A Model for Understanding Understanding in Mathematics

Although this article was written almost thirty years ago, its content is relevant to today’s conversations about the roles of conceptual understanding and procedural fluency in the teaching of mathematics.

Teachers strive to help students understand their subject. But what does it mean to understand something? Holt (1964), in trying to help teachers come to grips with understanding, offers the following:

It may help to have in our minds a picture of what we mean by understanding. I feel I understand something if and when I can do some, at least, of the following: (1) state it in my own words; (2) give examples of it; (3) recognize it in various guises and circumstances; (4) see connections between it and other facts or ideas; (5) make use of it in various ways; (6) foresee some of its consequences; (7) state its opposite or converse. This list is only a beginning; but it may help us in the future to find out what our students really know as opposed to what they can give the appearance of knowing, their real learning as opposed to their apparent learning. (pp. 136–37)

Holt’s operational definition of understanding is an attempt to guide teachers in planning instruction and in assessing learning. Holt feels his list is only a beginning.

Van Engen (1953, pp. 76–77) describes understanding as a process of organizing and integrating knowledge according to a set of criteria. But what are these criteria? Are the criteria the same for all kinds of knowledge?

Both Holt and Van Engen view understanding as a continuum. Students can have partial understandings. Surely teachers are helped when they know the status of a student’s understanding and the additional knowledge students need to acquire more complete understanding.

More recently, Smith (1969), Henderson (1971), and Cooney, Davis, and Henderson (1975) call attention to the kinds of logical things (moves) teachers do in teaching mathematics. It turns out, teachers make different kinds of moves in teaching different kinds of mathematical knowledge. Teachers behave differently, in a logical sense, when teaching a mathematical concept (e.g., place value, exponent, six, triangle) than when teaching children to become skilled at carrying out a procedure (e.g., long division, adding fractions, bisecting a line segment).

This article attempts to advance and demonstrate the proposition that moves in teaching mathematics can serve as a basis for defining understanding of mathematics. Since the moves are specific to teaching mathematics, it is hoped teachers will view them as ways to assess students’ knowledge and to plan instruction. Moves occur as a result of questions, exercises, problems, explanations, demonstrations,


A word on the editorial approach to reprinted articles: Obvious typographical errors have been silently corrected. Additions to the text for purposes of clarification appear in brackets. No effort has been made to reproduce the layouts or designs of the original articles, although the subheads are those that first appeared with the text. The use of words and phrases now considered outmoded, even slightly jarring to modern sensibilities, has likewise been maintained in an effort to give the reader a better feel for the era in which the articles were written.—Ed.
directions on task cards, and almost any other form of teacher-student interaction.

Moves also identify objectives that give a fairly complete picture of understanding. In addition to planning to make moves, teachers can use moves to evaluate and diagnose the degree of understanding held by a particular student for a given item of mathematical knowledge. Students can be said to have understanding to the extent that they can make a complete set of moves.

The work of Cooney, Davis, and Henderson has identified the following types of mathematical knowledge commonly taught in school mathematics: concepts, generalizations, procedures, and numerical facts. Understanding depends on the type of mathematical knowledge. Understanding a concept is different from understanding a procedure.

### Understanding Mathematical Concepts

CONCEPTS ARE FREQUENTLY DENOTED BY SINGLE TERMS OR PHRASES. **Addition, three-fourths, similar figures, equation, line, and greater than** are examples of mathematical concepts. Concepts are probably the most basic kind of subject matter in mathematics. Word lists at the end of a chapter and the index of a mathematics textbook are largely composed of concepts. In **figure 1**, the concept **prime number** will be used to illustrate the various moves.

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**Fig. 1**

<table>
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<tr>
<th>Level 1: Examples and nonexamples of the concept</th>
<th>Some sample questions to assess understanding of the concept prime number</th>
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<tr>
<td><strong>1. Give or identify examples of the concept</strong></td>
<td>1. Which of the following numbers are prime? 7, 9, 2, 17, 21? Can you find two prime numbers greater than 20 but less than 30?</td>
</tr>
<tr>
<td><strong>2. Defend choices of examples of the concept</strong></td>
<td>2. Why did you choose 17 as a prime number? Two is a prime number? I don't see how. Why do you think so?</td>
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<tr>
<td><strong>3. Give or identify nonexamples of the concept</strong></td>
<td>3. Two of these numbers are not prime, can you find them? 1, 2, 4, 5</td>
</tr>
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<td><strong>4. Defend choices of nonexamples of the concept</strong></td>
<td>4. You say 1 and 4 are not prime numbers. What are your reasons?</td>
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<th>Level 2: Characteristics of the concept</th>
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<tr>
<td><strong>5. Identify things that are necessarily true about examples of the concept</strong></td>
<td>5. Suppose I tell you that 97 is prime. What can you say for sure about 97? I have written a number on the back of this card. I promise it is a prime number. Can you tell me five things that are true about it?</td>
</tr>
<tr>
<td><strong>6. Determine properties sufficient to make something an example of the concept</strong></td>
<td>6. How can you tell for sure a number is prime? If a number is odd does it have to be prime? Why?</td>
</tr>
<tr>
<td><strong>7. Tell how one concept is like (or unlike) another concept</strong></td>
<td>7. What is true about both prime numbers and composite numbers? How are prime numbers different from composite numbers?</td>
</tr>
<tr>
<td><strong>8. Define the concept</strong></td>
<td>8. Who can tell me exactly what prime numbers are?</td>
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<tr>
<td><strong>9. Tell how we use the concept</strong></td>
<td>9. What can we do with prime numbers—how can they help us?</td>
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Understanding Mathematical Generalization

