5
Fractions

5-1
DEFINING FRACTIONS

For a number of reasons, we can describe this chapter by saying, "The plot thickens." The whole numbers, though infinite in extent, are at times quite meager in applying to real-life situations. A vote requires a two-thirds majority; a car's gas tank is one-quarter full; three girls wish to divide two pizzas evenly. These examples do not represent even one-millionth of all the uses for fractions. The plot indeed thickens: we need more than just the whole numbers.

Our first of several extensions of the whole numbers is to the fractions. We extend first to the fractions because it is consistent with the historical development of numbers and, more important, it is consistent with our personal experiences. Young children usually encounter fractions in and out of school well before they encounter negative and irrational numbers, which we will discuss later. Moreover, fractions (including decimals and percentages) play a much more important role in the elementary school curriculum; so we choose to give them more attention.

Some changes are taking place. With the move towards the metric system, there is less need for arithmetic with ordinary fractions. We will always have some use for common fractions, like those cited in the opening paragraph of this section, but many of the settings that give rise to doing arithmetic with fractions (such as adding 3/8 inches + 5/16 inches) will vanish altogether with increased use of the metric system. As the old units are phased out and the metric units become more universally accepted, there will be an emphasis instead on decimals, which are in fact just a specific type of fraction. The popularity of hand calculators also dictates a greater stress on decimals. Mathematically, the idea is the same whether we divide our units into sixteenths and twelfths or into tenths, though of course computation is usually easier in the latter case because we have a base ten number system.

After introducing the fraction concept, we will consider their ordering and certain other properties. In the remaining two sections of this chapter, we will develop the arithmetic of fractions. Certain important applications of whole numbers and fractions will be investigated in the next chapter.

Among the ways in which fractions arise in real-life situations are circumstances in which we consider part of a whole and those in which we wish to divide one counting number by another, but find that no counting number is a solution. The voting and gas tank problems are examples of the "part of" interpretation, while the pizza problem is an example of the "division" interpretation.

Whole numbers were defined in terms of sets. (How?) One of our interpretations of fractions will also depend on sets. In Figure 5-1 is a set of five flowers, three of which are tulips. We say in this situation that three-fifths of the flowers are tulips. The fraction 3/5 is then a pair of whole numbers that compares part of a set with all of the set. Since the tulips are a subset of the entire set, we could say that the pair 3/5 stands for the number of members in a certain set *preceded* by the number of members in one of its subsets. Simila interpretations could be given for any symbol of the form p/q, where p is any whole number less than or equal to a non-zero whole number q.

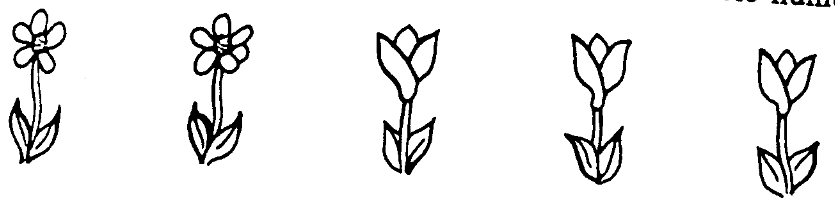


FIGURE 5-1

Sometimes we interpret fractions in terms of what we call **area models.** To show 3/5 we would start with one unit, preferably a square, divide it into five parts having the same size and shape, and shade three of them, as shown in Figure 5-2. The shaded area would then represent 3/5. (Though children might have little or no knowledge of area at the time fractions are introduced, it is not difficult for them to recognize this as simply three parts out of five.) We will find the area model to be very useful later.

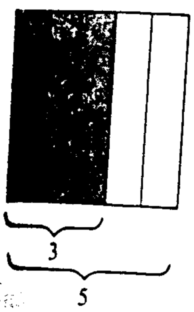


FIGURE 5-2

We alluded to another interpretation for fractions when we discussed division of whole numbers in Section 3-6. We saw that problems like 2 ÷ 3 and 3 ÷ 2 had no solution unless we introduced remainders. Yet the fact that when two pizzas are divided among three people each gets none with two left over (quotient = 0, remainder = 2) is of little help. In this case, it is more useful to know that each person gets two-thirds of a pizza. A practical way of accomplishing this is for each person to take one-third of each of the pizzas. If they were the same size, this would be equivalent to each person getting two-thirds of a pizza. The problem 3 ÷ 2 also has a solution if we admit fractions. Assuming that two people are sharing three pizzas, each hungry person gets one pizza and half of the third one (1½ pizzas), or half of each one (which is three halves in all). By similar methods, we can introduce fractions to give an answer to any problem in which one whole number is divided by another (non-zero) whole number.

In the "part of" interpretation, the symbol p/q had meaning only when p was no bigger than q. This restriction is removed in the division interpretation. We require only that p and q are whole numbers and that q is not 0. We call expressions of this form **fractions.** The numbers p and q are called the **numerator** and **denominator** respectively. (We will often use the familiar notation $\frac{3}{5}$ and $\frac{p}{q}$ to represent what we have been calling 3/5 and p/q respectively.) Notice that the whole numbers have representations in this set since according to the division interpretation, 0/1 = 0, 1/1 = 1, 2/1 = 2, etc. Technically we have not explicitly defined fractions, but have merely given a name to a set of ordered pairs of whole numbers and have chosen to write the ordered pair (p, q) either in the form p/q or in the form $\frac{p}{q}$.

We saw that the whole number 2 can be thought of as the fraction 2/1 in terms of the division interpretation. According to this interpretation, we also have 4/2 = 2, 6/3 = 2, 8/4 = 2, etc. Thus there are many fractions that represent the same whole number. The area model for fractions suggests that there are also many fractions that signify the same amount. This is shown in Figure 5-3, where we see that 2/3, 4/6, 6/9, and 8/12 each suggest the same part of the whole. Fractions like 2/3, 4/6, and 8/12, which name the same amount, are called **equivalent fractions**. Thus 2/1, 4/2, 6/3, and 8/4 are also equivalent fractions.

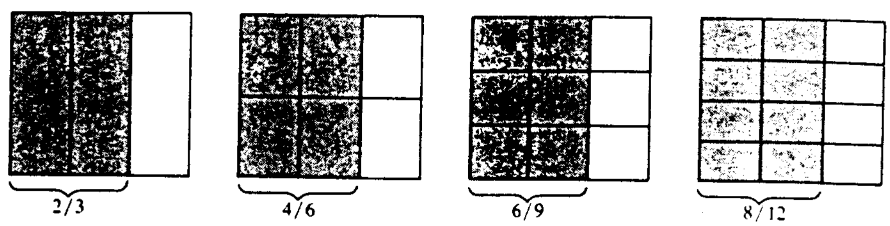


FIGURE 5-3

The area model strongly suggests, at least for fractions for which the numerator is less than or equal to the denominator, what we call

☐ The Reducing Fractions Principle

> If $\frac{p}{q}$ is any fraction and n is any whole number other than 0, then $\frac{p \times n}{q \times n}$ and $\frac{p}{q}$ are equivalent. $\left(\text{We will write } \frac{p \times n}{q \times n} = \frac{p}{q}.\right)$

For example, in saying that 8/12 = 2/3 we have $p = 2$, $q = 3$, and $n = 4$.

To argue that the reducing fractions principle also holds when p is larger than q, we can extend our area model. In order to represent an arbitrary fraction (like 13/5), we take a unit square, divide it into (five) parts of equal size and shape as before, and repeat this process successively until we have generated a number of parts equal to or greater than the numerator. Thus, in Figure 5-4, we see that three squares provide us with 15 fifths, of which we take 13. Figures 5-4 and 5-5 then demonstrate that 13/5 is equivalent to 39/15.

According to the reducing fractions principle, the essential idea is to find a common factor of the numerator and denominator. A fraction is said to be **reduced** or **in lowest terms** when the numerator and denominator have no common factor other than 1. (If there were any other common factor

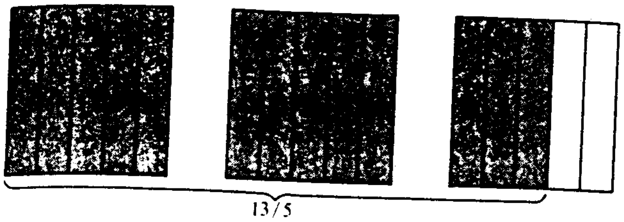


FIGURE 5-4

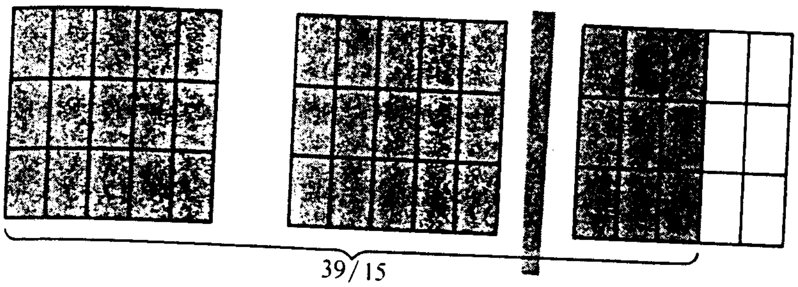


FIGURE 5-5

reducing fractions principle could be applied again to get an equivalent fraction in lower terms.) Hence we seek the greatest of all common factors of the numerator and denominator, which is precisely the greatest common divisor (GCD) of these two numbers as it was introduced in Chapter 4. Our next example shows how the techniques of Chapter 4 apply directly to reducing fractions.

EXAMPLE 5-1

Reduce (a) $\frac{20}{30}$ (b) $\frac{153}{204}$ (c) $\frac{2^5 \times 3^6 \times 11^8}{2^3 \times 3^{17} \times 7^2 \times 11}$ (d) $\frac{377}{261}$

SOLUTION

In Example 4-9 we found the GCD of each of these pairs of numbers. You may wish to refer to that example.

a) Since $20 = 2 \times 2 \times 5$ and $30 = 2 \times 3 \times 5$, GCD $(20, 30) = 2 \times 5 = 10$. Thus $\frac{20}{30} = \frac{2(\times 2 \times 5)}{3(\times 2 \times 5)} = \frac{2}{3}$. (This may have been obvious. The next example is less obvious.)

b) Common factors of 153 and 204 are not immediately apparent (except perhaps that each has a factor of 3), so we factor them into primes and learn that GCD (153, 204) is $3 \times 17 = 51$.

$$\frac{153}{204} = \frac{3(\times 3 \times 17)}{2 \times 2(\times 3 \times 17)} = \frac{3}{2 \times 2} = \frac{3}{4}$$

c) By the method of Example 4-9, the GCD is $2^3 \times 3^6 \times 11$. Hence

$$\frac{2^5 \times 3^6 \times 11^8}{2^3 \times 3^{17} \times 7^2 \times 11} = \frac{(2^2 \times 11^7)(\times (2^3 \times 3^6 \times 11))}{(3^{11} \times 7^2)(\times (2^3 \times 3^6 \times 11))} = \frac{2^2 \times 11^7}{3^{11} \times 7^2}$$

[redacted]

(d) When we found GCD (377, 261) in Example 4-9, we commented that 261 is easy to factor ($261 = 3^2 \times 29$) and that 377 is not so easy to factor. We learned after factoring 261 that the only possible common prime factors are 3 and 29. We immediately found that 29 divided 377; namely, $377 = 29 \times 13$. Hence,

$$\frac{377}{261} = \frac{13 \times 29}{3 \times 3 \times 29} = \frac{13}{3 \times 3} = \frac{13}{9}$$

It is unlikely that in real-life situations you will need to reduce fractions like most of the ones in Example 5-1. The reason for their inclusion here is to give you further insight into the *method* involved. We have deliberately exposed you to unfamiliar fractions to avoid the biases you may have in working with the familiar ones, much as we did arithmetic in other bases to give you better insight into the algorithms. To put it another way, you might know that 6/8 = 3/4 without understanding why; but to solve problems like 377/261 = 13/9, it is more likely that you need to be aware of what is going on.

