CONTEMPORARY MOTIVATED
MATHEMATICS
BOOK 2
INDEX AND SELECTED ANSWERS

Solutions to many of the strictly computational problems are obvious and are not included here at present. We have given some explanations, comments and solutions for those problems which we have considered essential or unusual. If you should find that solutions to some other problems are desirable we would appreciate hearing from you.

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CHESTNUT HILL
MASSACHUSETTS
02167
CONTEMPORARY MOTIVATED MATHEMATICS

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Problem 29

All the numerals on the diagonals that lie within the 5 x 5 square should not be moved. Move each numeral outside of the square as follows:

a) Move 5 to the cell 5 units down in the same column. Similarly, move 21 5 units up; 1 5 units right; 25 5 units left.

b) Move 4, 10 5 units down; 16, 22 5 units up; 2, 6 5 units right; 20, 24 5 units left.

<table>
<thead>
<tr>
<th>3</th>
<th>16</th>
<th>9</th>
<th>22</th>
<th>15</th>
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</table>

Magic Constant 65

Problem 30

All the numerals that lie within the 7 x 7 magic square should not be moved. Move each numeral outside of the square 7 cells down, up, to the right, or to the left.

<table>
<thead>
<tr>
<th>4</th>
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Magic Constant 175

Problem 31

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<td>60</td>
<td>6</td>
<td>7</td>
<td>64</td>
<td></td>
</tr>
</tbody>
</table>

Page 8  Problem 39

\[
\begin{align*}
34 + 4 \times 16 &= 34 + 64 = 98 \\
34 + 6 \times 16 &= 34 + 96 = 130 \\
34 + 8 \times 16 &= 34 + 128 = 162 \\
34 + 10 \times 16 &= 34 + 160 = 194 \\
34 + 13 \times 16 &= 34 + 208 = 242
\end{align*}
\]

Notice that the multiples of 16 which correspond to the number of rows omitted are added to the magic constant 34 of a normal 4 x 4 magic square.

Page 9  Problem 46

\[
\begin{align*}
65 + 4 \times 25 &= 65 + 100 = 165 \\
65 + 6 \times 25 &= 65 + 150 = 215 \\
65 + 9 \times 25 &= 65 + 225 = 290 \\
65 + 10 \times 25 &= 65 + 250 = 315 \\
65 + 13 \times 25 &= 65 + 325 = 390
\end{align*}
\]

Notice that the multiples of 25 which correspond to the number of rows omitted are added to the magic constant 65 of a normal 5 x 5 magic square.

Page 13  Problem 64

Even ordered magic squares are more difficult to form than odd ordered magic squares. In this and the next problem, we have combined two steps into one. The procedure for finding the 6 x 6 magic square is described below.

1. Write the natural numbers in consecutive order in the 6 x 6 square.
Table 1

2. Keep the diagonals of the consecutive number square from Figure 1.

Figure 2

3. We cannot take directly the 37's complement of the other numbers in the square in Figure 1 since some pairs of numbers must first be interchanged. Thus, interchange from the original consecutive number square in Figure 1 the following pairs: 4 and 34, 12 and 30, 23 and 17, 24 and 19, 27 and 28, 32 and 35. The other numbers remain unchanged. The result is the square below, which appears on page 13 of the text.

Figure 3
4. Take the 37's complement of each non-diagonal number in figure 3 in order to obtain the 6 x 6 magic square.

\[
\begin{array}{ccccccc}
1 & 35 & 34 & 3 & 32 & 6 \\
30 & 8 & 28 & 27 & 11 & 7 \\
24 & 23 & 15 & 16 & 14 & 19 \\
13 & 17 & 21 & 22 & 20 & 18 \\
12 & 26 & 9 & 10 & 29 & 25 \\
31 & 2 & 4 & 33 & 5 & 36 \\
\end{array}
\]

Page 13  Problem 65

As in the case of problem 64 so here, we cannot form the 8 x 8 magic square directly. Proceed as explained below.

1. Write the natural numbers in consecutive order in the 8 x 8 square.

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 \\
25 & 26 & 27 & 28 & 29 & 30 & 31 & 32 \\
33 & 34 & 35 & 36 & 37 & 38 & 39 & 40 \\
41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 \\
49 & 50 & 51 & 52 & 53 & 54 & 55 & 56 \\
57 & 58 & 59 & 60 & 61 & 62 & 63 & 64 \\
\end{array}
\]

Figure 1

2. Keep the diagonals of the consecutive number square from figure 1.

\[
\begin{array}{cccc}
1 & 8 \\
10 & 15 \\
19 & 22 \\
28 & 29 \\
36 & 37 \\
43 & 46 \\
50 & 55 \\
57 & 64 \\
\end{array}
\]

