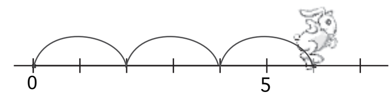
Multiple Models of Multiplication

What does *multiplication* mean? This question has many answers, because there are many ways of thinking about multiplication. In this activity you'll compare four such ways—multiplication as jumping, as grouping, as area, and as scaling.

MULTIPLICATION AS JUMPING

You can think of multiplication as jumping: Three jumps of two units each could be described by the multiplication problem 3 • 2. In this model, you will experiment with setting the number of jumps and the size of each jump.



1. Open **Multiplication Models.gsp.** Press the *Jump!* button to animate three jumps of two units each.
2. Press the *Reset* button, drag the circles to represent two jumps of five units each, and press the *Jump!* button again.

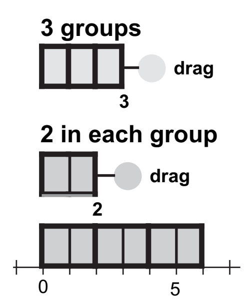
Q1 How many ways can you do jumps that end up at 6? Drag the green circles to try each way, and write down all the ways you found.

Q2 Change the number of jumps so that it's negative. What happens during the jumping? Make the number of jumps positive again, and make the size of each jump negative. What happens?

Q3 What happens if the number of jumps and the size of each jump are both negative? How can you explain this logically?

MULTIPLICATION AS GROUPING

You can also think of multiplication as grouping: 3 • 2 means three groups of two things each. In this model, you will group rectangles along a number line.



3. Go to the Grouping page. The objects in the sketch model the sentence "Put together three groups of two." The equation is $3 \cdot 2 = 6$.

Q4 Drag the circles to model each sentence below. On your paper, draw the bottom shape (the one on the number line) and write its equation.

a. Put together four groups of 2. b. Put together three groups of -3.

c. Put together one group of -8. d. Put together eight groups of -1.

How should you drag the top circle to represent "take away"?.

e. Take away two groups of 3.
f. Take away one group of 5.
g. Take away two groups of -3.
h. Take away eight groups of -1.

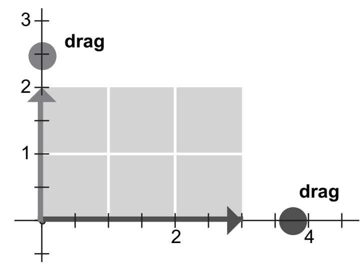
Q5 Model the following sentences and write their equations. How are they similar and how are they different?

a. Put together three groups of -4.
b. Put together four groups of -3.

Q6 Using 4's and 3's, write and model two "take away" sentences whose product is the same as the product in Q5.

MULTIPLICATION AS AREA

Another way to think about multiplication is in connection with the area of rectangles.



4. Go to the Area page. In this model, the height and width can be either positive or negative. When you start, both are positive.

Q7 Drag the width to model $-3 \cdot 2 = -6$. What happens to the rectangle when the width becomes negative? What does this change indicate about the area?

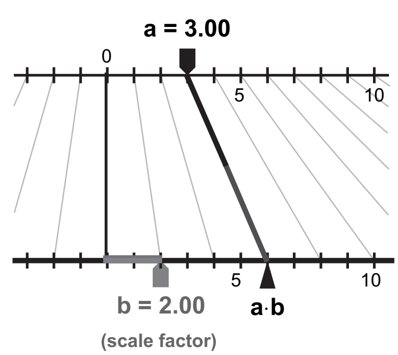
Q8 Model seven different problems in which the area equals –6. Write the problems on your paper.

Q9 Model and write down as many problems as you can in which the area equals 4.

Q10 The numbers 1, 4, 9, 16, . . . are called "squares." Explain why this makes sense given the area model of multiplication.

MULTIPLICATION AS SCALING

Whether you're drawing a scale model of your room or scaling a recipe to serve more people, you're using multiplication.



5. Go to the Scaling page. The *scale factor* (b) is 2, so every number on the top axis *maps* (corresponds) to a number twice as big on the bottom axis. Point a is at 3 and maps to 6, so the equation for this problem is $3 \cdot 2 = 6$.

6. Drag a to model $-3 \cdot 2 = -6$. Then drag b to model $-3 \cdot (-2) = 6$.

Q11 Describe what the gray mapping segments look like when:

a. b equals 1.

b. b is between 0 and 1.

c. b equals zero.

d. b is negative.