Our last remark on the topic of reducing fractions is that one does not always necessarily reduce a fraction just because it is possible. There are times, for instance, when 60/100 is a better form than 3/5. Can you think of such a time?

We saw in Figure 5-3 that two fractions can be equivalent though neither is a reduced form of the other. One such example is 6/9 and 8/12. We will now devise a test that applies to such cases as well. We will take the given fractions and replace them by equivalent fractions having the same denominator. By reducing 6/9 and 8/12 we see that they are each equivalent to 2/3, so they must be equivalent to each other:

$$\frac{6}{9} = \frac{2 \times 3}{3 \times 3} = \frac{2}{3} \quad \text{and} \quad \frac{8}{12} = \frac{2 \times 4}{3 \times 4} = \frac{2}{3}$$

Thus it worked in this example to reduce each fraction as far as possible and compare the reduced versions. Since *any* common denominator will facilitate direct comparison, there are other options. One number which is sure to be a common multiple of the two denominators is the least common multiple (LCM), which we investigated in Chapter 4. Since LCM (9, 12) = 36, we have

$$\frac{6}{9} = \frac{6 \times 4}{9 \times 4} = \frac{24}{36} \quad \text{and} \quad \frac{8}{12} = \frac{8 \times 3}{12 \times 3} = \frac{24}{36}$$

Finally, we mention that another common multiple of any two numbers is their product. Clearly 9×12 is a multiple of 9 and a multiple of 12. Hence,

$$\frac{6}{9} = \frac{6 \times 12}{9 \times 12} = \frac{72}{108} \quad \text{and} \quad \frac{8}{12} = \frac{8 \times 9}{12 \times 9} = \frac{72}{108}$$

While each of these options is quite acceptable, the last one is often the simplest to apply. Given 6/9 and 8/12, we see that one common denominator is simply 9×12 and the numerators are respectively 6×12 and 8×9. In general, given the fractions p/q and r/s, we have

$$\frac{p}{q} = \frac{p \times s}{q \times s} \quad \text{and} \quad \frac{r}{s} = \frac{r \times q}{s \times q} = \frac{r \times q}{q \times s}$$

Therefore, to compare $\frac{p}{q}$ and $\frac{r}{s}$ we need only compare $p \times s$ and $r \times q$. The fractions are equivalent if and only if $p \times s = r \times q$.

EXAMPLE 5-2

Test each pair of fractions to see if they are equivalent:

(a) $\frac{4}{5}$ and $\frac{7}{9}$ (b) $\frac{14}{91}$ and $\frac{22}{143}$

SOLUTION

(a) Since $4 \times 9 \neq 7 \times 5$, the fractions are not equivalent. (The underlying idea is that

$$\frac{4}{5} = \frac{4 \times 9}{5 \times 9} = \frac{36}{45} \quad \text{and} \quad \frac{7}{9} = \frac{7 \times 5}{9 \times 5} = \frac{35}{45}.)$$

(b) Since $14 \times 143 = 2002 = 22 \times 91$, the fractions are equivalent. (This would be a convenient place to use a calculator.) (The underlying idea is that

$$\frac{14}{91} = \frac{14 \times 143}{91 \times 143} = \frac{2002}{13013} \quad \text{and} \quad \frac{22}{143} = \frac{22 \times 91}{143 \times 91} = \frac{2002}{13013}.)$$

It should be clear that any fraction is either in lowest terms or is equivalent to exactly one fraction which is in lowest terms. This is an immediate consequence of the Unique Factorization Theorem. The explicit process of factoring the numerator and denominator into primes and "canceling" the common factors necessarily arrives at a well-defined answer. Hence the collection of all possible fractions is partitioned into various (non-overlapping) sets of equivalent fractions which we will call **classes**. Some of the possible classes are:

$$\left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \ldots\right\}$$

$$\left\{\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \frac{5}{15}, \ldots\right\}$$

$$\left\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \frac{10}{15}, \ldots\right\}$$

$$\left\{\frac{2}{13}, \frac{4}{26}, \frac{6}{39}, \frac{8}{52}, \ldots, \frac{14}{91}, \ldots, \frac{22}{143}, \ldots, \frac{2002}{13013}, \ldots\right\} \text{ (see Example 5-2)}$$

$$\left\{\frac{3}{1}, \frac{6}{2}, \frac{9}{3}, \frac{12}{4}, \frac{15}{5}, \ldots\right\}$$

$$\left\{\frac{4}{3}, \frac{8}{6}, \frac{12}{9}, \frac{16}{12}, \frac{20}{15}, \ldots\right\}$$

Up to now, we have thought of fractions as certain pairs of whole numbers. But our experience suggests that fractions themselves can be thought of as numbers which can be compared and which can be added, subtracted, multiplied, and divided. Since each member of a class represents the "same amount" (in terms of area models, such as Figure 5-3), an entire class gives us just one number. In accordance with common usage, we will call these numbers *fractions*. Thus the term *fraction* takes on both a numeral sense (as a representative for a number) and a number sense (as an amount). We do not wish to become overly concerned with the distinction between numerals and numbers. Consequently, we write "1/2 = 2/4" to mean both "the fractions (numerals) 1/2 and 2/4 are equivalent" and "the fractions (numbers) 1/2 and 2/4 are equal." (In other books you may encounter symbols like ~, ≃, or ≅ used to mean "is equivalent to.") It is important to remember that each number has (infinitely) many names.

Among the classes to which our new numbers have been assigned are the ones that correspond to whole numbers — that is, the classes for 0, 1, 2, . . . respectively:

$$\left\{\frac{0}{1}, \frac{0}{2}, \frac{0}{3}, \frac{0}{4}, \ldots\right\}$$

$$\left\{\frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \ldots\right\}$$

$$\left\{\frac{2}{1}, \frac{4}{2}, \frac{6}{3}, \frac{8}{4}, \ldots\right\}$$

$$\cdots$$

Thus the set of fractions includes the whole numbers as special cases. We will refer to this process of enlarging the set of whole numbers into a larger set as **extending** the whole numbers. We want to gain some new properties (like being able to divide 2 by 3) while preserving everything that worked for the whole numbers (ordering, addition and its properties, etc.).

One of the first things we did with whole numbers, even before we began to add and subtract them, was to order them. Given two whole numbers, the first was either less than, equal to, or greater than the second. (This was called the trichotomy property for comparing whole numbers.) Is there such a property for the fractions?

Given two fractions, they will be **equal** (the same number) if and only if their representatives come from the same class of equivalent fractions. Thus we can test to see if two such numbers are equal by checking if their representatives are equivalent fractions. To compare numbers which are not equal, we initiate the following definition:

> The fraction $\frac{p}{q}$ is **less than** the fraction $\frac{r}{s}$, written "$\frac{p}{q} < \frac{r}{s}$," if and only if $p \times s < r \times q$. $\frac{p}{q}$ is **greater than** $\frac{r}{s}$, written "$\frac{p}{q} > \frac{r}{s}$," if and only if $p \times s > r \times q$.

Note that we have defined < for fractions p/q and r/s in terms of < for the whole numbers $p \times s$ and $q \times r$. The idea here is the same as what we do to determine whether two fractions are equivalent. We express them in a form where they have a common denominator and compare their numerators. It may not be immediately obvious which is greater, 4/5 or 7/9, but expressing them in terms of fractions having a common denominator, it is clear that 36/45 > 35/45 (so 4/5 > 7/9). It turns out that we compared 36, which is 4×9, with 35, which is 7×5. Just as the definition of > says, $4/5 > 7/9$ since $4 \times 9 > 7 \times 5$.

In search of a trichotomy property for comparing fractions, suppose p/q and r/s are two such numbers. We compare them by looking at $p \times s$ and $r \times q$. Since p and s are whole numbers, so is $p \times s$. Similarly, $r \times q$ is a whole number. Now, by the trichotomy property for comparing *whole numbers*, exactly one of three things must be true about the whole numbers $p \times s$ and $r \times q$:

(1) $p \times s < r \times q$,

(2) $p \times s = r \times q$, or

(3) $p \times s > r \times q$.

But these three statements, respectively, tell us that

(1) $\frac{p}{q} < \frac{r}{s}$,

(2) $\frac{p}{q} = \frac{r}{s}$, or

(3) $\frac{p}{q} > \frac{r}{s}$.

Hence exactly one of these three statements about fractions must be true, and we have our **trichotomy property for comparing fractions**.

An interesting question arises because each non-negative rational number has many representations. We saw earlier that $4/5 > 7/9$. Now suppose we represent the same numbers in other ways. Is it necessarily true that $8/10 > 21/27$, for instance?! A check of this one case shows that $8 \times 27 > 21 \times 10$. Can we depend on being this fortunate in every case? The answer turns out to be yes. (See Problem 29.) If we think in terms of area models, the statement seems reasonable enough. If one figure has more area than another, dividing each of them into more (or fewer) pieces will have no effect. (See Problem 15.) Figure 5-6 illustrates our special case, that if 4/5 > 7/9, then 8/10 > 21/27. We use the term **well-defined** to refer to a definition that holds regardless of the choice of representation. Thus "greater than" as applied to fractions is well-defined.

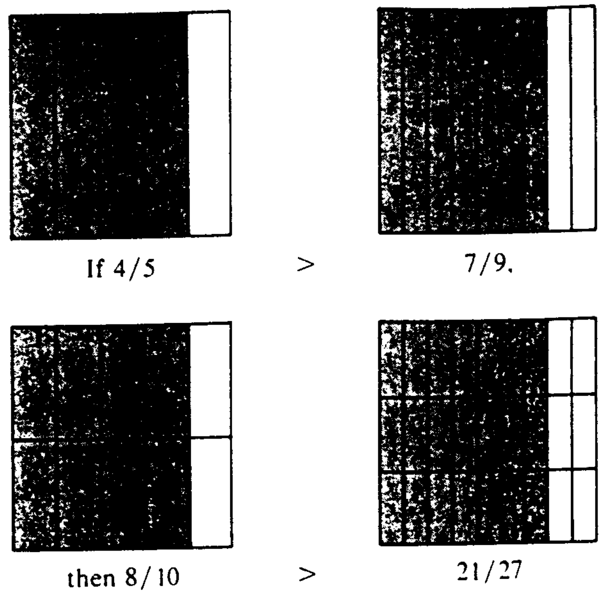


FIGURE 5-6

Having begun an extension of the whole numbers, we want to check to see whether our new rules are consistent with the old ones. For instance, $2 < 3$ if we think of them as whole numbers. It had better be the case that $2/1 < 3/1$ when we think of these numbers as fractions. This is easy to check, since $2/1 < 3/1$ is true if and only if $2 \times 1 < 3 \times 1$. Any time we have $p < q$, then $p/1 < q/1$ since $p \times 1 < q \times 1$. Making sure that definitions are well-defined and consistent is crucial throughout each of our extensions of the whole numbers.