Figure 2
3. We cannot take directly the 65's complement of the other numbers in the square in figure 1 since some pairs of numbers must first be interchanged. Thus, interchange from the original consecutive number square in figure 1 the following pairs: 4 and 61, 5 and 60, 11 and 54, 14 and 51, 18 and 47, 23 and 42, 25 and 40, 32 and 33. The other numbers remain unchanged. The result is the square below which appears on page 13 of the text book.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>61</th>
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<td>4</td>
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</table>

Figure 3

4. Take the 65's complement of each non-diagonal number in figure 3 to obtain the 8 x 8 magic square.

<table>
<thead>
<tr>
<th>1</th>
<th>63</th>
<th>62</th>
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<td>61</td>
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<td>64</td>
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</table>

This pattern always works for any three digit number abc. The procedure is expressed by

\[
\begin{align*}
(5(2a+5)+b) & \quad 10+c) - 250 \\
= (10a+25+b) & \quad 10+c) - 250 \\
= 100a + 250 + 10b + c - 250 \\
= 100a + 10b + c, \text{ the expanded notation for abc.}
\end{align*}
\]
January

Age 100

1
1 x 10 = 10
10 + 20 = 30
30 x 10 = 300
300 + 100 = 400
400 + 165 = 565
565 - 365 = 200

Dini Dunit's Magic 2 does not work in this case.
The pattern works for any age through 99. The procedure is expressed by

\[ mn \]
\[ mn0 \]
\[ mn0 + 20 \]
\[ mn00 + 200 \]
\[ mn00 + 200 + ab \]
\[ mn00 + 200 + ab + 165 \]
\[ mn00 + ab \]

But

\[ mn00 + ab = mn \ ab \]

If the person's age has 3 digits, the first digit of the age would be added to the last digit of the month and no longer would the first 2 digits be the month and the second 2 digits be the age.

<table>
<thead>
<tr>
<th>Sum of block of 9 numbers is</th>
<th>Divide sum by 9</th>
<th>Subtract 8</th>
<th>The block of 9 numbers on the calendar is</th>
</tr>
</thead>
<tbody>
<tr>
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<td>12</td>
<td>12 13 14 19 20 21 26 27 28</td>
</tr>
</tbody>
</table>

<table>
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<th>Sum of block of 9 numbers is</th>
<th>Divide sum by 9</th>
<th>Subtract 8</th>
<th>The block of 9 numbers on the calendar is</th>
</tr>
</thead>
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<tr>
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<td>162</td>
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<td>10</td>
<td>10 11 12 17 18 19 24 25 26</td>
</tr>
</tbody>
</table>
Page 22  Problem 89

The number 6174 is known as Kaprekar's constant after the mathematician who discovered the pattern in 1946.

Number of Subtractions

<table>
<thead>
<tr>
<th>Number</th>
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<td>8991</td>
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<tr>
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</tbody>
</table>

Page 23  Problem 90

Number of Subtractions

<table>
<thead>
<tr>
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<tbody>
<tr>
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<tr>
<td>1709</td>
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</tr>
</tbody>
</table>

Page 23  Text Question

If Dini Dunit's procedure is used on any 4 digit number where the digits are not all the same, the number 6174 will appear after at most 7 subtractions.

Page 24  Problem 91

\[
\begin{align*}
3^2 &= 9 \\
33^2 &= 1089 \\
333^2 &= 110889 \\
3333^2 &= 11110888889
\end{align*}
\]

Page 26  Problem 92

\[
\begin{align*}
9^2 &= 81 \\
99^2 &= 9801 \\
999^2 &= 998001 \\
9999^2 &= 99980001
\end{align*}
\]

Page 26  Problem 93

From 1 through 100 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 22, 33, 44, 55, 66, 77, 88, 99 Total Number is 19.

From 101 through 200 101, 111, 121, 131, 141, 151, 161, 171, 181, 191 Total Number is 10.

From 201 through 300 202, 212, 222, 232, 242, 252, 262, 272, 282, 292 Total Number is 10.
### Problem 94

**From 1 through 100**

- 2, 3, 5, 7, 11

Total Number is 5.

**From 101 through 200**

- 101, 131, 151, 181, 191

Total Number is 5.

**From 201 through 300**

There are none.

Total Number is 0.

**From 301 through 400**

- 313, 353, 373, 383

Total Number is 4.

**From 401 through 500**

There are none.

### Problem 99

#### Palindrome

<table>
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<tbody>
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<tr>
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<td>505</td>
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<td>Nicolas Copernicus</td>
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</table>

#### Number of Reversals

- Euclid: 1
- Archimedes: 4
- Eratosthenes of Cyrene: 2
- Hero of Alexandria: 1
- Ptolemy of Alexandria: 2
- Diophantus: 2
- Hypatia of Alexandria: 15
- Al-Khwarizmi: 2
- Mahavira: 3
- Abraham Ben Ezra: 1
- Omar Khayyam: 1
- Bhaskara: 1
- Fibonacci: 1
- Thomas Bradwardine: 2
- Regiomontanus: 1
- Leonardo da Vinci: 1
- Nicolas Copernicus: 2

### Problem 100

#### Starting number

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<th>Resulting number</th>
<th>Palindrome</th>
<th>Number of reversals</th>
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<td>101101</td>
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<tr>
<td>Base ten</td>
<td>Palindrome</td>
<td>Number of reversals</td>
<td>Base two</td>
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<td>------------</td>
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* Some students may say 0 reversals here since the given number 11011 is a palindrome.
CMM Book 2

Selected Answers

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* Some students may say 0 reversals here since the given number is a palindrome.

