announcing,

T: ***Here is some more information I received from the park ranger.***

Record this information on the board.

T: ***Out of eight measurements, the depth that occurred most frequently was 1 meter; that is, the mode was 1 meter. Also, the eight measurements ranged from 1 meter to 8 meters.***

8 measurements

Mean: 2 meters
Mode: 1 meter
Range: 1 to 8 meters

Before continuing, you may wish to return to the data already on the board and determine the range and mode for several sets of four measurements.

T: ***Use this new information to determine what the eight measurements could have been.***

Allow a short while for students to work (perhaps with a partner) on the problem. Record several responses and check them against the information given. There are many possible solutions, but students are likely to discover a variation of these measurements.

1, 1, 1, 2, 8, 1, 1, 1

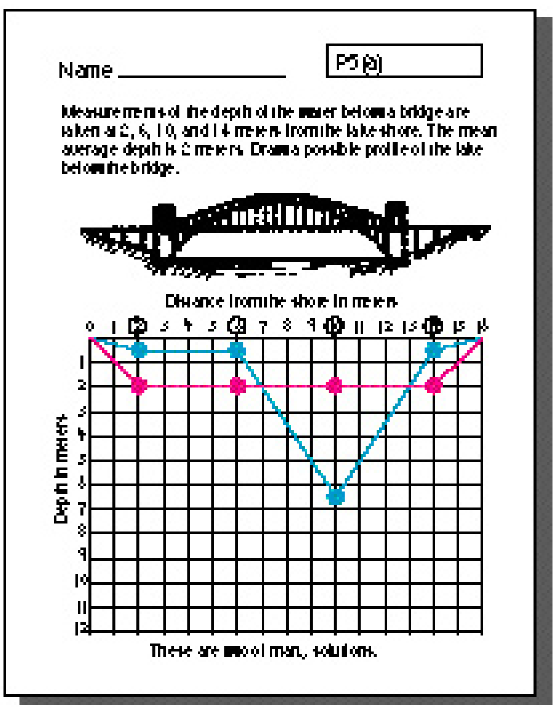
Ask students to draw a picture of the profile of the lake under the bridge on Worksheet P5(b). Briefly discuss the situation at the end of class.

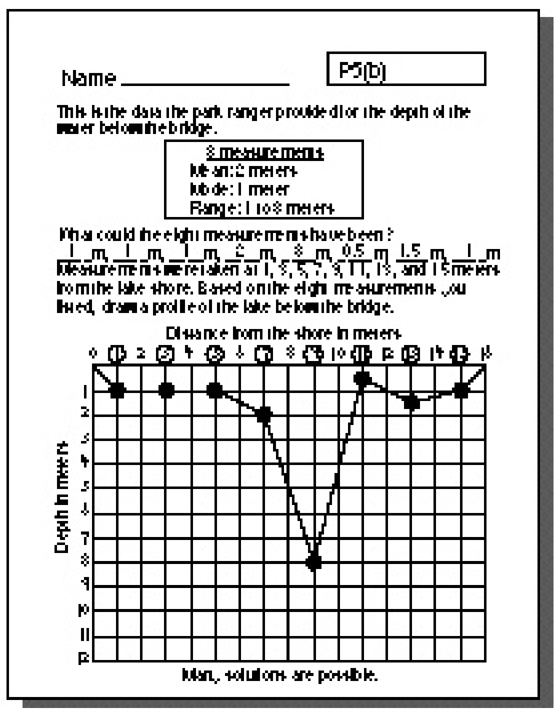
T: ***Do we know exactly what the profile of the lake looks like below the bridge?***

S: ***No, the measurements could be in any order.***

T: ***Would you dive in?***

S: ***No, the water is mostly shallow.***





Capsule Lesson Summary

Estimate the number of raisins in ½-oz. boxes and then find actual counts. Display the data from both estimates and actual counts in a back-to-back stem-and-leaf plot. Make a double bar graph of individual student data (estimates and actual counts.) Compare and contrast what information is communicated by the two visual representations.

Materials

Teacher
- Blackline P6

Student
- ½-oz. box of raisins
- Paper
- Colored pencils, pens, or crayons

Advance Preparation: Obtain ½-oz. boxes of raisins, enough for every student. As an alternative, use fun size packages of M&Ms or small containers with 1 tablespoon of dried beans in each. For the beans, you can use film canisters as containers and, at the appropriate time, show students the 1 tablespoon measure you used to measure the beans. Make several copies of Blackline P6 for group use in Exercise 2.

Description of Lesson

In this lesson, students should observe that different visual presentations of data allow different visual perspectives and readily answer different questions. This lesson is adapted and simplified from an activity in *Developing Graph Comprehension: Elementary and Middle School Activities* by Frances R. Curio.

Exercise 1

Give each student a ½-oz. box of raisins (or an alternative). Announce that they may not open the boxes yet.

T: ***Have you ever eaten raisins from a box this size?***
How many raisins do you think you get in one of these boxes?

Let students comment on the box size and how raisins are packed into the box. Then direct everyone to write on a piece of paper an estimate of the number of raisins they think will be in the box.

Call on each student in turn to give an estimate, and record the responses in a stem-and-leaf plot on the board. Explain to the class how you are recording their estimates.

For example:

S: *25.*
S: *32.*
S: *19.*
S: *20.*
S: *25.*

Tens	Ones
0	
1	9
2	5 0 5
3	2
4	
5	

Here the stem is the tens column and the leaves are the ones. Each estimate is represented by a leaf; the estimate of 25 is represented by 5 (ones) in a row next to 2 (tens) in the stem.

P6

After you record a few estimates, you may let students come to the board to record their own estimates in the stem-and-leaf plot. When estimates for all students are recorded, your stem-and-leaf plot may look similar to the one on the left below. At this point, suggest to the class that you order or rearrange the leaf data as in the plot on the right below.

Tens	Ones
0	
1	9 6 8
2	5 0 5 2 0 2 5 3 0 7
3	2 0 0 5 0 1 5 0
4	0 5 0 2
5	0 0

Tens	Ones
0	
1	6 8 9
2	0 0 0 2 2 3 5 5 5 7
3	0 0 0 0 1 2 5 5
4	0 0 2 5
5	0 0

Tell students that now they can open their boxes and count the raisins. Insist that everyone checks the count, and writes it on the same paper with his or her estimate. Suggest they label the numbers on the paper “estimate” and “actual.”

Call on several students to announce their actual counts, and record these results on the left side of the stem-and-leaf plot. Explain to the class that you are recording actual numbers back-to-back with the estimates. In this way you will construct a back-to-back stem-and-leaf plot, as illustrated here.

S: *38.*

S: *35.*

S: *40.*

S: *38.*

S: *39.*

Actual Ones	Tens	Estimate Ones
	0	
	1	6 8 9
	2	0 0 0 2 2 3 5 5 5 7
9 8 5 8	3	0 0 0 0 1 2 5 5
0	4	0 0 2 5
	5	0 0

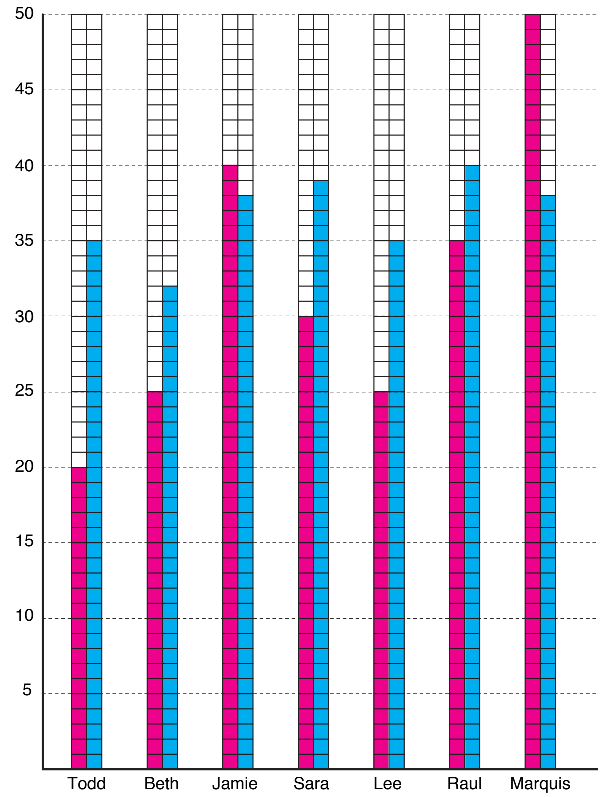
After you record a few actual counts, you may let students come to the board to record their own actual numbers in the stem-and-leaf plot. When actual counts for all students are recorded, rearrange the leaves on the actual side of the stem-and-leaf plot so it looks similar to one below.

Actual Ones	Tens	Estimate Ones
	0	
	1	6 8 9
9	2	0 0 0 2 2 3 5 5 5 7
9 9 9 9 8 8 8 8 5 5 5 4 4 2 2 2 2 1	3	0 0 0 0 1 2 5 5
1 1 1 1 0 0 0 0	4	0 0 2 5
	5	0 0

Hold a brief discussion about the results and, in particular, ask what students notice about the estimates compared to the actual counts. For example, the class may observe that the estimates are more spread out than the actual counts. Hold off from too much discussion of the data at this time so that you can have the discussion with two different visual displays after Exercise 2.

Exercise 2

Organize the class in groups of six or seven. Provide each group with one copy of Blackline P6, and explain that they are to present their individual estimates and actual counts in a double bar graph. Suggest that each student in a group write his or her name under a double column of little squares. On the left side of a column, the student colors a bar red to represent his or her estimate. Then, on the right side, the student colors a bar blue to represent his or her actual count. Students should have these numbers labeled on their papers. Here is a sample double bar graph from a group.



When a group completes and checks their double bar graph, ask them to work together to develop one other way to visually display their data or the class data.

Exercise 3

Collect the double bar graphs from all the groups. Make a display with the graphs next to each other on the board, separate but close to the back-to-back stem-and-leaf plot.

Begin a discussion with the class to interpret the two visual representations of the same data. This discussion should elicit comments about what information is better communicated by which type of graph. For example:

Back-to-Back Stem-and-Leaf Plot Graphs
- cannot see individual's estimate and actual count
- easy to find the frequency of a given estimate or actual count
- easy to see range and distribution of estimates and actual counts

Double Bar Graph
- readily compare an individual's estimate with actual count
- more difficult to find the frequency of a given estimate or actual count
- cannot easily see range and distribution of estimates and actual counts

You may ask specific questions to compare the two data displays. For each question, check both displays to see if one is easier to use in answering the question. For example:

- Which estimate was the highest? … lowest?
- What was the *range* of estimates (actual counts)?
- Which student(s) had the best estimate?
- Did most students estimate too high or too low?
- Which estimate (actual count) occurred most often (mode)?
- Which estimate (actual count) was in the middle (median)?
- How can we calculate the mean average of the actual counts?

If the groups developed other visual representations of the class data, you may at this point like to compare their displays to the double bar graph or the back-to-back stem-and-leaf plot.

Capsule Lesson Summary

Tell the history of a classic probability problem, the problem of points. Introduce the problem as a game played by two people. Calculate the chances of each player winning when the score is not tied.

Materials

Teacher
- *IG-IV Probability Poster* #1
- Meter stick
- Colored chalk
- One red and one blue marble
- One coin
- Tape

Student
- Paper
- Colored pencils, pens, or crayons
- Metric ruler
- Worksheet P7

Description of Lesson

Exercise 1

The following dialogue concerning the historical background of the central problem in this lesson is optional.

T: ***Most mathematics deals with certain events. Problems, whether easy or hard, usually have one or more definite answers. However, probability focuses on uncertain events. No one can consistently predict the roll of a die, the flip of a coin, or tomorrow's weather. We can only state probabilities for such events.***

Because of its uncertainty, mathematicians did not study probability for a long time. Only in the 1500s did interest in probability begin to grow. What activities do you think were the source of the first problems in probability?

S: ***Games and gambling that involved dice and cards.***

T: ***Today we will study a probability problem that arose in several of these two-person games. The game could be as simple as flipping a coin in which one player gets points for heads and the other player gets points for tails. The first player to get ten points wins.***

Select two students to play the game, one student for heads and the other for tails. Flip a coin and record one point for the winner. Continue until it is clear that most students understand the game. For example, here are the results of a three-flip sequence.

1. **Heads** - Player **A** gets a point; score: **A**-1, **B**-0
2. **Tails** - Player **B** gets a point; score: **A**-1, **B**-1
3. **Tails** - Player **B** gets a point; score: **A**-1, **B**-2

Write this information on the board.

A: 9
B: 8

$100

T: ***Here is the probability problem: Each player puts $50 into a pot and agrees that the first player to get ten points wins the $100 pot. However, the players have to stop the game when the score is* A*-9;* B*-8. How should they split the $100?***

Let students discuss the problem and offer possible solutions. For example:

- Divide the pot \$50-\$50 since the game is not finished.
- Give all of the money to **A** since **A** is leading.
- Give $\frac{9}{17}$ of the pot to **A** and $\frac{8}{17}$ to **B**.

For each solution, ask which player is most likely to object to the solution. For example, player **A** needs only one more point to win and might object to the pot being divided evenly or to getting $\frac{9}{17}$ of it, which is not much more than half. But player **B** is only one point behind **A** and might object to giving all of the pot to **A**.

T: ***An Italian mathematician, Tartaglia, claimed to have solved this problem in the 1550s. However, there was no record of his solution, and mathematicians continued to discuss the problem for about 100 years. In about 1650, the Chevalier de Mére, a member of the court of King Louis XIV of France and also a gambler, asked his friend Blaise Pascal for a solution. Pascal, a young mathematician, wrote about the problem to his older and more famous friend Pierre de Fermat. After about a year of correspondence, they solved the problem in three different ways. Their methods provided a beginning to the mathematical study of probability.***

The method that we will use to solve the problem is similar to one of the methods Pascal and Fermat used about 350 years ago.

Exercise 2

T: ***Rita and Bruce are playing a game. Instead of a coin, Rita has one red marble and one blue marble. Keeping her hands behind her back, she mixes the marbles and then puts one marble in each hand. Bruce chooses a hand. If he chooses the hand with the blue marble, he scores one point. Otherwise, Rita scores a point. The first player to get ten points wins.***

Select two students to play the roles of Rita and Bruce. Let them play the game in front of the class, recording the score of the game on the board. Let them continue without interruption until one player reaches six points. Here we assume that the score is 4–6.

- Rita: 4
- Bruce: 6

T: ***The score is now 4-6; let's always read Rita's score first. What could the next score be?***

S: ***5-6, if the red marble is chosen.***

S: ***4-7, if the blue marble is chosen.***

Let the game continue. After each point, ask students for the next two possible scores. Continue until a player wins by reaching ten points.

T: ***What is the shortest game Rita and Bruce could play?***

S: ***A game that ends 10-0 or 0-10.***

T: ***What is the longest game they could play?***

S: ***A game that ends 10-9 or 9-10.***

Exercise 3

Write the following information on the board. A convenient length for the line segment is 32 cm.

● Rita: 7
● Bruce: 8

T: ***One afternoon Rita and Bruce play the game until they have to stop. But the game is not over; they stop when the score is 7-8 and agree to continue the next day. When they continue the game the next day, who is favored to win?***

S: ***Bruce, because he is ahead.***

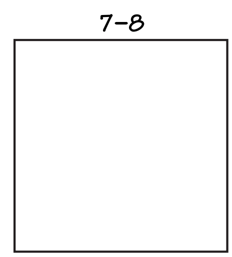
T: ***What do you think is Bruce's probability of winning the game? Draw a dot on the probability stick for your estimate.***

If necessary, remind students that a dot at 1 means that Bruce would always win and that a dot at 0 means that Bruce would never win. Let students estimate Bruce's probability of winning by drawing dots on the probability stick, as illustrated here.

Any dots placed at or below the midpoint should be challenged by other students. Since Bruce is leading, his probability of winning is definitely greater than ½.

Draw a large square on the board, as illustrated here.

● Rita
● Bruce



T: ***Let's use this square to calculate Bruce's chances of winning when the score is 7-8. What could the next score be?***

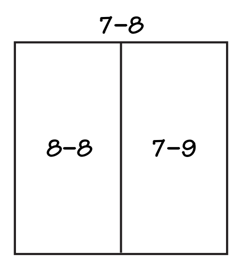
S: ***8-8 if the red marble is chosen, or 7-9 if the blue marble is chosen.***

T: ***Which next score is more likely, 8-8 or 7-9?***

S: ***They are equally likely, since there is one red marble and one blue marble.***

Divide the square in half; indicate half for 8-8 and half for 7-9.

● Rita
● Bruce



T: ***Now we must consider these two games, one with a score of 8-8, the other with a score of 7-9. First, let's pretend that the score is 7-9. If the score is 7-9, what could the next score be?***

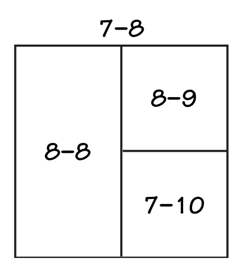
S: ***8-9 if the red marble is chosen, or 7-10 if the blue marble is chosen.***

T: ***These two scores, 8-9 and 7-10, are equally likely; therefore, let's divide the region for 7-9 in half with one half for 8-9 and the other half for 7-10.***

What do we know if the score in a game is 7-10?

S: ***Bruce has won. Color that region blue for Bruce.***

● Rita
● Bruce

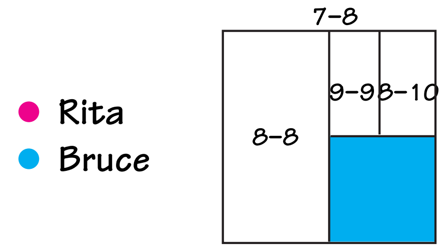


T: *If the score is 8-9, what could the next score be?*

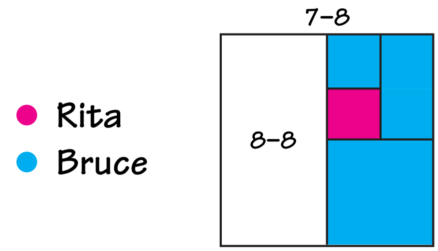
S: *9-9 or 8-10. Divide the region for 8-9 in half; one half 9-9 and the other half for 8-10.*

T: *What happens if the score is 8-10?*

S: *Bruce wins. Color that region blue.*



S: *If the score is 9-9, the next score could be 9-10 or 10-9. Divide the region for 9-9 in half and color one half red for Rita and one half blue for Bruce.*



T: *We still must consider a game when the score is 8-8.*

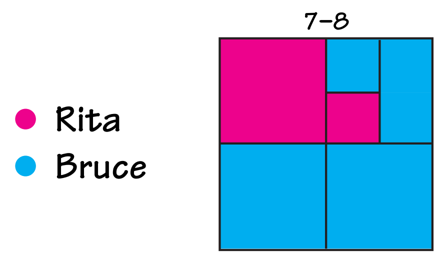
S: *The next score could be 9-8 or 8-9. Divide the region for 8-8 in half, one half 9-8 and the other half 8-9.*

T: *We could do that. When the score is 8-8, who is favored, Rita or Bruce?*

S: *Neither. Each player has a probability of $\frac{1}{2}$ of winning.*

T: *Therefore, we can divide the region for 8-8 in half and color one half red and the other half blue.*

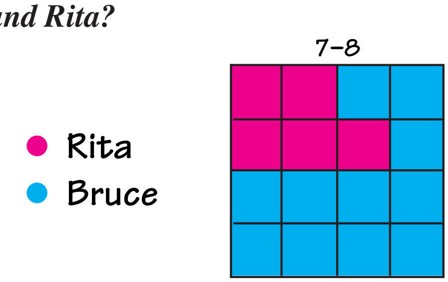
When the score in a game is 7-8, the blue regions represent Bruce's chances for winning and the red regions represent Rita's chances for winning. From the picture, can we tell who is more likely to win?



S: *Bruce, because more of the square is blue than red.*

T: *How can we calculate the chances of winning for Bruce and Rita?*

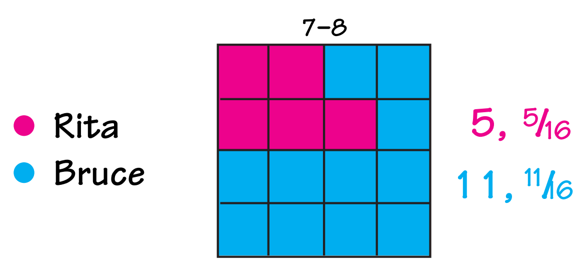
Invite class discussion. Encourage any suggestions to compare the areas shaded red and blue by choosing a small piece and counting how many of those pieces fit in the blue region and how many fit in the red region. It may be useful to subdivide the larger regions.



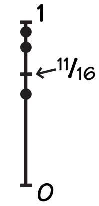
T: *The square shows that Rita has 5 chances of winning while Bruce has 11 chances of winning. Bruce is heavily favored to win.*

S: *Since 5 + 11 = 16, Rita's probability of winning is $\frac{5}{16}$ and Bruce's probability of winning is $\frac{11}{16}$.*

Record the chances and probabilities for winning near the square.



Refer back to the probability stick on the board on which students drew dots for their estimates of Bruce's probability of winning earlier in this lesson.



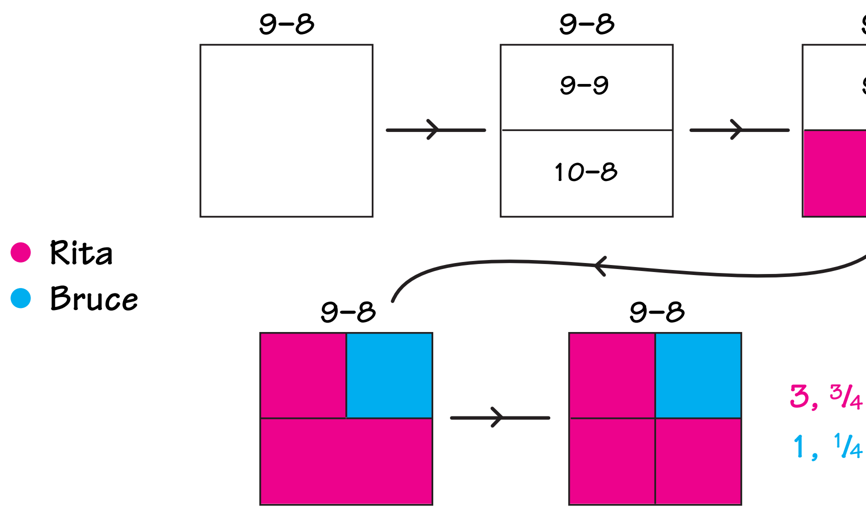
T: ***Whose dot is closest to $^{11}/_{16}$, Bruce's probability of winning?***

Let students discuss where $^{11}/_{16}$ is located on the line segment Invite students to use a meter stick to divide the segment into 16 pieces of the same length and to locate $^{11}/_{16}$. Determine whose estimate is closest.

Distribute copies of Worksheet P7.

T: ***Now let's solve the actual problem Pascal and Fermat worked on. Pretend Rita and Bruce each bet 50¢ on a game. The game stops when Rita leads 9-8. Use the square to calculate each player's probability of winning. Use these probabilities to determine how they should split the $1.00 pot.***

Allow several minutes for students to work independently. Then, using student suggestions, proceed gradually to solve the problem collectively at the board. Allow time between steps for students to complete the problem on their own. The squares drawn below show one procedure to calculate the chances. Your class might divide and color the square in a different way, but they should arrive at the same result.