Q12 For each problem below, set the scale factor b as listed, and then drag a so that $a \cdot b = 1$. (For example, if b were 0.5, you would make $a = 2$ because $0.5 \cdot 2 = 1$.)

a. $b = 4$; $a \cdot b = 1$; $a = ?$

b. $b = -0.5$; $a \cdot b = 1$; $a = ?$

c. $b = -1$; $a \cdot b = 1$; $a = ?$

d. $b = -10$; $a \cdot b = 1$; $a = ?$

Q13 Rewrite the answers to Q12 using fractions instead of decimals. What do you notice?

SUMMING UP

Q14 List one strength of each of the four models, perhaps something that each shows about multiplication better than the others.

Q15 Which of the four models do you think is most effective at showing why the product of two negatives is a positive? Defend your choice.

EXPLORE MORE

To copy from Sketchpad into a word processor, select the objects you want to copy, and resize the window to the desired size of your picture. Then choose **Edit | Copy,** and paste the result into your word processor.

Q16 The commutative property of multiplication says that it doesn't matter whether you multiply $3 \cdot 2$ or $2 \cdot 3$; you get the same answer using either order. Set up four pairs of multiplication problems, one for each model, to show this property. Copy the Sketchpad image for each of the eight problems and paste them into a word processor document. Which model do you think is most effective at showing why multiplication is commutative?

Multiple Models of Multiplication

Objective: Students work with four different models of multiplication and use each model to solve problems and investigate properties of multiplication. Students compare the four models, particularly with regard to how they make sense of negative operands.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: None. This will be a review topic for most Algebra 1 students, though perhaps it presents things in a new way.

Sketchpad Level: Easy. Students manipulate a pre-made sketch.

Activity Time: 40–50 minutes

Setting: Paired/Individual Activity or Whole-Class Presentation (use **Multiplication Models.gsp** in either setting)

This activity has two main purposes: to provide students with multiple models of multiplication and to give a variety of justifications for the rules for multiplying negatives.

By using multiple models of multiplication, students consider different ways of conceiving of this key operation and gain perspective on its meaning and uses. For this reason, don't allow students to do the problems in their heads without modeling them in Sketchpad—that would defeat the purpose.

As they work, students see how each model provides justification for the rules for multiplying negatives. A mental image provides a more solid foundation than a verbal rule for the idea that the product of two negative numbers is positive.

Keep in mind that these four models of multiplication aren't the only ones in this chapter. The Raz's Magic Multiplying Machine activity provides yet another model that challenges students to broaden their thinking about multiplication and gives compelling reasons for the rules for multiplying negatives. These two activities work especially well together.

MULTIPLICATION AS JUMPING

Q1 Jumps that end up at 6 include $3 \cdot 2$, $2 \cdot 3$, $6 \cdot 1$, $1 \cdot 6$, $-3 \cdot (-2)$, $-2 \cdot (-3)$, $-6 \cdot (-1)$, and $-1 \cdot (-6)$.

Q2 When the number of jumps is negative and each jump is positive, the rabbit faces right and jumps backward, moving to the left. When the number of jumps is positive but each jump is negative, the rabbit faces left and jumps forward, again moving to the left.

Q3 When both the number of jumps and the size of each jump are negative, the rabbit faces left and jumps backward, moving to the right. He faces left because the size of the jumps is negative, and he jumps backward because he's taking a negative number of jumps. By facing left and jumping backward, the rabbit moves in the positive direction along the number line.

MULTIPLICATION AS GROUPING

Q4 a. $4 \cdot 2 = 8$ b. $3 \cdot (-3) = -9$

c. $1 \cdot (-8) = -8$ d. $8 \cdot (-1) = -8$

e. $-2 \cdot 3 = -6$ f. $-1 \cdot 5 = -5$

g. $-2 \cdot (-3) = 6$ h. $-8 \cdot (-1) = 8$

Q5 a. $3 \cdot (-4) = -12$ b. $4 \cdot (-3) = -12$

These two results are similar in that they both give the same negative answer, -12. In both cases, one number is positive and one is negative. The biggest difference is that the reason for changing direction, from positive to negative, is completely different in the two cases.

Q6 "Take away three groups of 4" ($-3 \cdot 4 = -12$) and "take away four groups of 3" ($-4 \cdot 3 = -12$).

MULTIPLICATION AS AREA

Q7 When the width becomes negative, the rectangle flips over horizontally, the squares change color, and the area becomes negative. Some students may make a valuable logical connection between the flipping of the rectangle and the area becoming negative.

Q8 $-1 \cdot 6 = -6$ $-2 \cdot 3 = -6$

$-6 \cdot 1 = -6$ $1 \cdot (-6) = -6$

$2 \cdot (-3) = -6$ $3 \cdot (-2) = -6$

$6 \cdot (-1) = -6$

Q9 $1 \cdot 4 = 4$ $2 \cdot 2 = 4$

$4 \cdot 1 = 4$ $-1 \cdot (-4) = 4$

$-2 \cdot (-2) = 4$ $-4 \cdot (-1) = 4$

Q10 Every square number can be modeled with a square in the area multiplication model. For example, 4 can be modeled by $2 \cdot 2$ or $-2 \cdot (-2)$, both of which are squares.