Page 36  Problem 109

\[
\frac{22 \times 22}{1 + 2 + 1} = \frac{22 \times 22}{2^2} = 11 \times 11 = 121
\]

\[
\frac{333 \times 333}{1 + 2 + 3 + 2 + 1} = \frac{333 \times 333}{3^2} = 111 \times 111 = 12,321
\]

\[
\frac{4444 \times 4444}{1 + 2 + 3 + 4 + 3 + 2 + 1} = 1,111 \times 1,111 = 1,234,321
\]

\[
\frac{55,555 \times 55,555}{1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1} = 11,111 \times 11,111 = 123,454,321
\]

\[
\frac{666,666 \times 666,666}{1 + 2 + 3 + 4 + 5 + 6 + 5 + 4 + 3 + 2 + 1} = 111,111 \times 111,111 = 12,345,654,321
\]

\[
\frac{7,777,777 \times 7,777,777}{1 + 2 + 3 + 4 + 5 + 6 + 7 + 6 + 5 + 4 + 3 + 2 + 1} = 1,111,111 \times 1,111,111 = 1,234,567,654,321
\]

Page 37  Problem 110

\[
\frac{1 + (2 \times 3) + 4}{6} = \frac{1 + (2 \times 3) + 4}{6} = 6
\]

\[
\frac{[1 + (2 + 3) - 3] + [4 + 5 + 6]}{7} = \frac{8}{7} = 8
\]

\[
1 + (2 - 3) - (4 - 5) = 7
\]

\[
7 = [6 + (5 - 4)] + [(3 - 2) - 1]
\]

Page 38  Problem 111

\[
\begin{array}{c|c|c}
3 & 2 & 5 \\
5 & 4 & 28 \\
2 & 3 & 5 \\
16 & 7 & 10 \\
3 & 5 & 28 \\
16 & 35 & 72 \\
2 & 15 & 8 \\
15 & 8 & \\
\end{array}
\]

Page 38  Problem 112

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<tr>
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</tr>
</tbody>
</table>

Page 38  Problem 113

Smallest

\[
((1 + 2) + 3) + 4 = \frac{1}{24}
\]

Largest

\[
1 + ((2 + 3) + 4) = 6
\]
\[((1 + 2) \times 3) + 4) + 5 = \frac{1}{120} \quad \quad 1 + (((2 + 3) + 4) \times 5) = 30 \]
\[((1 + 2) + 3) + 4) + 5) \times 6 = \frac{1}{720} \quad \quad 1 + (((2 + 3) + 4) + 5) + 6) = 180 \]

Page 39  Text Comment

Some instructions for the Four Fours problem include the use of \( \frac{4}{4} \) or \( .4 \) where
\( .4 = .\overline{4} = .444444... = \frac{4}{9} \). If any of the students inquire about this you may suggest that they can use it. However, it is not needed for the numbers that are given here.

Page 40  Problem 114

5 = \frac{4}{4} + (\sqrt{4} + \sqrt{4}) \]
6 = (4 + \sqrt{4}) \times \frac{4}{4} \]
7 = 4 + 4 - \frac{4}{4} \]
8 = (4 + 4) \times \frac{4}{4} \]
9 = 4 + 4 + \frac{4}{4} \]
10 = (4 \times 4) - (4 + \sqrt{4}) \]
11 = \frac{4!}{\sqrt{4}} - \frac{4}{4} \]
12 = 4 + 4 + \sqrt{4} + \sqrt{4} \]
13 = \frac{4!}{\sqrt{4}} + \frac{4}{4} \]
14 = 4 \times 4 - \frac{4}{\sqrt{4}} \]
15 = 4 \times 4 - \frac{4}{4} \]
16 = 4 + 4 + 4 + 4 \]
17 = 4 \times 4 + \frac{4}{4} \]
18 = 4 \times 4 + \frac{4}{\sqrt{4}} \]
19 = 4! - 4 - \frac{4}{4} \]
20 = 4! - \frac{4 \times 4}{4} \]
21 = 4! - \sqrt{4} - \frac{4}{4} \]
22 = (4! - \sqrt{4}) \times \frac{4}{4} \]
23 = 4! - \frac{\sqrt{4} \times \sqrt{4}}{4} \]
24 = 4! \times \left(\frac{\sqrt{4} + \sqrt{4}}{4}\right) \]
25 = \frac{(4! \times 4) + 4}{4} \]
26 = (4! + \sqrt{4}) \times \frac{4}{4} \]
27 = 4! + \sqrt{4} + \frac{4}{4} \]
28 = (4! + 4) \times \frac{4}{4} \]
29 = 4! + 4 + \frac{4}{4} \]
30 = 4! + 4 + \frac{4}{\sqrt{4}} \]
31 = 4! + \frac{4! + 4}{4} \]
32 = 4! + \frac{4 \times 4}{\sqrt{4}} \]

