1.03 Multiplication of real numbers. — In telling what we mean by addition of real numbers, we gave, in terms of trips, interpretations of numerals such as

\[ 4 + 7, \quad 8 + 9, \quad 3 + 10, \quad \text{and} \quad 5 + 11. \]

In order to explain multiplication of real numbers, we shall give an interpretation of numerals such as

\[ 4 \times 3, \quad 8 \times 7, \quad 2 \times 5, \quad \text{and} \quad 4 \times -2. \]

This interpretation should help us find the product of each pair of real numbers.

We know that real numbers are numbers which can be used to measure trips. What are the characteristics of trips which make this possible? A trip involves

1. a change in position by a certain amount,
2. a change in position in one of two opposite directions.

In general, anything which involves an amount and one of two opposite directions can be measured by a real number. So, in looking for an interpretation of a numeral such as

\[ -4 \times 3, \]

we look for something which involves both an amount and a direction so that it can be measured by a real number.

**A PUMP, A TANK, AND A MOVIE**

Think of a pump which can pump water either into or out of a tank, and a camera which takes a movie of the tank while the pump is operating. Suppose the pump and camera are turned on together,
and that 4 gallons of water flow through the pipe each minute. After the pump and camera have run for 3 minutes, they are stopped. The film is then developed and projected on a screen. What change in the water-volume do you observe on the screen? It is easy to predict that the change observed on the screen will be a change of 12 gallons. But, will it be an increase? In order to answer this question, you need to know two more things.

One of the things you need to know is whether the flow of water was into the tank or out of the tank. Suppose that the flow was into the tank. Will the picture you see on the screen show an increase in water-volume? If your answer is 'yes' then you are probably assuming that the film is being run forward through the projector. But suppose the film were run backward through the projector. [Have you ever watched a comedy film in which a man seems to dive up out of the water and land on a diving board, or a film of a race horse running backward on a muddy track, picking up its footprints as it goes?] If the film were run backward, what change in water-volume would you see on the screen?

Now, suppose the water was being pumped out of the tank while the picture was being taken. If the film were run forward through the projector, what change in water-volume would you observe on the screen? If the film were run backward, what change would you see on the screen?

So, in order to predict what change you will observe on the screen, you need to know

(a) the amount of water per minute being pumped into or out of the tank, and

(b) the number of minutes the film is being run forward or backward.

Each of these things involves an amount and a direction, and therefore can be measured by real numbers. We can use real numbers to measure the rate at which the water is being pumped,
deciding to use positive numbers when water flows into the tank, and negative numbers when water flows out of the tank. Thus, if 4 gallons of water are being pumped into the tank each minute, we say that the rate is \(4\) gallons per minute. Explain what is meant by saying that the rate is \(-4\) gallons per minute.

Also, we can use real numbers to measure how long the film is being projected, deciding to use positive numbers when the film is run forward, and negative numbers when the film is run backward. So, if the film is running for \(3\) minutes, we know it is being projected forward (normally) for \(3\) minutes. Explain what is meant by saying that the film is running for \(-3\) minutes.

Now, how does all of this help us in interpreting a numeral such as \(4 \times 3\)?

We can think of \(4\) as measuring the rate at which water is being pumped. We can think of \(3\) as telling us how long the film is being projected. And, finally, we can think of the product \(4 \times 3\) as measuring the change in water-volume which we see on the screen. [Let's agree to use positive numbers to measure observed increases, and negative numbers to measure observed decreases.] In this case, since water is being pumped out of the tank at 4 gallons per minute and the film is being run forward for 3 minutes, the change in water-volume observed on the screen is a decrease of 12 gallons. So, the change is \(-12\) gallons. Since we agreed to think of the product as a measure of the change, we can say that

\[-4 \times -3 = -12.\]
Let's take another case:

\[ 9 \times -8 = ? \]

This number, \(9 \times -8\), measures the change in water-volume which you would observe on a screen. The first number, \(9\), measures the rate of pumping [is it filling or is it emptying?], and the second number, \(-8\), tells how long the film is being projected [is it being run forward or is it being run backward?]. Does \(9 \times -8\) measure an observed increase in water-volume or an observed decrease? Since a backward projection of the movie of a tank being filled shows a decrease in water-volume, we can say that

\[ 9 \times -8 = -72. \]

**EXERCISES**

**A.** The table below contains problems dealing with the pump-tank-movie interpretation. From each problem you can learn how to multiply a pair of real numbers. We have solved the first problem for you as a sample.