T: ***How should Rita and Bruce divide the $1.00 pot when the game stops at 9-8?***

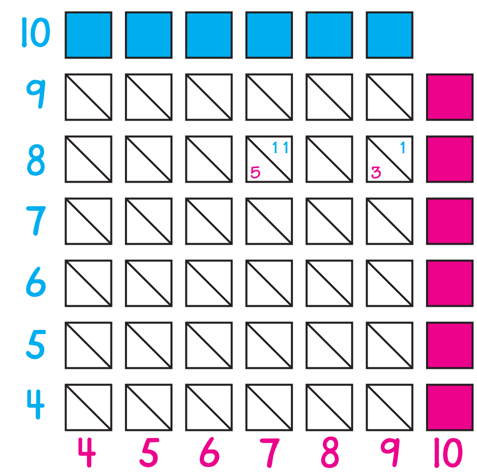
S: ***$0.75 for Rita and $0.25 for Bruce because $^{3}/_{4}$ x 1.00 = 0.75 and $^{1}/_{4}$ x 1.00 = 0.25.***

Display *IG-IV Probability Poster* #1. Record the results for the score 9-8 in the square that is in the 9-column (for Rita's score) and the 8-row (for Bruce's score). (See the next illustration.)

T: ***On this poster we can record the chances in our problems. In addition to the score 9-8, we also calculated Rita's and Bruce's chances of winning when Bruce leads 7-8. Who remembers the chances?***

S: ***When Bruce leads 7-8, he has 11 chances of winning and Rita has 5 chances of winning.***

Invite a student to record this result for 7-8 in the chart. Explain to the class that the result for 7-8 gets recorded in the square in the 7-column and the 8-row.



T: ***In later lessons, we will complete this chart by finding Bruce's and Rita's winning chances for many other starting scores. Hopefully, we can find some shortcuts and patterns so that we do not need to divide and color a square for every entry in the chart.***

Save the poster for use again in Lessons P8 and P9.

Capsule Lesson Summary

Review the story and results from Lesson P7 concerning a 16th century probability problem. Develop efficient techniques for determining the winning probabilities of the two players given different intermediate scores.

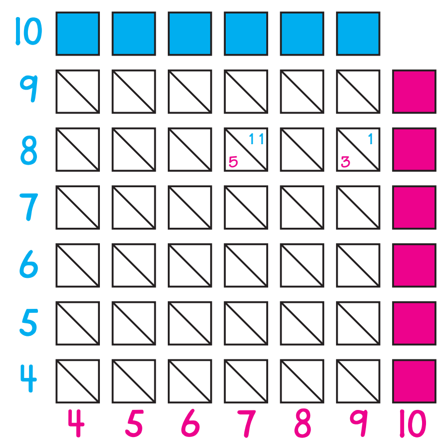
Materials

Teacher	• *IG-IV Probability Poster* #1 • One red and one blue marble • Calculator • Colored chalk	Student	• Paper • Colored pencils, pens, or crayons • Worksheets P8* and **

Description of Lesson

Exercise 1

Display *IG-IV Probability Poster* #1 with the entries made in Lesson P7. If necessary, make the following entries on a clean copy of the poster.



T: ***Who remembers the story about the famous probability problem Pascal and Fermat solved in the 1650s? What game did Rita and Bruce play that was similar to that problem?***

Let students recall details of the game and the history of the problem.

Students should mention the following aspects:

- The game for Rita and Bruce uses one red and one blue marble.
- Rita scores one point each time a red marble is chosen; Bruce scores one point each time a blue marble is chosen.
- The first player to get ten points wins.

Write this information on the board.

● Rita: 7
● Bruce: 6

T: ***Pretend that the score in a game is 7-6. What could the next score be?***

S: ***8-6 if a red marble is chosen, or 7-7 if a blue marble is chosen.***

Mix one red and one blue marble behind your back. Put one marble in each hand, and let a student select a hand, look at that marble, and give either Rita or Bruce a point. Record the new score on the board.

Continue in a similar manner until either Bruce or Rita wins by reaching ten points. At each turn, ask students for the next two possible scores.

P8

Direct the class's attention to the poster.

T: ***Last week we solved some problems and recorded the results on this poster.***

Point to the square for the score 7-8 (in the 7-column and 8-row).

T: ***What do red 5 and blue 11 in this square mean?***

S: ***If the score in a game is 7-8, Rita has 5 chances of winning as compared to Bruce's 11 chances of winning.***

T: ***What is each player's probability of winning?***

S: ***$^5/_{16}$ for Rita and $^{11}/_{16}$ for Bruce, since 11 + 5 = 16.***

T: ***How should Rita and Bruce split a prize pot if they stop the game at 7-8?***

S: ***Divide the amount in the pot into 16 equal parts. Give Rita 5 parts and Bruce 11 parts.***

S: ***Rita receives $^5/_{16}$ of the pot, and Bruce receives $^{11}/_{16}$ of the pot.***

At this point, you may like to choose an amount for the pot (say $1.00) and ask students to calculate Rita's share and Bruce's share. Depending on the amount chosen, students can use a calculator to divide the pot into 16 equal parts and then give 5 parts to Rita and 11 to Bruce. For example, with a $1.00 pot Rita's share would be 31¢ and Bruce's share 69¢.

Direct students' attention to the square for 9-8 on the poster.

T: ***If the score is 9-8, what are Bruce's and Rita's chances of winning?***

S: ***Bruce has 1 chance of winning and Rita has 3.***

T: ***If we know Rita's and Bruce's chances of winning when the score is 9-8, do we also know the results for another score?***

S: ***Yes, for 8-9. Rita has 1 chance and Bruce has 3 chances to win.***

T: ***Why?***

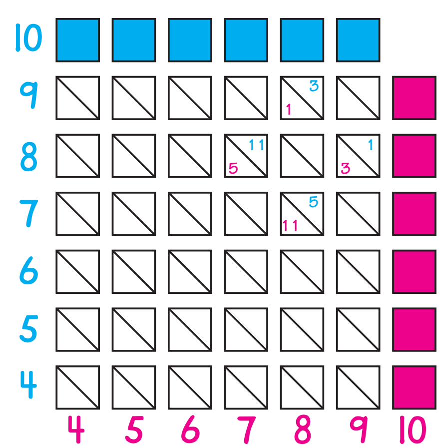
S: ***Since there is one red marble and one blue marble, Rita's advantage when the score is 9-8 is the same as Bruce's advantage when the score is 8-9.***

Invite a student to record the result for 8-9 on the poster.

T: ***Can we determine the result for another score?***

S: ***Yes, 8-7, because we already know the result for 7-8. When the score is 8-7, Rita has 11 chances of winning and Bruce has 5 chances.***

Invite a student to record this result for 8-7 on the poster.



Refer to the results for the scores 7-8 and 8-9 on the poster. You may rewrite them on the board.

7-8: 5 11
8-9: 1 3

T: ***Bruce is favored for each of these scores. Which of these two scores is better for Bruce?***

Encourage both intuitive answers based on the game situation and quantitative answers based on the chances.

T: ***8-9 is better for Bruce because in both situations Rita is one point behind Bruce. But when the score is 8-9, Rita has less time to catch up and win since Bruce is only one point from victory.***

S: ***For 8-9, Bruce has 3 chances out of 4 to win so his probability of winning is $\frac{3}{4}$. For 7-8, Bruce has 11 chances out of 16 to win, so his probability of winning is $\frac{11}{16}$. The 8-9 score is better for Bruce since $\frac{3}{4} = \frac{12}{16}$ and $\frac{12}{16}$ is greater than $\frac{11}{16}$.***

S: ***We calculated that for 8-9 Bruce would receive $0.75 of a $1.00 pot, while for 7-8 Bruce would receive about $0.69. So 8-9 is better for Bruce.***

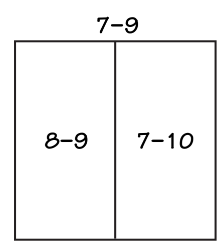
S: ***8-9 is the better score for Bruce, because he is three times more likely to win than Rita. When the score is 7-8, Bruce is only about two times as likely to win as Rita since 11 is just a little more than 2 x 5.***

Exercise 2

Draw a square on the board in preparation for calculating chances when the score is 7-9.

T: ***Let's compute Rita's and Bruce's probabilities of winning for some other scores at which they might stop playing. We will try to find some shortcuts as we solve these problems.***

One day Rita and Bruce stop playing when the score is 7-9. Let's calculate each player's chances of winning if they continue the game the next day. If the score is 7-9, what could the next score be?



S: ***8-9 if the red marble is chosen, or 7-10 if the blue marble is chosen. Divide the square in half, one half for 8-9 and the other half for 7-10.***

S: ***Bruce wins if the score is 7-10. Color that half blue.***

T: ***What do we know if the score is 8-9?***

S: ***The next score could be 9-9 or 8-10. We should divide the region for 8-9 in half.***

T: ***Yes, we could calculate Rita's and Bruce's winning chances when the score is 8-9 that way. But do we have to do all that work? What do we already know when the score is 8-9*** (refer to the poster square for 8-9)***?***

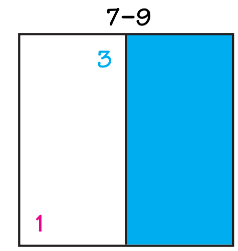
S: ***Bruce has 3 chances to win and Rita has only 1 chance.***

Invite a student to point to where this result is recorded on the poster.

T: ***So if we divided the region for 8-9 into smaller pieces, we know that we would get three blue pieces and one red piece.***

P8

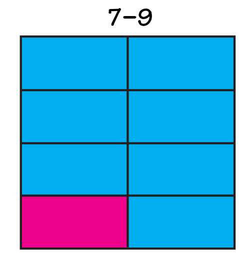
Record the information as shown here.



T: ***If we divide this uncolored region into three blue parts and one red part of the same size, what fraction of the whole square would be blue? Red?***

S: ***⅞ blue and ⅛ red.***

Invite a student to show this by dividing the uncolored region into four parts and coloring them appropriately. Divide the large blue region in the same way the first half was divided. For example:

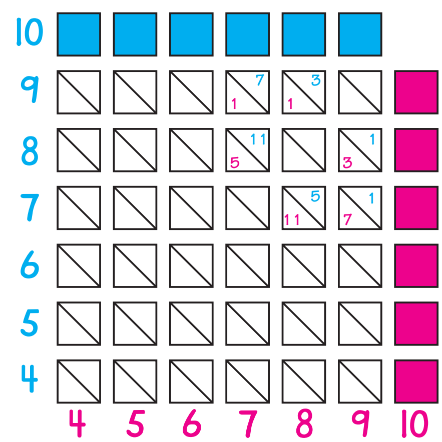


T: ***What are Rita's and Bruce's chances of winning when the score is 7-9?***

S: ***Bruce has 7 chances of winning and Rita has 1 chance.***

S: ***Bruce's probability of winning is ⅞; Rita's is ⅛.***

Ask a student to record this result for 7-9 on the poster.



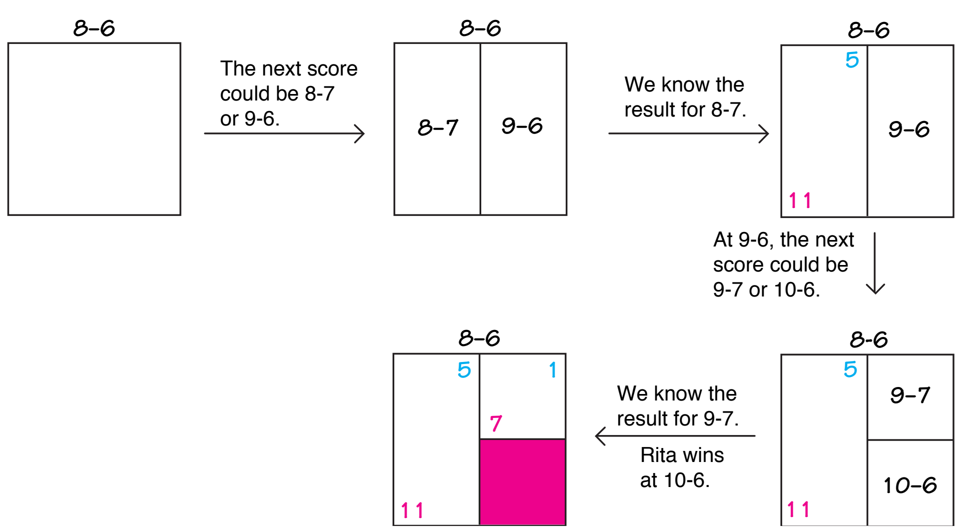
T: ***What other result can we record since we know the result for 7-9?***

S: ***9-7; Rita has 7 chances of winning and Bruce has 1.***

Invite a student to record the result for 9-7 on the poster.

Erase the board. Then draw a square on the board for a score of 8-6.

If your students are ready, let them solve this problem independently. Otherwise, let students lead you through the solution at the board. Insist that they use previous results when possible. The following illustration shows one procedure to calculate the probabilities. Key comments are included.



T: ***It should not be necessary to actually divide each region into small pieces, but let's imagine that we do it. How would we divide and color each region?***

S: ***Divide the region with blue 5 and red 11 into 16 equal-sized pieces; color five blue and 11 red.***

S: ***Divide the region with blue 1 and red 7 into eight equal-sized pieces; color one blue and seven red.***

S: ***Divide the red region into eight pieces, because that region is the same size as the region above it.***

Put 8 in red near the red region.

T: ***If we divide the regions in this way, would all of the small pieces be the same size?***

Let the class discuss this question until they agree that all of the pieces would be the same size.

T: ***In the whole square, how many small pieces would be blue?***

S: ***6 (5 + 1 = 6).***

T: ***How many would be red?***

S: ***26 (11 + 7 + 8 = 26).***

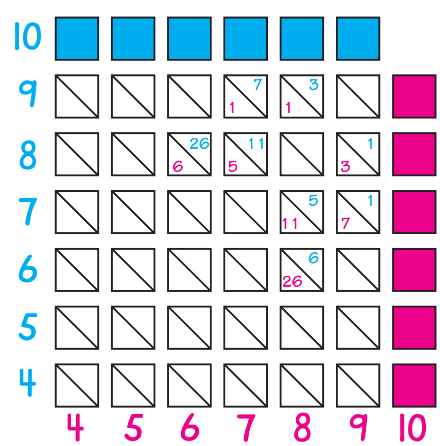
T: ***Therefore, when the score is 8-6, Rita has 26 chances of winning and Bruce has 6 chances. What is each player's probability of winning?***

S: ***$^{26}/_{32}$ for Rita and $^{6}/_{32}$ for Bruce since 6 + 26 = 32.***

T: ***Do we also know the chances of winning when the score is 6-8?***

S: ***For 6-8, Bruce has 26 chances of winning and Rita has 6 chances.***

Invite a student to record these results on the poster.

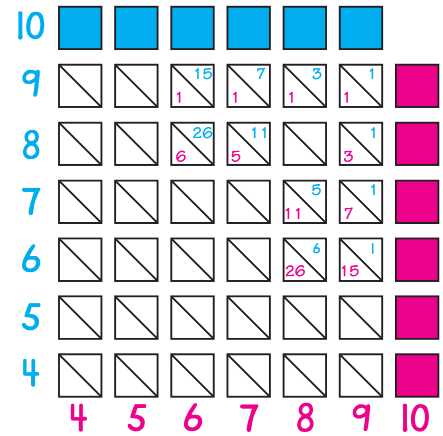


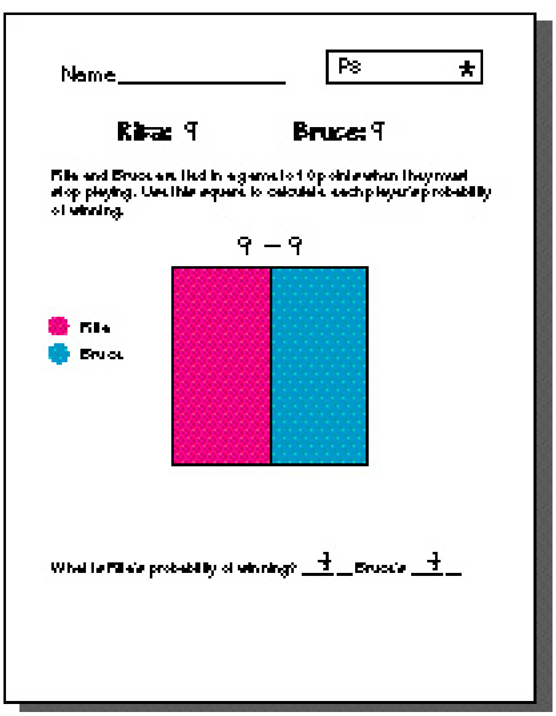
Distribute copies of Worksheets P8* and ** for individual work.

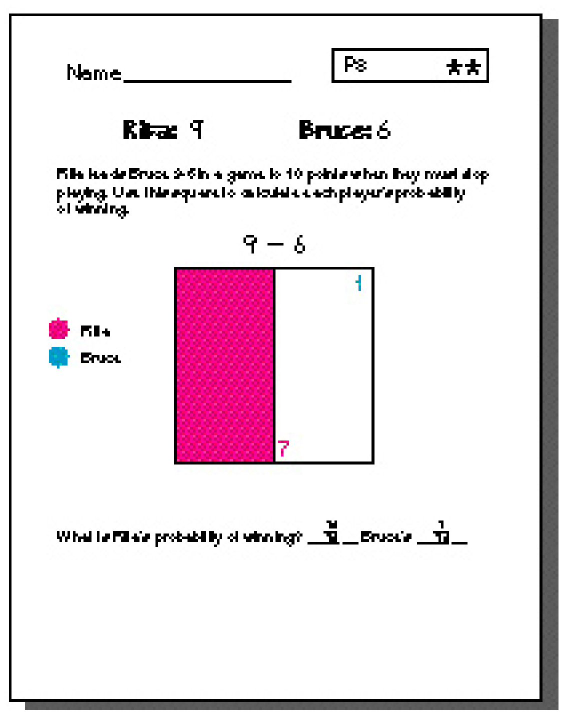
At the end of the lesson, invite students to record the results for 9-9, 9-6, and 6-9 on the poster.

T: ***Today we found a few shortcuts for filling in this chart. We learned that if we know the chances for one score, for example, 8-6, we also know the chances for 6-8. We also found out how to use previous results to help solve new problems. Next time we will complete the chart by finding some number patterns in this chart.***

Save the poster for use in Lesson P9.







Capsule Lesson Summary

Review the probability game from Lessons P7 and P8, and recall how to read the chart on the poster. Investigate some patterns in the chart and use them to complete the chart. Discuss and solve some problems about related probability games using the completed chart.

Materials

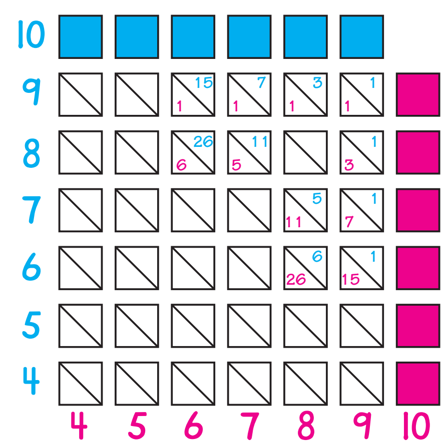
Teacher
- *IG-IV Probability Poster #1*

Student
- Colored chalk
- Paper
- Colored pencils, pens, or crayons
- Worksheet P9

Description of Lesson

Exercise 1

Distribute copies of Worksheet P9 and display *IG-IV Probability Poster #1*. The results from Lessons P7 and P8 should be recorded on the poster as they are on the worksheet.



T: ***Do you remember the game that Rita and Bruce were playing?***

Make certain that at least the following aspects of the game are mentioned.

- One red marble and one blue marble are used.
- Rita scores a point if the red marble is chosen, and Bruce scores a point if the blue marble is chosen.
- The first player to score ten points wins.

Point to the square for 6-8 on the poster.

T: ***What does this entry in the chart tell us?***

S: ***When Bruce leads 6-8 in the game, he has 26 chances of winning while Rita has only 6 chances.***

S: ***Bruce's probability of winning is ${}^{26}/_{32}$ and Rita's is ${}^{6}/_{32}$.***

T: ***How did we calculate these probabilities?***

Let students briefly describe how they used a square to calculate the probabilities.

T: ***When Rita leads 8-7, what are Rita's and Bruce's chances of winning?***

Let a student point to the square for 8-7 on the poster.

S: ***Rita has 11 chances to win and Bruce has 5 chances.***

S: ***Rita's probability of winning is $^{11}/_{16}$ and Bruce's is $^{5}/_{16}$.***

Point to the row of blue squares.

T: ***Why are these squares colored blue?***

S: ***Those squares are for scores of 4-10, 5-10, 6-10, and so on, all games that Bruce wins.***

Similarly, conclude that the column of red squares are for scores of games that Rita wins.

Exercise 2

Point to the square for 5-9 in the chart as you ask,

T: ***Let's try to find number patterns to predict what some other entries on the chart might be. For example, can you predict Rita's chances when the score is 5-9?***

S: ***Rita most likely has 1 chance of winning. There are only red 1s in that row.***

T: ***What do you think are Bruce's chances when the score is 5-9?***

S: ***31.***

T: ***What pattern did you use?***

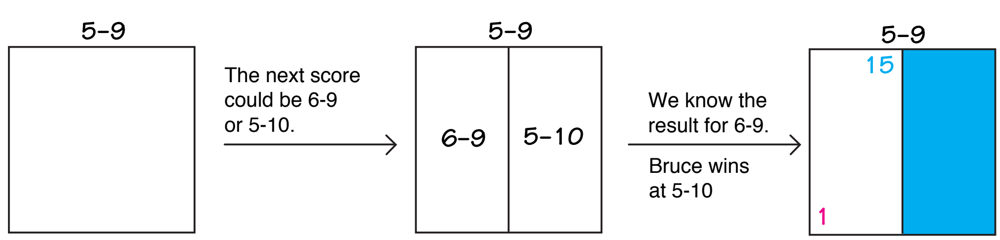
Elicit explanations from several students if they noticed different patterns.

S: ***The blue numbers in that row are 1, 3, 7, and 15. To get the next blue number, multiply the previous number by 2 and add 1. For example, (2 x 1) + 1 = 3, (2 x 3) + 1 = 7, and (2 x 7) + 1 = 15. Therefore, the next blue number is 31 since (2 x 15) + 1 = 31.***

S: ***I noticed that 1 + 2 = 3, 3 + 4 = 7, and 7 + 8 = 15. The number you add each time—2, 4, 8—is doubled. The next number should be 31, since 15 + 16 = 31 and 16 is the double of 8.***

S: ***I also noticed that 1 + 2 = 3, 3 + 4 = 7, and 7 + 8 = 15. The number you add each time is one more than the previous number. The next number should be 31, since 15 + 16 = 31 and 16 is one more than 15.***

Draw a square on the board for 5-9, and invite students to tell you how to use the square to calculate the chances and check the predictions. For example:



T: ***What are Rita's and Bruce's chances of winning?***

S: ***1 for Rita and 31 for Bruce.***

T: *Why?*

S: ***The left half of the square could be divided into 16 pieces: 1 red and 15 blue. To use the same size pieces, the right half must also be divided into 16 pieces, all blue. Then there would be 1 red piece and 31 (15 + 16 = 31) blue pieces.***

Invite students to record the results for 5-9 and 9-5 on the poster and on their worksheets.

Ask students to use patterns to predict the entry for 9-4.

S: ***Bruce has just 1 chance. All of the blue numbers in the 9-column are 1s.***

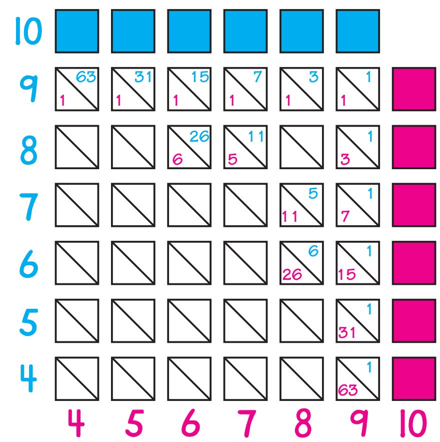
If students are using different patterns, encourage several explanations for Rita's chances.