Page 41 Problem 115

5 = (1 + 2 + 9) - 7 \]
6 = 9 - 2 - 1 \]
7 = 7! + (9 - 2 - 1)! \]
8 = 72 + (1 \times 9) \]
CMM Book 2

Selected Answers

\[
9 = 9 \times (2 - 1)^7 \\
10 = \left[ 9 \div (2 + 1) \right] + 7 \\
11 = \left[ (\sqrt{9} - 1) \times 2 \right] + 7 \\
12 = 29 - 17 \\
13 = (9 + 7) - (2 + 1)
\]

\[
14 = 7 \times 2 \times 1^9 \\
15 = 1 \times (7 - 2) \times \sqrt{9} \\
16 = (9 + 7) \times (2 - 1) \\
17 = (9 + 7) + (2 - 1) \\
18 = \left[ (9 + 7) + 2 \right] \times 1
\]

Here \( .1, .9, .7, .2, .19, .17 \) and so on are allowed. Also some students may want to use

\[
\overline{1} = .1 = .1111 \ldots = \frac{1}{9} \\
\overline{9} = .9 = .9999 \ldots = 1 \\
\overline{7} = .7 = .7777 \ldots = \frac{7}{9} \\
\overline{2} = .2 = .2222 \ldots = \frac{2}{9}
\]

which although acceptable are not necessary for the given numbers.

Page 44  Problem 118

\[
56 \times 79 = 4,424
\]

\[
47 \times 576 = 27,072
\]

\[
197 \times 596 = 117,412
\]

\[
86 \times 97 = 8,342
\]

\[
63 \times 689 = 43,407
\]

\[
857 \times 968 = 829,576
\]
CMM Book 2

Problem 119

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Problem 124

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Problem 125

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Problem 126

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</tbody>
</table>
Page 51  Problem 129
\[ 1^3 + 2^3 + 3^3 + 4^3 + \ldots + n^3 = (1 + 2 + 3 + 4 + \ldots + n)^2 \]

Page 52  Problem 130
Each term of the Fibonacci sequence, except the first two, is the sum of the preceding two Fibonacci numbers.

Page 52  Problem 131
Each term of the Lucas sequence, except the first two, is the sum of the preceding two Lucas numbers.

Page 52  Problem 132
\[
\begin{align*}
L_4 &= 7 & L_9 &= 76 \\
L_5 &= 11 & L_{10} &= 123 \\
L_6 &= 18 & L_{11} &= 199 \\
L_7 &= 29 & L_{12} &= 322 \\
L_8 &= 47 \\
\end{align*}
\]

Page 53  Problem 134
\[
1 + 3 + 4 + 7 + 11 + \ldots + L_n = \frac{L_{n+2} - 3}{2}
\]

Page 53  Problem 135
\[
\begin{align*}
\frac{L_4}{L_3} &= \frac{7}{4} = 1.75000 \\
\frac{L_{19}}{L_{18}} &= \frac{9349}{5778} = 1.61803 \\
\frac{L_5}{L_4} &= \frac{11}{7} = 1.57142 \\
\frac{L_{20}}{L_{19}} &= \frac{15127}{9349} = 1.61803 \\
\frac{L_{18}}{L_{17}} &= \frac{5778}{3571} = 1.61803 \\
\frac{L_{25}}{L_{24}} &= \frac{167761}{103682} = 1.61803 \\
\end{align*}
\]

Page 54  Problem 136

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<th>Sum of proper divisors</th>
<th>Amicable numbers</th>
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Page 56  Problem 140

Each pentagonal number except the first is the sum of a triangular number and a square number.

Page 57  Problem 143

Each hexagonal number except the first is the sum of a square number and twice a triangular number.