Statements of relationships between concepts are called generalizations. Most generalizations in school mathematics are statements we can show are true. Others are assumed to be true. Theorems are generalizations. The distributive law, the sum of the angles of a triangle is 180 degrees, and the sum of two odd numbers is an even number, are generalizations taught in elementary school mathematics. The last generalization is the focus of the questions in figure 2.

Level 1: Understanding what the generalization says

Children understand a generalization to the extent they can make the following moves:

1. Show understanding of the concepts in the generalization
2. State the generalization in their own words (paraphrase it)
3. Create or recognize instances of the generalization
4. Tell when the generalization can be used or state conditions under which it is true
5. Apply the generalization in exercises and simple problems

Some sample questions to assess understanding of the generalization, the sum of two odd numbers is an even number.

1. See concept moves and questions for even, odd, and sum
2. Can you say this in your own words? I don’t understand it as it is written in the book.
3. Can you show me two odd numbers whose sum is an even number?
4. When can you be sure the sum of two numbers will be an even number?
5. One of these sentences can’t be true. Can you find it without doing any adding?
   73 + 67 = 140
   37 + 121 = 158
   47 + 97 = 144
   73 + 125 = 195

Level 2: Understanding why the generalization is true

6. Give a plausible argument or proof why the generalization is true
7. Use physical or numerical examples to illustrate the rule
8. Recognize the applicability of the generalization in unfamiliar contexts

6. An odd number is one more than an even number. How can this help us to convince someone that two odd numbers add to make an even number?
7. Can you show me this generalization using only red Cuisenaire rods and white Cuisenaire rods? Remember, a train of all red rods will always be an even number.
8. There are 25 children in this room and everyone in line out in the hall except one child has a partner. If we all join the line outside could everyone have a partner?

Fig. 2

Understanding Mathematical Procedures

The elementary school mathematics curriculum includes many mathematical procedures. The algorithms for adding, subtracting, multiplying, and dividing whole numbers and rational numbers receive a great deal of instructional and learning time. The study of high school mathematics and college mathematics also contains many mathematical procedures. Solving equations, constructing proofs, geometric constructions, synthetic division, and differentiating are mathematical procedures. The distinction between Level 1 understandings and
Level 1: Understanding how the procedure works

Children understand a procedure to the extent they can make the following moves:

1. Accurately carry out the procedure
2. Show another student how to do the procedure
3. Paraphrase the procedure step by step
4. Tell when they can use the procedure
5. Execute the procedure rapidly and accurately
6. Understand prerequisite knowledge
7. Find errors in work

Some sample questions to assess understanding of the procedure for adding fractions

1. Can you add 2/3 and 3/4 for me?
2. Terry, you were here yesterday, will you show Al how to add these fractions?
3. Before you pick up your pencil, can you tell me everything you are going to do to add 5/3 and 4/5?
4. On which of these problems do we need to find a common denominator?
\[
\frac{2}{3} + \frac{4}{9} \quad \frac{3}{4} + \frac{7}{4}
\]
5. Can you work these two additions in only one minute?
6. See concept moves for: numerator, denominator, common denominator, fraction, adding, and equivalent fraction. The child would also need to understand the procedure for adding common fractions with the same denominator.
7. I made a mistake in my work. Can you find it?

Fig. 3

Level 2: Understanding why the procedure works

8. Show answers obtained as a result of the procedure are reasonable
9. Give a plausible argument or proof justifying the procedure
10. Recognize the applicability of the procedure in new contexts

8. You have an answer of 11/12 for 2/3 + 1/4. Can you use these Cuisenaire rods or the geoboard to show that 2/3 + 1/4 = 11/12 is true?
9. Let’s go over your work step by step. Can you convince me what is really happening in each step? Why do we find a common denominator?
10. Can you now add 3 2/3 and 5 1/4?