S: ***63, since (2 x 31) + 1 = 63.***

S: ***63, since 31 + 32 = 63 and 32 is the double of 16.***

S: ***63, since 31 + 32 = 63 and 32 is one more than 31.***

Invite students to record the entries for 9-4 and 4-9 both on the poster and on their worksheets.



T: ***Can you predict Rita's and Bruce's chances when the score is 8-8?***

S: ***1 for Rita and 1 for Bruce because 8-8 is a fair game.***

T: ***That's correct. We could record the result for 8-8 as 1 chance for Rita and 1 for Bruce. But are there other ways to record chances for a fair game?***

S: ***We could record 2 chances for each player or 3 chances for each, or any number of chances as long as it is the same for each player.***

T: ***Do you see any patterns in our chart that might suggest a way to record the results for the score 8-8?***

S: ***If we enter a red 4, the red numbers in the 8-row increase by one: 3, 4, 5, 6.***

S: ***If we enter a blue 4, the blue numbers in the 8-column also increase by one: 3, 4, 5, 6.***

T: ***To maintain these patterns, we'll record the results for the score 8-8 as 4 chances for Rita and 4 chances for Bruce.***

Record the result on the poster.

T: ***Can you predict Rita's chances when the score is 5-8?***

S: ***7 chances, because the red numbers in the 8-row increase by one: 3, 4, 5, 6, 7.***

T: ***What are Bruce's chances when the score is 5-8?***

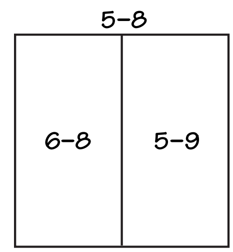
Record students' predictions and let them explain their answers.

T: ***On your own, draw a square on a piece of paper and calculate Bruce's and Rita's chances when the score is 5-8. Use previous results when possible. If you finish quickly, do the same for the score 8-4.***

You may need to assist some students and give hints to the class. After a while, collectively work with the class as follows.

T: ***Let's calculate the winning chances when the score is 5-8. How do we start?***

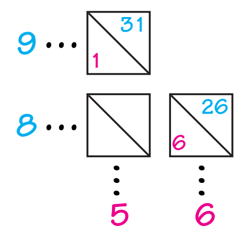
S: ***Divide the square for 5-8 in half; one half for 6-8 and the other half for 5-9.***



T: ***Do we know the results for the scores 6-8 and 5-9?***

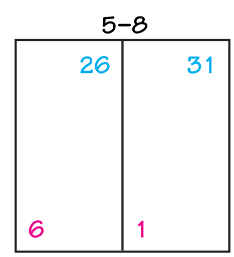
S: ***Yes, if the score is 6-8, Bruce has 26 chances to win Rita has 6 chances. If the score is 5-9, Bruce has 31 chances and Rita has 1 chance.***

Refer to this part of the chart as you point out the relationships among the squares for 5-8, 6-8, and 5-9.



T: ***To calculate the chances for 5-8, we can use the results recorded in this square for 6-8 and the results recorded in this square for 5-9.***

Record these results for 6-8 and for 5-9 in their respective regions of the large square on the board.



T: ***If we divided each half of the square into small pieces as indicated, would the pieces on both halves be the same size?***

S: ***Yes. There would be 32 (26 + 6 = 32 and 31 + 1 = 32) pieces on each side.***

T: ***What are the winning chances for Bruce and for Rita when the score is 5-8?***

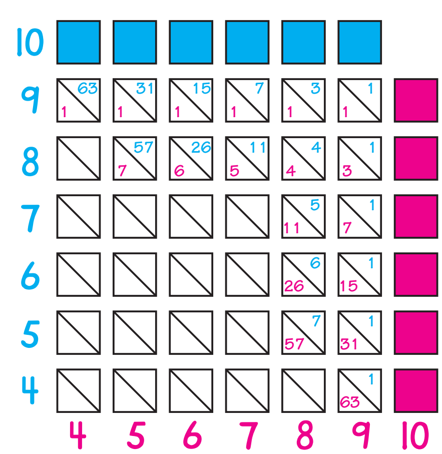
S: ***57 (26 + 31 = 57) for Bruce and 7 (6 + 1 = 7) for Rita.***

T: ***Which other result do we know immediately?***

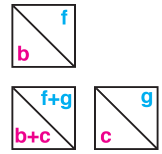
S: ***When the score is 8-5, Rita has 57 chances to win and Bruce has 7.***

Instruct students to record these results for 5-8 and 8-5 on their worksheets. Invite a student to fill in the same squares on the poster.

Check whether any students predicted 57 chances for Bruce. If so, ask them to explain how their predictions were made.

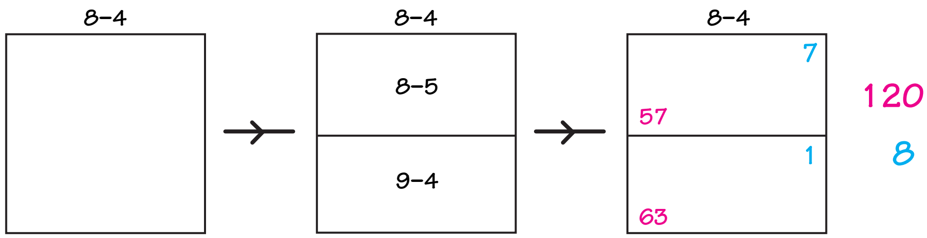


Note: The goal in the next part of the lesson is to lead students to observe a pattern that will allow them to quickly complete the rest of the chart. For your reference only, this illustration uses symbols to explain this pattern.

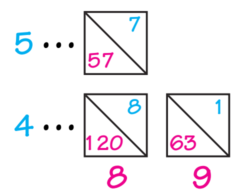


To find the results for a square when the results for the squares directly above and to the right are already known, simply add the numbers as indicated. The discussion of the score 5-8 has already suggested this technique. As soon as one or more students discover the rule, ask them to explain it to the class.

Allow a few more minutes for students to solve the problem for 8-4. Then solve the problem collectively at the board, and record the results on the poster. The following illustration indicates steps that can be followed.



Refer to this part of the poster, and emphasize that the solution was reached quickly because we knew the results for 8-5 and for 9-4.



Lead the class to observe that 57 + 63 is 120 (red numbers), 7 + 1 is 8 (blue numbers). That is, you can add the red numbers above and to the right, and the same for the blue numbers.

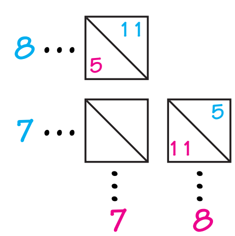
Encourage students to observe that this pattern works for these scores that have already been recorded: 5-8, 6-8, 8-4, 8-5, and 8-6.

T: ***Let's use this pattern to calculate the results for some more scores. For example, what are Rita's and Bruce's winning chances when the score is 7-7?***

S: ***Rita and Bruce both have 16 chances to win.***

T: ***Why?***

Direct a student to point to the appropriate part of the poster.

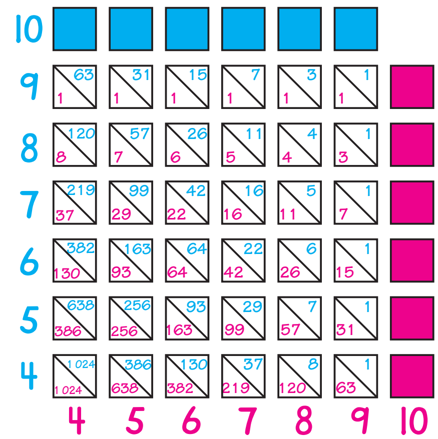


S: ***Add blue 11 and 5 to get Bruce's winning chances for 7-7, and add red 5 and 11 to get Rita's winning chances for 7-7.***

A student might observe that when the score is tied at 7-7, Rita and Bruce have the same chances to win and we could put 1 for Rita and 1 for Bruce. If a student suggests this, agree that it is a correct answer, but note that this formulation of the answer does not follow the pattern just discovered and will not help us complete other squares in the chart.

T: ***Why do Rita and Bruce have the same number of winning chances, namely 16, when the score is 7-7?***

S: ***The game is tied at 7-7. Rita and Bruce must each have the same number of chances to win.***



Invite the class to use the pattern to quickly calculate the results for a couple other scores such as 6-7 and 5-7. Then direct students to use the pattern to complete their worksheets. As students are working, invite students with correct answers to complete the chart on the poster.

Exercise 3 (optional)

T: ***Now that we have completed the chart, let's use it to solve some related problems concerning this game.***

Write this information on the board.

9-7: 7 1

8-6: 26 6

T: ***Rita is favored for both of the scores 9-7 and 8-6. If you were Bruce, would you prefer the score to be 9-7 or 8-6?***

S: ***8-6, because when the score is 8-6, Bruce's probability of winning is $^6/_{32}$. At 9-7, his probability of winning is $^1/_8$. $^6/_{32}$ is greater than $^1/_8$ since $^1/_8 = {}^4/_{32}$.***

S: ***8-6, because then Bruce would have more time to catch up. For both scores, Bruce is two points behind Rita.***

S: ***8-6, because if the score is 9-7, Rita is seven times more likely to win than Bruce. But if the score is 8-6, Rita is only about four times more likely to win, since 26 is about 4 x 6.***

Erase the board and write this information.

Game to 100
97-96

T: ***Rita and Bruce are now playing the same game, but the first to score 100 points wins. If the score is 97-96, can you use the same chart to find Rita's and Bruce's chances to win?***

S: ***When the score is 97-96 in a game to 100, Rita needs three points to win and Bruce needs four points to win. This situation is the same as when the score is 7-6 in the game to 10; Rita needs three points and Bruce needs four points. Therefore, when the score is 97-96, Rita has 42 chances to win and Bruce has 22.***

You may need to lead to this observation by asking,

T: ***If the score is 97-96 and the goal is 100, how many points does Rita need to win?*** (3) ***Bruce?*** (4)

In their old game to ten points, is there ever a situation when Rita needs three points to win and Bruce needs four points to win?

S: ***Yes, when the score is 7-6. Therefore, at both a score of 97-96 in a game to 100 and at a score of 7-6 in a game to 10, Rita has 42 winning chances and Bruce has 22 winning chances.***

Write this information on the board.

Game to 25
22–24

T: ***On another day, Rita and Bruce set a goal of 25. The score in their game is 22-24. What are Rita's and Bruce's chances to win?***

S: ***7 for Bruce and 1 for Rita. When the score is 22-24, Rita needs three points to win and Bruce needs one point. That is the same as when the score is 7-9 in the game to 10.***

Write this information on the board.

Game to Rita 15, Bruce 12
10–10

T: ***One day Rita was winning more often than Bruce. They decided that in their next game Rita would have to score 15 to win and Bruce would only need to score 12. If the score is 10-10, what are their chances of winning?***

S: ***57 for Bruce and 7 for Rita. When the score is 10-10 in this game, Rita needs five points to win and Bruce needs two points to win. That is the same as when the score is 5-8 in the game to 10.***

Exercise 4 (optional)

T: ***Let's look for some other number patterns in this chart.***

Write this series of numbers on the board.

Point to the blue numbers in the 7-column on the poster.

$\square$
7
11
16
22
29
37
$\square$

T: ***These are the blue numbers from the 7-column. Try to find a pattern to this sequence of numbers. According to your pattern, what number would come before 7 and what number after 37? Write your answers on a piece of paper.***

Check several answers before asking students to explain their answers.

S: ***46 is after 37. I saw that 7 + 4 = 11, 11 + 5 = 16, 16 + 6 = 22, 22 + 7 = 29, and 29 + 8 = 37. The number you add is one more each time. The next number should be 46 because 37 + 9 = 46.***

Highlight this pattern on the board.

S: ***4 is before 7. You must add 3 to some number to get 7. 4 + 3 = 7.***

$$\begin{array}{cl} \boxed{} & \\ 7 & \\ & +4 \\ 11 & \\ & +5 \\ 16 & \\ & +6 \\ 22 & \\ & +7 \\ 29 & \\ & +8 \\ 37 & \\ & +9 \\ \boxed{46} & \end{array}$$

Erase the board and write this sequence of numbers.

$$\boxed{},\quad \boxed{},\quad 120,\quad 57,\quad 26,\quad 11,\quad 4,\quad 1$$

T: ***These are the blue numbers from the 8-row. Can you find a pattern?***

S: ***Starting from 1 at the right, you double 1 and add 2 to get 4. Then double 4 and add 3 to get 11. Then double 11 and add 4 to get 26. This continues; each time, you double the number and add one more than you did the last time.***

Following students' suggestions, highlight a pattern on the board.

$$\boxed{},\quad \boxed{},\quad \underset{(2\times 57)+6}{120},\quad \overset{(2\times 26)+5}{57},\quad \underset{(2\times 11)+4}{26},\quad \overset{(2\times 4)+3}{11},\quad \underset{(2\times 1)+2}{4},\quad 1$$

T: ***On your paper, write what the next two numbers should be.***

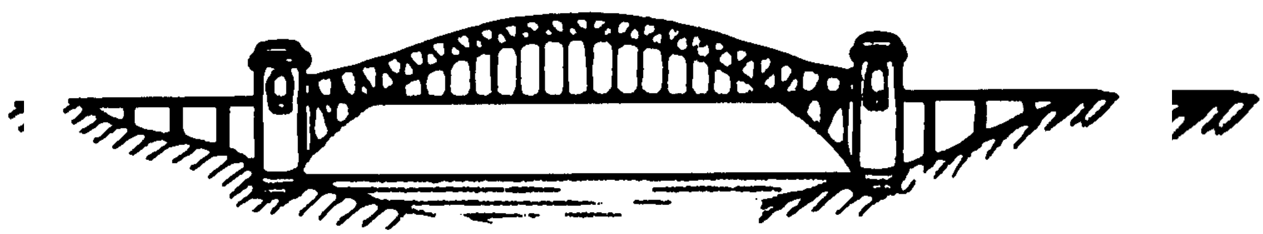
S: ***247 comes next because (2*** x ***120) + 7 = 247.***

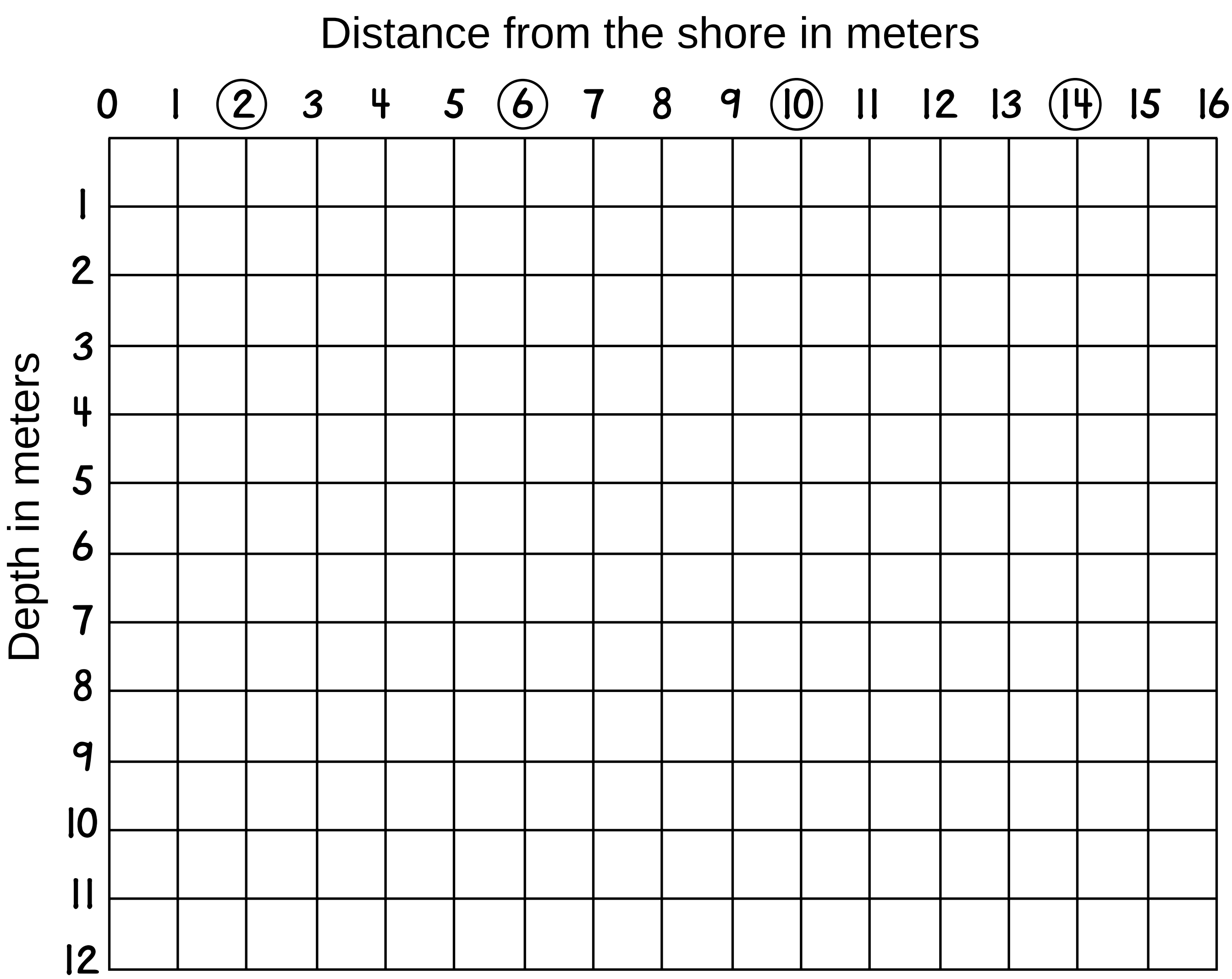
S: ***502 comes after 247 because (2*** x ***247) + 8 = 502.***

Name ________________

P5(a)

Measurements of the depth of the water below a bridge are taken at 2, 6, 10, and 14 meters from the lake shore. The mean average depth is 2 meters. Draw a possible profile of the lake below the bridge.





Name ________________

P5(b)

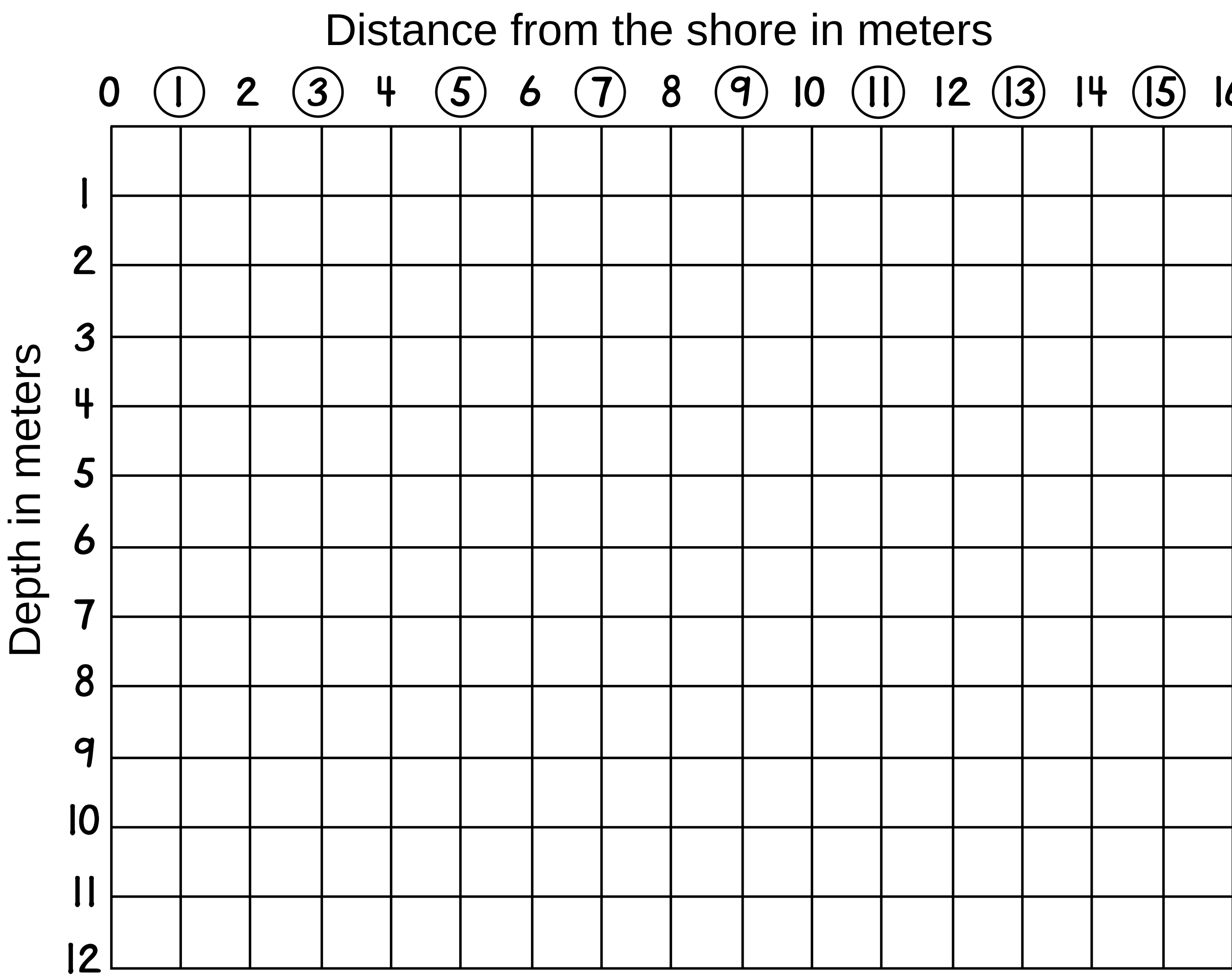
This is the data the park ranger provided for the depth of the water below the bridge.

<u>8 measurements</u>
Mean: 2 meters
Mode: 1 meter
Range: 1 to 8 meters

What could the eight measurements have been?

____m, ____m, ____m, ____m, ____m, ____m, ____m, ____m

Measurements were taken at 1, 3, 5, 7, 9, 11, 13, and 15 meters from the lake shore. Based on the eight measurements you listed, draw a profile of the lake below the bridge.



Name ____________________

P7

Rita: 9 Bruce: 8

Rita leads Bruce 9-8 in a game to 10 points when they must stop playing. Use this square to calculate each player's probability of winning.

9 – 8

What is Rita's probability of winning? ______ Bruce's ______

If Rita and Bruce each put 50¢ into a pot, how should they share the $1.00 when the game stops at 9-8? Rita ______ Bruce ______

Name ____________________

P8 *

Rita: 9 Bruce: 9

Rita and Bruce are tied in a game to 10 points when they must stop playing. Use this square to calculate each player's probability of winning.

9 – 9

What is Rita's probability of winning? ______ Bruce's ______

Name ____________________

P8 **

Rita: 9 Bruce: 6

Rita leads Bruce 9-6 in a game to 10 points when they must stop playing. Use this square to calculate each player's probability of winning.

9 – 6

What is Rita's probability of winning? ______ Bruce's ______

Name ____________________

P9

10							
9			1 / 15	1 / 7	1 / 3	1 / 1	
8			6 / 26	5 / 11		3 / 1	
7					11 / 5	7 / 1	
6					26 / 6	15 / 1	
5							
4							
	4	5	6	7	8	9	10

Probability and Statistics

PROBABILITY AND STATISTICS TABLE OF CONTENTS

PROBABILITY AND STATISTICS INTRODUCTION

In today's world, probabilistic and statistical methods have become a part of everyday life. They are an important part of industry, business, and both the physical and social sciences. Just being an intelligent consumer requires an awareness of basic techniques of descriptive statistics, for example, the use of averages and graphs. Probability and statistics are key to the present day modeling of our world in mathematical terms. The lessons in this strand utilize both fictional stories and real data as vehicles for presenting problems and applications. The problems and questions that arise focus attention on key concepts of probability and statistics such as randomness, equally likely events, and prediction.