Page 59  Problem 144

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<th>Happy number day</th>
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<td>23 + 5 + 4 + 14 + 5 + 19 + 4 + 1 + 25 = 100</td>
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<td>THURSDAY</td>
<td>20 + 8 + 21 + 18 + 19 + 4 + 1 + 25 = 116</td>
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<tr>
<td>FRIDAY</td>
<td>6 + 18 + 9 + 4 + 1 + 25 = 63</td>
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<td>SATURDAY</td>
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</tr>
</tbody>
</table>
Page 59  Problem 145

<table>
<thead>
<tr>
<th>Month</th>
<th>Sum of numbers associated with the letters in the word</th>
<th>Happy number month</th>
</tr>
</thead>
<tbody>
<tr>
<td>JANUARY</td>
<td>10 + 1 + 14 + 21 + 1 + 18 + 25 = 90</td>
<td>Yes</td>
</tr>
<tr>
<td>FEBRUARY</td>
<td>6 + 5 + 2 + 18 + 21 + 1 + 18 + 25 = 96</td>
<td>Yes</td>
</tr>
<tr>
<td>MARCH</td>
<td>13 + 1 + 18 + 3 + 8 = 43</td>
<td></td>
</tr>
<tr>
<td>APRIL</td>
<td>1 + 16 + 18 + 9 + 12 = 56</td>
<td></td>
</tr>
<tr>
<td>MAY</td>
<td>13 + 1 + 25 = 39</td>
<td></td>
</tr>
<tr>
<td>JUNE</td>
<td>10 + 21 + 14 + 5 = 50</td>
<td></td>
</tr>
<tr>
<td>JULY</td>
<td>10 + 21 + 12 + 25 = 68</td>
<td></td>
</tr>
<tr>
<td>AUGUST</td>
<td>1 + 21 + 7 + 21 + 19 + 20 = 89</td>
<td></td>
</tr>
<tr>
<td>SEPTEMBER</td>
<td>19 + 5 + 16 + 20 + 5 + 13 + 2 + 5 + 18 = 103</td>
<td></td>
</tr>
<tr>
<td>OCTOBER</td>
<td>15 + 3 + 20 + 15 + 2 + 5 + 18 = 78</td>
<td></td>
</tr>
<tr>
<td>NOVEMBER</td>
<td>14 + 15 + 22 + 5 + 13 + 2 + 5 + 18 = 94</td>
<td></td>
</tr>
<tr>
<td>DECEMBER</td>
<td>4 + 5 + 3 + 5 + 13 + 2 + 5 + 18 = 55</td>
<td></td>
</tr>
</tbody>
</table>

Page 60  Problem 148
The entries in the columns marked C are all the exact divisors of the particular composite number heading the column.

Page 61  Problem 150
Except for the prime numbers 2 and 3, all other prime numbers are in the first column and in the fifth column.

Page 61  Problem 151
Letting n take on the values 1, 2, 3, 4, 5 and so on, each prime number except 2 and 3 can be expressed as 6n + 1 or 6n - 1.

Page 62  Problem 154
Sum of pair of twin primes (each prime > 3) is divisible exactly by 12.

Page 63  Problem 155

<table>
<thead>
<tr>
<th>To find a</th>
<th>Take n</th>
<th>2n is</th>
<th>List one prime</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 digit prime</td>
<td>5 through 49</td>
<td>10 through 98</td>
<td>11</td>
</tr>
<tr>
<td>3 digit prime</td>
<td>50 through 499</td>
<td>100 through 998</td>
<td>101</td>
</tr>
</tbody>
</table>

Page 63  Problem 156

<table>
<thead>
<tr>
<th>n</th>
<th>2(n - 1)</th>
<th>Prime equal to n or between n and 2(n - 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>10</td>
<td>18</td>
<td>11</td>
</tr>
<tr>
<td>17</td>
<td>32</td>
<td>17</td>
</tr>
<tr>
<td>21</td>
<td>40</td>
<td>23</td>
</tr>
<tr>
<td>24</td>
<td>46</td>
<td>29</td>
</tr>
<tr>
<td>28</td>
<td>54</td>
<td>29</td>
</tr>
</tbody>
</table>

Problem 157

<table>
<thead>
<tr>
<th>n</th>
<th>2(n - 1)</th>
<th>Prime equal to n or between n and 2(n - 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>39</td>
<td>76</td>
<td>41</td>
</tr>
<tr>
<td>41</td>
<td>80</td>
<td>41</td>
</tr>
<tr>
<td>45</td>
<td>88</td>
<td>47</td>
</tr>
<tr>
<td>50</td>
<td>98</td>
<td>53</td>
</tr>
<tr>
<td>58</td>
<td>114</td>
<td>59</td>
</tr>
<tr>
<td>63</td>
<td>124</td>
<td>67</td>
</tr>
</tbody>
</table>
Page 70  Problem 172

A natural number $N$ is divisible exactly by 3 if the sum of the numbers represented by the digits in $N$ is exactly divisible by 3.

Page 70  Problem 173

A natural number $N$ is divisible exactly by 9 if the sum of the numbers represented by the digits in $N$ is exactly divisible by 9.

Page 71  Problem 175

A natural number $N$ is divisible exactly by 4 if the number represented by the last two digits of $N$ is divisible exactly by 4.