Understanding Number Facts

Many numerical facts are generalizations. Hence, moves for generalizations apply to number facts. However, since basic arithmetic facts such as 4 < 5, 6 = 6, 3 + 2 = 5, and 4 × 7 = 28 have played such a large role in school mathematics, an attempt is made to tailor moves for understanding generalizations specifically for number facts. The fact 5 + 4 = 9 is used in figure 4 in the sample questions.

Contexts for Making Moves

Bruner, Dienes, and others have been said to advise elementary school teachers to begin instruction with physical embodiments and progress to
Level 1: Understanding what the fact says

- Children understand a number fact to the extent they can make the following moves:
  1. Recall the fact
  2. Create or recognize embodiments of the fact
  3. Understand the concepts in the fact
  4. Use the fact in simple exercises
  5. Apply the fact

Some sample questions to assess understanding of the fact 5 + 4 = 9

- 1. 5 + 4 = ?
- 2. Can you use the number line, the rods, the counters, and your fingers to show 5 + 4 = 9?
- 3. See understanding moves for 5, 4, 9, +, =.
- 4. 
  \[
  \begin{array}{c}
  5 \\
  4 \\
  \hline \\
  \end{array}
  \]
- 5. Make up a story problem that asks you to use 5 + 4 = 9.

Level 2: Understanding why the fact is true and realizes its significance

- 6. Show the truth of the fact using objects, models, or other facts
- 7. Complete related statements of the fact

6. Suppose I say that 5 + 4 = 10. Can you show me I am wrong? Starting with 4 + 4 = 8, can you show me that 5 + 4 = 9?

7. Does 5 + 4 = 4 + 5? What makes 5 + \square = 9 true?

It may help to have in our minds a picture of what we mean by understanding (Holt 1964)

pictorial and then symbolic representation. It is advised here that care be taken in following this instructional sequence. Don’t stop at the symbolic level. Ask students working at the symbolic level to interpret their work by creating corresponding physical and pictorial representations. When students have completed a page working primarily with symbols, select an occasional item and require them to draw a picture or physically represent what they have done. Children should realize they are going to be required to use a picture or physically represent what they have done. Examples of such series of exercises in a primary level textbook and in a middle school text are shown in figure 5. In each case, a student can examine the sample exercise (or get help from a neighbor) and complete the exercises without being able to make any other understanding moves for the generalizations being taught.

Students can initiate the kinds of logical exchanges we call moves by asking questions. Task cards in mathematics laboratory settings can initiate moves by asking students to do such things as generate examples of concepts, recall facts, create or examine instances to detect a pattern, reflect on or discover the necessary or sufficient conditions of a concept, paraphrase a procedure, verify or defend a conclusion, and apply generalizations or procedures.

One way to study for an examination in mathematics is to identify the concepts, generalizations, and procedures taught and try to make as many of the moves as possible for each item of knowledge. Here the moves serve as a checklist for students to examine their understandings. A teacher often goes through a similar process when constructing a test.

Do not expect children to attain all Level 1 understandings before any Level 2 understandings are acquired. While instruction frequently moves from Level 1 to Level 2, and from physical to pictorial to symbolic representations, children do not always learn what the teacher intends. If they did, teaching would be easy.
Using Moves for Understanding

IT IS HOPED THAT THE SAMPLEx OUTLINES FOR understanding mathematical concepts, generalizations, procedures, and numerical facts will be seen to have practical value for teachers. Moves can easily be used to establish objectives for lessons and learning activities. To illustrate this claim, the moves listed for concepts, generalizations, procedures, and number facts were stated in objective format. From these objects, a teacher can receive guidance in planning lessons, activities, and tests. This can be done by using a listing of the moves and their corresponding questions to create a guide or checklist when planning a lesson or a test. The following is an example of such a checklist for the concept of division as commonly taught to children in grades 3 or 4. It comes directly from the list of moves for understanding a concept given earlier (fig. 1).

Can my students—

A. Create examples of division statements using objects or pictures? (from moves 1 and 2)
B. Select examples of division from a set of situations showing examples and nonexamples? (from moves 3 and 4)
C. Write corresponding multiplication (or subtraction) statements for given division statements? (from moves 5, 7, and 8)
D. Make up and solve story problems involving division? (from move 8)

Note that the foregoing checklist is not tied rigidly to each move. Items in the list were generalized from single or multiple moves by using common sense and a little creativity. The moves were used to stimulate the checklist and not to dictate the list. Another teacher might create a different and perhaps more complete checklist. The moves are advocated to help make a checklist “well-rounded” in covering the various aspects of understanding a concept.

A teacher need not feel limited to the moves given in this article for teaching a concept, generalization, procedure, or fact. Neither should a teacher feel compelled to use every move in every teaching situation. Moves are offered as one valuable resource in setting objectives, planning instruction, diagnosing, and evaluating understanding. They are not intended to replace common sense, creativity, or practices that a teacher has found to be successful.