In Chapter 3 it was helpful to pair the whole numbers with certain points on a line. We referred to this geometric representation as a number line and used it to illustrate addition and other operations with whole numbers. Is there a reasonable way to locate the fractions on a number line that will be suggestive later when we add, subtract, multiply, and divide them? We seek a way that would preserve order (1/5 should come "before 2/5," etc.) and would preserve distance (the distance from 1/5 to 2/5 should be the same as the distance from 2/5 to 3/5, etc.).

The fractions, whether interpreted on the number line or not, have a property not applicable to the whole numbers. Between any two consecutive whole numbers, such as 0 and 1, there is no other whole number, yet between two fractions there is always another. 37/100 and 38/100 may seem close together, but rewritten as 370/1000 and 380/1000 it is easy to see that 371/1000, 372/1000, 373/1000, . . . , 379/1000 are all between 37/100 and 38/100. Such is the case for any two fractions.

[redacted] Find a fraction between 1/8 and 1/7.

First find a common denominator and express each of these fractions using the common denominator: 1/8 = 7/56 and 1/7 = 8/56. Now get equivalent fractions by multiplying numerators and denominators by 10. 1/8 = 7/56 = 70/560 and 1/7 = 8/56 = 80/560. Then 71/560 is an answer.

We already have common *numerators*, but the denominators are "too close together." So, 1/8 = 2/16 and 1/7 = 2/14. Then 2/15 is an answer. In the problems at the end of this section we will show another way to get the answer 2/15.

Our number line as we would like to have it (so far) can be described as follows:

(1) We mark two points and identify them with the numbers 0 and 1. (1 is placed to the right of 0.) We lay off equally spaced points to the right using copies of the part of the line between 0 and 1 and match these points in the natural way with the numbers 2, 3, 4, ... These same points 0, 1, 2, 3, 4, . . . are also named 0/1, 1/1, 2/1, 3/1, ...

(2) We insert 1/2 halfway between 0/1 and 1/1 (by dividing the segment from 0 to 1 in half). Then we insert 1/3 and 2/3 one-third and two-thirds of the way between 0/1 and 1/1 respectively (by dividing the segment from 0 to 1 in thirds). Similarly, we insert 1/4, 2/4 = 1/2, and 3/4; then 1/5, 2/5, 3/5, 4/5, etc. A similar procedure is simultaneously carried out between 1/1 and 2/1, between 2/1 and 3/1, etc.

(3) By preserving order and distance, certain points to the right of the origin matched up with the fractions. It may seem as though we have completely used up all the points on this part of the line. However, there remain points to the right of the origin which have not been paired with any fractions. We will exhibit such points in Chapter 8. Thus the pairing of numbers and points is *not* yet complete. Of course, it is far from complete in that we have said nothing of the points to the left of the origin.

Though the story is not over yet, the plot has indeed thickened.

EXERCISES AND PROBLEMS 5-1

DEFINITION

1. Illustrate each of the following interpretations for 5/6:
   (a) set (b) area model (c) division (d) number line

2. (a) Make up some "real-life" division problems where we would prefer to have the answers expressed as quotients and remainders.
   (b) Make up some "real-life" division problems where we would prefer to have the answers expressed as fractions.

3. A company advertises its product on sale for "a fraction of the original cost." What information does this provide about the sale price?
4. Why might a rectangle area model be better than a circle area model for children?
5. Tell how to fold a string into
   (a) two equal parts (b) four equal parts (c) three equal parts.
6. Describe how to locate (precisely) 3/5 on a number line.
7. Change to base ten fractions:
   (a) $\frac{1}{10_{two}}$ (b) $\frac{1}{10_{three}}$ (c) $\frac{1}{10_{five}}$ (d) $\frac{1}{10_{twelve}}$
8. Ann drank a 16-ounce bottle of pop and Joyce drank a 12-ounce bottle. "I drank 1/3 more," said Ann. "I drank 1/4 less," said Joyce. Who was right?

EQUIVALENT FRACTIONS

9. Give five different fractions which name the whole number 6. Give five different fractions equivalent to 3/8.
10. Use an area model to show that 5/9 = 20/36.
11. Which pairs of fractions are equivalent?
    (a) 9/12 and 15/20 (b) 17/4 and 13/3
    (c) 4/9 and 2/3 (d) 18/6 and 42/14
12. Show that 5/3 = 1⅔ by dividing five pizzas among three people in two different ways.
13. Give a real-life example of when it is more convenient to think of 3/5 as 60/100.
14. Figure 5-3 shows that 6/9 and 8/12 are equivalent, though neither is a reduced form of the other. Find another such pair of fractions in Figure 5-3.
15. A guy ordered a pizza. Asked if he wanted it cut into four pieces or six, he replied "Four, I'm not that hungry." He was obviously unaware that what two fractions are equivalent?
16. In a story for children involving fractions, the "bad guy" 2/3 assumes a disguise as 4/6. What other disguises can he assume?
17. Reduce each of these fractions to lowest terms: (a) 6/8 (b) 4/10 (c) 4/9 (d) 34/51 (e) 52/65 (f) 253/207 (g) 477/1537 (h) 237/869
18. How many ways are there to reduce 18/24 (not necessarily completely)?
19. Use the Euclidean algorithm to reduce each to lowest terms: (a) 1007/1219 (b) 8051/13,363
20. Does $\frac{3 \times 2}{4 \times 2} = \frac{3}{4}$? Does $\frac{3 + 2}{4 + 2} = \frac{3}{4}$? Draw a conclusion about "canceling."
21. Note that $\frac{1\not{6}}{\not{6}4} = \frac{1}{4}$. Find another such example.
22. When possible, write each of the following as a fraction whose denominator is a power of 10; namely, 1, 10, 100, 1000, 10000, etc. Explain those cases that won't work.
    (a) $\frac{1}{5}$ (b) $\frac{1}{3}$ (c) $\frac{4}{25}$ (d) $\frac{3}{80}$
    (e) $\frac{5}{18}$ (f) $\frac{7}{60}$ (g) $\frac{1}{90}$ (h) $\frac{1}{11}$

(i) $\frac{3}{40}$ (j) $\frac{5}{140}$ (k) $\frac{7}{140}$ (Careful!)

(l) $\frac{65}{64}$ (m) $\frac{64}{65}$

23. Generalize Problem 22, characterizing all fractions that can be written with a denominator which is a power of 10.

ORDER

24. (a) Which is greater, 9/11 or 4/5? How can you tell?
(b) Find a fraction between 9/11 and 4/5.

25. Write in order, smallest to largest:
(a) $\frac{1}{4}, \frac{1}{5}, \frac{1}{6}$ (b) $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}$
(c) $\frac{2}{7}, \frac{3}{10}, \frac{5}{17}$ (d) $\frac{3}{5}, \frac{5}{8}, \frac{11}{18}$

26. Find a fraction between:
(a) $\frac{63}{100}$ and $\frac{64}{100}$ (b) $\frac{1}{12}$ and $\frac{1}{11}$
(c) $\frac{5}{6}$ and $\frac{6}{7}$ (d) $\frac{p}{q}$ and $\frac{r}{s}$

27. When Melissa tried to find a fraction between $\frac{1}{8}$ and $\frac{1}{7}$, she noticed that $\frac{1+1}{8+7} = \frac{2}{15}$ worked, but she didn't know why. "Will this always work?" she asked.
(a) Is $\frac{2+5}{7+9}$ between $\frac{2}{7}$ and $\frac{5}{9}$? Prove it.
(b) Try several more examples. Does this method work?
(c) Show that this works every time, or give an example to show it doesn't work.

28. For what values of p and q is $\frac{p}{q} < \frac{p+1}{q+1}$.

***29.** Show that "greater than" (applied to fractions) is well-defined.

***30.** Show that "greater than" (applied to fractions) is transitive.

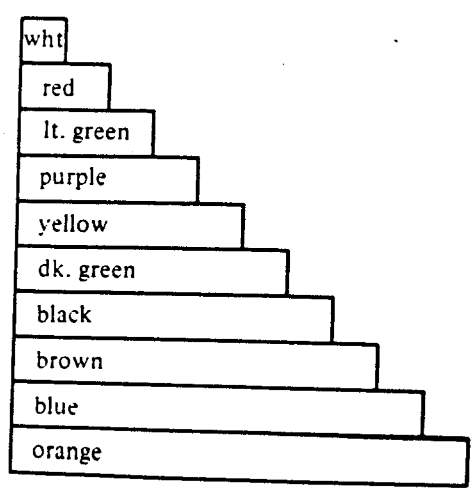
MATERIALS AND CALCULATORS

31. If a purple rod equals 1, what is the value of
(a) a white rod?
(b) a red rod?
(c) a yellow rod?

32. Repeat Problem 31 if a dark green rod equals 1.

33. Repeat Problem 31 if an orange rod equals 1.

34. (a) Explain how to use the rods to show that 1/2 is equivalent to 2/4.
(b) To show that 6/9 is equivalent to 2/3, which rod should you choose to represent 1? Why?



C35. Use the definition of "less than" for fractions and a calculator to determine which fraction in each pair is smaller:

(a) $\frac{243}{534}$ and $\frac{298}{654}$ (b) $\frac{79}{5464}$ and $\frac{119}{8231}$

(c) $\frac{87}{2842}$ and $\frac{3}{98}$ (d) $\frac{419}{2250}$ and $\frac{192}{1031}$

C36. Outline a procedure for comparing fractions using decimal notation. Use it to compare the fractions in the previous problem.

**P37.* Write a program that reduces fractions to lowest terms.

PROBLEM SOLVING EXTENSIONS

38. Solve this cryptarithm, where C > 0 and all values are different from 0.

$$\frac{\text{PORK}}{\text{CHOP}} = \text{C}$$

39. (a) If pizzas can be ordered with or without each of five ingredients, in how many different ways may a pizza be ordered?
*(b) Answer this question if each ingredient may be placed on all, half, or none of the pizza. For example, a pizza might be ordered with sausage and mushrooms, and anchovies on half.

*40. The **Farey sequence of order n** is the increasing sequence of all fractions $\frac{p}{q}$ such that $\frac{p}{q}$ is in lowest terms, $0 \le \frac{p}{q} \le 1$, and $q \le n$. For example, the Farey sequence of order 5 is $\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}$. Find the Farey sequence of order (a) 3 (b) 4 (c) 6 (d) 7. (Hint: These properties may be helpful: If a/b, c/d, e/f, are consecutive terms of a Farey sequence, then $bc - ad = 1$ and $\frac{c}{d} = \frac{a+e}{b+f}$.)

41. The sequence $\frac{1}{3}, \frac{1+3}{5+7}, \frac{1+3+5}{7+9+11}, \frac{1+3+5+7}{9+11+13+15}, \ldots$ was discovered by Galileo while studying free-falling bodies. Verify that

(a) $\frac{1+3}{5+7} = \frac{1}{3}$. (b) $\frac{1+3+5}{7+9+11} = \frac{1}{3}$. (c) $\frac{1+3+5+7}{9+11+13+15} = \frac{1}{3}$.

(d) What is the next term in the sequence? Does it equal 1/3?
*(e) What is the 54th term in the sequence? Does it equal 1/3?

5-2
ADDITION AND SUBTRACTION OF FRACTIONS

We saw in the previous section how fractions help fill certain gaps in the whole numbers. By incorporating them into our system, we can solve more problems involving whole numbers, for we are assured of an answer when we divide one whole number by another (non-zero) whole number. But a new question arises: Can we divide (and add, subtract, and multiply) these new numbers themselves, or have we merely created a different breed of undefined problems?