Page 76  Problem 186

\[ 72 = 2^3 \cdot 3^2 \]
\[ D_{72} = \{ 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72 \} \]

\[ 108 = 2^2 \cdot 3^3 \]
\[ D_{108} = \{ 1, 2, 3, 4, 6, 9, 12, 18, 27, 36, 54, 108 \} \]

\[ 1000 = 2^3 \cdot 5^3 \]
\[ D_{1000} = \{ 1, 2, 4, 5, 8, 10, 20, 25, 40, 50, 100, 125, 200, 250, 500, 1000 \} \]

\[ 3375 = 3^3 \cdot 5^3 \]
\[ D_{3375} = \{ 1, 3, 5, 9, 15, 25, 27, 45, 75, 125, 135, 225, 375, 675, 1125, 3375 \} \]
70 = 2·5·7
\[ D_{70} = \{ 1, 2, 5, 7, 10, 14, 35, 70 \} \]

105 = 3·5·7
\[ D_{105} = \{ 1, 3, 5, 7, 15, 21, 35, 105 \} \]

110 = 2·5·11
\[ D_{110} = \{ 1, 2, 5, 10, 11, 22, 55, 110 \} \]

231 = 3·7·11
\[ D_{231} = \{ 1, 3, 7, 11, 21, 33, 77, 231 \} \]

385 = 5·7·11
\[ D_{385} = \{ 1, 5, 7, 11, 35, 55, 77, 385 \} \]

715 = 5·11·13
\[ D_{715} = \{ 1, 5, 11, 13, 55, 65, 143, 715 \} \]
Page 79  Problem 188

$120 = 2^3 \cdot 3 \cdot 5$

$D_{120} = \{ 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120 \}$

$180 = 2^2 \cdot 3^2 \cdot 5$

$D_{180} = \{ 1, 2, 3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 30, 36, 45, 60, 90, 180 \}$

$360 = 2^3 \cdot 3^2 \cdot 5$

$D_{360} = \{ 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, 360 \}$
$900 = 2^2 \times 3^2 \times 5^2$

$D_{900} = \{ 1, 2, 3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 25, 30, 36, 45, 50, 60, 75, 90, 100, 150, 180, 225, 300, 450, 900 \}$

Page 81  Problem 190

Points 3
Lines 3

Points 5
Lines 10

Points 4
Lines 6

Points 6
Lines 15

Points $\frac{n \cdot n}{2}$
Lines $\frac{n(n - 1)}{2}$
### Problem 191

<table>
<thead>
<tr>
<th>Number of unit cubes forming the large cube</th>
<th>Unit cubes with 3 faces painted</th>
<th>Unit cubes with 2 faces painted</th>
<th>Unit cubes with 1 face painted</th>
<th>Unit cubes with 0 faces painted</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3^3 = 27$</td>
<td>8</td>
<td>12</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>$4^3 = 64$</td>
<td>8</td>
<td>24</td>
<td>24</td>
<td>8</td>
</tr>
<tr>
<td>$5^3 = 125$</td>
<td>8</td>
<td>36</td>
<td>54</td>
<td>27</td>
</tr>
<tr>
<td>$6^3 = 216$</td>
<td>8</td>
<td>48</td>
<td>96</td>
<td>64</td>
</tr>
<tr>
<td>$n^3$</td>
<td>$2^3$</td>
<td>$12(n - 2)$</td>
<td>$6(n - 2)^2$</td>
<td>$(n - 2)^3$</td>
</tr>
</tbody>
</table>

$n = 2, 3, 4, \ldots$

### Problem 192

2 x 3

- XabgY
- XaefgY
- XabfgY
- XaejY
- XdfjY
- XeijY
- XdijY
- XefjY
- XeijY
- XdeijY

Total: 10
The tree diagram can be drawn on the above pattern.

Total 35.

Page 84  Problem 193

The tree diagram can be drawn on the above pattern.

Total 6.

The tree diagram can be drawn on the above pattern.

Total 20.
The tree diagram can be drawn on the above pattern.

<table>
<thead>
<tr>
<th>12</th>
<th>21</th>
<th>31</th>
<th>41</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>23</td>
<td>32</td>
<td>42</td>
</tr>
<tr>
<td>14</td>
<td>24</td>
<td>34</td>
<td>43</td>
</tr>
</tbody>
</table>

There are 12 2-digit numbers.

\[
\frac{n!}{(n-r)!} = \frac{4!}{(4-2)!} = 12
\]

<table>
<thead>
<tr>
<th>123</th>
<th>213</th>
<th>312</th>
<th>412</th>
</tr>
</thead>
<tbody>
<tr>
<td>124</td>
<td>214</td>
<td>314</td>
<td>413</td>
</tr>
<tr>
<td>134</td>
<td>231</td>
<td>321</td>
<td>421</td>
</tr>
<tr>
<td>132</td>
<td>234</td>
<td>324</td>
<td>423</td>
</tr>
<tr>
<td>142</td>
<td>241</td>
<td>341</td>
<td>431</td>
</tr>
<tr>
<td>143</td>
<td>243</td>
<td>342</td>
<td>432</td>
</tr>
</tbody>
</table>