Probability stories fascinate most students and encourage their personal involvement in the situations. They often relate the probability activities to games they have encountered outside the classroom. This personal involvement builds students' confidence and encourages them to rely on their intuition and logical thinking to analyze the situations. In *IG-V*, students use number cubes and other devices to simulate a situation or to play a game. These activities help students understand the story and also form a basis for predicting the likelihood of particular outcomes. Yet simulations produce only estimates of the probabilities, leaving open the question of a true probability. Pictorial techniques make the analysis of theoretical probabilities accessible. This combination of simulation and analysis of situations demonstrates the strong interdependence between probability and statistics.

Some applications of mathematics are more easily investigated than others. The probability and statistics applications in this strand lean towards the easy end of the spectrum, both because of the small amount of theoretical knowledge required, and because the link between the situation and the mathematical model is easily perceived. In particular, the development of pictorial models for

s the ease of solutions.

Content Overview

Probability

There are many methods available for determining probabilities. The simplest techniques, though usually tedious, require listing all possible outcomes. Most powerful techniques rely on formulas involving the multiplication of probabilities. The lessons in this strand review and introduce several efficient pictorial techniques that elementary students can readily apply.

Three lessons this semester make use of the familiar area model for finding probabilities. In the first lesson, a story about making random selections of paths through a swamp and still getting to the other side within a specified amount of time uses the division of a square to reflect the choices and their consequences. In the second lesson, students find the probabilities involved in a fortune teller's method of deciding whether or not the time is right for a couple to marry. The method involves tieing the ends of three ropes (in different ways) together and hoping the result will be one long piece of rope. In the third lesson, students find that the analysis of a two-person game with four cubes yields a paradoxical result.

By studying the possible results of an extrasensory perception (ESP) test, students construct Pascal's triangle again (originally generated in the story-workbook *The Hidden Treasur*e in *IG-IV*). In Pascals's triangle, students discover patterns and predict the equivalence of certain tests.

PROBABILITY AND STATISTICS INTRODUCTION

While the major goal in these activities is the development of efficient and accessible pictorial techniques for determining probabilities, the lessons also reflect the continual development of other probabilistic themes; randomness, equally likely events, simulation, fair games, and predictions.

Lessons: P2, 3, 5, 6, and 7.

Statistics

Several lessons in this strand include descriptive statistics—the use of numerical and graphical techniques to summarize and compare sets of data. The activities continue to develop students' abilities to use averages, and to read, draw, and interpret bar graphs. The goal is to increase students' familiarity with these topics through rich experiences rather than to drill the techniques of computing an average or drawing a graph.

In Lesson P1 *Braille*, students use patterns and counting techniques to predict characters missing from a list of braille characters.

Another lesson this semester asks students to consider how three kinds of averages (mean, mode, and median) can be used to decide the person with the fastest reaction time. They discuss elements of a good class experiment on reaction time, conduct the experiment, and study the results. Advantages and disadvantages of the different methods for deciding the fastest reaction time evolve as part of the study.

In a third lesson on misleading advertisements, students investigate the meaning and visual effect of using different ways to graph data.

Lessons: P1, 4, and 8.

P1 BRAILLE

Capsule Lesson Summary

Use patterns to determine characters missing from a list of braille characters. Determine the number of possible characters in the revised and the original braille systems.

Materials

Teacher • Braille characters transparency

Student • Worksheets P1(a) and (b)

Advance Preparation: You may want to use Blackline P1(a) to make a transparency of braille characters for use on an overhead during the lesson.

Description of Lesson

Exercise 1

Distribute Worksheets P1(a) and (b).

Ask if anyone recognizes the symbols on Worksheet P1(a). Perhaps some students will realize that the symbols are braille characters. Ask a student to explain how the braille characters actually appear (raised dots on paper) and when they are used. Lead a short discussion on the history of braille. Here are a couple points you may wish to mention:

- Braille is one of several systems devised to allow the blind to read. Early attempts included letters carved in wood, pins inserted in cushions, and large wooden raised letters. For a long time the standard was to use raised letters on paper, however, this system was not effective with persons who were born blind.
- Louis Braille (1809–1852) invented a system of reading that proved effective with most blind people. At first the braille system met a great deal of resistance, but it was widely adopted after Louis Braille's death.

T: ***Look at Worksheet P1(a) and try to find some patterns that will help us to discover the braille characters for "Q," "the," and "W." Write these characters on the worksheet.***

Allow a few minutes for students to investigate patterns in the characters.

T: ***What do you notice about the arrangement of the dots for each character?***

S: ***There are at most three rows of dots for each character.***

S: ***There are at most two columns of dots for each character.***

T: ***A braille character is made up of six cells arranged in two columns and three rows. Let's label the cells to make it easier to explain patterns.***

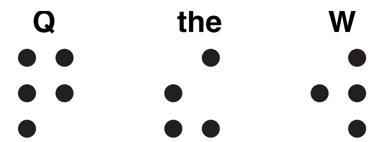
1	4
2	5
3	6

Draw a diagram on the board.

Let students describe patterns they discover. Perhaps they will mention the following:

- Each character in the second row is formed by adding a dot in cell 3 to the character directly above it in the first row. For example, the character for **L** (⠇) comes from the character for **B** (⠃) by adding a dot in cell 3.
- Each character in the third row is formed by adding a dot in cell 6 to the character directly above it in the second row. For example, **V** (⠧) comes from **L** (⠇).
- Each character in the fourth row is formed by removing a dot from cell 3 in the character directly above it in the third row. For example, **gh** (⠣) comes from **V** (⠧).

The class should find these missing characters.



Exercise 2

Pose a counting problem.

T: ***How many braille characters are shown on Worksheet P1(a)?*** (40)
Are more characters possible using this six-cell dot system? (Yes)
How many different characters are possible?

Encourage students to make predictions, and record some estimates on the board.

T: ***Let's see if we can determine how many characters are possible. What could be in each cell?***

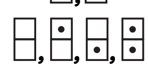
S: ***A blank or a dot.***

T: ***Suppose there were only one cell. How many characters would be possible?*** (2)

Begin a list of possibilities on the board.

T: ***Suppose there were two cells. How many different characters would be possible?***

Instruct students to use the first row of Worksheet P1(b) to draw all of the possible two-cell characters. After a few minutes, record the results on the board.

Number of Cells	Number of Characters	Characters
1	2	
2	4	

T: ***What happens when we add another cell?***

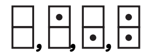
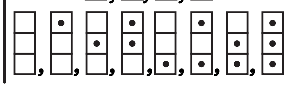
S: ***The number of characters doubles.***

Cover the bottom cell in each of the newly formed characters with a piece of paper.

T: ***What do you notice about the top cell of these characters and the characters with one cell?***

S: ***Each character with one cell is repeated twice.***

Extend the list on the board by asking the class to predict the number of possible characters using three cells. Then instruct students to use the second row on Worksheet P1(b) to draw all of the possible characters. Record the results on the board.

Number of Cells	Number of Characters	Characters
1	2	
2	4	
3	8	

T: ***What happens to the number of possible characters when we add another cell?***

S: ***The number of characters doubles again.***

Cover the bottom cell in each of the newly formed characters and ask the class to notice similarities between the new characters and the characters formed by using only two cells. Students should notice that each character with two cells is repeated twice. If necessary, ask a student to point to a previous character and the two new characters that are similar.

T: ***Why are each of the previous characters repeated twice?***

S: ***Because there are two possibilities for the third cell: a blank or a dot.***

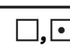
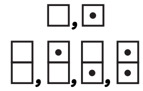
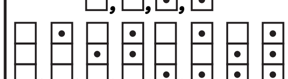
T: ***How many characters would be possible using four cells? Why?***

S: ***16, because the number of characters doubles each time we add another cell.***

S: ***16, because the characters will be the previous eight characters with a blank in the fourth cell and the previous eight characters with a dot in the fourth cell.***

Note: This last comment assumes that the arrangement of the four cells is like the arrangement for three cells with an additional cell attached.

Record this result in the list on the board, and continue in the same manner until you find the number of possible characters using six cells.

Number of Cells	Number of Characters	Characters
1	2	
2	4	
3	8	
4	16	
5	32	
6	64	

A class project could be to determine which of the 64 characters are missing on Worksheet P1(a) and then to research what they represent in the braille system.

Exercise 3

T: ***The original system that Louis Braille invented was inspired by Morse code. What kind of symbols are used in Morse code?***

S: ***Dots and dashes.***

T: ***Can you guess how the characters of Braille's original system differed from the characters on Worksheet P1(a)?***

S: ***He used dashes as well as dots.***

T: ***How many characters do you suppose were possible in the original system?***

Accept estimates and then proceed in the same manner as Exercise 2, considering the number of characters using only one cell and then two cells. The bottom half of Worksheet P1(b) is available for students to draw all of the possible characters using two cells. Make another list on the board.

Number of Cells	Number of Characters	Characters
1	3	□, ⊡, ⊟
2	9	□ □ □ ⊡ ⊡ ⊡ ⊟ ⊟ ⊟ □, ⊡, ⊟, □, ⊡, ⊟, □, ⊡, ⊟

T: ***What happens to the number of characters when we add another cell?***

S: ***It multiplies by 3.***

Cover the bottom cell in each of the two-cell characters and compare with the one-cell characters. The class should notice that each one-cell character is repeated three times.

T: ***How many characters would there be if we used three cells?***

S: ***27; because each two-cell character would be repeated three times, once with a blank in the third cell, once with a dot in the third cell, and once with a dash in the third cell. 3 x 9 = 27.***

Record the result in the list, and continue in the same manner until the number of possible characters using six cells is determined.

Number of Cells	Number of Characters	Characters
1	3	□, ⊡, ⊟
2	9	□ □ □ ⊡ ⊡ ⊡ ⊟ ⊟ ⊟ □, ⊡, ⊟, □, ⊡, ⊟, □, ⊡, ⊟
3	27	
4	81	
5	243	
6	729	

T: ***Why does the number of characters triple with this system when a cell is added and only double with the standard braille system?***

S: ***In the standard system, there are two possibilities for each cell. Therefore, each time a cell is added the number of characters doubles. In this system there are three possibilities for each cell, so each time a cell is added the number of characters triples.***

Extension Activity

Present students with Morse code and suggest they look for patterns. In Morse code, letters are given signaling elements of from one to four dots or dashes. The dot is a short duration electric current and the dash is a longer duration signal. How many possible signaling elements are possible with from one to four signals? Which signaling elements are not used in Morse code? Are there enough to include Morse code signaling elements for the ten digits 0 to 9? Why does Morse code use five signals for the digits? Blackline P1(b) has the elements for Morse code.

Writing Activity

You may like students to take less on notes on some, most, or even all their math lessons. The "Lesson Notes" section in Notes to the Teacher gives some suggestions and refers to forms in the Blacklines you may provide to students for this purpose. In this lesson, for example, students can note how to determine the number of characters possible in a code with a given number of cells and one or two symbols.

Capsule Lesson Summary

Tell a story about Sylvia who is invited to a beach party on an island. A problem arises since the last ferry leaves in 60 minutes which is not enough time for Sylvia to take her usual route. Find the probability that Sylvia will get to the ferry in time if she follows paths through an unfamiliar swamp.

Materials

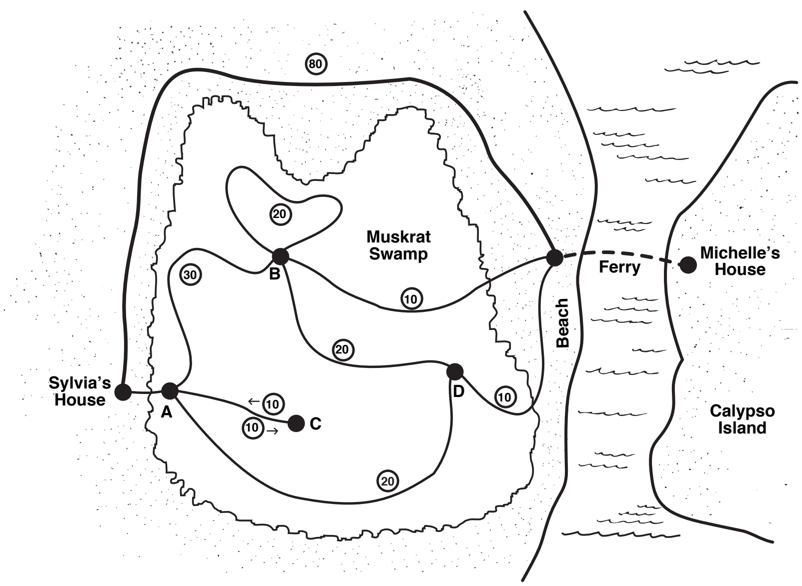
Teacher
- *IG-V Probability Poster* #1
- Colored chalk and pens
- Meter stick
- Blackline P2

Student
- Colored pencils
- Muskrat Swamp map
- Ruler
- Worksheets P2* and **

Advance Preparation: Use Blackline P2 to make copies of the Muskrat Swamp map for students. You may also want to make an overhead transparency.

Description of Lesson

Display the Muskrat Swamp map on *IG-V Probability Poster* #1 or with a transparency.



Refer to the map as you tell the following story about Sylvia's vacation near the coast. Make the story as interesting as possible, stopping occasionally to let students discuss the situation.

T: ***Sylvia enjoys the time she spends every summer with her aunt and uncle near the ocean. She often rides her bicycle to the beach, and she always rides on the main road around Muskrat Swamp.*** (Trace the heavy black path from **Sylvia's House** to the **Beach**.) ***It takes her 80 minutes if she does not stop. There are other paths through Muskrat Swamp on which Sylvia could ride, but she doesn't know which paths to take and is a little afraid of getting lost.***

Sylvia's friend, Michelle, lives on Calypso Island. To visit Michelle, Sylvia must take a ferry which runs only three times a day: morning, noon, and evening.

P2

Let students talk about the story; for example, they may have comments on the swamp, the ferry, the island, and so on.

T: ***Late one afternoon, Michelle calls Sylvia on the telephone and invites her to a party that evening on Calypso Island. They are planning to swim, build a bonfire on the beach, and Sylvia is invited to stay overnight. Sylvia is excited, but she realizes suddenly that it is almost 5:00 and the last ferry to Calypso Island leaves at 6:00. She tells Michelle, "It takes me 80 minutes to bicycle to the ferry so I will be too late. My aunt and uncle are not here now so I can't get a ride with them." Michelle tells Sylvia about the shortcuts through Muskrat Swamp. Sylvia is afraid of the swamp and of getting lost, but Michelle finally convinces her to try it.***

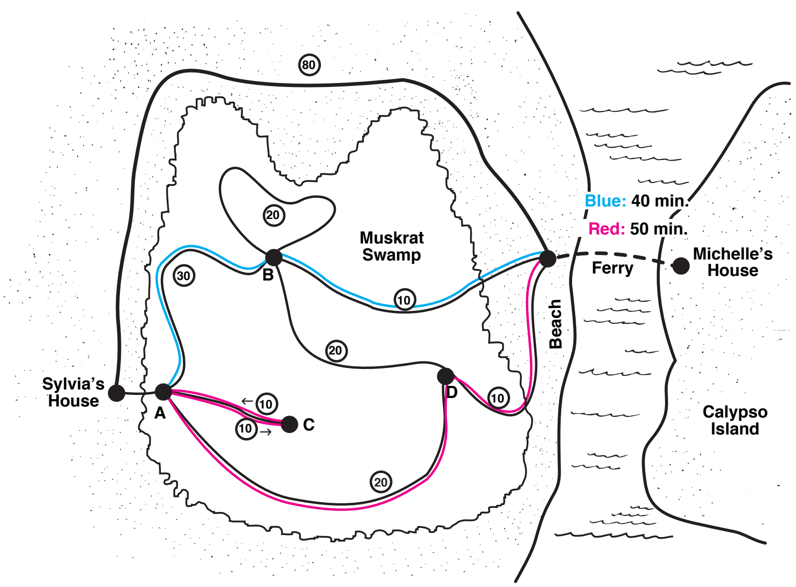
So Sylvia jumps on her bicycle and begins her ride through the swamp. Michelle could not give her good directions and there are no signs, so she has to guess which paths to follow. The number on each path tells how many minutes it takes to ride a bicycle along it. Sylvia has exactly one hour (60 minutes). Do you think she will get to the ferry on time?

Allow students to study the map and express their opinions. Be sure that both the dead-end path from **A** to **C** and the loop at **B** are noticed.

Distribute copies of the map (Blackline P2).

T: ***Who can trace a path that Sylvia could take and get to the ferry in 60 minutes or less?***

Invite several students to trace paths starting at Sylvia's house, ending at the ferry, and taking at most 60 minutes. Two such paths are shown below.

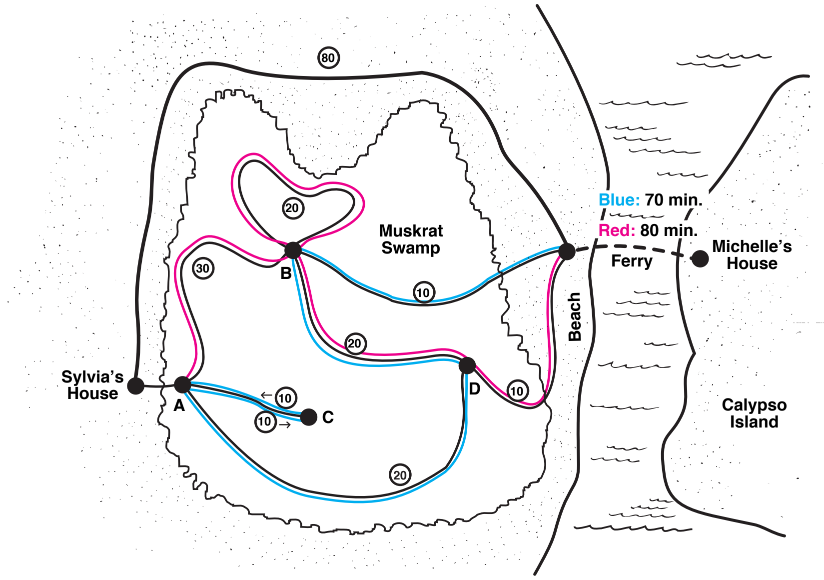


T: ***Do you think Sylvia would take the same path twice?***

Lead to the idea that if Sylvia is clever and marks each path she takes, she can avoid taking the same path twice.

T: ***Who can trace a path that Sylvia could take and not get to the ferry in 60 minutes or less?***

Invite students to trace paths starting at Sylvia's house, ending at the ferry, and taking more than 60 minutes. Two such paths are shown below.



T: ***Sylvia does get to the ferry on time and joins Michelle's party on Calypso Island. When Sylvia arrives, she tells Michelle, "I was really lucky to get to the ferry on time because there are a lot of paths through the swamp." Michelle answers, "There's not much luck involved. It's easy to ride through Muskrat Swamp in 60 minutes or less."***

What do you think? How lucky would Sylvia be if she always had to guess which path to take?

Encourage students to estimate Sylvia's chances or probability of getting through Muskrat Swamp in 60 minutes or less and to list their estimates on the board. Ask students if they feel Sylvia's probability
of arriving on time is low, say less than ½, or high, say more than ¾.

Draw a large square on the board.

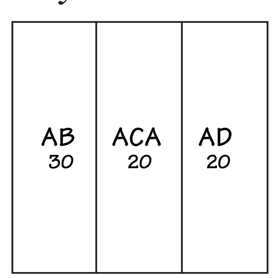
T: ***Let's use a square to calculate Sylvia's probability of riding through the swamp in 60 minutes or less. We must study all the paths Sylvia could take and all the choices she could make. Let's represent these paths and choices on the square.***

Sylvia enters the swamp here (point to **A** on the map)***. How many choices of paths does she have at A?***

S: ***Three; she could take the path to B, to D, or to the dead-end at C and back to A.***

T: ***How could I show on the square that Sylvia has three choices?***

S: ***Divide the square into thirds because each of the three paths is equally likely to be chosen by Sylvia.***



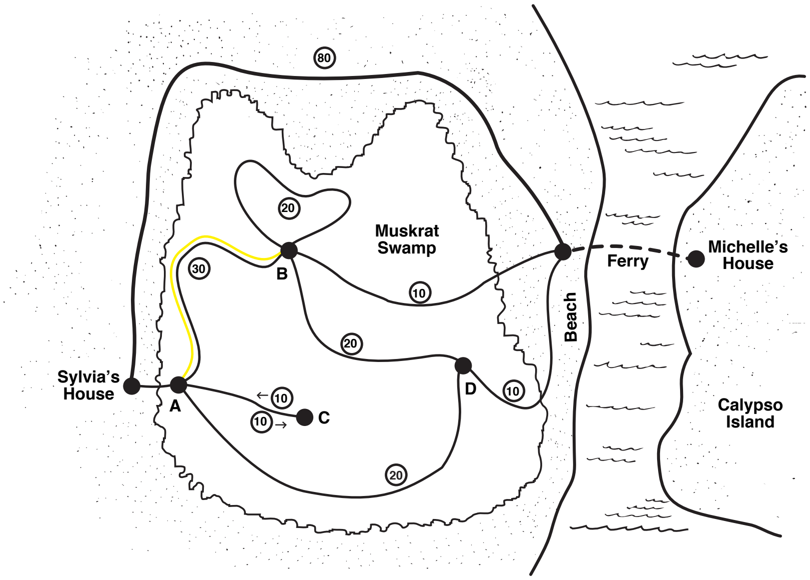
Invite a student to divide the square into thirds, using a meter stick for accuracy. For example, if the square is about 60 cm wide, each section should be about 20 cm wide.

Label each region for one of the paths and the time it would take to ride along that path.

P2

T: ***In each region, indicate a path Sylvia could choose and the number of minutes to ride that path. Suppose Sylvia chooses the path from* A *to* B** (point to this region)**.**

Highlight the path from **A** to **B** on the map.



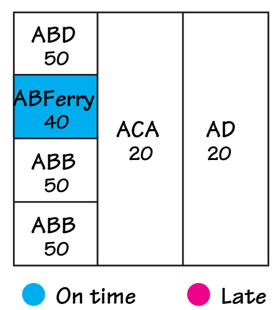
T: ***Now Sylvia is at* B*. How many paths could she take from* B*?***

S: ***Four; she could take the path from* B *to* D *or from* B *to the ferry. Also, there are two ways to go around the loop back to* B*. She won't go back to* A*.***

T: ***How do I show this on the square?***

S: ***Divide the region into four equal parts.***

Ask students to help you divide the region and then to label its four parts.



Note: The labels should indicate the entire path taken thus far and the total time.

Indicate that the path **ABFerry** gets Sylvia to the ferry on time. Choose a code for marking the regions, as illustrated here.

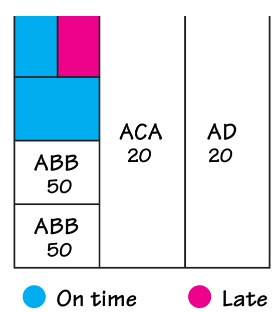
Point to the region for **ABD** and trace a path from **A** to **B** to **D**.

T: ***If Sylvia rides from* A *to* B *to* D*, she uses up 50 minutes. What are her chances of getting to the ferry on time?***

S: ***One out of two. She could take the path to the ferry and arrive just in time (60 minutes), or she could ride to* A *and use up too much time (70 minutes).***

T: ***How could I show that on the square?***

S: ***Divide the region in half; color one-half red and one-half blue.***



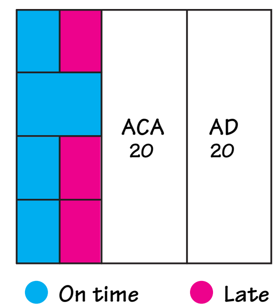
Point to the **ABB** regions and trace the two paths from **A** to **B** and around the loop (once in each direction) to **B**.