Our experience with elementary school mathematics tells us that it is possible to perform these operations, so our next goal is to develop our mathematical structure to include the arithmetic of fractions. Again we investigate *why* things are done as they are, beginning with addition.

First among our objectives is finding a suitable definition for addition of fractions. Since the whole numbers are special cases of fractions, we know what must happen in certain instances. For one thing, it had better be true that $\frac{3}{1} + \frac{2}{1} = \frac{5}{1}$, or for any whole numbers p and r that $\frac{p}{1} + \frac{r}{1} = \frac{p+r}{1}$. Our intuition, based on the arithmetic of whole numbers, also strongly suggests that three things added to two of the same things ought to be five of these things. So $\frac{3}{7} + \frac{2}{7}$ "should be" $\frac{5}{7}$, or in general $\frac{p}{q} + \frac{r}{q} = \frac{p+r}{q}$. The less obvious decision is what to do with fractions with unlike denominators, such as $\frac{3}{4} + \frac{2}{7}$. Here we take advantage of the fact that these fractions have other names. We list some of the names for the fractions $\frac{3}{4}$ and $\frac{2}{7}$:

$$\text{Names for } \frac{3}{4}: \quad \frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \frac{12}{16}, \frac{15}{20}, \frac{18}{24}, \frac{21}{28}, \frac{24}{32}, \ldots$$

$$\text{Names for } \frac{2}{7}: \quad \frac{2}{7}, \frac{4}{14}, \frac{6}{21}, \frac{8}{28}, \frac{10}{35}, \frac{12}{42}, \frac{14}{49}, \frac{16}{56}, \ldots$$

If we handle matters properly, the problem $\frac{3}{4} + \frac{2}{7}$ should give the same answer as the problem $\frac{21}{28} + \frac{8}{28}$. Once the original problem is replaced by one with equivalent fractions having a common denominator, we simply add the numerators as we did in the previous example, this time getting 29/28.

Each of these problems is illustrated by an area model in Figure 5-7. In the first two problems, we are merely combining like objects directly. In part (a) each region is one unit, so we read off the answer as a total of five units, or 5/1. In part (b) each region is 1/7, so we read off the answer as a total of five 1/7's, or 5/7. In part (c) the horizontal lines are drawn in to show that 3/4 is equivalent to 21/28 and that 2/7 is equivalent to 8/28. Then we read off the answer as a total of 29 1/28's, or 29/28. Finding a common denominator for the two fractions is done by taking the product of the given denominators. Geometrically this means that cutting fourths into seven like pieces yields the same result as cutting sevenths into four like pieces.

When adding fractions with unlike denominators, we capitalize on their having alternate representations. To give a definition for adding fractions, we need only tell how to add fractions with common denominators, for we can always convert a problem involving the addition of fractions into one in which the denominators are the same.

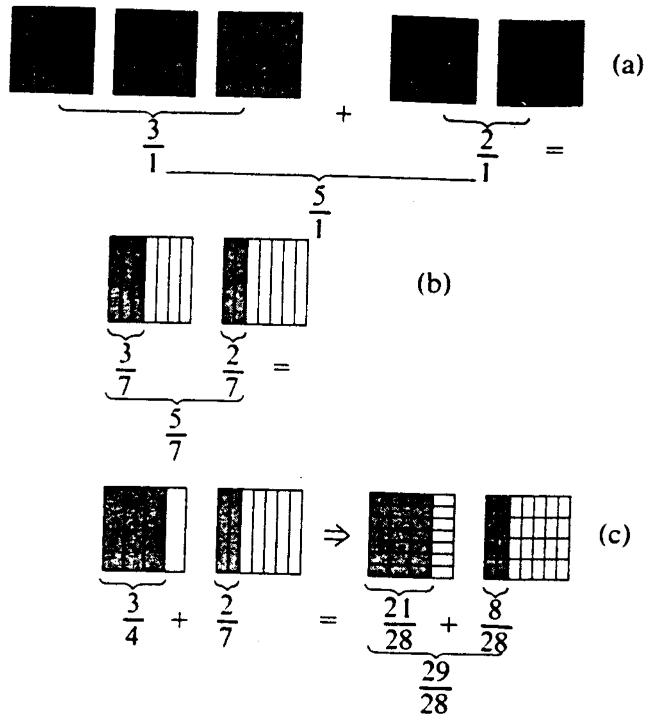


FIGURE 5-7

☐ Definition for Adding Fractions

> If $\frac{p}{q}$ and $\frac{r}{q}$ are fractions, then $\frac{p}{q} + \frac{r}{q}$ is called their **sum** and is defined to be $\frac{p + r}{q}$.

It may seem at first that little is accomplished by replacing $\frac{p}{q} + \frac{r}{q}$ by $\frac{p + r}{q}$. The point is that the "new" idea of having a plus sign between fractions is explained in terms of the "old" idea of having a plus sign between whole numbers.

We will now look at some examples, including several at a level beyond what is usually expected in elementary school mathematics.

Perform the following additions:

(a) $\frac{5}{8} + \frac{3}{4}$ (b) $\frac{1}{6} + \frac{2}{5} + \frac{3}{4}$ (c) $\frac{2}{153} + \frac{5}{204}$

(d) $\frac{1}{2^5 \times 3^6 \times 11^8} + \frac{1}{2^3 \times 3^{17} \times 7^2 \times 11}$

(a) Common denominators are easily recognizable. We could use 8, the LCM of 8 and 4; or we could use 32, the product of 8 and 4; or for that matter we could use 16 or 64 or many others. Using 8, we have $\frac{5}{8} + \frac{3}{4} = \frac{5}{8} + \frac{6}{8} = \frac{5+6}{8} = \frac{11}{8}$.

(b) When we have three or more fractions, it is easiest to find one common denominator for all the fractions. In this case LCM (6, 5, 4) = 60 is an easy-to-find common denominator. Thus $\frac{1}{6} + \frac{2}{5} + \frac{3}{4} = \frac{1 \times 10}{6 \times 10} + \frac{2 \times 12}{5 \times 12} + \frac{3 \times 15}{4 \times 15} = \frac{10}{60} + \frac{24}{60} + \frac{45}{60} = \frac{10 + 24 + 45}{60} = \frac{79}{60}$.

(c) One common denominator for 2/153 and 5/204 is the LCM of 153 and 204. Since $153 = 3^2 \times 17$ and $204 = 2^2 \times 3 \times 17$, LCM (153, 204) = $2^2 \times 3^2 \times 17 = 612$. So $\frac{2}{153} + \frac{5}{204} = \frac{2 \times 4}{153 \times 4} + \frac{5 \times 3}{204 \times 3} = \frac{8}{612} + \frac{15}{612} = \frac{8 + 15}{612} = \frac{23}{612}$. Alternatively, we could use the product of the denominators as a common denominator. Then $\frac{2}{153} + \frac{5}{204} = \frac{2 \times 204}{153 \times 204} + \frac{5 \times 153}{204 \times 153} = \frac{408}{31212} + \frac{765}{31212} = \frac{408 + 765}{31212} = \frac{1173}{31212}$. This answer is satisfactory, but may be reduced by a common factor of 51 to $\frac{23}{612}$.

(d) Again we can use the LCM $(2^5 \times 3^6 \times 11^8, 2^3 \times 3^{17} \times 7^2 \times 11) = 2^5 \times 3^{17} \times 7^2 \times 11^8$. Therefore,

$$\frac{1}{2^5 \times 3^6 \times 11^8} + \frac{1}{2^3 \times 3^{17} \times 7^2 \times 11}$$

$$= \frac{3^{11} \times 7^2}{2^5 \times 3^{17} \times 7^2 \times 11^8} + \frac{2^2 \times 11^7}{2^5 \times 3^{17} \times 7^2 \times 11^8}$$

$$= \frac{3^{11} \times 7^2 + 2^2 \times 11^7}{2^5 \times 3^{17} \times 7^2 \times 11^8}.$$

In our more careful look at the arithmetic of fractions, we should be awar- of a few concerns. We saw when adding whole numbers that the sum is necessarily a whole number, but when subtracting whole numbers that many problems (such as 2 − 6) are undefined. This suggests that we ought to check when our rule for adding fractions is defined. We admit as a fraction any :xpression which is a pair of whole numbers a/b where b is not 0. This means hat in our definition p, q, and r are whole numbers and that q isn't 0. Our rule ays that the sum is $\frac{p + r}{q}$. Is this always a fraction? Certainly q is an .cceptable denominator, since it is the denominator of the given fractions. The ıumerator $p + r$ is the sum of the two whole numbers and hence must be a 'hole number. Therefore $\frac{p + r}{q}$ is of the desired form and we can safely ay that the sum of two fractions is a fraction; or briefly, that the set of ·actions is closed under addition.

There is a second concern. Our definition applies to any two fractions having a common denominator. But there are many choices of common denominators for adding two arbitrary fractions. Will it matter which one we select? An illustration of this appeared in Example 5-4(c): $\frac{2}{153} + \frac{5}{204} = \frac{23}{612}$ when a common denominator of 612 is used, but $\frac{2}{153} + \frac{5}{204} = \frac{1173}{31212}$ when a common denominator of 31212 is used. Since $\frac{1173}{31212} = \frac{23 \times 51}{612 \times 51} = \frac{23}{612}$, the sums in this case are equivalent. We are asking now if we can depend on getting a well-defined sum for all choices of common denominators in each problem we do. To answer this question, we show first that the sum of two fractions using any common denominator is the same as the sum using the lowest common denominator. We use the fact that all common denominators are multiples of the lowest common denominator. (See Problem 16.) Suppose we begin with any two fractions, which when expressed in terms of a lowest common denominator are p/q and r/q. Let $\frac{p \times n}{q \times n}$ and $\frac{r \times n}{q \times n}$ be representations of these fractions with any other common denominator. We wish to show that the sum given by $\frac{p}{q} + \frac{r}{q}$ is equivalent to the sum given by $\frac{p \times n}{q \times n} + \frac{r \times n}{q \times n}$. Let's compute the second sum and show that it is the same as the first:

$\frac{p \times n}{q \times n} + \frac{r \times n}{q \times n} = \frac{p \times n + r \times n}{q \times n}$	Definition for adding fractions.
$= \frac{(p + r) \times n}{q \times n}$	Distributive property (for whole numbers)
$= \frac{p + r}{q}$	Reducing fractions principle
$= \frac{p}{q} + \frac{r}{q}$	Definition for adding fractions

This shows that when the definition is applied to two fractions using any common denominator we get the same result as when it is applied using the lowest common denominator. It follows that all choices of common denominator yield the same result. To say that we get the same sum regardless of choice of representation is to say that addition of fractions is well-defined.

Now that we see that our definition guarantees that all sums of fractions are defined (and well-defined), we can investigate for other properties. Since our definition applies only to fractions having a common denominator, we will assume that the fractions we are adding are in this form. This is not an unreasonable assumption, since we have just shown that the way in which the common denominator is selected does not alter the sum.