There are 24 3-digit numbers.

\[
\frac{n!}{(n-r)!} = \frac{4!}{(4-3)!} = 24
\]

<table>
<thead>
<tr>
<th>1234</th>
<th>2134</th>
<th>3124</th>
<th>4123</th>
</tr>
</thead>
<tbody>
<tr>
<td>1243</td>
<td>2143</td>
<td>3142</td>
<td>4132</td>
</tr>
<tr>
<td>1324</td>
<td>2314</td>
<td>3214</td>
<td>4213</td>
</tr>
<tr>
<td>1342</td>
<td>2341</td>
<td>3241</td>
<td>4231</td>
</tr>
<tr>
<td>1423</td>
<td>2413</td>
<td>3412</td>
<td>4312</td>
</tr>
<tr>
<td>1432</td>
<td>2431</td>
<td>3421</td>
<td>4321</td>
</tr>
</tbody>
</table>

There are 24 4-digit numbers.

\[
\frac{n!}{(n-r)!} = \frac{4!}{(4-4)!} = 24
\]
Problem 195

Pentagon with A, B, C, D, E
n = 5, r = 5

\[ \frac{n!}{(n-r)!} = \frac{5!}{(5-5)!} = 120 \]

Hexagon with A, B, C, D, E, F
n = 6, r = 6

\[ \frac{n!}{(n-r)!} = \frac{6!}{(6-6)!} = 720 \]

Problem 196

Number of different numbers

| Begin with | 1 | 4 | 5 | - | - | 2 |
| Begin with | 1 | 4 | - | - | - | 6 |
| Begin with | 1 | - | - | - | - | 24 |
| End with   | - | - | - | 1 | - | 24 |
| End with   | - | - | 4 | 1 | - | 6 |
| End with   | - | 5 | 4 | 1 | - | 2 |
| End with   | - | 8 | 5 | 4 | 1 | 1 |
| Like this  | - | 6 | 8 | 1 | - | 2 |
| Or this    | - | 4 | 5 | - | - | 6 |
| Or this    | - | 5 | - | - | - | 24 |
| Or this    | - | - | 9 | - | - | 24 |

Problem 197

Take 1 2 3 4 5

<table>
<thead>
<tr>
<th>1 2 3 4 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 2 3 4 5</td>
</tr>
<tr>
<td>5 1 4 2 3</td>
</tr>
<tr>
<td>3 5 2 1 4</td>
</tr>
<tr>
<td>4 3 1 5 2</td>
</tr>
<tr>
<td>2 4 5 3 1</td>
</tr>
<tr>
<td>1 2 3 4 5</td>
</tr>
</tbody>
</table>

5 orbits

Take 1 2 3 4 5 6

<table>
<thead>
<tr>
<th>1 2 3 4 5 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 2 3 4 5 6</td>
</tr>
<tr>
<td>6 1 5 2 4 3</td>
</tr>
<tr>
<td>3 6 4 1 2 5</td>
</tr>
<tr>
<td>5 3 2 6 1 4</td>
</tr>
<tr>
<td>4 5 1 3 6 2</td>
</tr>
<tr>
<td>2 4 6 5 3 1</td>
</tr>
<tr>
<td>1 2 3 4 5 6</td>
</tr>
</tbody>
</table>

6 orbits

Take 1 2 3 4 5 6 7

<table>
<thead>
<tr>
<th>1 2 3 4 5 6 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 2 3 4 5 6 7</td>
</tr>
<tr>
<td>7 1 6 2 5 3 4</td>
</tr>
<tr>
<td>4 7 3 1 5 6 2</td>
</tr>
<tr>
<td>2 4 6 7 5 3 1</td>
</tr>
<tr>
<td>1 2 3 4 5 6 7</td>
</tr>
</tbody>
</table>

4 orbits

Take 1 2 3 4 5 6 7 8

<table>
<thead>
<tr>
<th>1 2 3 4 5 6 7 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 2 3 4 5 6 7 8</td>
</tr>
<tr>
<td>8 1 7 2 6 3 5 4</td>
</tr>
<tr>
<td>4 8 5 1 3 7 6 2</td>
</tr>
<tr>
<td>2 4 6 8 7 5 3 1</td>
</tr>
<tr>
<td>1 2 3 4 5 6 7 8</td>
</tr>
</tbody>
</table>

4 orbits
CMM Book 2

Take 1 2 3 4 5 6 7 8 9

1 2 3 4 5 6 7 8 9

1st orbit 9 1 8 2 7 3 6 4 5
5 9 4 1 6 8 3 2 7
7 5 2 9 3 4 8 1 6
6 7 1 5 8 2 4 9 3
3 6 9 7 4 1 2 5 8
8 3 5 6 2 9 1 7 4
4 8 7 3 1 5 9 6 2
2 4 6 8 9 7 5 3 1
1 2 3 4 5 6 7 8 9