T: ***If Sylvia takes one of these paths and uses up 50 minutes, what are her chances of getting to the ferry on time?***

S: ***1 out of 2. She could either go from B directly to the ferry and arrive in time (60 minutes) or go from B to D and use up too much time (70 minutes). She wouldn't go back to A and wouldn't go around the loop again.***

T: ***How do I show this on the square?***

S: ***Divide each region into two pieces of the same size; color one piece red and one piece blue.***



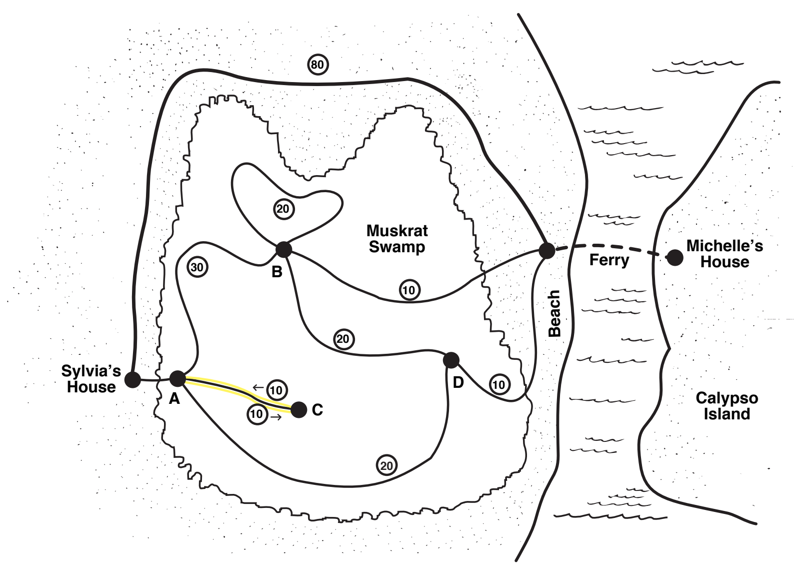
Refer to the third of the square that is colored.

T: ***If Sylvia first chooses the path from A to B, is she more likely to be on time or late for the ferry?***

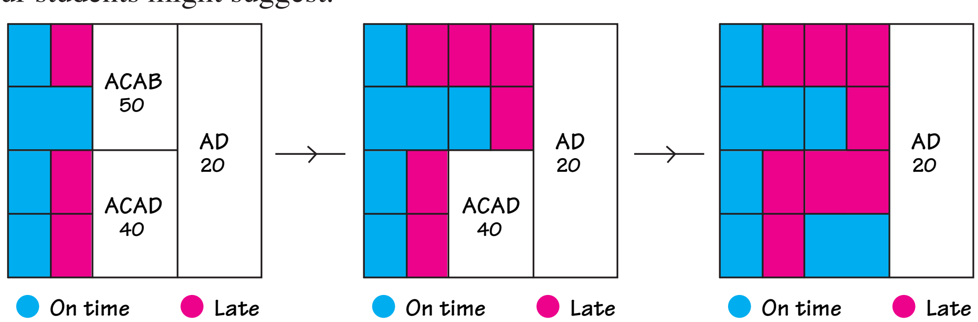
S: ***On time; more than half the region is colored blue.***

T: ***Let's calculate Sylvia's chances of being on time if she first chooses the dead-end path to C and then comes back to A*** (point to the region for **ACA**).

Highlight this path on the map.



Analyze this possibility in a similar manner as above. The following sequence of squares indicates the steps your students might suggest.

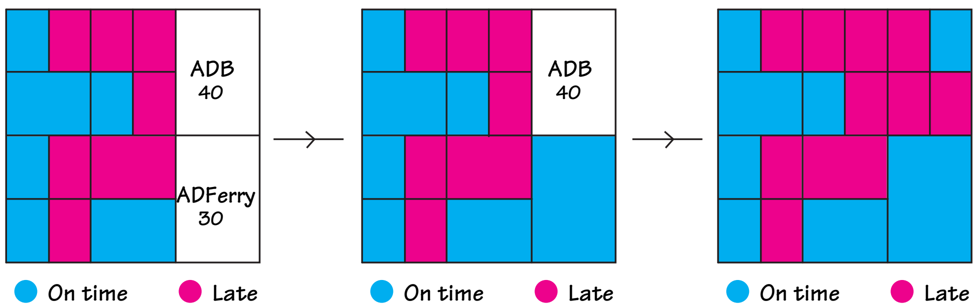


T: ***We still have to calculate Sylvia's chances of being on time if she first chooses the path from* A *to* D** (point to the region for **AD**).

Highlight a path from **A** to **D** on the map.

T: ***On your own paper, try to decide how we should color the last third of the square.***

Let students work (with a partner) for a few minutes, helping those who are having difficulty. Then complete the problem collectively at the board. The following sequence of squares indicates the steps your students might suggest.



T: ***We examined all the different paths Sylvia could take and determined for each if she would get to the ferry on time. Is she more likely to be late or on time?***

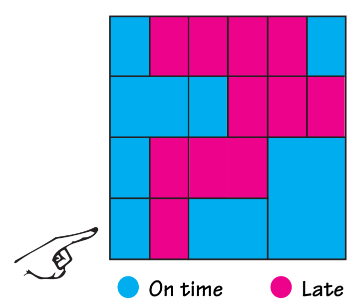
S: ***The chances are about the same because about half the square is red and about half is blue.***

T: ***Can we calculate exactly what Sylvia's chances are to be on time?***

Let students discuss this question. Encourage any suggestion to compare the number of small pieces which fit into the entire red region to the number of small pieces which fit into the entire blue region. If necessary, ask the following questions.

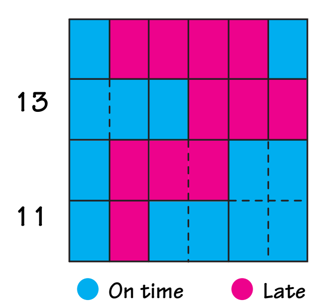
T: ***Let's compare the amount of the square colored red to the amount colored blue. Where is one of the smallest pieces in our division of the square?***

Invite a student to point to one of the smallest pieces; for example:



T: ***How many pieces of this size fit into the red region? … into the blue region?***

The class will likely find it helpful to subdivide the entire square into pieces all the same size.



S: ***13 red and 11 blue.***

T: ***What is Sylvia's probability of getting to the ferry on time?***

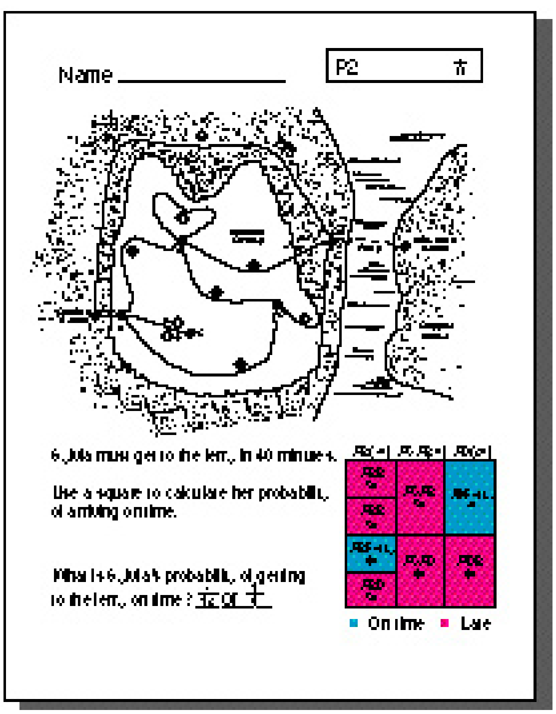
S: ***$^{13}/_{24}$ or 13 out of 24.***

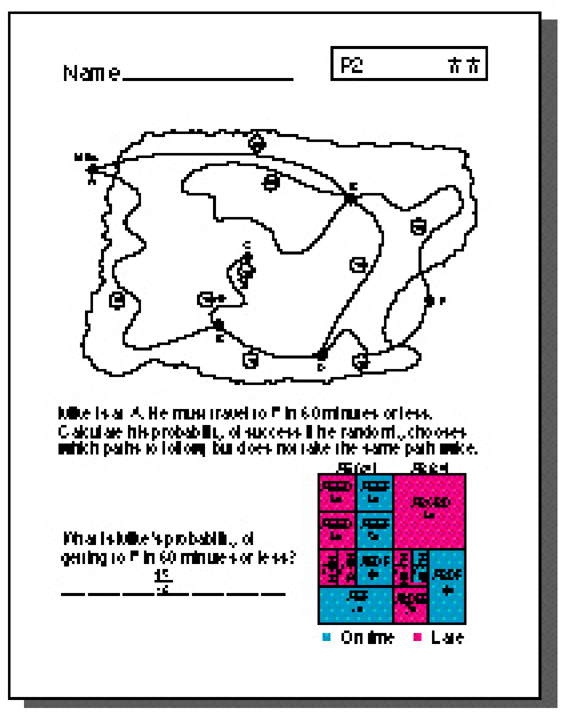
Compare the solution, $^{13}/_{24}$, with students' estimates of Sylvia's chances.

Worksheets P2* and ** are available for individual work. Point out that the map on Worksheet P2* is the same as the one used in class and that the problems are similar except that in this one Sylvia has only 40 minutes to get to the ferry.

Home Activity

Some students may like to take the map of Muskrat Swamp home to tell a family member about Sylvia's problem of getting to the ferry on time (60 minutes or less).





Capsule Lesson Summary

Calculate the probability of forming one long piece of rope when some of the ends of three ropes are joined by two knots. Vary how the ropes are held, and analyze the new situation that results.

Materials

Teacher		Student	
	• Three identical pieces of light rope or cord, each about one meter in length • Meter stick • Colored chalk		• Ruler • Colored pencils • Paper • Worksheets P3(a) and (b)

Description of Lesson

Initiate a discussion of how past civilizations often put much credibility in one individual for decision making or speculation about future events. Over time, these individuals have been known as oracles, sages, prophets, chiefs, wise people, fortune tellers, astrologers, medicine men, and elders, to name but a few.

T: ***Do you know any stories about fortune tellers? What kinds of questions do people ask fortune tellers? What might fortune tellers use to predict the future?***

Exercise 1

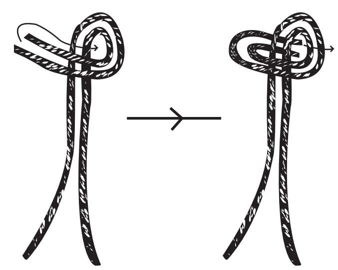
Once you have gained students' interest, tell the following story. Have the three pieces of rope on hand.

T: ***My story is about a fortune teller in a make-believe country. Many young couples planning to marry ask the fortune teller about their future together. For these romantic youths, the fortune teller has a special way to predict the future of proposed marriages using three identical pieces of rope like I have here.***

Show the class the three pieces of rope. Then fold the three ropes in half and hold them as shown here. Twist the strands lightly in your hand so that the students cannot tell which ends belong to the same piece of rope.

T: ***When young people in love ask whether or not a proposed marriage will be long and happy, this fortune teller holds three ropes like I am holding them now. The couple selects two of the six loose ends to tie together. Then they tie a second knot using two of the remaining loose ends. Let's try it.***

Invite two students to tie two knots. You may like to suggest using simple overhand knots to make it easier to untie the knots later.



P3

Note: An alternative would be to tie three knots with three pairs of rope ends. If this is done, the following description of what could happen will need minor adjustments. That is, a long piece would be a big loop with three ropes, a double piece would be a loop with two ropes, and a single piece would be a loop with one rope. In this case, there are three possibilities rather than four.

The corresponding analysis will, however, be similar.

After two knots are tied, there should still be two loose ends.

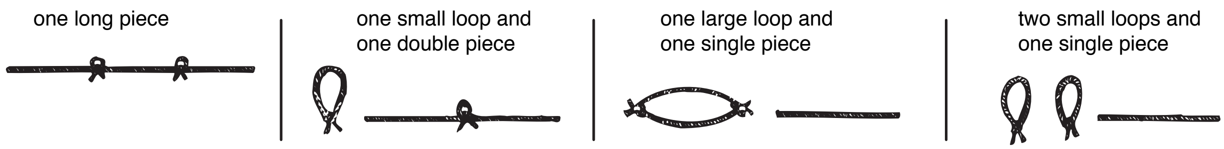
T: ***After two knots are tied, the fortune teller releases the ropes. What could happen?***

S: ***There might be one long piece of rope.***

S: ***There might be some loops.***

T: ***The fortune teller predicts the couple's future marriage will be long and happy if the result is one long piece of rope. Let's check our ropes.***

Release and untangle the ropes. Determine which of the following outcomes occurred.



Note whether or not the fortune teller would predict a long and happy marriage, that is, whether or not one long piece of rope occurs. If you wish, untie the ropes and repeat the experiment two or three more times.

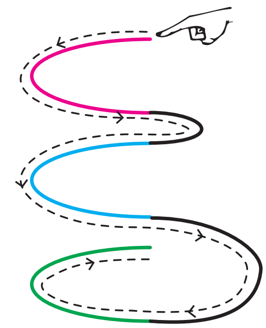
T: ***What do you think the probability is, or the chances are, that the rope will be one long piece after two knots are tied?***

List predictions of several students. Point out that the one trial (or the few trials) of the experiment made in class may not necessarily indicate the probability of success for one long piece.

Draw this picture on the board.

T: ***This is a picture of the three ropes. I've drawn each in a different color so that it is easier to talk about them. But, of course, the ropes are all the same so the couple will not know how to get a success.***

Draw two connectors, as shown here.

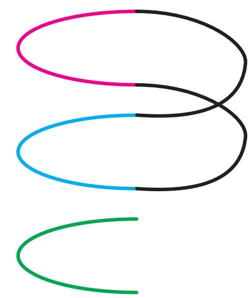


T: ***The connectors show which ends are tied in two knots. Would these two knots make one long piece of rope?***

S: ***Yes.***

Invite a student to trace the long piece of rope by starting at one loose end and following the ropes and connectors until the other loose end is reached. For example:

Change the connectors in the picture.

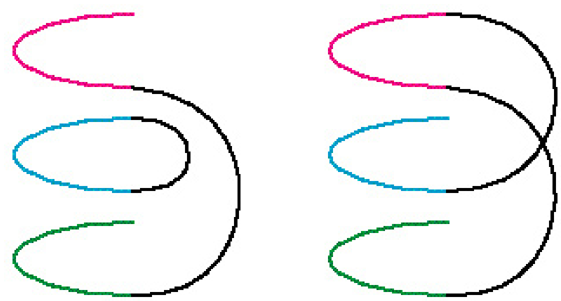


T: ***If these two knots were tied, would one long piece of rope result?***

S: ***No, the green rope is not attached.***

S: ***No, the red and blue ropes form a loop.***

Discuss one or two other possibilities in a similar manner, such as those illustrated here:



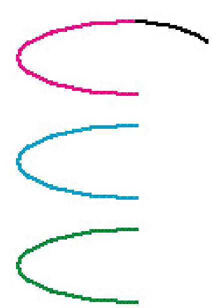
Erase any connectors from the picture and draw a square nearby.

T: ***Let's use this square to calculate the probability that one long piece of rope will result when two knots are tied randomly. We can analyze the knot-tiers' actions step-by-step. First, the couple selects one end to use in the first knot. Does it matter which end they start with?***

S: ***No.***

T: ***Since it doesn't matter which end they start with, suppose that the couple chooses an end of the red rope. Remember, the couple cannot distinguish the ropes; they are not really colored.***

Begin drawing a connector from one of the red ends.



T: ***Next, they tie this end to one of the other loose ends. In how many different ways can the first knot be completed?***

S: ***Five, since there are five other loose ends—one red, two blue, and two green.***

P3

Assuming the five possibilities are equally likely, divide the square into five parts of the same size. Use a meter stick for accuracy. Put colored dots above the parts to indicate which ropes are tied together (see the next illustration).

T: ***Would any of these five possible first knots be lucky or unlucky for the knot-tiers?***

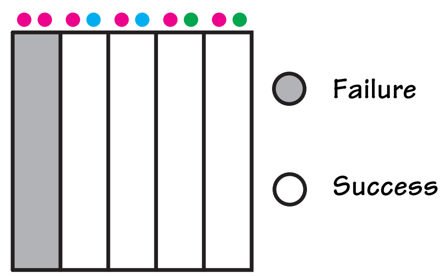
S: ***Tying the two red ends together would be unlucky since a loop would be formed.***

S: ***Tying the red end to a blue end or to a green end would be all right for the first knot since no loop is formed. But we don't know what will happen when a second knot is tied.***

T: ***What is the probability that the knot-tiers would be unlucky and would form a loop with the first knot?***

S: ***$\frac{1}{5}$; they have one chance out of five of forming a loop with the first knot.***

Indicate a code for failure or success (shaded or unshaded) next to the square, and shade the region for red-red.



T: ***We shade the first region to show that failure results if the first knot joins the two red ends. Now we must study what happens if the first knot joins a red end to one of the other four ends.***

Refer to the picture of the three ropes.

T: ***Which is luckier for the knot-tiers: to tie the red end to a blue end for the first knot or to tie the red end to a green end? Does it matter?***

S: ***It doesn't matter. There's no difference between the blue rope and the green rope.***

T: ***If the red end is tied to a blue end, does it matter which blue end it is tied to?***

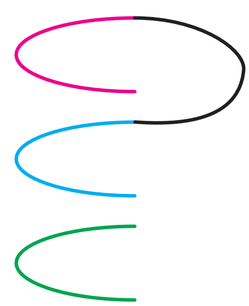
S: ***No.***

T (pointing to the four unshaded regions)**:** ***So the probability of success will be the same for each of these four knots. We need only determine how to shade one of these regions; the other three regions will be the same.***

Point to the left-most unshaded region.

T: ***Let's determine how to shade this region; that is, let's calculate the probability of success if the first knot joins the red end to one of the blue ends.***

Represent this situation by picturing a connection from a red end to a blue end.



T: ***How many different second knots could be tied?***

Invite students to the board to show all of the possibilities for the second knot. Continue until the class agrees the answer is six and students have shown the six different knots in a systematic way.

Distribute copies of Worksheet P3(a).

T: ***The six different ways the second knot could be tied are shown on this worksheet. For each picture, determine whether or not the two knots make one long piece of rope.***

Continue the collective lesson after most students have completed the worksheet.

T: ***Look at your worksheet. In how many of the six possibilities is one long piece of rope formed?***

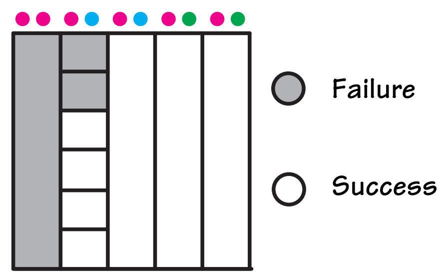
S: ***Four.***

T (pointing to the left-most unshaded region)**:** ***We are trying to shade this region. The first knot joins a red end to a blue end. What is the probability that the second knot will produce one long piece of rope?***

S: ***$\frac{4}{6}$ or $\frac{2}{3}$; there are two chances out of three.***

T: ***Explain your answer.***

S: ***We found that there were six different ways to tie the second knot, and four of those six ways resulted in one long piece of rope.***



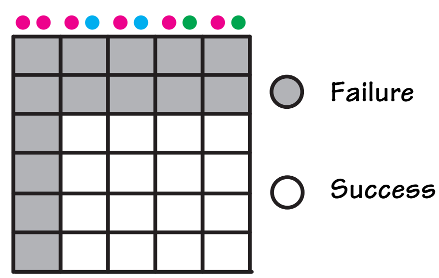
T: ***How should we shade the region?***

S: ***Divide the region into six parts of the same size; shade two parts for failure and leave four parts unshaded for success.***

S: ***It's the same if we divide that region into three parts, shade one part, and leave two parts unshaded.***

T: ***How should we shade the other three regions?***

S: ***The same as the region we just shaded, since we agreed earlier that the probability of success would be the same for each of those four regions.***



T: ***The shading for failure is complete, so now we can calculate the probability that one piece of rope will result when two knots are tied. What are the chances for success? What are the chances for failure?***

S: ***Sixteen chances for success and fourteen for failure.***

T: ***What is the probability of success?***

S: ***$\frac{16}{30}$ or $\frac{8}{15}$.***

T: ***Is success or failure more likely?***

S: ***Success, since 16 is more than 14.***

S: ***Success, since $\frac{16}{30}$ is greater than $\frac{1}{2}$.***

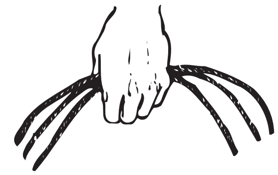
Compare earlier predictions to this result.

Exercise 2

Continue with the story about the fortune teller.

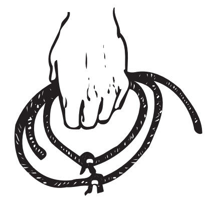
T: ***Sometimes a couple is very disappointed if the fortune teller predicts that their marriage will not be long and happy. They may ask to try again.***

Occasionally, the fortune teller agrees to give them another chance, using a different method. The three ropes are held like this.



Hold the three ropes as shown above. Twist the ropes slightly in your hand so that it is not easy to tell if two ends belong to the same rope.

T: ***The goal is the same: to form one long piece of rope by tying two knots. This time the ropes are held differently and each knot must join an end on one side with an end on the other. Who would like to try this new method?***



Select students to tie two knots according to your instructions.

Release and untangle the ropes. Determine if one long piece of rope is formed. If you wish, untie the ropes and repeat the experiment two or three more times.

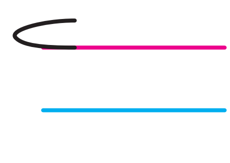
T: ***What do you think the probability of success is this time? Which of the two methods do you think has the higher probability of success?***

List some predictions on the board. Direct students to turn to Worksheet P3(b). Draw the picture from the worksheet on the board.

T: ***Let's calculate the probability of success using this second method of holding the ropes. I've drawn the three ropes straight this time to show how they are held. For the first knot, does it matter which end we start with?***

S: ***No.***

T: ***Let's start with a red end. Remember that the new method requires that each knot must join one end from the left side to one end from the right side. Now, on your worksheet, calculate the probability of success.***

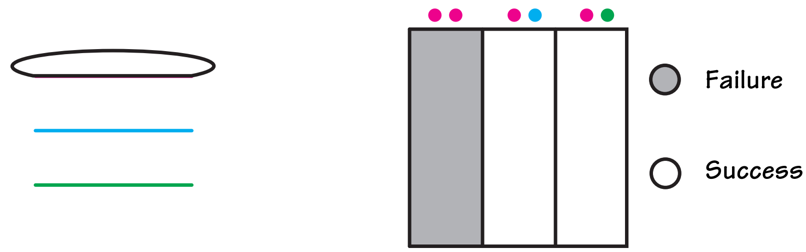


Let students work independently or with partners for a few minutes. Then do the first steps of the problem collectively.