In this problem you are told that a pump is filling the tank at the rate of 4 gallons per minute. Therefore, a '4' is written in the column headed 'Pump'. You learn from the second column that the movie has been run backward for 2 minutes. Therefore, you write a '-2' in this column. Now, we ask about the change in water-volume that would be observed on the screen. Since the pump is filling the tank (as indicated by the '4'), and since the film is run backward (as indicated by the '-2'), the volume of water appears to be decreasing. So, we observe on the screen a decrease in volume of 8 gallons. The number '-8' measures this observed change. Finally, we write the corresponding multiplication statement in the last row.
Complete the table.

<table>
<thead>
<tr>
<th>Pump</th>
<th>Movie</th>
<th>Observed Change in Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Filling 4 gal. per minute</td>
<td>Running backward 2 minutes</td>
<td>decrease of 8 gallons</td>
</tr>
<tr>
<td>+4</td>
<td>-2</td>
<td>-8</td>
</tr>
<tr>
<td>Corresponding multiplication statement:</td>
<td></td>
<td>+4 x -2 = -8</td>
</tr>
<tr>
<td>2. Emptying 4 gal. per minute</td>
<td>Running forward 2 minutes</td>
<td></td>
</tr>
<tr>
<td>Corresponding multiplication statement:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. Filling 4 gal. per minute</td>
<td>Running forward 2 minutes</td>
<td></td>
</tr>
<tr>
<td>Corresponding multiplication statement:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4. Emptying 4 gal. per minute</td>
<td>Running backward 2 minutes</td>
<td></td>
</tr>
<tr>
<td>Corresponding multiplication statement:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5. Filling 8 gal. per minute</td>
<td>Running forward 3 minutes</td>
<td></td>
</tr>
<tr>
<td>Corresponding multiplication statement:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6. Emptying 8 gal. per minute</td>
<td>Running forward 3 minutes</td>
<td></td>
</tr>
<tr>
<td>Corresponding multiplication statement:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7. Emptying 8 gal. per minute</td>
<td>Running backward 3 minutes</td>
<td></td>
</tr>
<tr>
<td>Corresponding multiplication statement:</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
[Note: In the rest of the problems you are given real numbers and you should fill in the corresponding blanks.]

<table>
<thead>
<tr>
<th>Pump</th>
<th>Movie</th>
<th>Observed Change in Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.</td>
<td>-5</td>
<td>-6</td>
</tr>
</tbody>
</table>

Corresponding multiplication statement: $-5 \times -6 =$

| 9.   | +7    | -3                        |

Corresponding multiplication statement: $+7 \times -3 =$

| 10.  | -8    | 0                         |

Corresponding multiplication statement: $-8 \times 0 =$

| 11.  | $-6 \frac{1}{2}$ | -4                       |

Corresponding multiplication statement: $-6 \frac{1}{2} \times -4 =$

B. Simplify. Use the pump-tank-film interpretation as long as you need to, but try to find a short cut.

1. $+5 \times +2$
2. $+6 \times +3$
3. $+8 \frac{1}{2} \times +8$
4. $+28 \times +6$
5. $+6 \times -2$
6. $-2 \times +6$
7. $-5 \times +7$
8. $+8 \times -8$
9. $-9 \times +10$
10. $+12 \times -10$
11. $-7 \times -8$
12. $-15 \times -3$
13. $-1 \times -1$
14. $-8 \times -12$
15. $+7 \times 0$
[1.03] 