9 orbits

Page 91  Problem 198

2 x 5

\[
\begin{array}{c|c}
2 & 5 \\
\hline
\end{array}
\]

\[
\begin{array}{c|c}
2 & R \ 1 \\
\hline
two & 2 \ R \ 0 \\
\end{array}
\]

\[
\begin{array}{c|c}
2 \times 2 & \text{two} \\
1 \times 1 & \\
\end{array}
\]

\[
\begin{array}{c|c}
3 \times 10 & \\
\hline
3 & R \ 1 \\
\hline
three & 3 \ R \ 0 \\
\end{array}
\]

\[
\begin{array}{c|c}
3 \times 3 & \text{three} \\
1 \times 1 & \\
\end{array}
\]

Page 92  Problem 199

Problem 200

Problem 201
Problem 208

\[
\begin{align*}
T & \ 4 \quad U \ 6 \\
Y & \ 2 \quad E \ 1 \\
S & \ 3 \quad A \ 5 \\
R & \ 0 \quad V \ 7 \\
& \ 8 \\
E & \ 4 \quad N \ 5 \\
R & \ 0 \quad I \ 6 \\
V & \ 7 \quad O \ 3 \\
U & \ 2 \quad F \ 8 \\
& \ 4 \\
H & \ 4 \quad U \ 2 \\
O & \ 3 \quad T \ 9 \\
N & \ 1 \quad M \ 8 \\
C & \ 6 \quad S \ 7 \\
& \ 5 \\
N & \ 0 \quad A \ 5 \\
I & \ 9 \quad H \ 7 \\
E & \ 6 \quad S \ 8 \\
U & \ 3 \quad R \ 2 \\
& \ 1 \\
\end{align*}
\]

Selected Answers

\[
\begin{align*}
M & \ 8 \quad I \ 1 \\
T & \ 6 \quad E \ 0 \\
N & \ 2 \quad W \ 7 \\
A & \ 3 \quad C \ 5 \\
& \ 4 \\
N & \ 0 \quad C \ 3 \\
E & \ 2 \quad A \ 9 \\
T & \ 8 \quad S \ 7 \\
O & \ 6 \quad M \ 5 \\
& \ 4 \\
H & \ 1 \\
T & \ 0 \quad I \ 2 \\
R & \ 6 \quad U \ 4 \\
N & \ 8 \quad P \ 7 \\
A & \ 5 \quad Q \ 1 \\
& \ 3 \\
Y & \ 4 \quad A \ 3 \\
E & \ 2 \quad U \ 6 \\
R & \ 5 \quad W \ 9 \\
T & \ 0 \quad L \ 7 \\
& \ 0 \\
\end{align*}
\]

Page 96

Problem 209

\[
\begin{align*}
(1 + 2) \times 3 &= 9 \\
(1 \div 2) + 3 &= 1 \\
(1 \div 2) \times 3 &= \frac{3}{8} \\
(1 \div 2) + 3 &= \frac{1}{3} \\
(1 \times 2) + 3 &= 5
\end{align*}
\]

Page 96

Problem 210

\[
\begin{align*}
[(1 + 2) - 3] \times 4 &= 0 \\
[(1 + 2) - 3] \div 4 &= 0 \\
[(1 + 2) \times 3] + 4 &= \frac{9}{4} \\
[(1 + 2) - 3] \times 4 &= 8 \\
[(1 - 2) + 3] \div 4 &= \frac{1}{2} \\
[(1 - 2) \times 3] + 4 &= 1 \\
[(1 + 2) \times 3] - 4 &= \frac{5}{4} \\
[(1 + 2) \div 3] - 4 &= \frac{3}{2} \\
[(1 + 2) \div 3] \times 4 &= \frac{1}{2} \\
[(1 - 2) + 3] \times 4 &= \frac{11}{3} \\
[(1 - 2) \times 3] \times 4 &= \frac{1}{3}
\end{align*}
\]
\[
\begin{align*}
  [(1 \times 2) + 3] - 4 &= \frac{1}{4} \\
  [(1 \times 2) + 3] ÷ 4 &= \frac{5}{4} \\
  [(1 \times 2) - 3] + 4 &= 3 \\
  [(1 \times 2) ÷ 3] + 4 &= \frac{4}{3} \\
  [(1 ÷ 2) + 3] - 4 &= -\frac{1}{2} \\
  [(1 ÷ 2) + 3] \times 4 &= 14 \\
  [(1 ÷ 2) - 3] + 4 &= \frac{3}{2} \\
  [(1 ÷ 2) ÷ 3] - 4 &= -\frac{1}{3} \\
  [(1 ÷ 2) - 3] \times 4 &\text{(S)} \quad 10 \\
  [(1 ÷ 2) ÷ 3] + 4 &= \frac{11}{2} \\
  [(1 ÷ 2) - 3] - 4 &= -\frac{5}{2}
\end{align*}
\]