T: ***What should we do first?***

S: ***Divide the square into three equal parts since the red end on the left can be joined to one of the three ends on the right.***

S: ***The region for red joined to red can be shaded since then the first knot forms a loop, which means failure.***



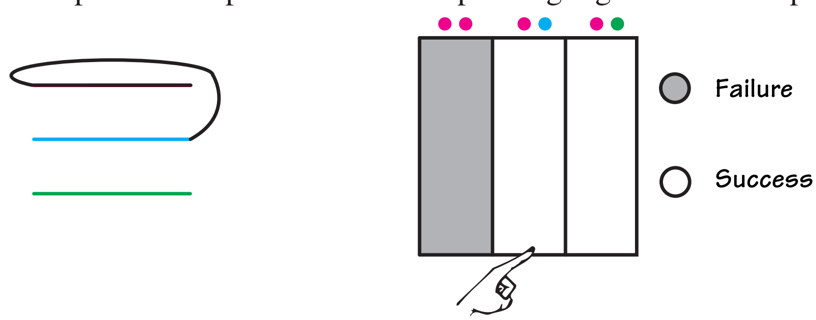
T: ***Complete the problem by determining what happens when the first knot joins a red end to either a blue end or a green end.***

After many students have finished, complete the solution collectively.

T: ***Does it matter whether the first knot joins the red end to a blue end or to a green end?***

S: ***No, the ropes are identical. The probability of success will be the same for both cases.***

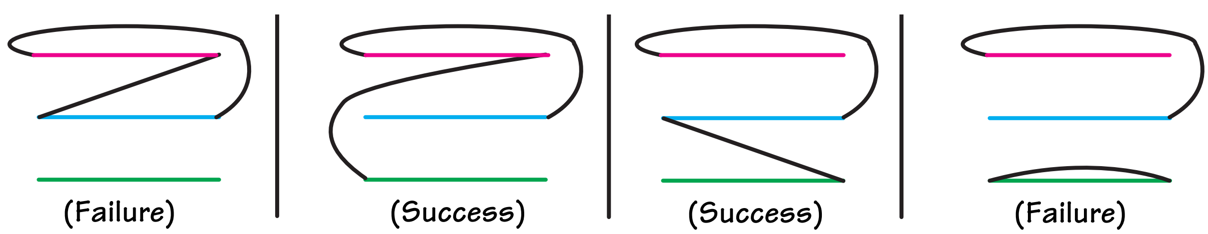
Add a connector to the picture and point to the corresponding region. For example:



T: ***Suppose that the first knot joins a red end to a blue end. How many different second knots could be tied?***

S: ***Four.***

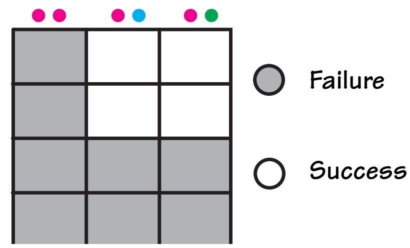
Invite a student to trace the four possible second knots, as shown below. You may wish to sketch the pictures on the board.



T: ***Which of these four possibilities result in success? … in failure?***

The class should determine that two possibilities result in success and two result in failure, as indicated in the preceding illustration. Invite a student to divide the middle region of the square, and shade it appropriately.

T: ***How should we shade this last region?***



S: *The same as the middle region, since it doesn't matter whether the first knot joins the red end to a blue end or to a green end.*

T: *What are the chances for success? … for failure?*

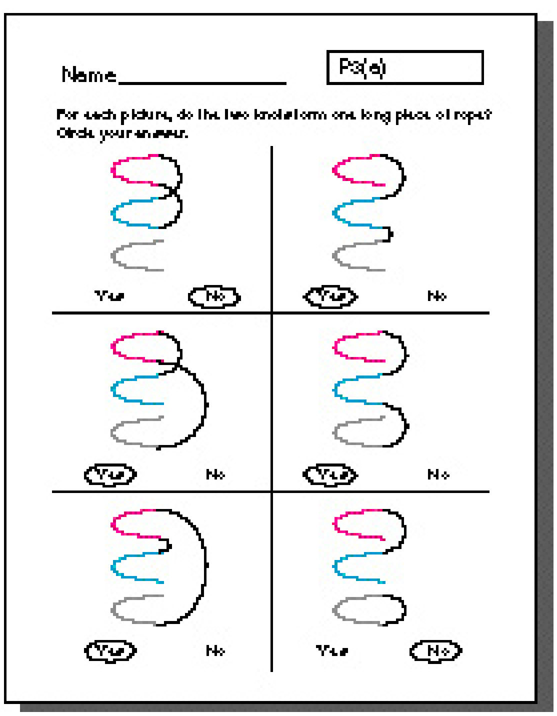
S: *Eight chances for failure and four chances for success.*

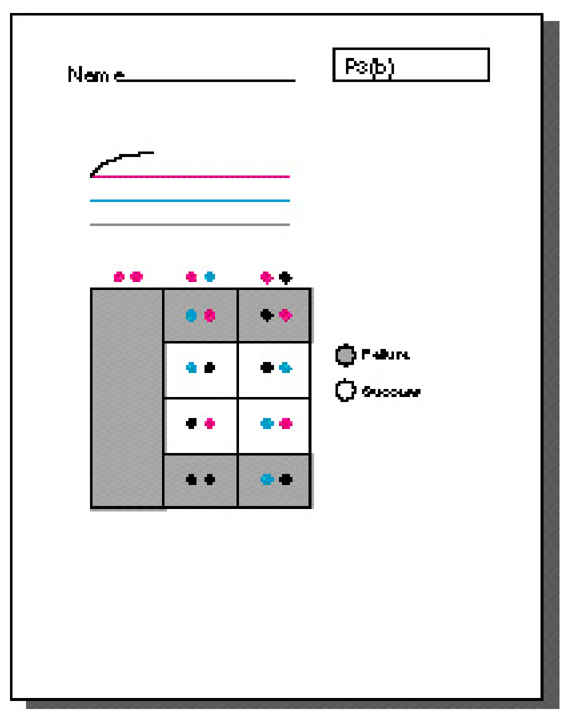
T: *What is the probability of success?*

S: $\frac{4}{12}$ *or* $\frac{1}{3}$.

T: *Which method of holding the ropes is better for a couple?*

S: *The first method;* $\frac{16}{30}$ *(about* $\frac{1}{2}$*) is greater than* $\frac{1}{3}$.





Capsule Lesson Summary

Conduct an experiment to measure students' reaction time when attempting to catch a falling ruler. Calculate the mean, median, and mode of each student's data in order to determine the student(s) with the fastest reaction time.

Materials

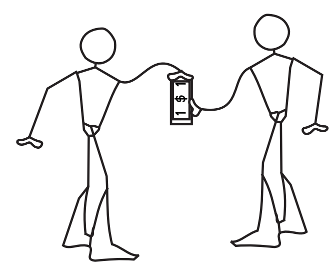
Teacher
- Crisp dollar bill

Student
- 30 cm ruler
- Calculator
- Worksheets P4(a), (b) and (c)

Description of Lesson

Note: In this lesson we make use of three statistical averages: mean, mode, and median. The lesson explains how to compute each of these numbers for a given set of data. The term *average* in everyday life often refers to the mean; however, in this lesson, use the specific terms *mean*, *mode*, and *median*. You may wish to discuss these distinctions with your students during the lesson.

Select a student and conduct the following experiment.



Hold the end of a crisp, new dollar bill so that it is hanging vertically. Tell the student to hold a thumb and forefinger apart on either side of the dollar bill near the center.

T: ***When I drop the bill, try to catch it by closing your thumb and forefinger. Do not move your hand down.***

Drop the bill and let the student try to catch it. Repeat the experiment with several other students. Most likely all, or nearly all, of the students will fail to catch the bill.

T: ***Why is the dollar bill so hard to catch?***

S: ***It falls too fast.***

S: ***We don't know when you will release the bill.***

T: ***Yes. It takes a certain amount of time for you to react and catch the bill. This is called*** **reaction time.**

Write this term on the board.

T: ***What must your brain and body do in order for you to catch the bill?***

Lead a discussion about the steps the brain and body perform:

1. The eyes see the bill start falling and send a message to the brain.
2. The brain receives that information and decides what to do.
3. The brain sends a message to the fingers telling them to close.

T: ***Each of these steps takes time, and the total amount of time is your reaction time.***

You are going to work with a partner and drop rulers to measure your reaction time. Why do you think we will use rulers instead of dollar bills?

S: ***You don't have enough dollar bills.***

S: ***Rulers fall more smoothly than dollar bills.***

S: ***Rulers are longer than dollar bills.***

S: ***With a ruler, we can measure how far it falls.***

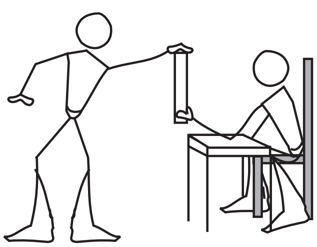
Pair students and equip each pair with a 30 cm metric ruler. Distribute Worksheet P4(a). Let students try the experiment a few times (without recording the results) before continuing.

T: ***We want to measure and compare your reaction times. In order to compare, we must all do the same thing. Who can suggest some rules for everyone to follow?***

Lead the class to establish rules similar to the following:

- The catcher's elbow must be on the table with the forearm held horizontally.
- The catcher's thumb and forefinger must be apart, away from the ruler about 2 cm on each side.
- The dropper holds the ruler with 0 at the bottom, even with the catcher's fingers.
- After a drop, measure how far the ruler drops to the nearest centimeter.
- If the catcher misses the ruler entirely and it falls to the floor, record that try as 30 cm.
- Repeat the experiment ten times and record the data on Worksheet P4(a).

Select one pair of students to demonstrate the technique three or four times at the front of the room. Draw a chart on the board to record the results, for example:



Trial	1	2	3	4	5	6	...
Drop (cm)	17	30	21	15			

The student pairs should each take a turn at being the catcher, repeating the experiment ten times. On Worksheet P4(a), a student should record only the data from his/her ten tries at being the catcher.

T: ***When you finish ten trials each, graph your own results. Connect the dots on your graph with line segments to form a line graph.***

Observe the activity to confirm that students perform the experiment correctly. As necessary, assist students who need help drawing a line graph of their data.

When most students have finished, continue the collective discussion.

T: ***We want to know who in this class has the fastest reaction time. How can we determine who is fastest?***

Encourage students to suggest several methods, for example, finding the lowest total or the lowest mean or the single smallest measurement.

Call attention to Worksheet P4(b).

T: ***Before determining who in our class has the fastest reaction time, let's look at the data on Worksheet P4(b) for four imaginary students. Each of these students thinks he or she has the fastest reaction time, but each for a different reason. Try to find the reason why each student believes he or she has the fastest reaction time.***

Let students work at least five minutes on their own or with their partners. Be sure students know that they may use calculators. While students are working, draw the table from the next illustration on the board, leaving room to draw another column on the left.

T: ***Select one of the four people and tell me why you think that person believes his or her reaction time is the fastest.***

Your students might suggest two or three of the following reasons. Explain other reasons yourself. As you consider each reason, compute the appropriate number for the four people and add one line of data to the table on the board.

S: ***Arnold might believe that he has the fastest reaction time because his drop of 5 cm is the single best measurement amongst all of the data.***

Direct students to find the single best result for each of the four students, and enter that result in the table. (See the first line in the table below.)

S: ***Lucy might believe that she has the fastest reaction time because her total of 156 cm and her mean of 15.6 cm (156 ÷ 10) are the lowest.***

Review how to compute the mean (add the ten numbers and divide by 10). Ask students to compute the mean for all four students.

Reason	Arnold	Lucy	Pierre	Michelle
Best result (cm)	5	7	14	11
Mean (cm)	16.9	15.6	16.3	16.5

S: ***Pierre might believe that he has the fastest reaction time because his most common result, 15 cm, is lower than anyone else's most common result.***

T: ***This number is called the mode. Pierre feels that he is most consistent.***

Ask students to find modes for all four students (see the third line in the following table). Note that Lucy has three modes.

S: ***Michelle might believe that she has the fastest reaction time because she has the fastest middle result. She arranges her ten results in order (11, 12, 12, 13, 13, 16, 16, 16, 28, 28) and looks at the average of the middle two results:*** $\frac{13+16}{2}$ ***= 14.5.***

T: ***This number is her median. She has the best median of the four people.***

Ask students to compute medians for all four students. Stress that you put the ten numbers for a person in order before looking at the two middle results.

Reason	Arnold	Lucy	Pierre	Michelle
Best result (cm)	5	7	14	11
Mean (cm)	16.9	15.6	16.3	16.5
Mode (cm)	19	16 or 17 or 18	15	16
Median (cm)	18.5	16.5	15	14.5

T: ***Each person has a good reason for claiming to have the fastest reaction time. Do you understand their reasons? Who do you think has the best claim for being fastest?***

Discuss student comments and compare the four reasons.

Distribute Worksheet P4(c).

T: ***Let's use each of these four methods to determine who in this class has the fastest reaction time. To do this, use the data from your ten trials to compute these four numbers (best result, mean, mode, and median).***

After allowing time for individual students or partners to complete Worksheet P4(c), ask students for their best result and determine the best single result in the class. Also find the lowest mean, mode, and median. Record this information in a column added to the left in the table on the board. For example:

In our class		Reason
Nathan	6 cm	Best result (cm)
Carla	11.2 cm	Mean (cm)
Melinda & Ameed	14 cm	Mode (cm)
Carla	11.5 cm	Median (cm)

Let students express opinions about who they believe has the fastest reaction time. There is no one correct answer to this question; it is a matter of interpretation.

For your information, some advantages and disadvantages of using each method are listed below. Discuss some of these, especially if mentioned by students.

Method	Advantages	Disadvantages
Single Best Result	Many winners, especially in athletics (for example, the long jump or shot put), are determined by the one best effort.	Only one measurement counts for each person.
Mean	All data count equally.	One or two bad (or good) results strongly affect the mean.
Mode	To some extent, the mode reflects consistency.	The mode often reflects only two or three values and is easily changed by new data.
Median	All data affect the median, but extreme data does not affect it strongly.	Very inconsistent data may push the median one way.

Instruct students to look at the line graphs of their data.

T: ***If a person has a consistently fast reaction time, what will the line graph look like?***

S: ***The dots and lines will be near the bottom of the graph.***

T: ***What will the graph for a person with a consistently slow reaction time look like?***

S: ***The dots and lines will be near the top of the graph.***

T: ***If a person's reaction time improves during the ten trials, what will the line graph look like?***

S: ***The dots and lines will go from the upper left to the lower right.***

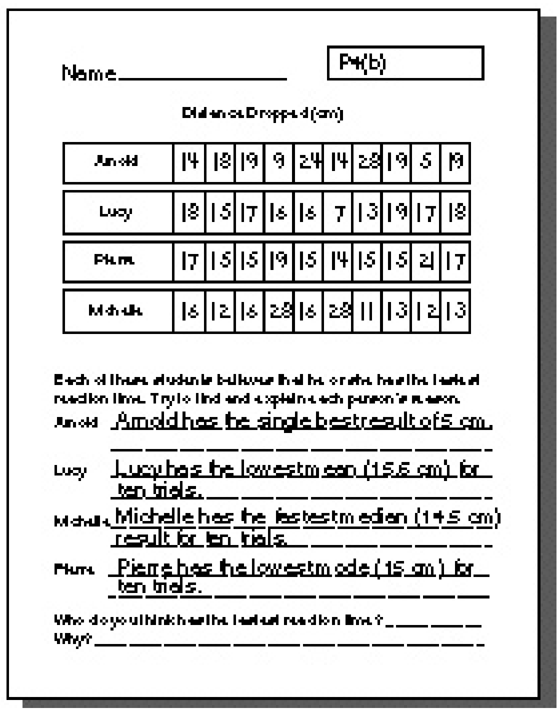
Note: The highly erratic results in this experiment indicate that this is not a very reliable technique for measuring reaction time. Still, this lesson is useful to illustrate reaction time, it is simple to perform, and it requires students to collect and analyze data.

Writing Activity

Suggest that students write about the line graphs for their reaction time.

Home Activity

Some students may like to use the experiment from this lesson to determine who in their family has the fastest reaction time.



Name________________

P4(b)

Distance Dropped (cm)

Arnold	14	18	19	9	24	14	28	19	5	19
Lucy	18	15	17	16	16	7	13	19	17	18
Pierre	17	15	15	19	15	14	15	15	21	17
Michelle	16	12	16	28	16	28	11	13	12	13

Each of these students believes that he or she has the fastest reaction time. Try to find and explain each person's reason.

Arnold Arnold has the single best result of 5 cm.

Lucy Lucy has the lowest mean (15.6 cm) for ten trials.

Michelle Michelle has the fastest median (14.5 cm) result for ten trials.

Pierre Pierre has the lowest mode (15 cm) for ten trials.

Who do you think has the fastest reaction time? __________

Why? ______________________________

Capsule Lesson Summary

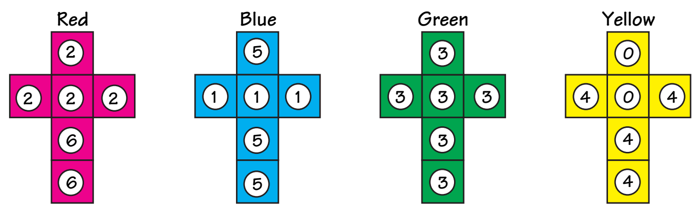
Analyze a two-person game played with four cubes to determine which player is favored and to find the best strategies. Discover some surprising results about the game, results that may be contraintuitive.

Materials

Teacher
- Colored chalk
- Four number cubes

Student
- Paper
- Ruler
- Worksheets P5(a) and (b)

Advance Preparation: Before the lesson begins, put number labels on four colored cubes, as shown below.



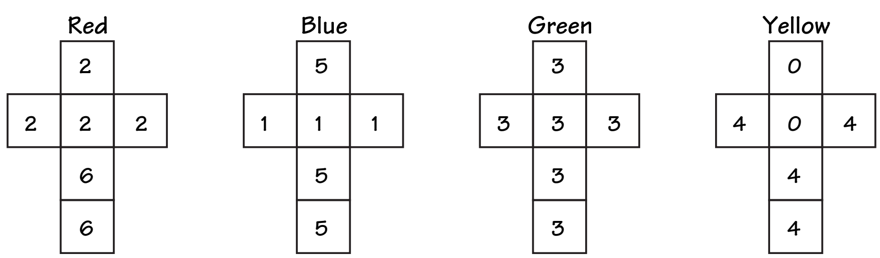
Description of Lesson

Note: Control the length of discussions and pace of the lesson so that students begin worksheets by the middle of the class period.

Draw four maps of a cube on the board. Label one for each of the four colors.

T: ***I have a game to show you. We play it with these four cubes.***

Hold up each of the four cubes for the class to observe. Let four students describe the cubes and direct other students to label each map of a cube on the board. (The face positions of the numbers on a map is not important for this lesson.)



Select a student to be your opponent in the first game.

T: ***I will choose one of the four cubes and then you*** (student opponent) ***can select any one of the other three cubes. We roll the cubes we've chosen and whoever gets the higher number wins.***

Play the game several times. Each time you, the teacher, select a cube first and then let your ~~opponent select a cube~~. Encourage players to comment on why they select the cubes they do.

†These cubes were designed by Bradley Efron, a statistician at Stanford University. See Martin Gardner's *Mathematical Games*, Scientific American, December 1970.

T: ***Do you think this is a fair game?***

Encourage students to explain why they think that the game is fair, or why they think the game favors either the teacher or the student. Students may suggest ways to make the game fair. Accept such suggestions, but remind the class that your question concerns whether *this* game is fair. After the discussion, you may wish to poll the class on their opinions.

T: ***If you feel that the teacher is favored, what is the teacher's best strategy? If you feel that the student is favored, what is the student's best strategy?***

Discuss strategies students feel each player should use.

T: ***Let's use probability methods to determine who is favored in this game and to learn each player's best strategy.***

Draw a square on the board

T: ***Let's use this square to analyze the game, assuming that the teacher selects the red cube and the student selects the blue cube. Does it matter who rolls first?*** (No)

Suppose the teacher rolls first. What are the possible outcomes?

S: ***The teacher could roll a 2 or a 6.***

T: ***How can we show this on the square?***

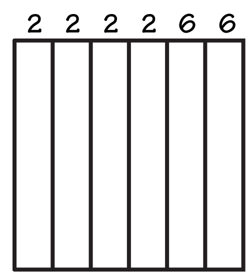
S: ***Divide the square into six equal parts for the cube's six faces. Label four parts 2 and the other two parts 6.***

S: ***You could divide the square into three equal parts and label two parts 2 and one part 6.***

T: ***Yes, either way would be okay. I'll use six parts to remind us of the six faces of the cube.***

What happens if the teacher rolls a 6?

S: ***The teacher wins for sure.***



Teacher: Red
Student: Blue

Put **T** in the two regions for 6, and point to one of the regions for 2.

T: ***What happens if the teacher rolls a 2?***

S: ***The student wins about half the time, since the blue cube has three 1s and three 5s.***

T: ***How can we show that on the square?***

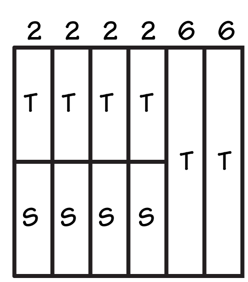
S: ***Divide each region for 2 in half. Put* T*, for teacher, in one half and* S*, for student, in the other half.***

T: ***Who is favored if the teacher selects the red cube and the student selects the blue cube?***

S: ***Teacher, since* T *has $\frac{2}{3}$ of the square and* S *has only $\frac{1}{3}$.***

T: ***What is each player's probability of winning?***

S: ***$\frac{8}{12}$ or $\frac{2}{3}$ for the teacher; $\frac{4}{12}$ or $\frac{1}{3}$ for the student.***



Teacher: Red
Student: Blue

You may divide the regions for 6 in half to get pieces all the same size. Then eight pieces are for **T** and four pieces are **S**. Record the probabilities on the board.

T: ***The probability of red beating blue is ⅔.***

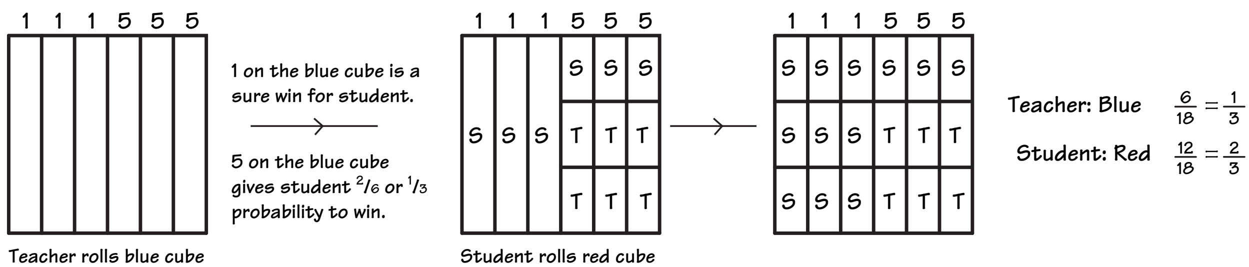
Record this information on the board as indicated here:

p (R, B) = ⅔

T: ***If the teacher chooses the blue cube and the student chooses the red cube, what are their probabilities of winning?***

S: ***Switch the* T*s and* S*s in the square. The student would have a probability of ⅔ of winning; the teacher, ⅓.***

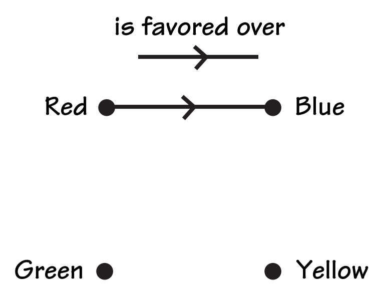
Note: If students suggest an analysis assuming the teacher rolls the blue cube first, the division of the square could look different, but the winning probabilities still will be: teacher ⅓, student ⅔.



T: ***The probability of blue beating red is ⅓.***

p (B, R) = ⅓

Draw this picture on the board.