16. \(0 \times -6\)  
17. \(0 \times 0\)  
18. \(-12 \times 0\)  
19. \(-16 \times \frac{-1}{4}\)  
20. \(-100 \times \frac{5}{2}\)  
21. \(*5 \times \frac{-3}{10}\)  
22. \(*3 \times -15\)  
23. \(-7 \times *12\)  
24. \(-3 \times -8\)  
25. \(*6 \times -4\)  
26. \(-3 \times -2\)  
27. \(-17 \times *2\)  
28. \(*47 \times -58\)  
29. \(*27 \times -65\)  
30. \(*705 \times *15\)  
31. \(-86 \times -75\)  
32. \(-1.83 \times -1.81\)  
33. \(*9.65 \times -7.48\)  
34. \((+2 \times -3) \times -4\)  
35. \((+2 \times -7) \times -3\)  
36. \((+5 \times -3) \times *4\)  
37. \((+6 \times -2) \times -3\)  
38. \(*4 \times (-3 \times -7)\)  
39. \(-6 \times (+2 \times -5)\)  
40. \((+73 \times -81) \times 0\)  
41. \((+5 \times -17) \times (+3 \times 0)\)  

[Supplementary exercises are in Part E on page 1-133.]  

C. Simplify. [Be careful not to confuse addition signs with multiplication signs.]  

1. \((+5 \times -3) \times -7\)  
2. \((+3 \times -4) + -6\)  
3. \((+8 \times -3) + -5\)  
4. \((+12 + -11) \times -3\)  
5. \((+1 + +1) + *1\)  
6. \((+1 \times +1) + *1\)  
7. \((-1 + -1) + -1\)  
8. \((-1 \times -1) \times -1\)  
9. \((-4 \times -2) + (-5 \times +6)\)  
10. \((-3 \times -7) + (-8 \times -4)\)  
11. \((+71 + -11) \times (6 + -4)\)  
12. \((+8 + 0) \times (4 \times +2)\)  

D. Fill in the blanks to make true sentences.  

1. \(*5 \times \underline{\hspace{1cm}} = -20\)  
2. \(-3 \times \underline{\hspace{1cm}} = +18\)  
3. \(*7 \times \underline{\hspace{1cm}} = *21\)  
4. \(-7 \times \underline{\hspace{1cm}} = *21\)  
5. \(\underline{\hspace{1cm}} + -3 = *9\)  
6. \(\underline{\hspace{1cm}} \times -3 = *9\)  
7. \(*8 + \underline{\hspace{1cm}} = -8\)  
8. \(*8 \times \underline{\hspace{1cm}} = *8\)  
9. \(*3 \times \underline{\hspace{1cm}} = *1\)  
10. \(*3 + \underline{\hspace{1cm}} = 0\)  
11. \(*3 \times \underline{\hspace{1cm}} = 0\)  
12. \(*3 + \underline{\hspace{1cm}} = *1\)  
13. \(*8 \times -2 = *8 \times (8 \times \underline{\hspace{1cm}})\)  
14. \(-5 \times \underline{\hspace{1cm}} = *(5 \times 2)\)
EXPLORATION EXERCISES

A. Consider the table of pairs of numbers at the right. One of the interesting features of this table is that you can carry out some computations with the numbers listed in one column by doing computations with the corresponding numbers listed in the other column. For example, suppose you want to find the sum of, say, 267 and 445, two numbers listed in the lefthand column.

To simplify 

'267 + 445'

merely simplify 

'3 + 5'.

3 and 5 correspond with 267 and 445, respectively. '3 + 5' simplifies to '8', and 8 corresponds with 712. And we find that

267 + 445 = 712.

Here is another example.

\[
\begin{array}{c}
534 \quad + \\
801 \quad + \\
\hline
\end{array}
\]

? \quad 15

15 corresponds with 1335, and it turns out that 534 + 801 = 1335.

Use the table and the illustrated procedure to simplify each of the following. Check your results by carrying out the simplification directly.

1. 356 + 534  
2. 712 + 979  
3. 1068 + 267  
4. 445 + 1157  
5. 1424 - 1068  
6. 1157 - 178  

7. Is it possible to multiply pairs of numbers listed in the lefthand column by multiplying the corresponding numbers listed in the righthand column? Try simplifying '178 \times 534' that way.