One property we can easily check is commutativity: Does the order in which we add affect the sum? For given fractions p/q and r/q, does $\frac{p}{q} + \frac{r}{q} = \frac{r}{q} + \frac{p}{q}$? According to the definition for adding fractions, this is the same as asking whether $\frac{p+r}{q} = \frac{r+p}{q}$. But this is a simple consequence of the commutative property for adding *whole numbers*, so we are done. If you prefer, this could be written out in this form:

$\frac{p}{q} + \frac{r}{q} = \frac{p+r}{q}$	Definition for adding fractions
$= \frac{r+p}{q}$	Commutative property for adding whole numbers
$= \frac{r}{q} + \frac{p}{q}$	Definition for adding fractions

It should not be surprising that the commutative property for adding fractions depends on the commutative property for adding whole numbers, since addition of fractions is defined in terms of addition of whole numbers.

Another thing to check is whether there is an identity element for addition in the set of fractions, a member of the set which when added to any fraction gives that fraction as the sum. We know that 0 is the identity for adding whole numbers, so the likely candidate among fractions is 0/1, a fraction that corresponds to the whole number 0. We find that for any fraction p/q we have $\frac{p}{q} + \frac{0}{1} = \frac{p}{q} + \frac{0 \times q}{1 \times q} = \frac{p}{q} + \frac{0}{q} = \frac{p+0}{q} = \frac{p}{q}$. Since we know addition of fractions is commutative, we also have $\frac{0}{1} + \frac{p}{q} = \frac{p}{q}$, so (since it has the desired property whether added on the right or on the left) 0/1 is an identity for adding fractions. Notice that in saying $\frac{p+0}{q} = \frac{p}{q}$ we invoked the identity property for whole numbers, so again a property for fractions depends on the corresponding property for whole numbers.

Three properties for adding fractions which are analogs of properties of adding whole numbers — closure, commutativity, and identity — have been mentioned so far. Can you think of any other properties that might exist? Do you see a pattern developing? Once we establish a definition for addition of fractions in terms of addition of whole numbers, all the "new" properties are simple consequences of the "old" ones.

This highly desirable feature of our work encourages us to define subtraction of fractions in a similar way: express the given fractions in terms of a common denominator and then define $\frac{p}{q} - \frac{r}{q} = \frac{p-r}{q}$. Examples:

$$\frac{3}{4} - \frac{1}{4} = \frac{3-1}{4} = \frac{2}{4}$$

$$\frac{7}{10} - \frac{4}{10} = \frac{7-4}{10} = \frac{3}{10}$$

$$\frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6} = \frac{3-2}{6} = \frac{1}{6}$$

$\frac{1}{8} - \frac{7}{8} = \frac{1-7}{8}$ is undefined, since $1 - 7$ is undefined.

Something else is inherited from whole number arithmetic! Since we cannot subtract a larger whole number from a smaller one (and get a whole number answer), neither can we subtract a larger fraction from a smaller one (and get a fraction for an answer). Thus $\frac{p}{q} - \frac{r}{q}$ is defined only when $\frac{p}{q} \geq \frac{r}{q}$. Since it is not always possible to subtract one arbitrary fraction from another, we say that the set of fractions is *not* closed under subtraction. (Recall that the whole numbers are not closed under subtraction, either.)

What about other properties for subtraction of fractions? We will investigate the existence of an identity and let you decide for yourself about the others. Again we get a clue from whole number arithmetic, where we found that 0 is an identity "on the right" (for instance, $17 - 0 = 17$) but not "on the left" ($0 - 17$ is undefined). We try 0/1 as a candidate for the identity. As we might expect, for an arbitrary fraction $\frac{p}{q}$, $\frac{p}{q} - \frac{0}{1} = \frac{p}{q} - \frac{0 \times q}{1 \times q} = \frac{p}{q} - \frac{0}{q} = \frac{p-0}{q} = \frac{p}{q}$, so it serves the purpose on the right, but $\frac{0}{1} - \frac{p}{q} = \frac{0 \times q}{1 \times q} - \frac{p}{q} = \frac{0}{q} - \frac{p}{q} = \frac{0-p}{q}$, which is not defined unless p is 0. Since it fails to work on the left, 0/1 is not an identity element for subtraction. Actually, the restriction on the definition (that larger fractions cannot be subtracted from smaller ones) eliminates the possibility of a two-sided identity. For instance, if a/b is the identity we must have (among other things) $\frac{1}{2} - \frac{a}{b} = \frac{1}{2}$ and $\frac{a}{b} - \frac{1}{2} = \frac{1}{2}$. But if $\frac{1}{2} - \frac{a}{b}$ is defined, a/b must be no more than 1/2, and if $\frac{a}{b} - \frac{1}{2}$ is defined, a/b must be no less than 1/2. The only possibility is that $a/b = 1/2$, in which case $\frac{1}{2} - \frac{a}{b} = \frac{1}{2}$ fails.

Both addition and subtraction of fractions can be represented on the number line with a few modifications of what we did for the whole numbers.

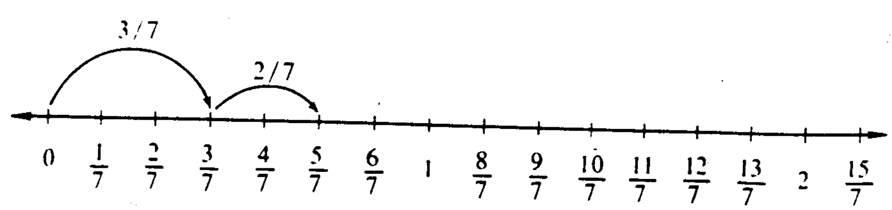
EXAMPLE 5.5

Represent each of these on the number line:

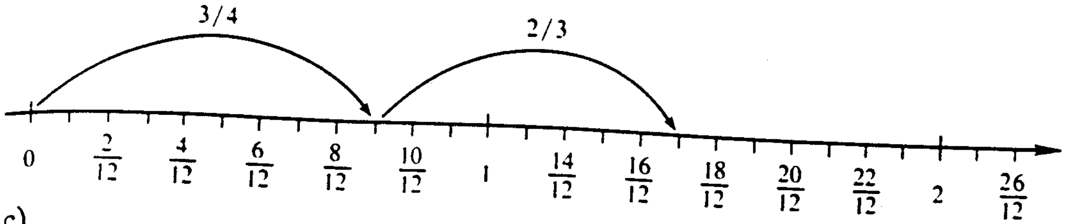
(a) $\frac{3}{7} + \frac{2}{7}$ (b) $\frac{3}{4} + \frac{2}{3}$ (c) $\frac{3}{4} - \frac{1}{4}$

SOLUTION

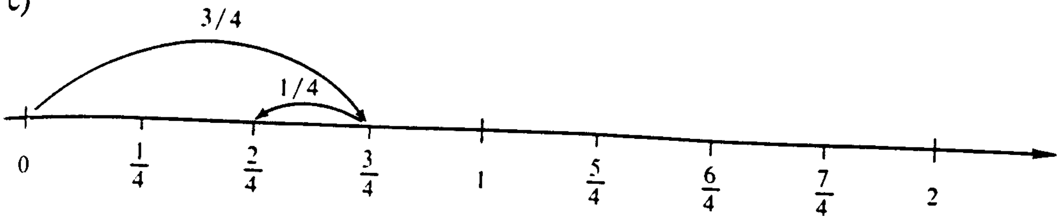
(a)



(b) To display this problem, it is easiest to use first our knowledge of equivalent fractions to express $\frac{3}{4} + \frac{2}{3}$ in terms of a common denominator as $\frac{9}{12} + \frac{8}{12}$, and then to mark off the number line in twelfths.



c)



We summarize this section by noting that addition and subtraction of ·actions are defined in terms of addition and subtraction of whole numbers, sing the fact that fractions can be expressed in terms of common enominators. The fractions inherit all the properties the whole numbers ossess with respect to addition and lack the properties the whole numbers lack ith respect to subtraction. A few properties remain to be checked in the ·oblems.

EXERCISES AND PROBLEMS 5-2

DEFINITION

1. Give three different common denominators for each pair of fractions:
   (a) $\frac{2}{3}, \frac{5}{6}$ (b) $\frac{4}{5}, \frac{7}{9}$ (c) $\frac{p}{q}, \frac{r}{s}$
2. Tell in each case whether the "+" refers to the addition of fractions or to the addition of whole numbers: (a) $\frac{p+r}{q}$ (b) $\frac{p}{q} + \frac{r}{s}$ (c) $\frac{p}{q+s}$
3. Perform the following additions:
   (a) $\frac{3}{7} + \frac{1}{7}$ (b) $\frac{1}{2} + \frac{1}{4}$ (c) $\frac{1}{6} + \frac{2}{7} + \frac{3}{8}$
   (d) $\frac{9}{21} + \frac{1}{39}$ (e) $\frac{3}{10} + \frac{7}{100}$ (f) $\frac{1}{10} + \frac{31}{1000}$
   (g) $\frac{7}{869} + \frac{2}{237}$ (h) $\frac{1}{2^3 \times 3^7} + \frac{1}{2^5 \times 3^2}$
   (i) $\frac{1}{2^3 \times 3^{20} \times 7^{11}} + \frac{1}{3^4 \times 5^3 \times 7^{15}}$
4. Display the problem $\frac{3}{4} + \frac{2}{5}$ using (a) an area model. (b) a number line.
5. The discussion about the sum of two fractions says that "the sum of two fractions is a fraction." Yet $\frac{1}{4} + \frac{3}{4} = 1$, so the sum of two fractions is sometimes a whole number. Explain. (Four words should be sufficient.)

6. Perform the following subtractions (when possible):
   (a) $\frac{3}{7} - \frac{1}{7}$ (b) $\frac{1}{2} - \frac{1}{4}$ (c) $\frac{2}{91} - \frac{1}{39}$
   (d) $\frac{1}{39} - \frac{2}{91}$ (e) $\frac{7}{10} - \frac{3}{100}$ (f) $\frac{1}{10} - \frac{31}{1000}$
   (g) $\frac{1}{8051} - \frac{1}{13{,}363}$

7. Find a value of b such that $\frac{1}{10} - \frac{963}{b}$ is defined. Find a value of a such that $\frac{1}{10} - \frac{a}{8}$ is defined.

8. Split this set into two sets so that the sums of the fractions in the two sets are the same.

$\left\{ \frac{1}{3}, \frac{1}{2}, \frac{1}{12}, \frac{1}{4}, \frac{1}{6} \right\}$

PROPERTIES

9. We found that addition of fractions satisfies closure, commutative, and identity properties. Name one or more other possible properties and check to see if they hold, giving evidence to support your answer.

10. 0/1 is an identity for adding fractions. Are there any others? How many different *numbers* are there that serve as an identity for the addition of fractions?

11. How does the answer obtained when applying the definition of subtracting fractions to $\frac{a \times n}{b \times n} - \frac{c \times n}{b \times n}$ compare with the answer obtained when the definition is applied to $\frac{a}{b} - \frac{c}{b}$?

12. Formulate a "missing addend" description for the definition of subtraction. Show that it is equivalent to the definition given.

13. We found that subtraction of fractions lacks the closure and identity properties. Name one or more other possible properties and check to see if they hold, giving evidence to support your answer.

14. Sometimes we do arithmetic with "mixed numbers," such as $2\frac{3}{8}$.
   (a) Give a definition for expressions of this type.
   (b) Give a definition for addition of mixed numbers. Verify that the sum obtained by your definition agrees with the sum obtained by using the definition $\frac{p}{q} + \frac{r}{q} = \frac{p + r}{q}$.