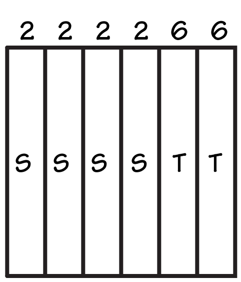


T: ***This arrow picture records the fact that the red cube is favored over the blue cube. Do you think that the green cube or the yellow cube would be a better choice for the student when the teacher selects the red cube?***

S: ***I think that the green cube is favored over the red cube.***

T: ***Let's analyze the game supposing that the teacher selects the red cube and the student selects the green cube.***

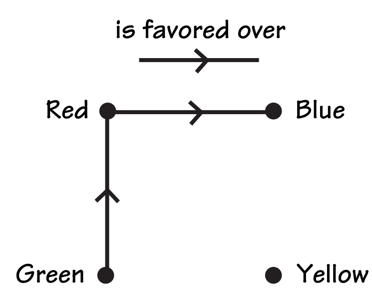
Analyze this game in a manner similar to the previous game, and record the result in the arrow picture.



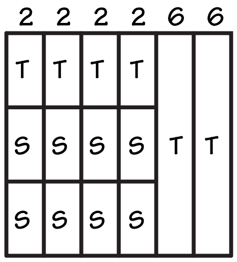
Teacher: Red $\frac{2}{6} = \frac{1}{3}$

Student: Green $\frac{4}{6} = \frac{2}{3}$

$p(G,R) = \frac{2}{3}$ $p(R,G) = \frac{1}{3}$

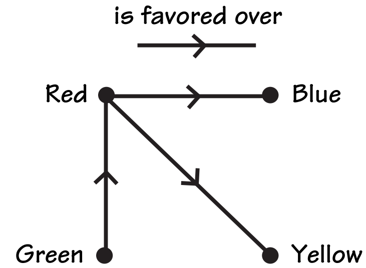


In a similar manner, analyze the game in the case in which the teacher selects the red cube and the student selects the yellow cube. Draw an arrow from red to yellow to record the result.



Teacher: Red $\frac{10}{18} = \frac{5}{9}$

Student: Yellow $\frac{8}{18} = \frac{4}{9}$



$p(R,Y) = \frac{5}{9}$ $p(Y,R) = \frac{4}{9}$

T: ***In this situation, is it a good strategy for the teacher to select the red cube?***

S: ***No, because the student could select the green cube and be favored to win.***

T: ***Which cube might be better for the teacher?***

S: ***The green cube, because the green cube is favored over the red cube.***

Students might suggest that you draw an arrow from green to blue since green is favored over red and red is favored over blue. Also, they might suggest drawing an arrow from green to yellow. Indicate that these are interesting ideas, but that these suggestions need to be checked by further analysis.

Distribute copies of Worksheets P5(a) and (b). Divide the class into three groups, and write this information on the board.

Group 1	Group 2	Group 3
Teacher selects the blue cube	Teacher selects the green cube	Teacher selects the yellow cube
p(B,G) =_____	p(G,B) =_____	p(Y,B) =_____
p(B,Y) =_____	p(G,Y) =_____	p(Y,G) =_____

T: ***I want you to complete the analysis of this game on the worksheets. Each group has two problems to solve, one problem for each side of the worksheet. Those in group 1 should assume that the teacher selects the blue cube. Then they should calculate the probability of the blue cube winning over the green cube and the probability of the blue cube winning over the yellow cube.***

In a similar manner, explain the responsibilities of group 2 and group 3.

T: ***On your worksheets, write your group number and use the squares to solve the two problems assigned to your group.***

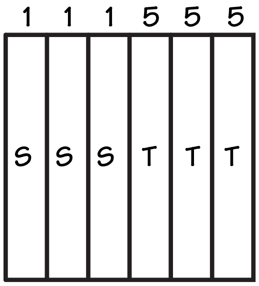
Note: You may want to form six groups and assign two groups to each pair of problems.

As you observe the group, you may need to help some get started on their problems. If some groups finish quickly, suggest they solve the problems from another group. The following pictures indicate possible solution methods for the six problems.

Group 1

Teacher selects the blue cube (1, 1, 1, 5, 5, 5).

Student selects the green cube (3, 3, 3, 3, 3, 3).



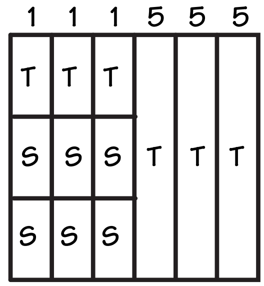
Teacher: $\frac{3}{6} = \frac{1}{2}$

Student: $\frac{3}{6} = \frac{1}{2}$

$p(B,G) = \frac{1}{2}$

$p(G,B) = \frac{1}{2}$

Student selects the yellow cube (0, 0, 4, 4, 4, 4).



Teacher: $\frac{12}{18} = \frac{2}{3}$

Student: $\frac{6}{18} = \frac{1}{3}$

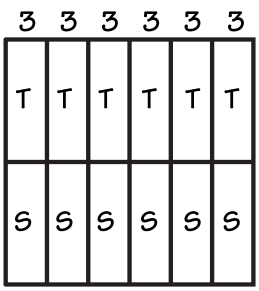
$p(B,Y) = \frac{2}{3}$

$p(Y,B) = \frac{1}{3}$

Group 2

Teacher selects the green cube (3, 3, 3, 3, 3, 3).

Student selects the blue cube (1, 1, 1, 5, 5, 5).



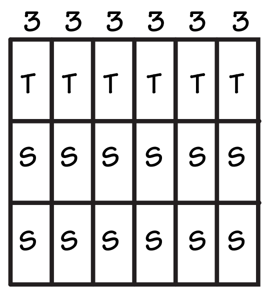
Teacher: $\frac{6}{12} = \frac{1}{2}$

Student: $\frac{6}{12} = \frac{1}{2}$

$p(G,B) = \frac{1}{2}$

$p(B,G) = \frac{1}{2}$

Student selects the yellow cube (0, 0, 4, 4, 4, 4).



Teacher: $\frac{6}{18} = \frac{1}{3}$

Student: $\frac{12}{18} = \frac{2}{3}$

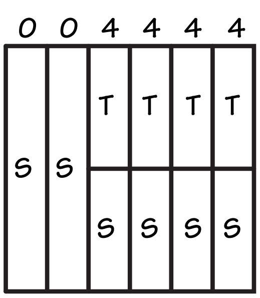
$p(G,Y) = \frac{1}{3}$

$p(Y,G) = \frac{2}{3}$

Group 3

Teacher selects the yellow cube (0, 0, 4, 4, 4, 4).

Student selects the blue cube (1, 1, 1, 5, 5, 5).



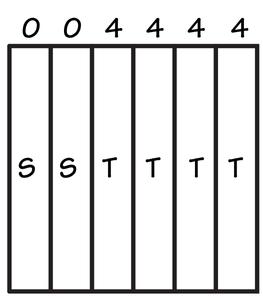
Teacher: $\frac{4}{12} = \frac{1}{3}$

Student: $\frac{8}{12} = \frac{2}{3}$

$p(Y,B) = \frac{1}{3}$

$p(B,Y) = \frac{2}{3}$

Student selects the green cube (3, 3, 3, 3, 3, 3).



Teacher: $\frac{4}{6} = \frac{2}{3}$

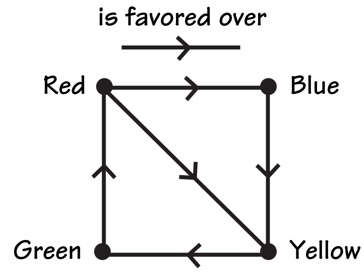
Student: $\frac{2}{6} = \frac{1}{3}$

$p(Y,G) = \frac{2}{3}$

$p(G,Y) = \frac{1}{3}$

During the last ten minutes of the class period, collect and discuss the results. Ask each group for the solutions to their two problems. If there is not a consensus on the correct solution to a problem, quickly solve the problem at the board. Record the results on the board, and invite students to add arrows to the arrow picture whenever possible.

Group 1	Group 2	Group 3
$p(B,G) = \frac{1}{2}$	$p(G,B) = \frac{1}{2}$	$p(Y,B) = \frac{1}{3}$
$p(B,Y) = \frac{2}{3}$	$p(G,Y) = \frac{1}{3}$	$p(Y,G) = \frac{2}{3}$



Note: Since **p (B,G) = ½**, an arrow cannot be drawn between the dots for the blue cube and the green cube.

As the results are recorded, students may make several observations:

- p (B,Y) = $\frac{2}{3}$ and p (Y,B) = $\frac{1}{3}$ both imply that an arrow can be drawn from blue to yellow. Similarly, p (Y,G) = $\frac{2}{3}$ and p (G,Y) = $\frac{1}{3}$ both imply that an arrow can be drawn from yellow to green.
- p (B, Y) + p (Y, B) = $\frac{2}{3} + \frac{1}{3} = 1$; p (Y, G) + p (G, Y) = $\frac{2}{3} + \frac{1}{3} = 1$; p (B, G) + p (G, B) = $\frac{1}{2} + \frac{1}{2} = 1$.
- Even though green is favored over red and red is favored over blue, green is not favored over blue. Similarly, green is favored over red and red is favored over yellow, but green is not favored over yellow.

T: ***We have analyzed all of the possible games with these four cubes. Who is favored, the teacher or the student?***

S: ***The student, because no matter which cube the teacher selects the student can choose a cube that is favored over the teacher's cube.***

T: ***Look at the arrow picture. If I select the red cube, which cube would you select?*** (Green)
If I select the green cube, which cube would you select? (Yellow)
If I select the yellow cube, which cube would you select? (Red or blue)
If I select the blue cube, which cube would you select? (Red)

Writing Activity

Suggest that students write directions to an absent student on how to select a cube in this game with four cubes.

P6 PASCAL'S TRIANGLE #1

Capsule Lesson Summary

Conduct an extrasensory perception (ESP) test with the class, and discuss the probability of getting certain results by just guessing. Assume a probability of ½ that a person guessing is correct. Construct Pascal's triangle, using the idea of an ESP test with a given number of items administered to a given number of people. Predict the number of correct responses per person.

Materials

Teacher
- Colored chalk
- Large piece of paper
- Ten index cards

Student
- Paper
- Colored pencils, pens, or crayons

Advance Preparation: Before teaching this lesson, prepare five index cards with **TRUE** written on one side and five index cards with **FALSE** written on one side.

Description of Lesson

Exercise 1

Lead a short discussion of extrasensory perception (ESP). Include the idea that ESP might be tested using a set of cards with two different words written on them. The tester could look at a card and then concentrate on the word; the person being tested could try to read the tester's mind and tell what word is on the card.

Show the class the ten index cards, five with **TRUE** and five with **FALSE** on them. Tell students that they will all take a five-item true-false test, and ask them to number 1 to 5 on their papers in preparation for the five test items. Explain that for each test item you will shuffle the cards, select a card, and then concentrate on the word that is on the card. They should try to read your mind and guess what word is on the card. Keep track yourself of the correct answers to the test.

When you finish the test, suggest students exchange papers to grade each other's tests. Then collect results from the class to see how many students had 0, 1, 2, 3, 4, or 5 correct. For example:

Correct test items	0	1	2	3	4	5
Number of students	2	4	10	9	5	0

T: ***Do you think anyone in our class has ESP?***

S: ***Certainly not the ones with no correct items.***

S: ***Maybe those who got four out of five correct.***

T: ***Is this a good test for ESP?***

S: ***No; a person could be just guessing and still give several correct responses.***

T: ***Let's determine the probability of getting these results if people are just guessing.***

P6

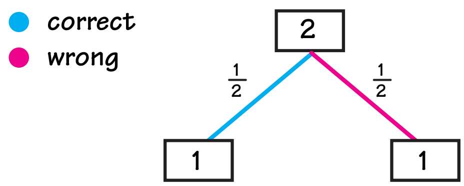
Begin a tree diagram on the board.

T: ***If we give the test with each person making only one guess, what might we expect the results to be?***

S: ***About one half of the people would make a correct guess and one half would make a wrong guess.***

T: ***So for every two people taking the test we might expect that one would be correct and one would be wrong.***

Indicate how the tree diagram can display this information, and begin a table for recording the results.

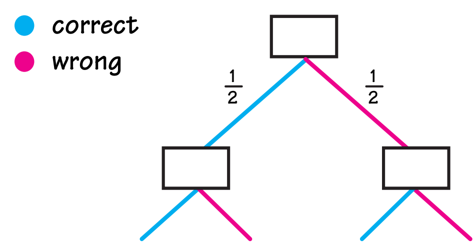


Number of guesses per person	Number of people	Number of people with correct guesses: 0	1	2	3	4	5
1	2	1	1				

Erase the numbers from the tree diagram.

T: ***Now let's look at what could happen if each person were given two guesses. What could the results be for one person?***

As a student describes a possible result, add to the tree diagram to show this possibility. In the illustration below, the student responses are given in an order that adds branches to the tree from left to right.



S: ***Both guesses are correct.***

S: ***The first guess is correct and the second is wrong.***

S: ***The first guess is wrong and the second is correct.***

S: ***Both guesses are wrong.***

T: ***For the first guess we expect about one half of the people to be correct and one half to be wrong. What about for the second guess?***

S: ***We expect the same, about one half to be correct and one half to be wrong.***

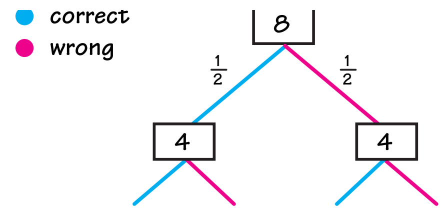
T: ***What number of people should we consider taking a two-item ESP test (with two guesses per person), if we want to see what results to expect?***

Put whatever number the students suggest in the top box of the tree diagram. If the number is not a positive multiple of 4, it will necessitate the use of fractions in the tree. When this becomes apparent, ask the class to suggest another number of people so as to avoid using fractions.

Perhaps a student will suggest giving the test to eight people.

T: ***How many of these eight people might we expect to be correct and how many to be wrong on their first guesses? Why?***

S: *Four and four, because correct and wrong guesses are equally likely.*



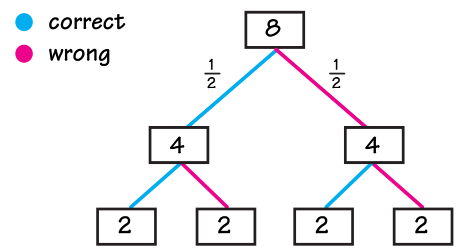
Point to the four people who were correct on the first guess.

T: *How many of these four people would we expect to be correct on their second guesses? How many wrong?*

S: *Two correct and two wrong.*

T: *How many of the four people who were wrong on the first guess would we expect to be correct on the second guess? How many wrong?*

S: *The same; two correct and two wrong.*

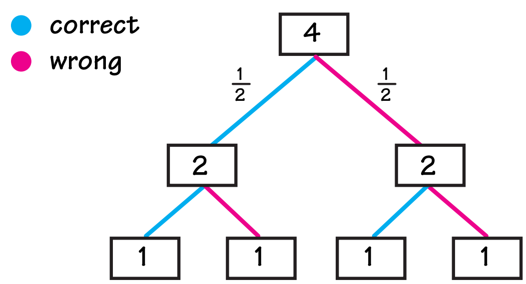


T: *Could we use more or fewer people in this probability tree?*

S: *Yes, we could use a large number of people.*

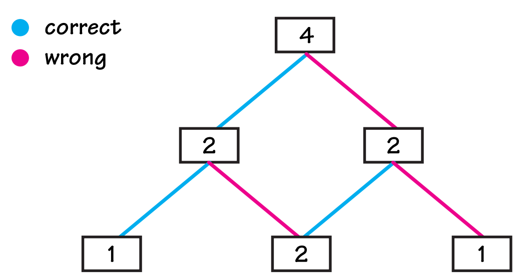
S: *Yes, four would be a good number to use.*

Discuss how to change the other numbers if the top number (in this case 8) is different. Erase all of the numbers in the tree diagram, and repeat the activity assuming four people are taking the test. Then let students use the information in the probability tree to tell you how to make another entry in the table.

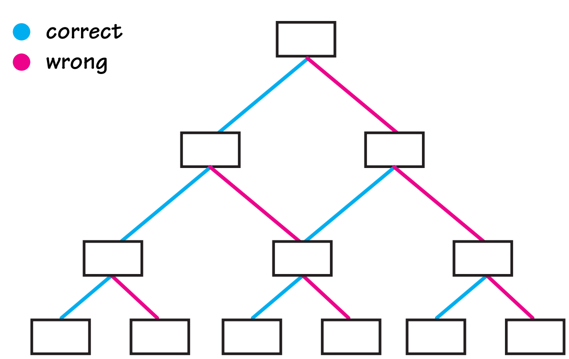


Number of guesses per person	Number of people	Number of people with 0 correct guesses	1	2	3	4	5
1	2	1	1				
2	4	1	2	1			

With the class, discuss that the probability tree shows two possible ways to get the same result (one correct guess). Since it is not important which one of the two guesses is correct, alter the probability tree to make it clear that there are really only three possible results for a two-item test.



Erase all of the numbers in the tree diagram and ask the class to help extend it to show the possible results of a three-item test, i.e., three guesses per person. Perhaps the tree the students suggest will have six boxes in the bottom row.

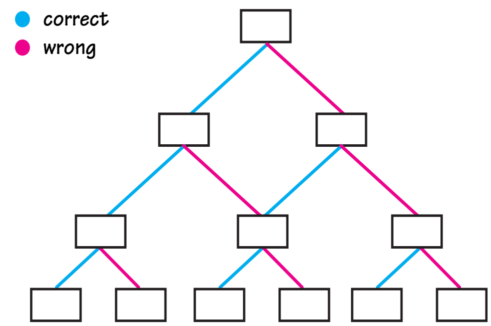


T: *This probability tree suggests that there are six different results for a three-item test. Are all of these results really different?*

S: ***There are two boxes for the same result, one wrong and two correct guesses. We could make just one box for this result.***

S: ***There are also two boxes for one correct and two wrong guesses. We could have just one box for this result too.***

Change the probability tree to have boxes for four possible results: zero, one, two, or three wrong.

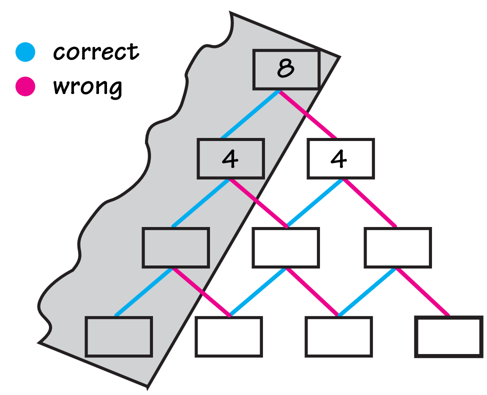


T: ***How many people could we use in this probability tree if we want to see what results to expect from a three-item test? Try to use a small number of people. Do you see a pattern in the number of people?***

S: ***Eight people. Each time we add an item to the test we double the number of people taking it.***

If necessary, mention to the class that since two was a good number of people to use for a one-item test, and four was a good number for a two-item test, we might predict that eight would be a good number for a three-item test.

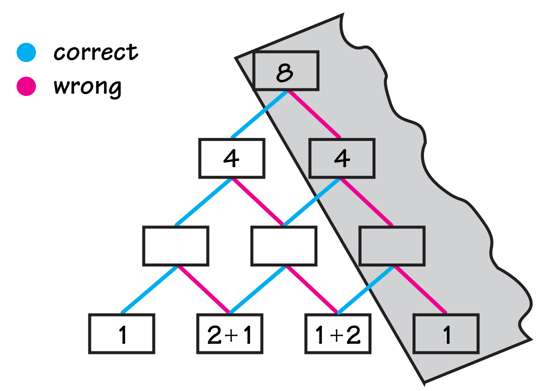
Write 8 in the top box and ask how many out of eight people we expect to be correct (four) and wrong (four) on the first guess. Add this information to the probability tree, and then cover part of the picture with a large sheet of paper, as shown here.



T: ***Look at this part of the picture. How many people are guessing?*** (Four)
How many guesses are left? (Two each)
Perhaps the table can help us fill in the bottom row of boxes.

S: ***If four people have two guesses each, the numbers in the bottom row of boxes would be 1, 2, and 1.***

Record 1, 2, and 1 in the appropriate boxes and repeat by covering the other side of the picture.



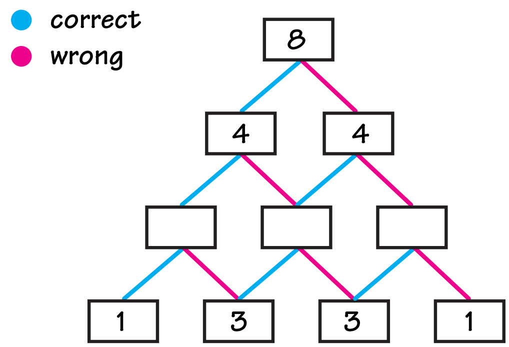
T: ***What numbers should we write in the bottom row of boxes?***

S: ***It is the same as before. If four people have two guesses each, the results could be 1, 2, and 1 respectively, for the three possibilities. We must add these numbers to those already in the boxes.***

S: ***Put 1 in the first box, 3 in the second box because 1 + 2 = 3, and 3 in the third box because 2 + 1 = 3.***

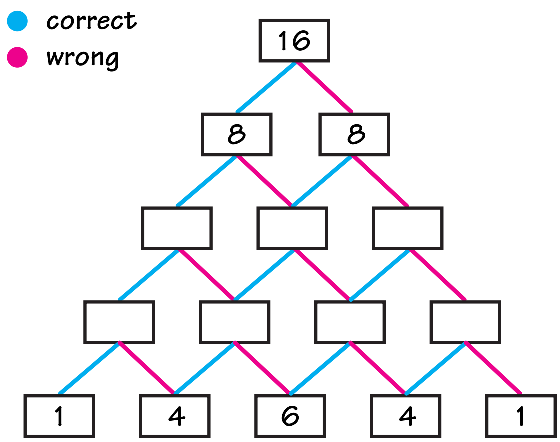
T: ***Let's use the information in this probability tree to make another entry in our table.***

Invite students to tell you what data to enter in the table on the board.



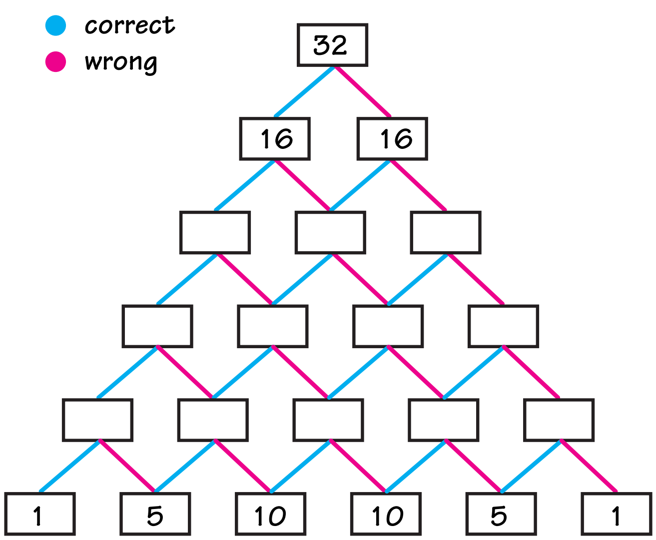
Number of guesses per person	Number of people	Number of people with correct guesses					
		0	1	2	3	4	5
1	2	1	1				
2	4	1	2	1			
3	8	1	3	3	1		

Erase all of the numbers in the tree, and repeat the activity for a four-item test. Let students tell you how to extend the tree diagram; suggest that 16 would be a good number of people to use with a four-item test. Determine that on the first guess we expect eight people to be correct and eight to be wrong. Then cover part of the tree diagram to show eight people with three guesses each, and use the previous results in the table to fill in the bottom row of boxes. Finally, enter this data in the table.