(A number such as -4 can also be modeled with squares, $2 \cdot (-2)$ or $-2 \cdot 2$. However, in these squares, the base and height are not equal. This can be interpreted as a weakness of this model, or it might represent an opportunity for a sneak preview of imaginary numbers.)

MULTIPLICATION AS SCALING

Q11 a. The mapping segments point straight down, parallel to each other. Every number maps to itself. For example, $2 \cdot 1 = 2$, $-3 \cdot 1 = -3$, $0 \cdot 1 = 0$, etc.

b. The mapping segments point inward toward the bottom. Every number maps to a number whose absolute value is less than its own absolute value (or equal to, in the case of 0), but whose sign is the same. For a scale factor of 0.5, for example, $2 \cdot 0.5 = 1$, $-3 \cdot 0.5 = -1.5$, $0 \cdot 0.5 = 0$, etc.

c. The mapping segments all point to zero, so every number maps to zero. For example, $2 \cdot 0 = 0$, $-3 \cdot 0 = 0$, $0 \cdot 0 = 0$, etc.

d. The mapping segments cross between the two number lines. Every number maps to a number with the opposite sign (except for 0, which points to itself). For a scale factor of -2, for example, $2 \cdot (-2) = -4$, $-3 \cdot (-2) = 6$, $0 \cdot 0 = 0$, etc.

Q12 a. $a = 0.25$ b. $a = -2$

c. $a = -1$ d. $a = -0.1$

Q13 a. $b = 4$; $a = 1/4$ b. $b = -1/2$; $a = -2/1$

c. $b = -1/1$; $a = -1/1$ d. $b = -10$; $a = -1/10$

In each pair, the numbers are reciprocals of each other. For example, in part a, $b = 4/1$ and $a = 1/4$.

SUMMING UP

Q14 There are many possible answers. We feel that Jumping and Grouping are particularly effective as an introduction to multiplication. They correspond with most people's basic conception of multiplication and so are a good place to start. Area is particularly effective at showing the "dimensionality" of multiplication—how multiplying two one-dimensional objects produces a two-dimensional object. Scaling is good for showing how multiplication affects an entire set of objects, including non-integers. It also serves as a great introduction to dynagraphs.

Q15 Jumping, Grouping (especially when using the terms "put together" and "take away"), and Scaling are effective at demonstrating the rules of multiplication for negatives. Area is less effective for this, in our view, because there is no compelling reason why the rectangles in the first and third quadrants are blue and those in the second and fourth quadrants are red.

EXPLORE MORE

Q16 Students should model pairs of equations, such as $2 \cdot (-5) = -10$ and $-5 \cdot 2 = -10$. Area may be especially useful for demonstrating commutativity because it's so easy to see that the two rectangles have the same area and sign.

WHOLE-CLASS PRESENTATION

The whole-class presentation of this activity substantially follows the steps of the student activity sheet. Refer to the Presenter Notes for tips to follow and adjustments to make so that the presentation can be as useful to students as possible.

You can present any of the models in this activity independently, though it's valuable to present at least two models in succession. Students will get the greatest benefit from this activity when they compare the behaviors of several different models.

Follows the steps in the student activity sheet, with the adjustments described below.

MULTIPLICATION AS JUMPING

Be sure to elicit answers from a number of students.

When the rabbit first jumps, ask students how the rabbit's motion illustrates the multiplication problem shown before adjusting the numbers (leaving both positive) and doing another example.

Before changing the number of jumps to be negative, ask students to predict what the rabbit will do.

Be sure to make the *jumps* value positive again before you make the *units* value negative.

Similarly, before making the size of each jump negative, ask students to predict what the rabbit will do. And ask again for predictions before making both numbers negative at the same time.

MULTIPLICATION AS GROUPING

In the grouping part of the presentation, ask students to make up problems using particular combinations of negative and positive ("put together" and "take away") rather than using the specific ones from Q4. Be sure to show how a "put together" problem and a "take away" problem can give the same result.

MULTIPLICATION AS AREA

When presenting multiplication as area, you may want to emphasize that the color of the rectangle indicates the sign of the result, without too much emphasis on the idea of "negative area." Don't let a discussion of negative area become a distraction.

MULTIPLICATION AS SCALING

Unlike the other models, the numbers don't need to be integers, so this page shows a continuous model of multiplication.

CONCLUSION

Finish the class discussion by asking students to compare the various models, particularly with regard to how they show the product of two negative numbers.