15. Suppose we defined addition of fractions by $\frac{p}{q} \oplus \frac{r}{s} = \frac{p + r}{q + s}$. (We use the symbol "$\oplus$" to distinguish it from ordinary addition of fractions.) By this definition $\frac{1}{2} \oplus \frac{3}{4} = \frac{4}{6}$.
   (a) Is $\oplus$ defined for all fractions?
   (b) Is it well-defined? (Do we get an equivalent answer when fractions are replaced by equivalent fractions?)

(c) Does it agree with the arithmetic of whole numbers? (For instance, does 2 + 3 still equal 5 when interpreted as fractions?)
(d) Does $\oplus$ satisfy a commutative property?
(e) Associative property?
(f) Identity property?
(g) Bill calls this operation "baseball addition." Why? (Hint: Think of "hits" and "at bats.")

16. Show that every common denominator is a multiple of the lowest common denominator. (Hint: See Problem 18, Section 4-3.)

MATERIALS AND CALCULATORS

17. To illustrate $\frac{1}{3} + \frac{1}{2}$ using Cuisenaire rods (see Problem 31, Section 5-1), which single rod should be used to represent
(a) 1? (b) 1/3? (c) 1/2?
(d) Explain how to get the answer in terms of the rods. (Be sure to give the answer!)

18. Explain how Cuisenaire rods might be used to illustrate subtraction of fractions.

19. Consult several elementary school mathematics series (texts). Identify which ones use area models for fractions; number line models.

20. Suppose a child writes "$\frac{2}{3} + \frac{4}{5} = \frac{6}{8}$." How can you convince him this is incorrect?

P21. Write programs to do the following for fractions, giving *exact* answers. (For example, for $\frac{1}{2} + \frac{1}{3}$ the program gives 5/6, not a decimal approximation.) The answers need *not* be in lowest terms.
(a) the sum of two fractions
(b) the difference of two fractions

P22. Write programs to generate
(a) the geometric series of Problem 27
(b) the harmonic series of Problem 28.

PROBLEM SOLVING EXTENSIONS

23. Find the largest fraction that has a denominator of 17 and when added to 1/3 gives an answer less than 1.

24. (a) Express each of the following as a simple fraction:

$$\frac{1}{1},\quad \frac{1}{1+\frac{1}{1}},\quad \frac{1}{1+\cfrac{1}{1+\frac{1}{1}}},\quad \frac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{1}}}}$$

(b) What are the next three simple fractions in the sequence?

25. (a) The divisors of 6 are 1, 2, 3, and 6. Compute $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$.
(b) The divisors of 28 are 1, 2, 4, 7, 14, and 28. Compute $\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{7} + \frac{1}{14} + \frac{1}{28}$.

*(c) Recall from Problem 23 in Section 4-1 that 6 and 28 are "perfect numbers." Will a similar result hold for other perfect numbers? Explain.

*26. In ancient times, only **unit fractions** (fractions with numerators of 1) like $\frac{1}{2}, \frac{1}{3}$, and $\frac{1}{8}$ were used. Other fractions were expressed as sums of different unit fractions. Examples: $\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$; $\frac{3}{10} = \frac{1}{4} + \frac{1}{20}$; $\frac{5}{7} = \frac{1}{2} + \frac{1}{5} + \frac{1}{70}$. Express each of the following as sums of different unit fractions:

(a) $\frac{3}{4}$ (b) $\frac{6}{11}$ (c) $\frac{2}{7}$ (d) $\frac{4}{13}$ (e) $\frac{53}{87}$

27. The unending series $\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ is an example of an infinite geometric series, one in which each term is a fixed multiple (in this case 1/2) of the previous term. The sum of the first three terms is $\frac{1}{1} + \frac{1}{2} + \frac{1}{4} = \frac{7}{4} + 1\frac{3}{4}$.
(a) Find the sum of the first four terms, first five terms, first six terms.
(b) Guess the sum of the unending series.
(c) Guess the sum of $\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000} + \ldots$

*28. The unending series $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$ is called the **harmonic series**. The sum of the first three terms is $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} = \frac{11}{6} = 1\frac{5}{6}$.
(a) Find the sum of the first four terms, first five terms, first six terms.
(b) Guess the sum of the unending series.
(c) Note the following.

$$\frac{1}{1} + \frac{1}{2} = 1\frac{1}{2}$$

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > \frac{1}{1} + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}} = 2$$

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$$

$$> \frac{1}{1} + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}} = 2\frac{1}{2}$$

Show that the sum of the first 16 terms exceeds 3.
(d) Guess the sum of the unending series in light of part (c).

5-3
MULTIPLICATION AND DIVISION OF FRACTIONS

So far in our extension of the whole numbers to the fractions we have proceeded straightforwardly and encountered no difficulties. We have gained the ability to divide one whole number by another (non-zero) whole number, which is one of the chief reasons for inventing fractions, without encumbering the addition and subtraction processes. The remaining task is to develop satisfactory methods for multiplication and division of fractions. Our approach will closely parallel what we did in the preceding section, with a few important new ideas introduced along the way.

To introduce multiplication and division of fractions we will once again call upon area models. Just as the product 3×4 is the number of square units (or area) in a rectangle whose dimensions are 3 units by 4 units, we can interpret $\frac{2}{3} \times \frac{5}{7}$ as the number of square units (of area) in a rectangle whose dimensions are 2/3 units by 5/7 units, as shown in Figure 5-8.

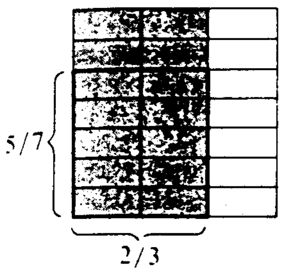


FIGURE 5-8

We see that if we use vertical lines to divide the original unit square into three like pieces and take two of these pieces, we have a (shaded) rectangle with a width of 2/3. If we use horizontal lines to divide the shaded rectangle into seven like pieces and take five of these pieces, we have (inside the heavy line) a rectangle whose dimensions are 2/3 by 5/7. It is easy to determine what fraction of the unit area this (heavy line) rectangle represents. It contains 2×5 of the smallest pieces, while there are 3×7 of these pieces in the entire unit square. Thus its area is $\frac{2 \times 5}{3 \times 7}$, so in terms of our area model $\frac{2}{3} \times \frac{5}{7} = \frac{2 \times 5}{3 \times 7}$. Since we have taken *part* (5/7) of the shaded area, which is *part* (2/3) of the original square, we can think of the product of fractions in terms of a "part-of-part" model.

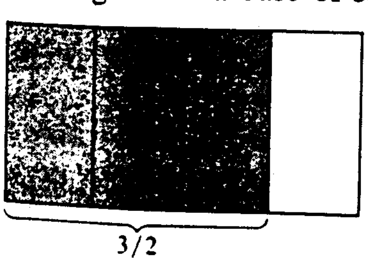
Only slight modification is needed when one or both of the fractions exceed 1.

EXAMPLE 5-6

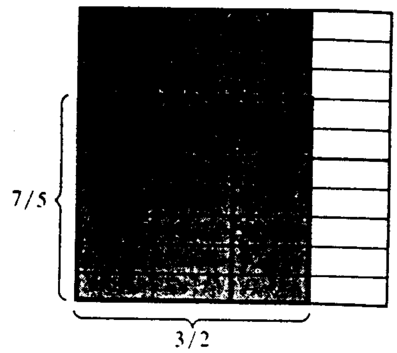
Represent the product $\frac{3}{2} \times \frac{7}{5}$ using an area model.

SOLUTION

To show 3/2, we need more than one unit. Since we are going to build a rectangle with a base of 3/2, we place two unit squares as shown.



To show a height of 7/5, we need to extend the shaded rectangle upward by adding two more unit squares.



Now we have inside the heavy line a rectangle with dimensions 3/2 by 7/5. Each unit square has been cut up into 2 × 5 pieces. In all, we have 3 × 7 of these pieces. Thus the area of the rectangle (inside the heavy line) is the fraction $\frac{3 \times 7}{2 \times 5}$. Our model suggests that $\frac{3}{2} \times \frac{7}{5} = \frac{3 \times 7}{2 \times 5}$. From this it seems natural to define the product of two arbitrary fractions p/q and r/s by $\frac{p}{q} \times \frac{r}{s} = \frac{p \times r}{q \times s}$.

This product is defined for all fractions p/q and r/s, for if p and r are whole numbers, so is $p \times r$; and if q and s are non-zero whole numbers, so is $q \times s$. Is it well-defined — namely, do we get the same answer regardless of representation? First let's try $\frac{3}{5} \times \frac{2}{5}$. By our new definition, $\frac{3}{5} \times \frac{2}{5} = \frac{3 \times 2}{5 \times 5} = \frac{6}{25}$. Applying this definition to the equivalent problem, $\frac{6}{10} \times \frac{4}{10} = \frac{6 \times 4}{10 \times 10} = \frac{24}{100} = \frac{6}{25}$, so at least it is agreeable for this example. This is quite easy to show in the general setting. First we show that the product of any two fractions is equivalent to the product of their reduced representatives. Suppose p/q and r/s are reduced fractions and that $\frac{p \times m}{q \times m}$ and $\frac{r \times n}{s \times n}$ are respective equivalent fractions. Then we begin by applying the definition to the unreduced representatives and show that this is equivalent to the product determined by the reduced fractions:

$$\frac{p \times m}{q \times m} \times \frac{r \times n}{s \times n} = \frac{(p \times m) \times (r \times n)}{(q \times m) \times (s \times n)} = \frac{(p \times r) \times (m \times n)}{(q \times s) \times (m \times n)}$$

$$= \frac{p \times r}{q \times s} = \frac{p}{q} \times \frac{r}{s}$$

Since each representation gives an answer equivalent to the result when the product is written in terms of reduced fractions, it follows that all representations give the same answer.

Since this definition of multiplying fractions is defined for all fractions, yields equivalent results for equivalent problems, and agrees with our area model (in the typical examples we have investigated), we are persuaded to adopt it.

Definition for Multiplying Fractions □

> If $\frac{p}{q}$ and $\frac{r}{s}$ are fractions, then $\frac{p}{q} \times \frac{r}{s}$ is called their **product** and is defined to be $\frac{p \times r}{q \times s}$.

Observe that once again a "new" idea (multiplying fractions) is defined in terms of an "old" idea (multiplying whole numbers). This will make the questions of commutativity, associativity, and identity for multiplying fractions easy to deal with.

Another old friend, distributivity, can also be probed. Does multiplication distribute over addition? As before, we can safely assume that the fractions being added have the same denominator, as the process of changing to a common denominator has no bearing on the results. Recall the distributive law for whole numbers a, b, and c:

$$a \times (b + c) = a \times b + a \times c$$

The analog for fractions p/q, r/s, and u/s is:

$$\frac{p}{q} \times \left(\frac{r}{s} + \frac{u}{s}\right) = \frac{p}{q} \times \frac{r}{s} + \frac{p}{q} \times \frac{u}{s}$$

Our technique will be to evaluate both sides of this equation and compare. We begin with the expression on the left side and apply the definition for adding fractions (which the parentheses indicate is to be done first) followed by the definition for multiplying fractions:

$$\frac{p}{q} \times \left(\frac{r}{s} + \frac{u}{s}\right) =$$
$$\frac{p}{q} \times \frac{r+u}{s} =$$
$$\frac{p \times (r + u)}{q \times s}$$

We arrive at a fraction whose numerator and denominator are expressed in terms of sums and products of whole numbers. Evaluating the right side of the original expression involves first multiplying, then adding:

$$\frac{p}{q} \times \frac{r}{s} + \frac{p}{q} \times \frac{u}{s} =$$
$$\frac{p \times r}{q \times s} + \frac{p \times u}{q \times s} =$$
$$\frac{p \times r + p \times u}{q \times s}$$

...again arrive at a fraction whose numerator and denominator are expressed ...erms of sums and products of whole numbers. How do the fractions $\frac{\ldots(r + u)}{\ldots q \times s}$ and $\frac{p \times r + p \times u}{q \times s}$ compare? Since they have the same ...ominator, they will be equivalent if their numerators are equal; namely, if ...$\times (r + u) = p \times r + p \times u$. But this is merely a statement of the distributive ...perty for whole numbers, so the fractions are clearly equal.