Number of guesses per person	Number of people	Number of people with correct guesses					
		0	1	2	3	4	5
1	2	1	1				
2	4	1	2	1			
3	8	1	3	3	1		
4	16	1	4	6	4	1	

Ask students to predict the entries in the table for a five-item test based upon patterns they observe; record students' predictions. If necessary, record more than one prediction for a single number. Then extend the probability tree and determine the entries in the table for a five-item test with 32 people taking the test. Decide which patterns in the table can be used to predict the next entries.



Number of guesses per person	Number of people	Number of people with correct guesses					
		0	1	2	3	4	5
1	2	1	1				
2	4	1	2	1			
3	8	1	3	3	1		
4	16	1	4	6	4	1	
5	32	1	5	10	10	5	1

Compare your class results on the five-item test to the expected results for 32 people on a five-item test.

Tell the class that you will return to the question of whether the test was a good test for ESP in another lesson. You may want to save your class test results for reference in Lesson P7.

Home Activity

Suggest that students give a five-item true-false test to family members, friends, and neighbors. They can use a method similar to the one used at the beginning of this lesson, but administer the test individually. Then, if they get results for about thirty people, they can compare the actual results to the expected results.

Capsule Lesson Summary

Use the table constructed in Lesson P6 *Pascal's Triangle #1* to determine the probability of guessing correctly a given number of times on an extrasensory perception (ESP) test. Use a tree diagram to determine the probability of always guessing correctly on two separate tests. Discover that giving two five-item tests is the same experiment as giving a ten-item test.

Materials

Teacher
- Colored chalk
- Test results from Lesson P6

Student
- Calculator
- Colored pencils
- Paper

Description of Lesson

Exercise 1

Draw the table from Lesson P6 *Pascal's Triangle #1* on the board.

Number of guesses per person	Number of people	Number of people with 0 correct guesses	1	2	3	4	5
1	2	1	1				
2	4	1	2	1			
3	8	1	3	3	1		
4	16	1	4	6	4	1	
5	32	1	5	10	10	5	1

Lead a short discussion of ESP and the test used in Lesson P6. Review how the table was constructed, and again compare your class test results to the table.

T: ***Let's use the table to determine the probabilities of some possible results from ESP tests. If we give a five-item test (each person making five guesses) to 32 people, we can expect five people to make exactly one correct guess. What is the probability that by just guessing someone will make exactly one correct guess?***

S: ***5 out of 32, or $\frac{5}{32}$.***

Write this information on the board. Continue finding probabilities for the other possible results. For example:

p (0 correct out of 5) = $\frac{1}{32}$ p (1 correct out of 5) = $\frac{5}{32}$ p (2 correct out of 5) = $\frac{10}{32}$

p (5 correct out of 5) = $\frac{1}{32}$ p (4 correct out of 5) = $\frac{5}{32}$ p (3 correct out of 5) = $\frac{10}{32}$

T: ***Which results do we expect most often?***

S: ***Either two or three correct guesses.***

T: ***What is the probability of guessing correctly either two or three times on a five-item test?***

S: ***20/32. There are 10 chances out of 32 of guessing correctly twice and 10 chances out of 32 of guessing correctly three times; 10 + 10 = 20.***

T: ***What is the probability of guessing correctly at least three times? Why?***

S: ***16/32 or 1/2. There are 10 out of 32 chances of getting three guesses correct, 5 out of 32 chances of getting four guesses correct, and 1 out of 32 chances of getting all five guesses correct. 10 + 5 + 1= 16, so there are 16 out of 32 chances of guessing correctly at least three times.***

T: ***What is the probability of guessing correctly at least four times?*** (6/32)
What result would you want before you would believe that someone has ESP?

S: ***All five guesses correct.***

T: ***What is the probability that all five guesses will be correct?*** (1/32)
So when we gave the five-item test to everyone in this class, it was likely that one of you would guess correctly all five times. Do you think this is a reliable test for ESP?

S: ***No.***

Exercise 2

Make calculators available for this exercise.

T: ***What could we do to make the test for ESP more reliable?***

S: ***Give the test several times.***

S: ***Put more items (guesses) on the test.***

S: ***Include more choices of answers on the cards; that is, make it a multiple choice test.***

If necessary, suggest some of these possibilities yourself.

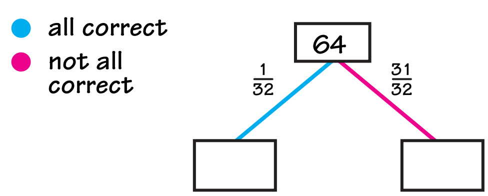
T: ***Let's determine what the probabilities would be if we gave the five-item test twice. How many correct guesses would you want a person to have each time to say that they have ESP?***

S: ***Five correct guesses on each test.***

T: ***What is the probability that a person would guess correctly five times on the first test?*** (1/32)

If a person does not guess correctly all five times, then there would be at least one wrong guess. What is the probability of having at least one wrong guess? (31/32)

Begin a probability tree on the board.



T: ***If 64 people took a five-item test, how many of them would we expect to have all their guesses correct? Why?***

S: ***2; 64 ÷ 32 = 2.***

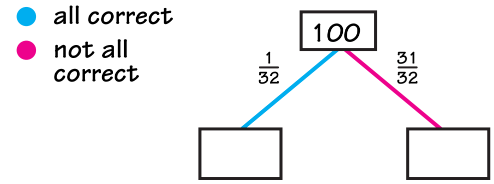
T: ***How many would we expect to have at least one wrong guess? Why?***

S: ***62; 64 – 2 = 62.***

S: ***62; 31 x 64 = 1 984 and 1984 ÷ 32 = 62.***

Put 2 and 62 in the appropriate boxes.

Erase the numbers in the boxes and start with 100 in the top box.



T: ***If 100 people took a five-item test, how many of them would we expect to get all five guesses correct? Why?***

S: ***3; because 100 ÷ 32 = 3.125 which is close to 3.***

T: ***How many of them would we expect to get at least one guess wrong? Why?***

S: ***97; 100 – 3 = 97.***

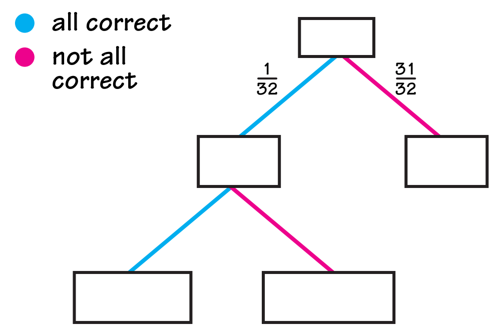
S: ***97; 31*** x ***100 = 3 100 and 3100 ÷ 32 = 96.875 which is close to 97.***

S: ***97; 31*** x ***3.125 = 96.875 which is close to 97.***

Record 3 and 97 in the appropriate boxes.

T: ***Now let's see how many people we would expect to always guess correctly if we gave them the five-item test twice.***

Erase the numbers in the boxes and extend the tree diagram. Point to each of the bottom boxes in turn and ask the class what it represents.



S: ***Blue followed by blue represents guessing correctly all five times on both tests.***

S: ***Blue followed by red represents guessing correctly all five times on the first test but not all five times on the second test.***

T: ***Do we need to extend this tree diagram on the right side in order to find the chances of guessing correctly all five times on both of two five-item ESP tests?***

S: ***No, because red represents having a wrong guess at least once.***

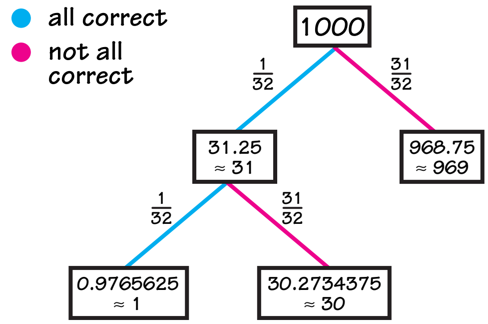
T: ***What is the probability of guessing correctly all five times on the second test?*** ($\frac{1}{32}$) ***Of not guessing correctly all five times?*** ($\frac{31}{32}$)

Show this information on the probability tree by labeling the bottom blue and red cords, as in the next illustration.

T: ***Copy this picture on your paper. Suppose 1000 people take two five-item ESP tests. Put 1 000 in the top box, and fill in the other boxes.***

P7

Allow students to work independently or with partners for a few minutes. Then invite students to help you fill in the boxes at the board.



T: ***What is the probability of guessing correctly all five times on both tests?***

S: ***About $^1/_{1000}$ because we would expect one person out of 1000 to guess correctly all five times on both tests.***

Save this tree diagram for comparison later in the lesson.

T: ***You also suggested that we could put more items on the ESP test to make it more reliable.***

Point to the table on the board, and ask students to consider extending it to include a six-item ESP test. Encourage them to predict the next row of entries by noticing patterns in the table. Students might observe various patterns, but, if necessary, point out the following pattern yourself.

Number of guesses per person	Number of people	Number of people with 0 correct guesses	1	2	3	4	5
1	2	1 +	1				
2	4	1	2 +	1			
3	8	1 +	3	3	1		
4	16	1	4	6 +	4	1	
5	32	1	5	10	10	5	1

The sum of two adjacent entries in a row gives an entry in the next row, as shown.

Use this pattern to extend the table until it includes information for a ten-item ESP test.

Number of guesses per person	Number of people	Number of people with 0 correct guesses	1	2	3	4	5	6	7	8	9	10
1	2	1	1									
2	4	1	2	1								
3	8	1	3	3	1							
4	16	1	4	6	4	1						
5	32	1	5	10	10	5	1					
6	64	1	6	15	20	15	6	1				
7	128	1	7	21	35	35	21	7	1			
8	256	1	8	28	56	70	56	28	8	1		
9	512	1	9	36	84	126	126	84	36	9	1	
10	1024	1	10	45	120	210	252	210	120	45	10	1

T: ***What is the probability that a person will have ten correct guesses on a ten-item test?***

S: ***$^1/_{1024}$.***

Point to the tree diagram on the board.

T: ***We estimated that the probability of always guessing correctly on two five-item ESP tests was about $^1/_{1000}$. Let's compare the probabilities of always guessing correctly on one ten-item test and on two five-item tests.***

Erase the numbers in the boxes of the tree diagram and start with 1 024 in the top box.

T: ***How many people would we expect to always guess correctly if 1 024 people each took two five-item ESP tests?***

S: ***We expect 32 people to guess correctly all five times on the first test because 1 024 ÷ 32 = 32.***

Record 32 in the appropriate box.

S: ***We expect one of those 32 people to guess correctly all five times on the second test.***

Record 1 in the appropriate box. The students should conclude that the probability of always guessing correctly on two five-item tests or on one ten-item test is the same, $\frac{1}{1024}$.

T: ***Would either of these tests be a good indicator of ESP?***

Allow students to express their opinions freely, but note that either method would be much more reliable than the results of one five-item ESP test.

Extension Activity

Make copies of Blackline P7 (Pascal's triangle arranged differently) and suggest that students look for number patterns.

Investigate other probability situations where Pascal's triangle may be used. For example, what is the probability that a family with five children will have four boys and one girl? This problem assumes that the probability of having a boy or a girl is the same.

Capsule Lesson Summary

Use students' knowledge of percent in everyday life to relate certain fractions with percents (for example, $\frac{1}{2}$ is equivalent to 50% and $\frac{3}{4}$ is equivalent to 75%). Find misleading information in a cereal advertisement. Compare the visual impact of two different bar graphs for the same data. Investigate methods for comparing the prices of two items when the given prices are for different quantities.

Materials

Teacher
- Overhead
- *Nutribest* advertisement
- Blackline P8

Student
- Calculator
- Colored pencils
- Paper
- Worksheets P8(a), (b), (c), and (d)

Advance Preparation: Use Blackline P8 to make a copy of the *Nutribest* advertisement for display in Exercise 2.

Description of Lesson

Exercise 1

Note: The goal of this exercise is to relate certain fractions with percents by using students' intuitive knowledge and concepts of percent from everyday life. Attempt to rely on students' experiences rather than presenting formal techniques yourself.

T: ***When we have 100% of the class here, what does that mean?***

S: ***Every student in the class is here.***

T: ***When 50% of the students in a class are girls, what does that mean?***

S: ***One-half of the students in the class are girls.***

Begin a table on the board.

Fraction	Percent
$\frac{1}{2}$	50%

T: ***Yes, 50% is another name for $\frac{1}{2}$. Do you know the percent name for $\frac{1}{4}$?***

S: ***25%. $\frac{1}{2}$ or $\frac{2}{4}$ is the same as 50%, so $\frac{1}{4}$ is the same as 25%.***

S: ***25%, since $\frac{1}{4} \times 100 = 25$.***

S: ***25%. 1 out of 4 is the same as 25 out of 100.***

Record this information in the table.

Fraction	Percent
$\frac{1}{5}$	20%
$\frac{1}{4}$	25%
$\frac{2}{5}$	40%
$\frac{1}{2}$	50%
$\frac{3}{5}$	60%
$\frac{3}{4}$	75%
$\frac{4}{5}$	80%

Similarly, lead students to find equivalent percent names for $\frac{3}{4}$, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, and $\frac{4}{5}$, and record the results in the table.

Point out that in every line of the table the fraction could be written as $\frac{\square}{100}$. For example, $\frac{3}{5} = \frac{60}{100} = 60\%$.

T: ***What is a fractional name for 100%?***

S: ***$^{100}/_{100}$.***

S: ***$^5/_5$. $^1/_5$ = 20%, $^2/_5$ = 40%, $^3/_5$ = 60%, $^4/_5$ = 80%, and so $^5/_5$ = 100%.***

S: ***$^4/_4$. $^1/_4$ = 25%, $^2/_4$ = 50%, and $^3/_4$ = 75%. So $^4/_4$ = 100%.***

S: ***$^2/_2$. $^1/_2$ = 50%, so $^2/_2$ = 100%.***

S: ***$^{100}/_{100}$. $^5/_5$, $^4/_4$, $^2/_2$ are all the same as 1.***

Students may be surprised that 1 is another name for 100%.

Exercise 2

Display the advertisement on Blackline P8.

T: ***Advertisers rarely outright lie to the public, but they often try to mislead you. This is an advertisement for Nutribest cereal. Let's assume that everything in this ad is true, but that the advertisers have used many "tricks" to convince us to buy Nutribest cereal. What are some of these tricks?***

Encourage students to find many, but not necessarily all, of the following misleading ideas in the advertisement.

Picture

- The pictures suggest that *Nutribest* will make you muscular.
- The *Nutribest* boy is smiling while the other boy is frowning.

Quotation

- Did they ask only five people?
- Who were the people they asked? Parents? Employees?
- In what way do people prefer *Nutribest*? Maybe Brand X tastes horrible. Is Brand X even a breakfast cereal?

Graphs

- The *Nutribest* graph has thicker bars.
- Are the scales for the two graphs the same?

Ingredients

- How much vitamin A is good for you? It may be harmful to get too much of some vitamins.
- Sodium usually comes from salt (sodium chloride), and many Americans are trying to reduce their sodium intake.
- Does Brand X have important nutrients that *Nutribest* does not have?

Price

- How much cereal does $1.38 and $1.60 buy? Maybe *Nutribest* comes in a small box.
- Maybe Brand X includes prizes or coupons.

Exercise 3

Distribute Worksheets P8(a), (b), and (c).

T: ***Based on tests and research, the United States Department of Agriculture has determined the recommended daily allowance of various vitamins and minerals. The graph on Worksheet P8(a) gives the percent of each allowance in Nutribest and in Brand X. Was the advertisement true for vitamin A, protein, and sodium?***

S: ***Yes, Nutribest has more vitamin A, protein, and sodium than Brand X.***

T: ***What about the vitamins and minerals not listed on the advertisement?***

S: ***Brand X has more vitamin C, vitamin D, calcium, niacin, and riboflavin than Nutribest. Nutribest has more thiamine and iron.***

T: ***Was the advertisement fair?***

S: ***No, it made Nutribest look much better than it really is.***

T: ***On the advertisement, the bars for the Nutribest graph are wider than the bars for Brand X. Let's look at another trick advertisers often use when making bar graphs.***

Assign half of the class to do the following activity on Worksheet P8(b) and the other half to do the same activity on Worksheet P8(c).

T: ***Draw a bar graph for the amount of thiamin, iron, and vitamin A in both Nutribest and Brand X.***

After a while, pair students to trade and compare Worksheets P8(b) and (c).

T: ***What do you notice about the two graphs?***

S: ***On one graph, Nutribest appears to have much more thiamine, iron, and vitamin A. On the other graph, Nutribest appears to have only a little more of each item.***

T: ***Why?***

S: ***The scales on the two graphs are different.***

S: ***One graph starts at 0%, the other starts at 20%.***

T: ***Are both graphs correct?***

S: ***Yes, although they look very different.***

T: ***Which company, Nutribest or Brand X, is more likely to use the graph on Worksheet P8(b)? On Worksheet P8(c)?***

S: ***The graph on Worksheet P8(b) makes Nutribest look better.***

S: ***Brand X might prefer the graph on Worksheet P8(c) since the two cereals appear about the same. But Brand X would probably use different vitamins or minerals on a graph.***

Exercise 4

Write the following information on the board. Make calculators available.

300 grams of Nutribest: $1.38
400 grams of Brand X: $1.60

T: ***This information gives the weight of the boxes of cereal with the prices in the advertisement. Which is the better buy?***

S: ***A box of Nutribest costs less, but you get less cereal.***

T: ***Try to calculate which cereal is cheaper.***

S: ***Brand X is a better buy. 400 grams of Brand X cost $1.60 so 100 grams cost $1.60 ÷ 4 or $0.40. 300 grams of Nutribest cost $1.38 so 100 grams cost $1.38 ÷ 3 or $0.46.***

S: ***Brand X is cheaper. Three boxes of Brand X weight the same as four boxes of Nutribest. But three boxes of Brand X cost 3 x $1.60 or $4.80, whereas four boxes of Nutribest cost 4 x $1.38 or $5.52.***

S: ***One gram of Brand X costs $1.60 ÷ 400 or $0.004, while one gram of Nutribest costs $1.38 ÷ 300 or $0.0046. So Brand X is cheaper.***

Worksheet P8(d) is available for individual work.

Exercise 5

Begin this table on the board.

People Interviewed	People who Preferred Nutribest
5	4
10	

T: ***When Nutribest advertises "four out of five people prefer Nutribest," they might mean that they questioned many people and that four out of every five people preferred Nutribest to Brand X. If so, how many out of ten people would prefer Nutribest?***

S: ***Eight. There are two groups of five and 2 x 4 = 8.***

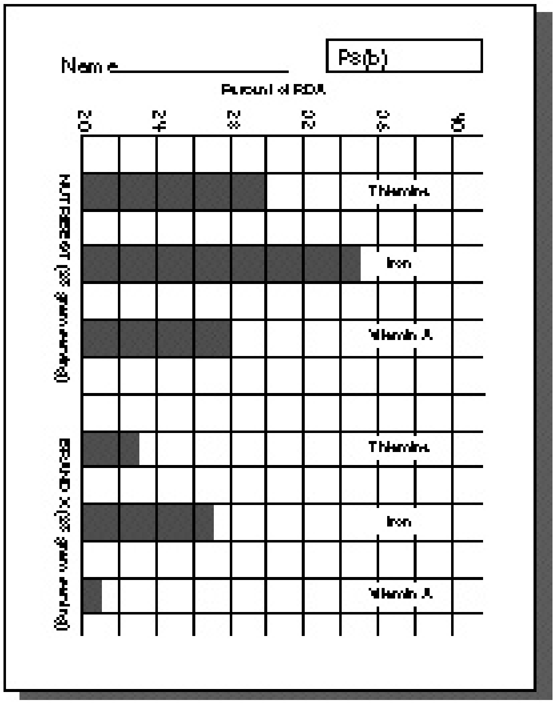
In a similar manner, continue to add several more lines to the table. (Answers are in boxes.)

People Interviewed	People who Preferred Nutribest
5	4
10	$\boxed{8}$
15	$\boxed{12}$
20	$\boxed{16}$
50	$\boxed{40}$
100	$\boxed{80}$
1000	$\boxed{800}$

T: ***If Nutribest really interviewed 1 000 people and 800 preferred Nutribest to Brand X, that is much more impressive than just four out of five people. Still, we should know who these 1000 people were.***

Extension Activity

As a class project, you may wish to ask students to look for misleading advertisements in newspapers and magazines and to share them with the class. Another project could be to collect nutrition information from cereal boxes and to compare the cereals.



Name ____________ P8(b)

Percent of RDA

Thiamine

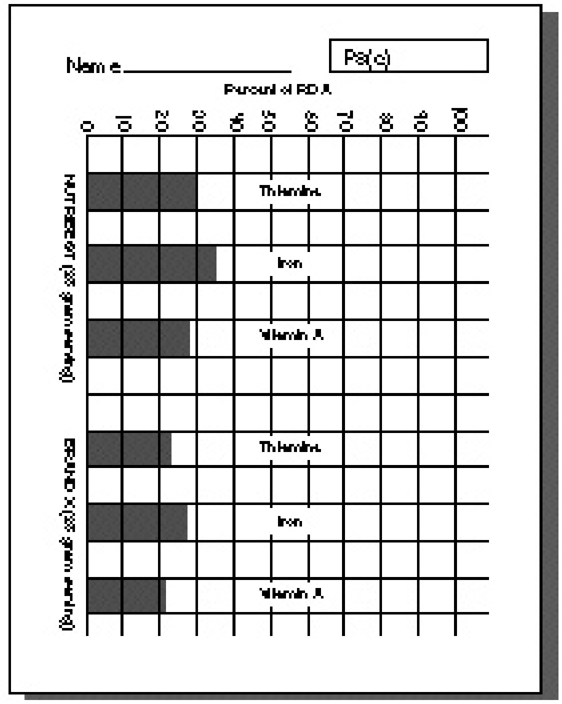
Iron

Vitamin A

Thiamine

Iron

Vitamin A



Name ____________ P8(c)

Percent of RDA

Thiamine

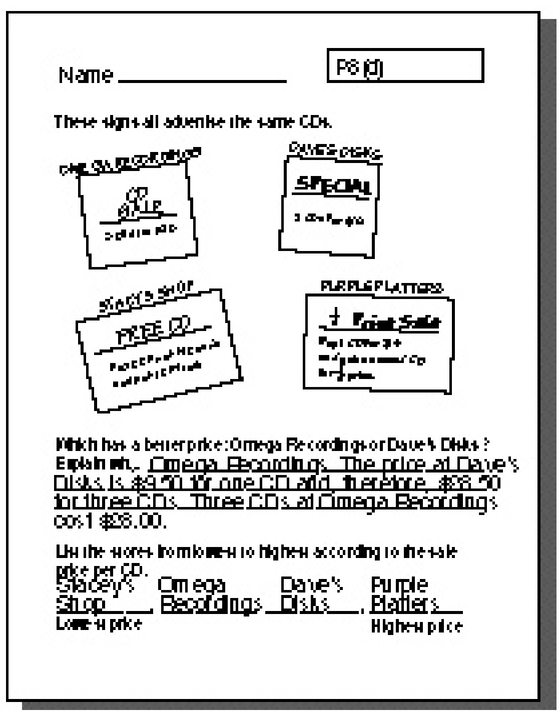
Iron

Vitamin A

Thiamine

Iron

Vitamin A



Name ____________ P8(d)

These signs all advertise the same CDs.

Which has a better price: Omega Recordings or Dave's Disks?
Explain why. Omega Recordings. The price at Dave's Disks is $9.50 for one CD and, therefore, $28.50 for three CDs. Three CDs at Omega Recordings cost $28.00.

List the stores from lowest to highest according to the sale price per CD.
Stacey's Shop, Omega Recordings, Dave's Disks, Purple Platters
Lowest price ... Highest price