Moves can help a teacher generate questions in instructional as well as testing situations. When a teacher needs to diagnose a student’s understanding of a mathematical concept, generalization, procedure, or numerical fact, moves can be called on to provide a comprehensive picture of the student’s knowledge. A list of questions similar to those in figures 1–4 can be posed to diagnose understanding. Before beginning remediation, a teacher may want to ask questions based on a number of the moves to ascertain what the student understands relative to a particular item of knowledge.

Moves could also be selected by a teacher to give directions to an aide or tutor working with a student or a small group. Aides and tutors need to “zero in” to make efficient use of instructional time. Teachers do not give an aide or tutor very specific directions to

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**Fill in the blanks**

3 tens and 7 ones is 37
3 tens and 6 ones is ___
2 tens and 9 ones is ___
9 tens and 1 one is ___

**Now try these**

___ tens and ___ ones is 47
___ tens and ___ ones is 62
___ tens and ___ ones is 50
___ tens and ___ ones is 33

**Fill in the blanks**

3 • (4 + 6) = 3 • 4 + 3 • 6
5 • (8 + 2) = 5 • 8 + ___
7 • (1 + 9) = ___ + ___
12 • (15 + 2) = ___

**Now try these**

4 • (___ + ___) = 4 • 5 + 4 • 8
___ • (___ + ___) = 7 • 2 + 7 • 9
___ • (___ + ___) = 3 • 1 + 3 • 5
___ • (___ + ___) = 4 • 8 + 4 • 6

**Fig. 5**

**Moves in teaching mathematics can serve as a basis for defining understanding of mathematics**
follow with comments like “Alice doesn’t understand division please work with her,” or “Please help Dan with the rule for finding the area of a triangle, he just doesn’t seem to get it.” Contrast these directions with “Help Alice create examples for division statements like fifteen divided by five,” and “Dan is not sure what he needs in order to apply the rule for the area of a triangle. Give him some help by reviewing the concepts of base and altitude and by going through some instances. Then see if he can apply the rule on his own.”

**Research on Using Moves in Teaching for Understanding**

**SINCE TEACHERS CAN USE ALL OR SOME OF THE moves for teaching a particular concept, generalization, procedure, or numerical fact, when planning lessons and in assessing understandings, it is possible to consider moves in developing a theory of teaching mathematics. Teachers must act. They need prescriptive principles. Consider some of Bruner’s (1966) comments on teaching:**

1. A theory of instruction is prescriptive in the sense that it sets forth rules concerning the most effective way of achieving knowledge or skill. (p. 40)
2. But theories of learning and of development are descriptive rather than prescriptive. They tell us what happened after the fact: for example, that most children of six do not yet possess the notion of reversibility. A theory of instruction, on the other hand, might attempt to set forth the best means of leading the child toward the notion of reversibility. A theory of instruction, in short, is concerned with how what one wishes to teach can best be learned, with improving rather than describing learning. (p. 40)
3. A theory of instruction must specify the ways in which a body of knowledge should be structured so that it can be most readily grasped by the learner. (p. 41)
4. A theory of instruction should specify the most effective sequences in which to present the materials to be learned. (p. 41)

If moves are used to specify and control teacher actions it is reasonable to consider or call for research to see if prescriptions for teaching types of mathematical knowledge can be found. Can effective instructional sequences be described in terms of moves? Do moves help structure knowledge in ways most readily grasped by the learner?

Studies by Dossey (1976) indicate teachers are not constrained to choose a particular sequence of moves in teaching mathematical concepts. However, teaching in Dossey’s studies was confined to printed programmed material. Students could read over each sequence of moves a number of times—something that does not always characterize classroom teaching. Swank’s work (1976) indicates teachers judged by their peers to be superior make approximately twice as many moves in teaching concepts as inexperienced teachers and teachers rated lowest by their peers. Swank’s results also show some support for concept teaching strategies containing a high number of moves with a high level of student/teacher interaction. While Swank investigated teaching in a classroom setting, he cautions against premature generalization of this finding since it did not hold for students of low ability (p. 109).

Research on classroom teaching is confounded by many variables. Moves in teaching mathematics hold promise of one way to design and control teacher actions so effects of teacher behavior and achievement can be studied.

Research aside, Bruner (1968) sees another area of concern in teaching:

Perhaps, in discussing the functions of teaching, we should make a special place for . . . teaching people to listen to what they have been doing. (p. 10)

If the set of moves in teaching students to understand types of mathematics is used to record logical interactions in teaching mathematics, we may develop an instrument that can be used by teachers to “listen” to the logical dimension of their teaching.

**References**


Swank, Earl W. “An Empirical Comparison of Teaching


Note: Other research on issues raised by Bruner and others is being coordinated by the Teaching Strategies Project of the Georgia Center for the Study of Learning and Teaching Mathematics (Cooney 1976, p. 5).

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