This completes our proof that multiplication distributes over addition. The distributive property for fractions is seen to depend on the distributive property for whole numbers, still another example of the mathematical structure underlying arithmetic.

Recapping our treatment so far, we see that introducing fractions gains the advantage of being able to divide one whole number by another (non-zero) whole number and that these fractions themselves can be added, subtracted, and multiplied according to new rules which are consistent with the old rules governing whole numbers. The final challenge is to arrange for division of fractions to fit reasonably into the scheme.

The usual approach of inverting the divisor and multiplying can be obtained by directly paralleling the alternative definition for dividing whole numbers. Recall the missing factor interpretation for division of whole numbers:

> If a and b are whole numbers (and b is not 0), then the quotient of a and b is the (whole number) answer to the question $b \times \square = a$.

If we rewrite this in terms of fractions, we have:

> If p/q and r/s are fractions (and r/s is not 0), then the quotient of p/q and r/s is the (fraction) answer to the question $\frac{r}{s} \times \square = \frac{p}{q}$.

First we'll do a specific example involving numbers. Consider the problem $\frac{2}{3} \div \frac{4}{5} = \square$. It is equivalent to the problem $\frac{4}{5} \times \square = \frac{2}{3}$. We now show that the solution is $\frac{2}{3} \times \frac{5}{4}$.

$$\frac{4}{5} \times \boxed{\frac{2}{3} \times \frac{5}{4}} = \frac{4}{5} \times \frac{2 \times 5}{3 \times 4}$$

$$= \frac{4 \times (2 \times 5)}{5 \times (3 \times 4)}$$

$$= \frac{2 \times (4 \times 5)}{3 \times (4 \times 5)}$$

$$= \frac{2}{3}$$

This shows that $\frac{2}{3} \times \frac{5}{4}$ is precisely the solution to $\frac{2}{3} \div \frac{4}{5} = \square$.

We now proceed to show that $\frac{p}{q} \times \frac{s}{r}$, which is obtained from $\frac{p}{q} \div \frac{r}{s}$ by inverting the divisor, is precisely the quantity that satisfies $\frac{r}{s} \times \square = \frac{p}{q}$, or, in other words, satisfies $\frac{p}{q} \div \frac{r}{s} = \square$. Let's fill in $\frac{p}{q} \times \frac{s}{r}$ for $\square$ and evaluate $\frac{r}{s} \times \square$.

$\frac{r}{s} \times \left(\frac{p}{q} \times \frac{s}{r}\right) = \frac{r}{s} \times \frac{p \times s}{q \times r}$ (Definition for multiplying fractions)

$= \frac{r \times (p \times s)}{s \times (q \times r)}$ (Definition for multiplying fractions)

$= \frac{p \times (r \times s)}{q \times (r \times s)}$ (Commutative and associative properties for multiplying whole numbers)

$= \frac{p}{q}$ (Reducing fractions principle)

This shows that $\frac{p}{q} \times \frac{s}{r}$ is the desired quotient according to the missing factor interpretation, which we applied to division of fractions. It leads to the

Definition for Dividing Fractions ☐

> If $\frac{p}{q}$ and $\frac{r}{s}$ are fractions (and $\frac{r}{s}$ is not 0), then $\frac{p}{q} \div \frac{r}{s}$ is called their **quotient** and is defined to be $\frac{p}{q} \times \frac{s}{r}$.

This definition tells us that to divide one fraction by another we can invert (the second fraction) and multiply (by the first fraction). So long as r/s is a fraction different from 0, s/r will be a fraction and $\frac{p}{q} \times \frac{s}{r}$ will be defined. This tells us that $\frac{p}{q} \div \frac{r}{s}$ is defined for all fractions p/q and non-zero fractions r/s. Since multiplication of fractions has been shown to yield an unambiguous answer regardless of choice of representation, we are guaranteed that $\frac{p}{q} \div \frac{r}{s}$ is also well-defined.

The fraction s/r obtained by inverting r/s is called the **reciprocal** or **multiplicative inverse** of r/s. Notice that the reciprocal of r/s is the fraction which when multiplied by r/s gives 1:

$$\frac{r}{s} \times \frac{s}{r} = \frac{r \times s}{s \times r} = \frac{r \times s}{r \times s} = \frac{1}{1} = 1$$

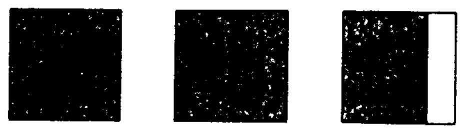
Some examples of reciprocals are:

Number	*Reciprocal*
2/3	3/2
6	1/6
0	does not exist

0 has no reciprocal, since there is no number which can be multiplied by 0 to give 1. All other fractions do have reciprocals.

This chapter concludes with a brief discussion of arithmetic with so-called mixed numbers, numbers like $2\frac{3}{4}$. If we interpret $2\frac{3}{4}$ as two whole units plus $\frac{3}{4}$ of another unit,

we can convert it to an ordinary fraction by thinking of everything in terms of fourths.



Then, $2\frac{3}{4}$ is easily recognizable as $\frac{11}{4}$.

Given a problem involving mixed numbers, we can readily convert it to one involving ordinary fractions. For instance, to add $2\frac{3}{4} + 5\frac{2}{7}$ we can first rewrite $2\frac{3}{4}$ as $\frac{11}{4}$ and rewrite $5\frac{2}{7}$ as $\frac{37}{7}$. Then:

$$\begin{aligned} 2\frac{3}{4} + 5\frac{2}{7} &= \frac{11}{4} + \frac{37}{7} \\ &= \frac{77}{28} + \frac{148}{28} \\ &= \frac{77 + 148}{28} \\ &= \frac{225}{28} \\ &= 8\frac{1}{28} \end{aligned}$$

In practice it is not necessary when adding (or subtracting) mixed numbers to change to fractions. Another way of doing this problem is to think of $2\frac{3}{4}$ as $\frac{2}{1} + \frac{3}{4}\left(\text{or simply } 2 + \frac{3}{4}\right)$ and to think of $5\frac{2}{7}$ as $\frac{5}{1} + \frac{2}{7}$ $\left(\text{or simply } 5 + \frac{2}{7}\right)$. Then:

$$\begin{aligned} 2\frac{3}{4} + 5\frac{2}{7} &= 7 + \left(\frac{3}{4} + \frac{2}{7}\right) \\ &= 7 + \frac{29}{28} \\ &= 7 + 1\frac{1}{28} \\ &= 8\frac{1}{28} \end{aligned}$$

It is always true that to add mixed numbers we can add the whole number parts and add the fractional parts and form a new mixed number.

To multiply or divide mixed numbers, it is usually easiest to convert them to ordinary fractions. For example, to find $3\frac{2}{5} \div 4\frac{3}{7}$ we change these mixed numbers to $\frac{17}{5}$ and $\frac{31}{7}$ respectively:

$$\begin{aligned} 3\frac{2}{5} \div 4\frac{3}{7} &= \frac{17}{5} \div \frac{31}{7} \\ &= \frac{17}{5} \times \frac{7}{31} \\ &= \frac{17 \times 7}{5 \times 31} \\ &= \frac{119}{155} \end{aligned}$$

Since we can readily think of mixed numbers as ordinary fractions, they are worthy of no further discussion. Once again we are able to dispense with a new form of arithmetic in terms of something mastered earlier. In fact, the next example illustrates a whole chain of converting given problems to easier ones.

Problem	*Type*
$3\frac{2}{5} \div 4\frac{3}{7}$	Division of mixed numbers
$\frac{17}{5} \div \frac{31}{7}$	Division of fractions
$\frac{17}{5} \times \frac{7}{31}$	Multiplication of fractions
$\frac{17 \times 7}{5 \times 31}$	Multiplication of whole numbers

In the next chapter we will consider ratios, decimals, and percents, all of which are closely related to fractions. Following that, we will continue extending our system of numbers to remedy other deficiencies.

EXERCISES AND PROBLEMS 5-3

DEFINITION

1. Perform the following operations:

(a) $\frac{2}{3} \times \frac{5}{8}$ (b) $\frac{3}{10} \times \frac{7}{100}$ (c) $\frac{5}{9} \times \frac{3}{7} + \frac{5}{9} \times \frac{4}{7}$

(d) $\frac{2}{3} + \frac{4}{5} \times \frac{6}{7}$ (e) $\frac{2}{3} \times \frac{4}{5} + \frac{6}{7}$ (f) $\frac{2}{3} \times \left(\frac{4}{5} + \frac{6}{7}\right)$

(g) $3\frac{1}{8} \times 6\frac{2}{3}$ (h) $\frac{2}{3} \div \frac{5}{6}$ (i) $2\frac{3}{7} \div 3\frac{4}{11}$

(j) $\frac{13}{10} \div \frac{7}{100}$

2. Expressions of the form $\frac{p/q}{r/s}$, called **complex fractions**, occur occasionally in arithmetic. They can be changed to ordinary fractions as follows:

$$\frac{\frac{p}{q}}{\frac{r}{s}} = \frac{\frac{p}{q} \times q \times s}{\frac{r}{s} \times q \times s} = \frac{p \times s}{r \times q}$$

Example:

$$\frac{\frac{2}{3}}{\frac{5}{7}} = \frac{\frac{2}{3} \times 3 \times 7}{\frac{5}{7} \times 3 \times 7} = \frac{2 \times 7}{5 \times 3} = \frac{14}{15}$$

Express each of these as ordinary fractions:

(a) $\dfrac{\frac{5}{8}}{\frac{3}{7}}$ (b) $\dfrac{1\frac{2}{3}}{5\frac{3}{4}}$

3. Find reciprocals of (a) 9/10 (b) 11.

4. Which fraction(s) have no reciprocal?

5. (a) Find the reciprocal of the reciprocal of 5/8.
(b) Find the reciprocal of the multiplicative inverse of 5/8.

6. (a) Does $1\frac{2}{3} + 4\frac{5}{6} = 1\frac{5}{6} + 4\frac{2}{3}$? Explain.
(b) Does $1\frac{2}{3} \times 4\frac{5}{6} = 1\frac{5}{6} \times 4\frac{2}{3}$? Explain.

7. Determine the following, answering in mixed numbers reduced to lowest terms.
(a) $6\frac{1}{5} + 4\frac{2}{3}$ (b) $6\frac{1}{5} - 4\frac{2}{3}$ (c) $6\frac{1}{5} \times 4\frac{2}{3}$ (d) $6\frac{1}{5} \div 4\frac{2}{3}$

PROPERTIES

8. Suppose for the fractions $\frac{p}{q}$ and $\frac{r}{s}$ that $\frac{p}{q} < \frac{r}{s}$. What can you say about the order of their reciprocals?

9. Is multiplication of fractions
(a) commutative? (b) associative?
(c) Is there a (two-sided) identity?

10. Is division of fractions
(a) commutative? (b) associative?
(c) Is there a (two-sided) identity?

11. Are fractions closed under
(a) addition? (b) subtraction?
(c) multiplication? (d) division?

12. Are the positive fractions closed under
(a) addition? (b) subtraction?
(c) multiplication? (d) division?

13. Display $\frac{2}{5} \times \frac{3}{4}$ using an area model. Do the same for $\frac{9}{7} \times \frac{5}{8}$.

14. Supply reasons for each statement in this proof of the distributive property for fractions:

$$\begin{aligned}\frac{p}{q} \times \left(\frac{r}{s} + \frac{u}{s}\right) &= \frac{p}{q} \times \left(\frac{r+u}{s}\right)\\ &= \frac{p \times (r+u)}{q \times s}\\ &= \frac{p \times r + p \times u}{q \times s}\\ &= \frac{p \times r}{q \times s} + \frac{p \times u}{q \times s}\\ &= \frac{p}{q} \times \frac{r}{s} + \frac{p}{q} \times \frac{u}{s}\end{aligned}$$

15. The usual distributive law is referred to by saying "multiplication (of fractions) distributes over addition (of fractions)." Which of these forms of the distributive law holds for arbitrary fractions?
(a) multiplication over subtraction
(b) multiplication over division
(c) addition over multiplication
(d) division over addition

16. Does the following distributive property hold for fractions? Explain why.

$$\frac{p}{q} \div \left(\frac{r}{s} + \frac{u}{s}\right) = \left(\frac{p}{q} \div \frac{r}{s}\right) + \left(\frac{p}{q} \div \frac{u}{s}\right)$$

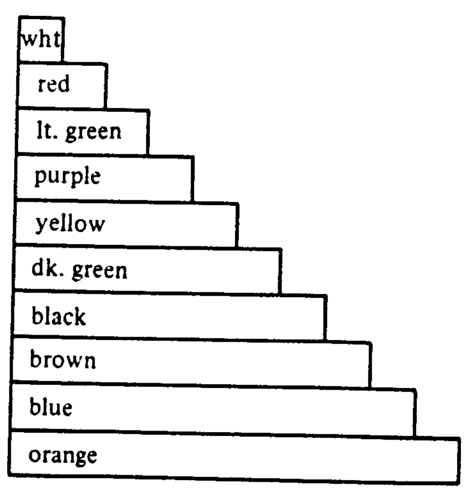
17. Suppose a, b, c, d, e, and f are all positive whole numbers. Write $\frac{a}{b} \times \left(\frac{c}{d} + \frac{e}{f}\right)$ as a single fraction involving these six letters.

18. Extending the whole numbers to include the fractions makes division (except by 0) possible. Can you anticipate what will be accomplished by extending later to include negative numbers?

19. The positive fractions satisfy a property not satisfied by the whole numbers. What is this property?

MATERIALS AND CALCULATORS

20. We wish to illustrate the division problem $\frac{3}{5} \div \frac{1}{2}$ using Cuisenaire rods. (See Problem 31, Section 5-1.)
(a) Which single rod should we use to represent 1? Why?
(b) Which single rod should we use to represent 1/2? Why?
(c) How do we represent 3/5?
(d) Finish explaining how to use the rods to solve $\frac{3}{5} \div \frac{1}{2}$, giving the answer to the problem and telling how to get the answer with the rods.



C21. If we use a calculator to multiply fractions by first converting them to decimals, is the answer usually exact or usually approximate? Give an example of each.

P22. Write programs to do the following for fractions giving *exact* answers, *not* decimal approximations (the answers need *not* be in lowest terms):
(a) the product of two fractions (b) the quotient of two fractions

PROBLEM SOLVING EXTENSIONS

23. Gwen multiplies her fractions like this:

$$\frac{1}{3} \times \frac{2}{3} = \frac{2}{3} \qquad \frac{1}{2} \times \frac{1}{3} = \frac{3}{6} \times \frac{2}{6} = \frac{6}{6} = 1 \qquad \frac{1}{4} \times \frac{3}{8} = \frac{2}{8} \times \frac{3}{8} = \frac{6}{8} = \frac{3}{4}$$

(a) What would you expect her to get for the problem $\frac{1}{6} \times \frac{2}{3}$?
(b) Why do you think she multiplies fractions this way?
(c) How can you help her?

24. (a) What is Jennifer's error in the work shown below? Describe the error pattern.

$$\begin{array}{r} 6\frac{7}{9} \\ -3\frac{2}{9} \\ \hline 3\frac{5}{9} \end{array} \qquad \begin{array}{r} 8\frac{1}{9} \\ -2\frac{4}{9} \\ \hline 5\frac{7}{9} \end{array} \qquad \begin{array}{r} 15\frac{2}{7} \\ -8\frac{6}{7} \\ \hline 6\frac{6}{7} \end{array} \qquad \begin{array}{r} 7\frac{1}{10} \\ -2\frac{3}{10} \\ \hline 4\frac{8}{10} \end{array} = 4\frac{4}{5}$$

(b) What would you expect her to get for this problem? $\begin{array}{r} 12\frac{1}{8} \\ -7\frac{5}{8} \\ \hline \end{array}$

(c) Make up another problem she'd probably get right.
(d) How can you help her?

25. Ms. Kazee had a student who divided $\frac{1}{6}$ by $\frac{1}{2}$ as follows:

$$\frac{1}{6} \div \frac{1}{2} = \frac{1 \div 1}{6 \div 2} = \frac{1}{3}$$

(a) Was the student just lucky, or is it always true that $\frac{p}{q} \div \frac{r}{s} = \frac{p \div r}{q \div s}$?
(b) Is this process most like the way we add, subtract, or multiply fractions?
(c) Is this way of dividing fractions well-defined? (Do we get the same answer regardless of choice of representation?)

26. By multiplying the sun protection factor (SPF) of a suntan lotion by one-half, you can determine how many hours you are protected from the sun. For example, if a lotion has an SPF of 4 you can lie in the sun for two hours.
(a) What should the SPF be if you want to spend 1½ hours in the sun with one application of lotion?
(b) If you use lotion with a SPF of 6, how many times must you apply the lotion to be protected for eight hours?

27. If a box of laundry detergent contains 20 cups, how many loads can be washed if each load requires three-quarters of a cup? (Before working the problem, estimate whether your answer should be more or less than 20.)

28. A certain outboard motor requires that a half-pint of oil be mixed with each gallon of gas. How much oil should be mixed with 3½ gallons of gas?

29. (a) Two adults and a child took a round-trip flight together between Columbus, Ohio, and Wilkes-Barre, Pennsylvania. Each adult paid full fare, and the child paid two-thirds fare. If the total (round-trip) fare was $264, how much of the fare was for the child?
(b) The flight between Columbus and Pittsburgh took 30 minutes. After a plane change, the Pittsburgh to Wilkes-Barre flight took 40 minutes. What portion of the (one-way) flight does the Columbus to Pittsburgh leg represent?
(c) The airlines deprived the child of a seat on an overbooked flight (asking that he share a seat with a parent) on the return trip from Pittsburgh to Columbus. The travelers sought a refund for the child's part of the fare for the Pittsburgh to Columbus flight. To how much refund were they entitled?

*30. Fred and Minnie Hill are each 90 years old. Joe Smith, on the other hand, is half again as old as he was when he lacked 20 years of being eight-ninths as old as he is now. How old is Joe Smith?

*31. Find the smallest positive fraction which is exactly divisible (a whole number of times) by both
(a) 4/15 and 6/25 (b) 2/3 and 5/8 (c) $\frac{2^3 \times 3 \times 5^9}{7^9 \times 11^3}$ and $\frac{3^6 \times 5^2}{7^4 \times 11}$.

32. Fractions of the form $2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{5}}}$ are called **continued fractions.**

(a) Show that the given continued fraction has a value of $\frac{157}{68}$.
Evaluate the following continued fractions:

(b) $1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}$ (c) $5 + \cfrac{1}{4 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{1}}}}$

33. An ordinary fraction such as 21/16 can be expressed as a continued fraction where all the numerators are 1's, as follows:

$$\frac{21}{16} = 1 + \frac{5}{16} = 1 + \cfrac{1}{\frac{16}{5}} = 1 + \cfrac{1}{3 + \cfrac{1}{5}}$$

Another example:

$$\frac{157}{41} = 3 + \frac{34}{41} = 3 + \cfrac{1}{\frac{41}{34}} = 3 + \cfrac{1}{1 + \cfrac{7}{34}} = 3 + \cfrac{1}{1 + \cfrac{1}{\frac{34}{7}}} = 3 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{6}{7}}}$$

$$= 3 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{\frac{7}{6}}}} = 3 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{6}}}}$$

Express each of the following as a continued fraction where the numerators are 1's:
(a) 5/6 (b) 41/33 (c) 131/15

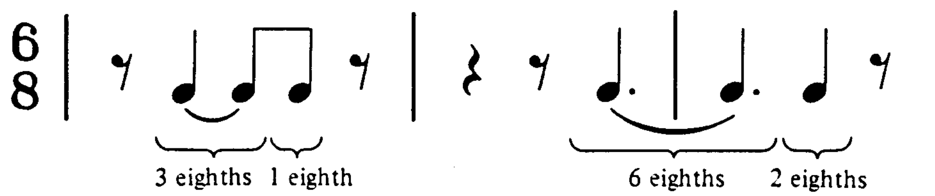
34. A man stipulated that upon his death one-half of his camels be given to his oldest child, one-third be given to his second child, and one-ninth be given to his youngest child. When he died, he had 17 camels. His lawyer, a clever woman, borrowed a camel from a neighbor to make a total of 18. She then distributed them as follows:

Oldest child	$\frac{1}{2} \times 18 =$	9
Second child	$\frac{1}{3} \times 18 =$	6
Youngest child	$\frac{1}{9} \times 18 =$	2
Total		17

She then returned the borrowed camel, and everyone was satisfied except the local arithmetic teacher. Aside from the fact that she got no camels, what bothered her?

35. If there are four quarter notes in one beat of music, how many eighth notes are there in
(a) 1 beat? (b) 3 beats? (c) 1½ beats?

36. When playing the Horn Trio of Brahms (Opus 40), Virginia, Bill, and Jim noticed this rhythm:



They noticed how the lengths of the notes were doubled. This is an example of **augmentation**, a process in which lengths of notes are multiplied, usually by 2. Another process is **diminution**, in which lengths of notes are decreased in value.

Tell what kind of note results when the length of
(a) a half note is multiplied by 2.
(b) a quarter note is multiplied by 2.
(c) an eighth note is multiplied by 2.
(d) a quarter note is multiplied by 3.
(e) a half note is multiplied by 1/2.
(f) a quarter note is multiplied by 1/2.

37. If microwaves lose half their energy for each 2 cm of food they penetrate, what fraction of the original energy will reach the bottom of a meatloaf 10 cm thick?

38. A certain cookbook recommends that when doubling a recipe for a microwave oven, the cooking time be multiplied by 1½. For example, if doubling a recipe requiring 10 minutes' cooking time, the new time is 1½ × 10 = 15 (minutes). Tell how long to cook something if we
(a) double a recipe which requires 7 minutes' cooking time.
*(b) triple a recipe which requires 12 minutes' cooking time.
(c) Will this use of fractions vanish with calculators and the metric system?